Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning
Main Track

A Gödel Calculus for Linear Temporal Logic

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Abstract

We consider Gödel temporal logic (GTL), a variant of linear temporal logic based on Gödel–Dummett propositional logic. In recent work, we have shown this logic to enjoy natural semantics both as a fuzzy logic and as a superintuitionistic logic. Using semantical methods, the logic was shown to be PSPACE-complete. In this paper we provide a deductive calculus for GTL, and show this calculus to be sound and complete for the above-mentioned semantics.

1 Introduction

Despite their potential usefulness in areas such as spatio-temporal reasoning (Artemov, Davoren, and Nerode 1997) or vague temporal reasoning (Davies 1996), the combination of linear temporal logic with a modal or non-classical base tends to lead to high computational complexity (Balbiani et al. 2018) or even undecidability (Konev et al. 2006; Vidal 2021). Nevertheless, our recent work (Aguilera et al. 2022) provides a promising avenue for fuzzy temporal reasoning. There, we show that linear temporal logic (LTL) over a Gödel–Dummett base is PSPACE-complete, which is optimal for a logic interpreting classical LTL.

The methods used in (Aguilera et al. 2022) are model-theoretic and leave open the question of whether a proof-theoretic approach is possible. Here, we aim to close this gap by employing techniques used to establish completeness of an intuitionistic LTL in (Boudou, Diéguez, and Fernández-Duque 2017) with 'eventually' but without 'henceforth.' As we will see, the complications that led to omitting 'henceforth' in that work can be solved by incorporating the *dual implication* into our language, a connective that is to implication what disjunction is to conjunction. Aside from the technical advantages it afford us, it has been argued by Rauszer (1980) that dual implication is useful for reasoning with incomplete or inconsistent information.

Previously, LTL based on the intermediate logic of here-and-there (Heyting 1930) was axiomatized by Balbiani and Diéguez (2016). This logic allows for three truth values and is the basis for temporal answer set programming (Aguado et al. 2021; Aguado et al. 2013). Another combination of here-and-there, modal logic and Rauszer's co-implication has been studied in (Balbiani and Diéguez 2018).

Gödel logics and their extensions with possibility theory (Dubois, Lang, and Prade 1987) have been extensively studied in (Dellunde, Godo, and Marchioni 2011). These extensions have applications in the field of logic programming (Alsinet and Godo 2013; Blandi, Godo, and Rodríguez 2005). Aside from this, a version of the 'next' fragment of intuitionistic LTL was axiomatized by (Kojima and Igarashi 2011) and a logic with 'next' and 'eventually' (but not henceforth) by (Boudou, Diéguez, and Fernández-Duque 2017). Intuitionistic LTL with 'henceforth' has not been axiomatized, but (Boudou et al. 2021) showed that logics with the latter tense are more sensitive to choice of semantics than those without it and (Chopoghloo and Moniri 2021) provided a strongly complete infinitary calculus.

We recently showed that GTL possesses two natural semantics, corresponding to whether it is viewed as a fuzzy logic or as a superintuitionistic logic (Aguilera et al. 2022). As a fuzzy logic, propositions take values in $[0, 1]$, and truth values of compound propositions are defined using standard operations on the reals. As a superintuitionistic logic, models consist of bi-relational structures equipped with a partial order to interpret implication intuitionistically and a function to interpret the LTL tenses. We showed that the set of validities for either of these semantics coincides, and in fact coincides with the set of validities for a third class of structures we call *non-deterministic quasimodels.* Similar structures were used to prove upper complexity bounds for dynamic topological logic (Fernández-Duque 2009) and intuitionistic temporal logic (Fernández-Duque 2018). In the setting of GTL, they can be used to prove that the validity problem is decidable: while the logic does not enjoy the finite model property for either the fuzzy or the superintuitionistic semantics, it *does* enjoy the finite quasimodel property.

(Diéguez and Fernández-Duque 2018; Fernández-Duque 2012) have shown that quasimodels also come in handy in completeness proofs. There are two main reasons for this. First, as quasimodels are somewhat more flexible than proper models, it is easier to construct them. Thus our task is to construct a quasimodel falsifying a given non-derivable formula. Once constructed, we can use unwinding techniques to produce a proper bi-relational model from our quasimodel. The second advantage is that it allows us to use techniques normally available only for logics enjoying the finite model property, such as fully characterizing a given structure using a finite formula. In fact, contrary to the classical case, we will assign two characteristic formulas to each state w of our quasimodels, $\chi^+(w)$ and $\chi^-(w)$, characterizing the 'positive' and 'negative' information available in w.

Aside from these points, our completeness proof involves several stages that should be standard to those familiar with temporal logic: a *canonical model* $\mathfrak{C}$ is built, which correctly interprets $\bigcirc$ but not $\Diamond$ and $\Box$. In order to remedy this, a modified filtration $\mathfrak{C}/\Sigma$ is built, which *does* interpret all tenses correctly when restricted to the set of formulas Σ, but fails to have a deterministic successor relation. Finally, an unwinding procedure is applied to $\mathfrak{C}/\Sigma$ to obtain a bi-relational model. This unwinding is somewhat more complex than in the classical case, but fortunately we may appeal to results of (Aguilera et al. 2022) to conclude that any formula falsified in $\mathfrak{C}/\Sigma$ is also falsified in a suitable unwinding of $\mathfrak{C}/\Sigma$.

2 Syntax and Semantics

In this section we first introduce the temporal language we work with and then two possible semantics for this language: real semantics and bi-relational semantics.

Fix a countably infinite set $\mathbb{P}$ of propositional variables. Then the **Gödel temporal language** $\mathcal{L}$ is defined by the grammar (in Backus–Naur form):

$$\varphi, \psi := p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \varphi \Leftarrow \psi \mid \bigcirc\varphi \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \mathbb{P}$. Here, $\bigcirc$ is read as 'next', $\Diamond$ as 'eventually', and $\Box$ as 'henceforth'. The connective $\Leftarrow$ is dual (or co-) implication and represents the operator dual to implication (Wolter 1998). We also use $\bot$ (respectively $\top$) as a shorthand for $p \Leftarrow p$ (respectively $p \Rightarrow p$) for some fixed variable p, and we use $\neg\varphi$ (respectively $\sim\varphi$) as a shorthand for $\varphi \Rightarrow \bot$ (respectively $\top \Leftarrow \varphi$), and $\varphi \Leftrightarrow \psi$ (not related to dual implication) as a shorthand for $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.

We now introduce the first of our semantics for the Gödel temporal language: *real semantics*, which views $\mathcal{L}$ as a fuzzy logic (enriched with temporal modalities). In the definition, $[0, 1]$ denotes the real unit interval.

Definition 1 (real semantics)**.** A **flow** is a pair $\mathcal{T} = (T, S)$, where T is a set and $S\colon T \to T$ is a function. A **real valuation** on $\mathcal{T}$ is a function $V\colon \mathcal{L} \times T \to [0, 1]$ such that, for all $t \in T$, the following equalities hold.

$$\begin{array}{rcl}
V(\bot, t) & = & 0 \\
V(\varphi \wedge \psi, t) & = & \min\{V(\varphi, t), V(\psi, t)\} \\
V(\varphi \vee \psi, t) & = & \max\{V(\varphi, t), V(\psi, t)\} \\
V(\varphi \Rightarrow \psi, t) & = & \begin{cases} V(\psi, t) & \text{if } V(\varphi, t) > V(\psi, t) \\ 1 & \text{if } V(\varphi, t) \leq V(\psi, t) \end{cases} \\
V(\varphi \Leftarrow \psi, t) & = & \begin{cases} V(\varphi, t) & \text{if } V(\varphi, t) > V(\psi, t) \\ 0 & \text{if } V(\varphi, t) \leq V(\psi, t) \end{cases} \\
V(\bigcirc\varphi, t) & = & V(\varphi, S(t)) \\
V(\Diamond\varphi, t) & = & \sup_{n<\omega} V(\varphi, S^n(t)) \\
V(\Box\varphi, t) & = & \inf_{n<\omega} V(\varphi, S^n(t))
\end{array}$$

A flow $\mathcal{T}$ equipped with a valuation V is a **real (Gödel temporal) model**.

The second semantics, *bi-relational semantics*, views $\mathcal{L}$ as an intuitionistic logic (temporally enriched).

Definition 2 (bi-relational semantics)**.** A **(Gödel temporal) bi-relational frame** is a quadruple $\mathcal{F} = (W, T, \leq, S)$ where $(W, \leq)$ is a linearly ordered set and (T, S) is a flow. A **bi-relational valuation** on $\mathcal{F}$ is a function $[\![\cdot]\!] : \mathcal{L} \to 2^{W \times T}$ such that, for each $p \in \mathbb{P}$, the set $[\![p]\!]$ is *downward closed* in its first coordinate, and the following equalities hold.

$$\begin{array}{rcl}
[\![\bot]\!] & = & \varnothing \\
[\![\varphi \wedge \psi]\!] & = & [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![\varphi \vee \psi]\!] & = & [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\varphi \Rightarrow \psi]\!] & = & \{(w, t) \in W \times T \mid \forall v \leq w((v, t) \in [\![\varphi]\!] \\
 & & \qquad \text{implies } (v, t) \in [\![\psi]\!])\} \\
[\![\varphi \Leftarrow \psi]\!] & = & \{(w, t) \in W \times T \mid \exists v \geq w((v, t) \in [\![\varphi]\!] \\
 & & \qquad \text{and } (v, t) \notin [\![\psi]\!])\} \\
[\![\bigcirc\varphi]\!] & = & (\mathrm{id}_W \times S)^{-1}\, [\![\varphi]\!] \\
[\![\Diamond\varphi]\!] & = & \bigcup_{n<\omega} (\mathrm{id}_W \times S)^{-n}\, [\![\varphi]\!] \\
[\![\Box\varphi]\!] & = & \bigcap_{n<\omega} (\mathrm{id}_W \times S)^{-n}\, [\![\varphi]\!]
\end{array}$$

where $(\mathrm{id}_W \times S)$ is the function such that $(\mathrm{id}_W \times S)(w, t) = (w, S(t))$. Given $(w, t) \in W \times T$, we say that $S((w, t)) \stackrel{\text{def}}{=} (w, S(t))$. A bi-relational frame $\mathcal{F}$ equipped with a valuation $[\![\cdot]\!]$ is a **(Gödel temporal) bi-relational model**.

This semantics combines standard semantics for the implications based on $\leq$ (read downward) and for the tenses based on S: for example, $(w, t) \in [\![\Diamond\varphi]\!]$ if and only if there exists $n \geq 0$ such that $(w, S^n(t)) \in [\![\varphi]\!]$. Note that, by structural induction, the valuation of *any* $\varphi \in \mathcal{L}$ is downward closed in its first coordinate, in the sense that if $(w, t) \in [\![\varphi]\!]$ and $v \leq w$, then $(v, t) \in [\![\varphi]\!]$.

Validity of $\mathcal{L}$-formulas is defined in the usual way.

Definition 3 (validity)**.** Given a real model $\mathcal{X} = (T, S, V)$ and a formula $\varphi \in \mathcal{L}$, we say that φ is **globally true** on $\mathcal{X}$, written $\mathcal{X} \models \varphi$, if for all $t \in T$ we have $V(\varphi, t) = 1$. Given a bi-relational model $\mathcal{X} = (\mathcal{F}, [\![\cdot]\!])$ and a formula $\varphi \in \mathcal{L}$, we say that φ is **globally true** on $\mathcal{X}$, written $\mathcal{X} \models \varphi$, if $[\![\varphi]\!] = W \times T$.

If $\mathcal{X}$ is a flow or a bi-relational frame, we write $\mathcal{X} \models \varphi$ and say φ is **valid** on $\mathcal{X}$, if φ is globally true for every valuation on $\mathcal{X}$. If Ω is a class of flows, frames, or models, we say that $\varphi \in \mathcal{L}$ is **valid** on Ω if, for every $\mathcal{X} \in \Omega$, we have $\mathcal{X} \models \varphi$. If φ is not valid on Ω, it is **falsifiable** on Ω.

We define the logic $\mathsf{GTL}_{\mathbb{R}}$ to be the set of $\mathcal{L}$-formulas that are valid over the class of all flows and the logic $\mathsf{GTL}_{\mathrm{Rel}}$ to be the set of $\mathcal{L}$-formulas that are valid over the class of all bi-relational frames. The main theorem of (Aguilera et al. 2022) is that these logics coincide:

Theorem 4. $\mathsf{GTL}_{\mathbb{R}} = \mathsf{GTL}_{\mathrm{Rel}}$. *That is, for each $\varphi \in \mathcal{L}$, φ is valid over the class of real Gödel temporal models if and only if it is valid over the class of all Gödel temporal bi-relational models.*

3 The Calculus

We begin by establishing our basic calculus for logics over $\mathcal{L}$. It is obtained by adapting the standard axioms and inference rules of LTL (Lichtenstein and Pnueli 2000), as well as their dual versions.

Definition 5. The logic GTL is the least set of $\mathcal{L}$-formulas closed under the following axioms and rules.

I **All (substitution instances of) intuitionistic tautologies (see e.g. (Mints 2000))**

II **Axioms and rules of H-B logic:**

A $\varphi \Rightarrow (\psi \vee (\varphi \Leftarrow \psi))$

B $\dfrac{\varphi \Rightarrow \psi}{(\varphi \Leftarrow \theta) \Rightarrow (\psi \Leftarrow \theta)}$

C $\dfrac{\varphi \Rightarrow \psi \vee \gamma}{(\varphi \Leftarrow \psi) \Rightarrow \gamma}$

III **Linearity axioms:**

A $(\varphi \Rightarrow \psi) \vee (\psi \Rightarrow \varphi)$

B $\neg ((\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi))$

IV **Temporal axioms:**

A $\neg \bigcirc \bot$

B $\bigcirc (\varphi \vee \psi) \Rightarrow (\bigcirc\varphi \vee \bigcirc\psi)$

C $(\bigcirc\varphi \wedge \bigcirc\psi) \Rightarrow \bigcirc (\varphi \wedge \psi)$

D $\bigcirc (\varphi \Rightarrow \psi) \Leftrightarrow (\bigcirc\varphi \Rightarrow \bigcirc\psi)$

E $\Box (\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$

F $\Box (\varphi \Rightarrow \psi) \Rightarrow (\Diamond\varphi \Rightarrow \Diamond\psi)$

G $\Box\varphi \Rightarrow \varphi \wedge \bigcirc\Box\varphi$

H $\varphi \vee \bigcirc\Diamond\varphi \Rightarrow \Diamond\varphi$

I $\Box(\varphi \Rightarrow \bigcirc\varphi) \Rightarrow (\varphi \Rightarrow \Box\varphi)$

J $\Box(\bigcirc\varphi \Rightarrow \varphi) \Rightarrow (\Diamond\varphi \Rightarrow \varphi)$

V **Back–up confluence axiom:**

$\bigcirc (\varphi \Leftarrow \psi) \Rightarrow (\bigcirc\varphi \Leftarrow \bigcirc\psi)$

VI **Standard modal rules:**

A $\dfrac{\varphi,\ \varphi \Rightarrow \psi}{\psi}$ B $\dfrac{\varphi}{\bigcirc\varphi}$ C $\dfrac{\varphi}{\Box\varphi}$

Axiom group II concerns the relationship between $\Rightarrow$ and $\Leftarrow$. In particular, Axiom II.A is used in C. Rauszer's axiomatization of intuitionistic logic with co-implication (called H-B logic) (Rauszer 1974). The Gödel–Dummett axiom III.A and its order dual III.B are used to force the connectives $\Rightarrow$ and $\Leftarrow$ to be implemented on locally linear posets (i.e. posets that are a disjoint union of linear orders).

Axioms IV.C, IV.D, and IV.E are standard modal axioms (viewing $\bigcirc$ as a box-type modality). In particular they hold in any normal modal logic, although of course GTL is not itself normal by virtue of being strictly sub-classical. Axiom IV.F is a dual version of IV.E; such dual axioms are often needed in intuitionistic modal logic, since $\Diamond$ and $\Box$ are not typically interdefinable. The axioms IV.A and IV.B have to do with the passage of time being deterministic in linear temporal logic: IV.A characterises seriality and IV.B characterises (partial) functionality, thus together they constrain temporal accessibility to be a total function.

The co-inductive axiom IV.G states that if something will henceforth be the case, then it is true now and, moreover, in the next moment, it will still henceforth be the case, and IV.I is successor induction, as time is interpreted over the natural numbers. Axioms IV.H and IV.J are their duals. Note that 'henceforth' is interpreted reflexively. All rules of group VI.A are standard modal logic deduction rules, and in particular any normal modal logic is closed under these rules.

Most of the axioms are either included in the axiomatization of intuitionistic LTL (Boudou, Diéguez, and Fernández-Duque 2017) or a variant of one of them (e.g. a contrapositive). From this, we easily derive the following.

Proposition 6. *The above calculus is sound for the class of real models, as well as for the class of bi-relational models.*

Proof. The rules II.B and II.C are readily seen to preserve validity. We check Axioms V and III.B; all other rules or axioms have been shown to be sound for intuitionistic or bi-relational models in the literature (see e.g. (Balbiani et al. 2018; Rauszer 1974)).

For Axiom V, it suffices to check its validity on the class of bi-relational models. Let $\mathcal{M} = (W, T, \leq, S, [\![\cdot]\!])$ be a bi-relational model and suppose that $(w, t) \in [\![\bigcirc (\varphi \Leftarrow \psi)]\!]$; we must show that $(w, t) \in [\![\bigcirc\varphi \Leftarrow \bigcirc\psi]\!]$. From $(w, t) \in [\![\bigcirc (\varphi \Leftarrow \psi)]\!]$ we see that $(w, S(t)) \in [\![\varphi \Leftarrow \psi]\!]$; hence there is $(v, S(t)) \geq (w, S(t))$ with $(v, S(t)) \in [\![\varphi]\!] \setminus [\![\psi]\!]$. But then $(v, S(t)) \in [\![\bigcirc\varphi]\!] \setminus [\![\bigcirc\psi]\!]$, witnessing that $(w, t) \in [\![\bigcirc\varphi \Leftarrow \bigcirc\psi]\!]$.

For Axiom III.B, let us assume towards a contradiction that $\neg ((\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi))$ is not valid with respect to bi-relational models, so we can find $\mathcal{M}$ as above and $(w, t) \in \mathcal{M}$ such that $(w, t) \notin [\![\neg ((\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi))]\!]$. Therefore, there exists $(v, t) \leq (w, t)$ such that $(v, t) \in [\![(\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi)]\!]$. Therefore, $(v, t) \in [\![\varphi \Leftarrow \psi]\!]$ and $(v, t) \in [\![\varphi \Leftarrow \psi]\!]$. Therefore, there exists $(v', t) \geq (v, t)$ and $(v'', t) \geq (v, t)$ such that $(v', t) \in [\![\varphi]\!] \setminus [\![\psi]\!]$ and $(v'', t) \in [\![\psi]\!] \setminus [\![\varphi]\!]$. Since $(W, \leq)$ is a linear order, either $(v', t) \leq (v'', t)$ or $(v', t) \geq (v'', t)$. In the former case we get that $(v', t) \in [\![\psi]\!]$ and in the latter case we get that $(v'', t) \in [\![\varphi]\!]$; in any case we reach a contradiction. $\Box$

Our main objective is to show that our calculus is indeed complete; proving this will take up the remainder of this paper.

As we show next, we can also derive the converses of some of these axioms. Below, for a set of formulas Γ we define $\bigcirc\Gamma = \{\bigcirc\varphi : \varphi \in \Gamma\}$, and empty conjunctions and disjunctions are defined by $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$.

Lemma 7. *Let $\varphi \in \mathcal{L}$ and $\Gamma \subseteq \mathcal{L}$ be finite. Then the following formulas belong to* GTL.

1. $\bigcirc \bigvee \Gamma \Leftrightarrow \bigvee \bigcirc\Gamma$
2. $\bigcirc \bigwedge \Gamma \Leftrightarrow \bigwedge \bigcirc\Gamma$
3. $\Diamond\varphi \Rightarrow \varphi \vee \bigcirc\Diamond\varphi$
4. $\varphi \wedge \bigcirc\Box\varphi \Rightarrow \Box\varphi$
5. $(\varphi \Leftarrow \varphi) \Rightarrow \psi$
6. $(\varphi \Leftarrow \psi) \Rightarrow \varphi$

4 Labelled Systems and Quasimodels

Quasimodels will be a central tool in our completeness proof. These were originally introduced in (Fernández-Duque 2009) for *dynamic topological logic*, a classical predecessor of *intuitionistic temporal logic*, for which quasimodels were also used in (Fernández-Duque 2018). In this section we will introduce labelled spaces, labelled systems, and finally, quasimodels. Quasimodels can be viewed as a sort of nondeterministic generalisation of bi-relational models. Quasimodels are a great advantage to us since GTL has the finite quasimodel property (any falsifiable formula is falsifiable in a finite quasimodel), despite not having the finite model property for either the real or the bi-relational semantics (Aguilera et al. 2022).

Definition 8. Let $\Sigma \subseteq \mathcal{L}$ be closed under subformulas and $\Phi^+, \Phi^- \subseteq \Sigma$. We say that the pair $\Phi = (\Phi^+, \Phi^-)$ is a **two-sided Σ-type** if:

1. $\Phi^- \cap \Phi^+ = \varnothing$,
2. if $\varphi \wedge \psi \in \Phi^+$, then $\varphi, \psi \in \Phi^+$,
3. if $\varphi \wedge \psi \in \Phi^-$, then $\varphi \in \Phi^-$ or $\psi \in \Phi^-$,
4. if $\varphi \vee \psi \in \Phi^+$, then $\varphi \in \Phi^+$ or $\psi \in \Phi^+$,
5. if $\varphi \vee \psi \in \Phi^-$, then $\varphi, \psi \in \Phi^-$,
6. if $\varphi \Rightarrow \psi \in \Phi^+$, then $\varphi \in \Phi^-$ or $\psi \in \Phi^+$,
7. if $\varphi \Rightarrow \psi \in \Phi^-$, then $\psi \in \Phi^-$,
8. if $\varphi \Leftarrow \psi \in \Phi^-$, then $\varphi \in \Phi^-$ or $\psi \in \Phi^+$,
9. if $\varphi \Leftarrow \psi \in \Phi^+$, then $\varphi \in \Phi^+$,
10. if $\Diamond\varphi \in \Phi^-$, then $\varphi \in \Phi^-$,
11. if $\Box\varphi \in \Phi^+$, then $\varphi \in \Phi^+$.

If moreover $\Sigma = \Phi^- \cup \Phi^+$, we may say that Φ is **saturated**. The set of saturated two-sided Σ-types will be denoted $\mathbb{T}_\Sigma$. Given $\Phi, \Psi \in \mathbb{T}_\Sigma$, we write

$$\Phi \leq_\Sigma \Psi \text{ if and only if } \Phi^- \subseteq \Psi^- \text{ and } \Phi^+ \supseteq \Psi^+.$$

Often we want Σ to be finite, in which case we write $\Sigma \Subset \mathcal{L}$ to indicate that $\Sigma \subseteq \mathcal{L}$ and Σ is finite and closed under subformulas. We remark that if $\Phi \in \mathbb{T}_\Sigma$, then $\Phi^- = \Sigma \setminus \Phi^+$ (and vice-versa), but it is convenient to view Φ as a pair, since both the 'positive' and 'negative' information will play an important role.

A partially ordered set $(A, \leq)$ is **locally linear** if it is a disjoint union of linear posets. If $a, b \in A$, we write $a \lesseqgtr b$ if $a \leq b$ or $b \leq a$. We call the set $\{b \in A : b \lesseqgtr a\}$ the **linear component** of a; by assumption, linear components partition A.

Definition 9. Let $\Sigma \subseteq \mathcal{L}$ be closed under subformulas. A **Σ-labelled space** is a triple $\mathcal{W} = (|\mathcal{W}|, \leq_\mathcal{W}, \ell_\mathcal{W})$, where $(|\mathcal{W}|, \leq_\mathcal{W})$ is a locally linear poset and $\ell\colon |\mathcal{W}| \to \mathbb{T}_\Sigma$ a monotone function, in the sense that

$$w \leq_\mathcal{W} v \text{ implies } \ell_\mathcal{W}(w) \leq_\Sigma \ell_\mathcal{W}(v),$$

and such that for all $w \in |\mathcal{W}|$:

- whenever $\varphi \Rightarrow \psi \in \ell^-_\mathcal{W}(w)$, there is $v \leq_\mathcal{W} w$ such that $\varphi \in \ell^+_\mathcal{W}(v)$ and $\psi \in \ell^-_\mathcal{W}(v)$;
- whenever $\varphi \Leftarrow \psi \in \ell^+_\mathcal{W}(w)$, there is $v \geq_\mathcal{W} w$ such that $\varphi \in \ell^+_\mathcal{W}(v)$ and $\psi \in \ell^-_\mathcal{W}(v)$.

The Σ-labelled space $\mathcal{W}$ **falsifies** $\varphi \in \mathcal{L}$ if $\varphi \in \ell_\mathcal{W}(w)^-$ for some $w \in W$. The **height** of $\mathcal{W}$ is the supremum of all n such that there is a chain $w_1 <_\mathcal{W} w_2 <_\mathcal{W} \ldots <_\mathcal{W} w_n$.

If $\mathcal{W}$ is a labelled space, elements of $|\mathcal{W}|$ will sometimes be called **worlds**. When clear from context we will omit subscripts and write, for example, $\leq$ instead of $\leq_\mathcal{W}$.

Recall that a subset S of a poset $(P, \leq)$ is **convex** if $s \in S$ whenever $a, b \in S$ and $a \leq s \leq b$. A **convex relation** between posets $(A, \leq_A)$ and $(B, \leq_B)$ is a binary relation $R \subseteq A \times B$ such that for each $x \in A$ the image set $\{y \in B \mid x \ R \ y\}$ is convex with respect to $\leq_B$, and for each $y \in B$ the preimage set $\{x \in A \mid x \ R \ y\}$ is convex with respect to $\leq_A$. The relation R is **fully confluent** if it validates the four following conditions:

Forth–down
if $x \leq_A x' \ R \ y'$ there is y such that $x \ R \ y \leq_B y'$,

Forth–up
if $x' \geq_A x \ R \ y$ there is y' such that $x' \ R \ y' \geq_B y$,

Back–down
if $x' \ R \ y' \geq_B y$ there is x such that $x' \geq_A x \ R \ y$,

Back–up
if $x \ R \ y \leq_B y'$ there is x' such that $x \leq_A x' \ R \ y'$.

In other words, R is fully confluent if $\leq_A \circ R = R \circ \leq_B$ and $\geq_A \circ R = R \circ \geq_B$.

Definition 10. Let $\Sigma \subseteq \mathcal{L}$ be closed under subformulas. Suppose that $\Phi, \Psi \in \mathbb{T}_\Sigma$. The ordered pair (Φ, Ψ) is **sensible** if it satisfies the following conditions:

1. If $\bigcirc\varphi \in \Phi^+$, then $\varphi \in \Psi^+$.
2. If $\bigcirc\varphi \in \Phi^-$, then $\varphi \in \Psi^-$.
3. If $\Diamond\varphi \in \Phi^+$, then $\varphi \in \Phi^+$ or $\Diamond\varphi \in \Psi^+$.
4. If $\Diamond\varphi \in \Phi^-$, then $\varphi \in \Phi^-$ and $\Diamond\varphi \in \Psi^-$.
5. If $\Box\varphi \in \Phi^+$, then $\varphi \in \Phi^+$ and $\Box\varphi \in \Psi^+$.
6. If $\Box\varphi \in \Phi^-$, then $\varphi \in \Phi^-$ or $\Box\varphi \in \Psi^-$.

A pair (w, v) of worlds in a labelled space $\mathcal{W}$ is **sensible** if $(\ell(w), \ell(v))$ is sensible. A relation $S \subseteq |\mathcal{W}| \times |\mathcal{W}|$ is **sensible** if every pair in S is sensible. Further, S is **ω-sensible** if

- whenever $\Diamond\varphi \in \ell^+_\mathcal{W}(w)$, there are $n \geq 0$ and v such that $w \ S^n \ v$ and $\varphi \in \ell^+_\mathcal{W}(v)$;
- whenever $\Box\varphi \in \ell^-_\mathcal{W}(w)$, there are $n \geq 0$ and v such that $w \ S^n \ v$ and $\varphi \in \ell^-_\mathcal{W}(v)$.

Recall that a binary relation is said to be **serial** if every element of the domain is related to some element of the codomain.

A **labelled system** is a labelled space $\mathcal{W}$ equipped with a serial, fully confluent, convex sensible relation $R_\mathcal{W} \subseteq |\mathcal{W}| \times |\mathcal{W}|$. If moreover $R_\mathcal{W}$ is ω-sensible, we say that $\mathcal{W}$ is a **Σ-quasimodel.**

Any bi-relational model can be regarded as a Σ-quasimodel: If $\mathcal{X} = (W, T, \leq, S, [\![\cdot]\!])$ is a bi-relational model and $x \in W \times T$, we can assign a Σ-type $\ell_\mathcal{X}(x)$ to x given by

$$\ell_\mathcal{X}(x)^+ = \{\psi \in \Sigma \mid x \in [\![\psi]\!]\}$$
$$\ell_\mathcal{X}(x)^- = \{\psi \in \Sigma \mid x \notin [\![\psi]\!]\}.$$

Note that this assignment of types is $\leq_\Sigma$-monotone. We also set $R_\mathcal{X} = \{((w,t),(w,S(t))) \mid w \in W, t \in T\}$; it is obvious that $R_\mathcal{X}$ is ω-sensible. Henceforth we will tacitly identify $\mathcal{X}$ with its associated Σ-quasimodel.

The following is proved in (Aguilera et al. 2022).

Theorem 11. *Given $\varphi \in \mathcal{L}$, the following are equivalent:*

1. *φ is falsifiable.*
2. *φ is falsifiable in a quasimodel.*
3. *φ is falsifiable in a finite quasimodel.*

5 The Canonical Model

In this section we construct a standard canonical model for GTL. In the presence of $\Diamond$ and $\Box$, the standard canonical model is only a labelled system, rather than a proper bi-relational model. Nevertheless, it will be a useful ingredient in our completeness proof. Since we are working over an intermediate logic, the role of maximal consistent sets will be played by complete types, as defined below. The notation $\vdash$ always refers to derivability in the calculus defined in Section 3. Below, recall that by convention, $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$.

Definition 12. Given two sets of formulas $\Gamma, \Delta \subseteq \mathcal{L}$, we say that Δ is a consequence of Γ, denoted by $\Gamma \vdash \Delta$, if there exist finite (possibly empty) $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash \bigwedge \Gamma' \Rightarrow \bigvee \Delta'$ (i.e. $\bigwedge \Gamma' \Rightarrow \bigvee \Delta' \in \mathsf{GTL}$).

We say that a pair of sets $\Phi = (\Phi^+, \Phi^-) \in \mathcal{L} \times \mathcal{L}$ is **consistent** if $\Phi^+ \not\vdash \Phi^-$. A saturated, consistent pair is a **complete type**. The set of complete types will be denoted $\mathbb{T}_\infty$.

Note that we are using the standard interpretation of $\Gamma \vdash \Delta$ in Gentzen-style calculi. When working within a turnstile, we will follow the usual proof-theoretic conventions of writing Γ, Δ instead of $\Gamma \cup \Delta$, and writing φ instead of $\{\varphi\}$. Observe that there is no clash in terminology regarding the use of the word *type*.

Lemma 13. *If Φ is a complete type then Φ is a saturated two-sided $\mathcal{L}$-type.*

Proof. Let Φ be a complete type. Observe that Φ is already saturated by definition, so it remains to check that it satisfies all conditions of Definition 8. Condition 1 follows from the consistency of Φ. For condition 10 we use Axiom IV.H: if $\Diamond\varphi \in \Phi^-$ and $\varphi \in \Phi^+$ we would have that Φ is inconsistent; hence $\varphi \in \Phi^-$. Condition 11 is proved using Axiom IV.G. The remaining conditions are left to the reader. □

As with maximal consistent sets, complete types satisfy a Lindenbaum property. Below, if (Γ, Δ) and (Γ', Δ') are pairs of sets of formulas, we say that (Γ', Δ') **extends** (Γ, Δ) if $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.

Lemma 14 (Lindenbaum lemma)**.** *Let $\Gamma, \Delta \subseteq \mathcal{L}$. If $\Gamma \not\vdash \Delta$, then there exists a complete type Φ extending (Γ, Δ).*

Proof. The proof is standard, but we provide a sketch. Let $\varphi \in \mathcal{L}$. Note that either $\Gamma, \varphi \not\vdash \Delta$ or $\Gamma \not\vdash \Delta, \varphi$, for otherwise by a cut rule (which is intuitionistically derivable) we would have $\Gamma \vdash \Delta$. Thus we can add φ to Γ or to Δ, and by repeating this process for each element of $\mathcal{L}$ (or using Zorn's lemma) we can find a suitable Φ. □

Before defining the canonical model, recall that for a set of formulas Γ, we have $\bigcirc\Gamma \stackrel{\text{def}}{=} \{\bigcirc\varphi \mid \varphi \in \Gamma\}$. We also define

$$\ominus\Gamma \stackrel{\text{def}}{=} \{\varphi \mid \bigcirc\varphi \in \Gamma\}.$$

Given a set A, let $\mathbb{I}_A$ denote the identity function on A. The canonical model $\mathfrak{C}$ is defined as the labelled structure

$$\mathfrak{C} = (|\mathfrak{C}|, \leq_\mathfrak{C}, S_\mathfrak{C}, \ell_\mathfrak{C}),$$

where $|\mathfrak{C}| = \mathbb{T}_\infty$ is the set of complete types, $\Phi \leq_\mathfrak{C} \Psi$ if $\Phi \leq_\mathcal{L} \Psi$ (i.e., if $\Phi^- \subseteq \Psi^-$ and $\Phi^+ \supseteq \Psi^+$), $S_\mathfrak{C}(\Phi) = (\ominus\Phi^+, \ominus\Phi^-)$, and $\ell_\mathfrak{C}(\Phi) = \Phi$. We will usually omit writing $\ell_\mathfrak{C}$, as it has no effect on its argument.

Next we show that $\mathfrak{C}$ is an $\mathcal{L}$-labelled system. We begin by showing that it is based on a labelled space.

Lemma 15. $(|\mathfrak{C}|, \leq_\mathfrak{C}, \ell_\mathfrak{C})$ *is a $\mathcal{L}$-labelled space.*

Proof. We know that $\leq_\mathcal{L}$ is a partial order and restrictions of partial orders are partial orders, so $\leq_\mathfrak{C}$ is a partial order. Moreover, $\ell_\mathfrak{C}$ is the identity, so $\Phi \leq_\mathfrak{C} \Psi$ implies that $\ell_\mathfrak{C}(\Phi) \leq_\mathcal{L} \ell_\mathfrak{C}(\Psi)$.

To prove that $(|\mathfrak{C}|, \leq_\mathfrak{C})$ is locally linear, assume towards a contradiction that it is not. We consider two cases:

1. There exist Φ, Ψ and Θ such that $\Phi \leq_\mathcal{L} \Psi$ and $\Phi \leq_\mathcal{L} \Theta$, but $\Psi \not\leq_\mathcal{L} \Theta$ and $\Theta \not\leq_\mathcal{L} \Psi$. By definition, there exist two formulas $\varphi \in \Theta^+ \setminus \Psi^+$ and $\psi \in \Psi^+ \setminus \Theta^+$. It is easy to see that $\varphi \Rightarrow \psi \notin \Theta^+$ and $\psi \Rightarrow \varphi \notin \Psi^+$. This would imply that Axiom III.A does not belong to Φ^+—a contradiction.
2. There exist Φ, Ψ and Θ such that $\Phi \geq_\mathcal{L} \Psi$ and $\Phi \geq_\mathcal{L} \Theta$, but $\Psi \not\geq_\mathcal{L} \Theta$ and $\Theta \not\geq_\mathcal{L} \Psi$. Then it is easy to see that there exist two formulas $\varphi \in \Psi^+ \setminus \Theta^+$ and $\psi \in \Theta^+ \setminus \Psi^+$ such that $\varphi \Leftarrow \psi \in \Psi^+$ and $\psi \Leftarrow \varphi \in \Theta^+$. From $\Phi \geq_\mathcal{L} \Psi$, $\Phi \geq_\mathcal{L} \Theta$, and some intuitionistic reasoning we conclude that $(\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi) \in \Phi^+$, which contradicts Axiom III.B.

We finish by considering the conditions on $\Rightarrow$ and $\Leftarrow$. Let us consider $\Phi \in |\mathfrak{C}|$:

- If $\varphi \Rightarrow \psi \in \Phi^-$ then, by Condition 7 of Definition 8, $\psi \in \Phi^-$. Let us define $u = (\Phi^+ \cup \{\varphi\}, \{\psi\})$, and let us assume by contradiction that u is not consistent. This means that there exists $\gamma \in \Phi^+$ such that $\gamma \wedge \varphi \Rightarrow \psi \in \mathsf{GTL}$. By propositional reasoning, $\gamma \Rightarrow (\varphi \Rightarrow \psi) \in \mathsf{GTL}$. Since $\gamma \in \Phi^+$ and Φ is consistent, $\varphi \Rightarrow \psi \notin \Phi^-$—a contradiction. Therefore, u is consistent and, by Lemma 14, it can be extended to a complete type Ψ. From the definition of u we can conclude that $\Psi \leq_\mathcal{L} \Phi$, $\varphi \in \Psi^+$, and $\psi \in \Psi^-$ as required.
- If $\varphi \Leftarrow \psi \in \Phi^+$ then, by Condition 9, $\varphi \in \Phi^+$. Let us define $u = (\{\varphi\}, \Phi^- \cup \{\psi\})$, and let us assume by contradiction that u is not consistent. This means that there exists $\gamma \in \Phi^-$ such that $\varphi \Rightarrow \psi \vee \gamma \in \mathsf{GTL}$. By Rule II.C, we get $(\varphi \Leftarrow \psi) \Rightarrow \gamma \in \mathsf{GTL}$. Since $\gamma \in \Phi^-$, we deduce that $\varphi \Leftarrow \psi \notin \Phi^+$—a contradiction. By Lemma 14, u can be extended to a complete type Ψ. It is easy to check that $\Phi \leq_\mathcal{L} \Psi$, $\varphi \in \Psi^+$, and $\psi \in \Psi^-$ as required. □

Lemma 16. $S_\mathfrak{C} \colon |\mathfrak{C}| \to |\mathfrak{C}|$ *is well defined.*

Proof. Let $\Phi \in |\mathfrak{C}|$ and $\Psi = S_\mathfrak{C}(\Phi)$; we must check that $\Psi \in |\mathfrak{C}| = \mathbb{T}_\infty$. Recall that $\Psi^+ = \ominus\Phi^+$ and $\Psi^- = \ominus\Phi^-$. To see that Ψ is saturated, let $\varphi \in \mathcal{L}$ be so that $\varphi \notin \Psi^-$. It follows that $\bigcirc\varphi \notin \Phi^-$, but Φ is saturated, so $\bigcirc\varphi \in \Phi^+$ and thus $\varphi \in \Psi^+$. Since φ was arbitrary, $\Psi^- \cup \Psi^+ = \mathcal{L}$. Next we check that Ψ is consistent. If not, let $\Gamma \subseteq \Psi^+$ and $\Delta \subseteq \Psi^-$ be finite and such that $\bigwedge \Gamma \Rightarrow \bigvee \Delta \in \mathsf{GTL}$. Using VI.B and IV.D we see that $\bigcirc \bigwedge \Gamma \Rightarrow \bigcirc \bigvee \Delta \in \mathsf{GTL}$, which in view of Lemma 7 implies that $\bigwedge \bigcirc\Gamma \Rightarrow \bigvee \bigcirc\Delta \in \mathsf{GTL}$ as well. But $\bigcirc\Gamma \subseteq \Phi^+$ and $\bigcirc\Delta \subseteq \Phi^-$, contradicting the fact that Φ is consistent. We conclude that $\Psi \in |\mathfrak{C}|$. □

Lemma 17. *$S_\mathfrak{C}$ is fully confluent.*

Proof. We check the four conditions:

Forth–down, forth–up: Let Φ, Ψ be such that $\Phi \leq_\mathcal{L} \Psi$. Since $S_\mathfrak{C}$ is a function, these properties amount to showing that $S_\mathfrak{C}(\Phi) \leq_\mathcal{L} S_\mathfrak{C}(\Psi)$. If $\varphi \in \big(S_\mathfrak{C}(\Psi)\big)^+$ then $\bigcirc\varphi \in \Psi^+$, which since $\Phi \leq_\mathcal{L} \Psi$ implies that $\bigcirc\varphi \in \Phi^+$ and hence $\varphi \in \big(S_\mathfrak{C}(\Phi)\big)^+$. Similarly we can check that if $\varphi \in \big(S_\mathfrak{C}(\Phi)\big)^-$ then $\varphi \in \big(S_\mathfrak{C}(\Psi)\big)^-$, so that $S_\mathfrak{C}(\Phi) \leq_\mathcal{L} S_\mathfrak{C}(\Psi)$, as needed.

Back–up: Let Φ, Ψ, and Θ be such that $\Psi = S_\mathfrak{C}(\Phi)$ and $\Psi \leq_\mathcal{L} \Theta$, and let us define $u = (\ominus\Theta^+, \Phi^- \cup \ominus\Theta^-)$. Assume toward a contradiction that u is not consistent. Therefore, there exist $\gamma \in \Phi^-$, $\varphi \in \Theta^+$, and $\psi \in \Theta^-$ such that $\bigcirc\varphi \Rightarrow (\gamma \vee \bigcirc\psi) \in \mathsf{GTL}$. By Lemma 7(6, we get that $(\bigcirc\varphi \Leftarrow \bigcirc\psi) \Rightarrow \gamma \in \mathsf{GTL}$. Since $\gamma \in \Phi^-$ and Φ is consistent and saturated, we have $\bigcirc\varphi \Leftarrow \bigcirc\psi \in \Phi^-$. By Axiom V, $\bigcirc(\varphi \Leftarrow \psi) \Rightarrow (\bigcirc\varphi \Leftarrow \bigcirc\psi) \in \Phi^+$. Since $\bigcirc\varphi \Leftarrow \bigcirc\psi \in \Phi^-$ and Φ is consistent and saturated, we have $\bigcirc(\varphi \Leftarrow \psi) \in \Phi^-$. Since $\Psi = S_\mathfrak{C}(\Phi)$, we have $\varphi \Leftarrow \psi \in \Psi^- \subseteq \Theta^-$. Thus either $\varphi \in \Theta^-$ or $\psi \in \Theta^+$—a contradiction. By Lemma 14, u can be extended to a complete type Υ, which satisfies $\Phi \leq_\mathcal{L} \Upsilon$ and $\Theta = S_\mathfrak{C}(\Upsilon)$, as required.

Back–down: Let Φ, Ψ, and Θ be such that $\Psi = S_\mathfrak{C}(\Phi)$ and $\Theta \leq_\mathcal{L} \Psi$ and define $u = (\Phi^+ \cup \ominus\Theta^+, \ominus\Theta^-)$ and assume that u is not consistent. This means that there exists $\gamma \in \Phi^+$, $\varphi \in \Theta^+$, and $\psi \in \Theta^-$ such that $\gamma \wedge \bigcirc\varphi \Rightarrow \bigcirc\psi \in \mathsf{GTL}$. By propositional reasoning, $\gamma \Rightarrow (\bigcirc\varphi \Rightarrow \bigcirc\psi) \in \mathsf{GTL}$, so $\bigcirc\varphi \Rightarrow \bigcirc\psi \in \Phi^+$. By Axiom IV.D, $\bigcirc(\varphi \Rightarrow \psi) \in \Phi^+$. Since $\Psi = S_\mathfrak{C}(\Phi)$, we have $\varphi \Rightarrow \psi \in \Psi^+ \subseteq \Theta^+$. Therefore $\psi \in \Theta^+$—a contradiction. By Lemma 14, u can be extended to a complete type Υ. It can be checked that $\Upsilon \leq_\mathcal{L} \Phi$ and $\Theta = S_\mathfrak{C}(\Upsilon)$, as required. □

Lemma 18. *$S_\mathfrak{C}$ is a convex relation.*

Proof. Since $S_\mathfrak{C}$ is a function, images of points are singletons, hence automatically convex. Thus we need only prove that preimages are convex. We proceed by contradiction. Let us take $\Upsilon \in |\mathfrak{C}|$ and let us define $\mathcal{A} = S_\mathfrak{C}^{-1}(\Upsilon)$ and let us assume that $\mathcal{A}$ is not convex. This means that there exist $\Phi, \Psi, \Theta \in |\mathfrak{C}|$ such that $\Phi, \Psi \in \mathcal{A}$ and $\Phi \leq_\mathcal{L} \Theta \leq_\mathcal{L} \Psi$, but $\Theta \notin \mathcal{A}$. Since $\Phi, \Psi \in \mathcal{A}$ and $\Theta \notin \mathcal{A}$, it follows that $S_\mathfrak{C}(\Phi) = S_\mathfrak{C}(\Psi) = \Upsilon \neq S_\mathfrak{C}(\Theta)$. We consider two cases:

- there exists $\bigcirc\varphi \in \Theta^+$ such that $\varphi \notin \Upsilon^+$. Then $\bigcirc\varphi \in \Phi^+$, so $\varphi \in \Upsilon^+$—a contradiction.
- there exists $\bigcirc\varphi \in \Theta^-$ such that $\varphi \notin \Upsilon^-$. Then $\bigcirc\varphi \in \Psi^-$ so $\varphi \in \Upsilon^-$—a contradiction. □

Lemma 19. *$S_\mathfrak{C}$ is sensible.*

Proof. Let us consider Φ, Ψ such that $\Psi = S_\mathfrak{C}(\Phi)$. We consider the conditions for (Φ, Ψ) to be sensible.

If $\bigcirc\varphi \in \Phi^+$ then $\varphi \in \Psi^+$ by the definition of $S_\mathfrak{C}$. If $\bigcirc\varphi \notin \Phi^+$ then $\bigcirc\varphi \in \Phi^-$ and, by definition, $\varphi \in \Psi^-$.

If $\Diamond\varphi \in \Phi^+$ and $\varphi \notin \Phi^+$, it follows that $\varphi \in \Phi^-$. By Lemma 7, $\Diamond\varphi \Rightarrow \varphi \vee \bigcirc\Diamond\varphi \in \mathsf{GTL}$, so we cannot have that $\bigcirc\Diamond\varphi \in \Phi^-$, and hence $\bigcirc\Diamond\varphi \in \Phi^+$, so that $\Diamond\varphi \in \Psi^+$. Similarly, if $\Diamond\varphi \in \Phi^-$ we have that $\bigcirc\Diamond\varphi \in \Phi^-$, for otherwise we obtain a contradiction from IV.H. Therefore, $\Diamond\varphi \in \Psi^-$ as well.

If $\Box\varphi \in \Phi^+$ then, by Axiom IV.G we get $\varphi, \bigcirc\Box\varphi \in \Phi^+$. Since $\Psi = S_\mathfrak{C}(\Phi)$, we get $\Box\varphi \in \Psi^+$. Conversely, assume that $\Box\varphi \in \Phi^-$. By Lemma 7, $\varphi \wedge \bigcirc\Box\varphi \in \Phi^-$, so either $\varphi \in \Phi^-$ or $\bigcirc\Box\varphi \in \Phi^-$ (giving in the second case $\Box\varphi \in \Psi^-$). In either case we reach the desired conclusion. □

We remark the general fact that given a Σ_1-labelled system and a subformula-closed $\Sigma_2 \subseteq \Sigma_1$, one can restrict the labelling to Σ_2 in the natural way (by replacing its output at any point by its intersection with Σ_2). Doing so yields a Σ_2-labelled system. This is easily verifiable from the definitions.

Proposition 20. *The canonical model $\mathfrak{C}$ is an $\mathcal{L}$-labelled system. Restricting the labelling to any subformula-closed $\Sigma \subseteq \mathcal{L}$ yields a Σ-labelled system.*

Proof. For the first claim, we need for the following three properties to hold: 1. $(|\mathfrak{C}|, \leq_\mathfrak{C}, \ell_\mathfrak{C})$ is a labelled space; 2. $S_\mathfrak{C}$ is a serial, fully confluent, convex sensible relation; and 3. $\ell_\mathfrak{C}$ has $\mathbb{T}_\mathcal{L}$ as its codomain. The first item is Lemma 15. $S_\mathfrak{C}$ is serial since it is a well defined function by Lemma 16, and it is a fully confluent, convex, sensible relation by Lemmas 17, 18, and 19. Finally, if $\Phi \in |\mathfrak{C}|$ then $\ell_\mathfrak{C}(\Phi) = \Phi$, which is an element of $\mathbb{T}_\mathcal{L}$ by Lemma 13.

The second claim follows from the observation preceding the proposition. □

6 The Canonical Quasimodel

In this section we describe a finite quotient $\mathfrak{C}/\Sigma$ of the canonical labelled system $\mathfrak{C}$ constructed in Section 5, and we show that $\mathfrak{C}/\Sigma$ is a Σ-labelled system. Later, in Section 8, we will show that $\mathfrak{C}/\Sigma$ is also ω-sensible and thus a quasimodel.

We obtain $\mathfrak{C}/\Sigma$ from $\mathfrak{C}$ in two steps. First, we will take a bisimulation quotient to obtain a finite Σ-labelled space equipped with a fully confluent sensible relation. The second step will be to extend the sensible relation to be convex, yielding a finite Σ-labelled system.

We describe the quotient explicitly, noting afterwards that it is a particular type of bisimulation quotient. The assumption that Σ is finite is only needed at the end: if Σ is finite then $\mathfrak{C}/\Sigma$ will be finite. So for now let Σ be any subformula-closed subset of $\mathcal{L}$, and let $\mathfrak{C} = (|\mathfrak{C}|, \leq_\mathfrak{C}, S_\mathfrak{C}, \ell_\mathfrak{C})$ be the canonical labelled system, which by Proposition 20 is a Σ-labelled system when $\ell_\mathfrak{C}$ is restricted to a Σ-labelling, which we assume (and henceforth denote by ℓ).

For $\Phi \in |\mathfrak{C}|$, define $L(\Phi) = \big\{\ell(\Psi) \mid \Psi \lesseqgtr \Phi\big\}$. We define the binary relation $\sim$ on $|\mathfrak{C}|$ by

$$\Phi \sim \Psi \iff (\ell(\Phi), L(\Phi)) = (\ell(\Psi), L(\Psi)).$$

If Σ is finite, then clearly $|\mathfrak{C}|/{\sim}$ is finite.

Note that $\sim$ is the largest relation that is simultaneously a bisimulation with respect to the relations $\leq$ and $\geq$, with Σ

treated as the set of atomic propositions that bisimilar worlds must agree on.

Now define a partial order $\leq_Q$ on the equivalence classes $|\mathfrak{C}|/\sim$ of $\sim$ by

$$[\Phi] \leq_Q [\Psi] \iff L(\Phi) = L(\Psi) \text{ and } \ell(\Phi) \geq \ell(\Psi),$$

noting that this is well-defined and is indeed a partial order.

Since each set $L(\Phi)$ can be linearly ordered by inclusion and $\ell(\Phi) \in L(\Phi)$, the poset $(|\mathfrak{C}|/\sim, \leq_Q)$ is a disjoint union of linear orders. By defining ℓ_Q by

$$\ell_Q([\Phi]) = \ell(\Phi)$$

we obtain a Σ-labelled space $(|\mathfrak{C}|/\sim, \leq_Q, \ell_Q)$; it is not hard to check that this labelling is inversely monotone and that the clauses for $\Rightarrow$ and $\Leftarrow$ hold with this labelling.

Now define the binary relation R_Q on $|\mathfrak{C}|/\sim$ to be the smallest relation such that $[\Phi]\ R_Q\ [S(\Phi)]$, for all $\Phi \in |\mathfrak{C}|$.

Lemma 21. *The relation R_Q is fully confluent and sensible.*

Proof. It is clear that R_Q is sensible. For confluence, suppose $[\Phi]\ R_Q\ [S(\Phi)]$. To see that the forth–up condition holds, suppose further that $[\Phi] \leq_Q [\Psi]$. Then as $\ell(\Phi) \in L(\Phi) = L(\Psi)$ there is some $\Theta \geq \Phi$ with $[\Psi] = [\Theta]$. Then we have $[\Theta]\ R_Q\ [S(\Theta)]$ and $[S(\Phi)] \leq_Q [S(\Theta)]$, as required for the forth–up condition. The proofs of the remaining three confluence conditions are entirely analogous. $\square$

As promised, we now have a Σ-labelled space equipped with a fully confluent sensible relation. We now transform this labelled space into a Σ-labelled system by making the additional relation convex by fiat.

Define R_Q^+ by $X\ R_Q^+\ Y$ if and only if there exist $X_1 \leq_Q X \leq_Q X_2$ and $Y_1 \leq_Q Y \leq_Q Y_2$ such that $X_2\ R_Q\ Y_1$ and $X_1\ R_Q\ Y_2$. Now define ${}^{\mathfrak{C}}/_{\Sigma} = (|\mathfrak{C}|/\sim, \leq_Q, R_Q^+, \ell_Q)$.

Lemma 22. *The structure ${}^{\mathfrak{C}}/_{\Sigma}$ is a Σ-labelled system.*

Proof. We already know that $(|\mathfrak{C}|/\sim, \leq_Q, \ell_Q)$ is a Σ-labelled space. First we must check R_Q^+ is still fully confluent and sensible.

For the forth–down condition, suppose $X \leq_Q X'\ R_Q^+\ Y'$. Then by the definition of R_Q^+, there are some $X_2 \geq_Q X'$ and $Y_1 \leq_Q Y'$ such that $X_2\ R_Q\ Y_1$. Since $X \leq_Q X' \leq_Q X_2$, by the forth–down condition for R_Q there is some $Y \leq_Q Y_1$ with $X\ R_Q\ Y$ and therefore $X\ R_Q^+\ Y$. Since $Y \leq_Q Y_1 \leq_Q Y'$, we are done. The proof that the forth–up condition holds is just the order dual of that for forth–down. The proofs of the back–down and back–up conditions are similar.

To see that R_Q^+ is sensible, suppose $X\ R_Q^+\ Y$ and that $\bigcirc\varphi \in \Sigma$. Take $X_1 \leq_Q X \leq_Q X_2$ and $Y_1 \leq_Q Y \leq_Q Y_2$ such that $X\ R_Q\ Y_1$. Then

$$\begin{aligned} \bigcirc\varphi \in \ell_Q(X) &\implies \bigcirc\varphi \in \ell_Q(X_1) \\ &\implies \varphi \in \ell_Q(Y_2) \\ &\implies \varphi \in \ell_Q(Y) \\ &\implies \varphi \in \ell_Q(Y_1) \\ &\implies \bigcirc\varphi \in \ell_Q(X_2) \quad \implies \bigcirc\varphi \in \ell_Q(X), \end{aligned}$$

so $\bigcirc\varphi \in \ell_Q(X) \iff \varphi \in \ell_Q(Y)$. The $\Diamond$ and $\Box$ cases are similar.

Finally, we show that R_Q^+ is convex. Firstly, for the image condition, if $X\ R_Q^+\ Y_1$ and $X\ R_Q^+\ Y_2$ with $Y_1 \leq_Q Y \leq Y_2$, then by the definition of R_Q^+ we can find $X_2 \geq_Q X$ and $Y_1' \leq_Q Y_1$ with $X_2\ R_Q\ Y_1'$, and similarly $X_1 \leq_Q X$ and $Y_2' \geq_Q Y_2$ with $X_1\ R_Q\ Y_2'$. Since then $X_1 \leq_Q X \leq_Q X_2$ and $Y_1' \leq_Q Y \leq_Q Y_2'$, by the definition of R_Q^+ we conclude that $X\ R_Q^+\ Y$. The preimage condition is completely analogous. This completes the proof that ${}^{\mathfrak{C}}/_{\Sigma}$ is a Σ-labelled system. $\square$

Lemma 23. *Suppose Σ is finite, and write $\|\Sigma\|$ for its cardinality. Then the height of ${}^{\mathfrak{C}}/_{\Sigma}$ is bounded by $\|\Sigma\| + 1$, and the cardinality of the domain $|\mathfrak{C}|/\sim$ of ${}^{\mathfrak{C}}/_{\Sigma}$ is bounded by $(\|\Sigma\| + 1) \cdot 2^{\|\Sigma\|(\|\Sigma\|+1)+1}$*

Proof. Each element of the domain of ${}^{\mathfrak{C}}/_{\Sigma}$ is a pair (ℓ, L) where L is a (nonempty) subset of $\wp\Sigma$ and $\ell \in L$. Since L is linearly ordered by inclusion, it has height at most $\|\Sigma\| + 1$. There are $(2^{\|\Sigma\|})^i$ subsets of $\wp\Sigma$ of size i, so there are at most $\sum_{i=1}^{\|\Sigma\|+1} (2^{\|\Sigma\|})^i$ distinct L. The sum is bounded by $2^{\|\Sigma\|(\|\Sigma\|+1)+1}$. The factor of $\|\Sigma\|+1$ corresponds to choice of an $\ell \in L$, for each L. $\square$

Thus we have an exponential bound on the size of ${}^{\mathfrak{C}}/_{\Sigma}$. Later, once we prove ${}^{\mathfrak{C}}/_{\Sigma}$ is a quasimodel, the decidability of GTL can be inferred from this bound. See (Aguilera et al. 2022) for a more direct proof of decidability using the same quotient construction. However, for our purposes, it suffices to observe that ${}^{\mathfrak{C}}/_{\Sigma}$ is finite.

7 Characteristic Formulas

Next we show that there exist formulas defining points in the canonical quotient, i.e. to each $w \in |{}^{\mathfrak{C}}/_{\Sigma}|$ we assign formulas 'distinguishing' w. In fact, we need two versions of such formulas, as we can define them to be either true or false outside of the linear component of w. First, we define a formula $\chi_\Sigma^+(w)$ (or $\chi^+(w)$ when Σ is clear from context) such that for all $\Gamma \in |\mathfrak{C}|$, $\chi^+(w) \in \Gamma$ if and only if $w = [\Gamma']$ for some $\Gamma' \geq \Gamma$. Dually, we define $\chi^-(w) = \chi_\Sigma^-(w)$ so that for all $\Gamma \in |\mathfrak{C}|$, $\chi^-(w) \notin \Gamma$ if and only if $w = [\Gamma']$ for some $\Gamma' \leq \Gamma$. Compared to (Diéguez and Fernández-Duque 2018), these formulas require dual implication, as they must look 'up' and 'down' the model. In this section, we write ${}^{\mathfrak{C}}/_{\Sigma} = (|{}^{\mathfrak{C}}/_{\Sigma}|, \leq, R, \ell)$. We will omit subindices on the ℓ and L functions.

Definition 24. Fix $\Sigma \Subset \mathcal{L}$. Given $\Delta \in \mathbb{T}_\Sigma$, define $\overrightarrow{\Delta} = \bigwedge \Delta^+ \Rightarrow \bigvee \Delta^-$ and $\overleftarrow{\Delta} = \bigwedge \Delta^+ \Leftarrow \bigvee \Delta^-$. Given $w \in |{}^{\mathfrak{C}}/_{\Sigma}|$, we define a formula $\chi^0(w) = \chi_\Sigma^0(w)$ by

$$\chi^0(w) := \bigwedge_{\Delta \in L(w)} {\sim}\overrightarrow{\Delta} \wedge \bigwedge_{\Delta \notin L(w)} \neg\overleftarrow{\Delta}.$$

Then define $\chi^+(w) = \chi_\Sigma^+(w)$ by

$$\chi^+(w) = \overleftarrow{\ell(w)} \wedge \chi^0(w)$$

and $\chi^-(w) = \chi^-_\Sigma(w)$ by

$$\chi^-(w) = \chi^0(w) \Rightarrow \overrightarrow{\ell(w)}.$$

Proposition 25. *Given $w \in |\mathfrak{C}/\Sigma|$ and $\Gamma \in |\mathfrak{C}|$,*

1) $\chi^0(w) \in \Gamma^+$ if and only if $L(\Gamma) = L(w)$,
2) $\chi^+(w) \in \Gamma^+$ if and only if $[\Gamma] \leq w$, and
3) $\chi^-(w) \in \Gamma^-$ if and only if $[\Gamma] \geq w$.

Proof. Let $w \in |\mathfrak{C}/\Sigma|$ and $\Gamma \in |\mathfrak{C}|$.

1) First assume that $\chi^0(w) \in \Gamma^+$, so that $\bigwedge_{\Delta \in L(w)} \sim\overrightarrow{\Delta} \in \Gamma^+$ and $\bigwedge_{\Delta \notin L(w)} \neg\overleftarrow{\Delta} \in \Gamma^+$. Let $\Delta \in L(w)$. From $\sim\overrightarrow{\Delta} \in \Gamma^+$, we obtain $\Phi \geq \Gamma$ such that $\overrightarrow{\Delta} \notin \Phi^+$. Hence there is $\Phi_\Delta \leq \Phi$ with $\bigwedge \Delta^+ \in \Phi^+_\Delta$ and $\bigvee \Delta^- \in \Phi^-_\Delta$, i.e. $\ell(\Phi_\Delta) = \Delta$. From local linearity we see that $\Phi_\Delta \lesseqgtr \Gamma$; hence $\Delta = \ell(\Phi_\Delta) \in L(\Gamma)$.

Similarly, if $\Delta \in \mathbb{T}_\Sigma \setminus L(w)$, for any $\Psi \leq \Gamma$ we have that $\overleftarrow{\Delta} \notin \Psi^+$, so that there is no $\Psi_\Delta \geq \Psi$ with $\bigwedge \Delta^+ \in \Psi^+_\Delta$ and $\bigvee \Delta^- \in \Psi^-_\Delta$. Thus there is no $\Psi_\Delta \lesseqgtr \Gamma$ with $\bigwedge \Delta^+ \in \Psi^+_\Delta$ and $\bigvee \Delta^- \in \Psi^-_\Delta$, i.e. $\Delta \notin L(\Gamma)$ (for the $\Psi_\Delta \geq \Gamma$ case, set $\Psi = \Gamma$; for $\Psi_\Delta \leq \Gamma$ set $\Psi = \Psi_\Delta$).

The converse follows by similar reasoning. Assume that $L(\Gamma) = L(w)$. Then from $\Delta \in L(\Gamma)$ we readily obtain $\sim\overrightarrow{\Delta} \in \Gamma^+$, and similarly from $\Delta \notin L(\Gamma)$ we obtain $\neg\overleftarrow{\Delta} \in \Gamma^+$, from which we obtain by propositional reasoning $\chi^0(w) \in \Gamma^+$.

2) If $\chi^+(w) \in \Gamma^+$ then $\chi^0(w) \in \Gamma^+$, hence $L(\Gamma) = L(w)$, while $\overleftarrow{\ell(w)} \in \Gamma^+$ implies that there is some $\Gamma' \geq \Gamma$ with $\ell(\Gamma') = \ell(w)$. This shows that $w = [\Gamma'] \geq [\Gamma]$, as claimed.

3) This item is similar to the previous, except that we observe that if $L(\Gamma) \neq L(w)$, then $\chi^-(w) \notin \Gamma^-$. □

Remark 26. Note that the formula $\chi^\pm_\Sigma(w)$ makes essential use of dual implication, as properties of $w \geq [\Gamma]$ do not affect truth values in Γ in the language with $\Rightarrow$ alone. In contrast, the formulas χ^-_Σ are similar to the formulas $\mathrm{Sim}(w)$ of (Diéguez and Fernández-Duque 2018), although we remark that dual implication is still needed to describe the full linear component of w.

Next we establish some provable properties of each of χ^+_Σ and χ^-_Σ. We begin with the former.

Proposition 27. *Given $w \in |\mathfrak{C}/\Sigma|$ and $\psi \in \Sigma$:*

1) If $\psi \in \ell^-(w)$, then $\vdash \chi^+(w) \Rightarrow (\chi^+(w) \Leftarrow \psi)$.
2) If $\psi \in \ell^+(w)$, then $\vdash \chi^+(w) \Rightarrow \psi$.
3) For any $w \in |\mathfrak{C}/\Sigma|$, $\vdash \chi^+(w) \Rightarrow \bigcirc \bigvee\limits_{wRv} \chi^+(v)$.

Proof. 1) Let $\Gamma \in |\mathfrak{C}|$ and assume that $\psi \in \ell^-(w)$ and $\chi^+(w) \in \Gamma^+$; by properties of the canonical model, it suffices to show that $(\chi^+(w) \Leftarrow \psi) \in \Gamma^+$. From $\chi^+(w) \in \Gamma$ and Proposition 25 we obtain $\Delta \leq \Gamma$ such that $[\Delta] = w$, hence $\chi^+(w) \in \Delta^+$ and $\psi \in \Delta^-$, yielding $(\chi^+(w) \Leftarrow \psi) \in \Gamma$.

2) If $\psi \in \ell^+(w)$, as above, let $\Gamma \in |\mathfrak{C}|$ be such that $\chi^+(w) \in \Gamma^+$, and $\Delta \leq \Gamma$ with $[\Delta] = w$. Then $\psi \in \Delta^+$, yielding $\psi \in \Gamma^+$.

3) Let Γ be such that $\chi^+(w) \in \Gamma$, so that there is $\Delta \leq \Gamma$ with $[\Delta] = w$. Then $w \ R \ [S_\mathfrak{C}(\Delta)]$ by definition, and moreover $\chi^+([S_\mathfrak{C}(\Delta)]) \in S_\mathfrak{C}(\Delta^+)$ implies that $\bigcirc\chi^+([S_\mathfrak{C}(\Delta)]) \in \Delta^+$. Thus $\bigcirc\chi^+([S_\mathfrak{C}(\Delta)]) \in \Gamma^+$ by downward persistence, so that $\bigvee_{wRv} \chi^+(v) \in \Gamma^+$. □

The formula χ^-_Σ behaves 'dually', as established below.

Proposition 28. *Given $w \in |\mathfrak{C}/\Sigma|$ and $\psi \in \Sigma$:*

1) If $\psi \in \ell^-(w)$, then $\vdash \psi \Rightarrow \chi^-(w)$.
2) If $\psi \in \ell^+(w)$, then $\vdash \big(\psi \Rightarrow \chi^-(w)\big) \Rightarrow \chi^-(w)$.
3) For any $w \in |\mathfrak{C}/\Sigma|$, $\vdash \bigcirc \bigwedge\limits_{wRv} \chi^-(v) \Rightarrow \chi^-(w)$.

Proof. 1) Assume that $\psi \in \ell^-(w) \cap \Gamma^+$ and write $w = [\Delta]$. Then $\psi \in \Delta^-$, which means we cannot have $\Gamma \geq \Delta$. Hence Proposition 25 implies that $\chi^-(w) \notin \Gamma^-$, i.e. $\chi^-(w) \in \Gamma^+$.

2) Suppose that $\psi \in \ell^+(w)$ and proceed to prove the claim by contrapositive. If $\chi^-(w) \in \Gamma^-$ for some $\Gamma \in |\mathfrak{C}|$, then there is $\Delta \leq \Gamma$ such that $w = [\Delta]$. But then $\chi^-(w) \in \Delta^-$ and $\psi \in \Delta^+$, which implies that $\big(\psi \Rightarrow \chi^-(w)\big) \in \Delta^-$, hence also $\big(\psi \Rightarrow \chi^-(w)\big) \in \Gamma^-$, as required.

3) Proceed by contrapositive. If $\chi^-(w) \in \Gamma^-$ for some $\Gamma \in |\mathfrak{C}|$, then there is $\Delta \leq \Gamma$ such that $w = [\Delta]$. We have that $w \ R \ [S_\mathfrak{C}(\Delta)]$ by definition. Letting $v = [S_\mathfrak{C}(\Delta)]$, we have that $\chi^-(v) \in S^-_\mathfrak{C}(\Delta)$, hence $\bigcirc\chi^-(v) \in \Delta^-$, and by downward persistence, $\bigcirc\chi^-(v) \in \Gamma^-$. Hence $\bigcirc \bigwedge_{wRv} \chi^-(v) \in \Gamma^-$. □

8 Completeness

The formulas $\chi^\pm_\Sigma$ are fundamental in our completeness proof; specifically, we will use them to show that $\mathfrak{C}/\Sigma$ is ω-sensible, hence a quasimodel. Since validity over the class of quasimodels is equivalent to real validity by Theorem 11, completeness will follow. The following lemma is the first step towards establishing ω-sensibility. Once again, we write $\mathfrak{C}/\Sigma = (|\mathfrak{C}/\Sigma|, \leq, R, \ell)$, and as usual R^* is the transitive, reflexive closure of R.

Lemma 29. *If $\Sigma \Subset \mathcal{L}$ and $w \in |\mathfrak{C}/\Sigma|$, then*

1. $\vdash \bigvee_{wR^*v} \chi^+(v) \Rightarrow \bigcirc \bigvee_{wR^*v} \chi^+(v)$, and
2. $\vdash \bigcirc \bigwedge_{wR^*v} \chi^-(v) \Rightarrow \bigwedge_{wR^*v} \chi^-(v)$.

Proof. The first item follows from Proposition 27(3), as for any $v \in R^*(w)$ we have that

$$\vdash \chi^+(v) \Rightarrow \bigcirc \bigvee_{vRu} \chi^+(u).$$

Since $v \ R \ u$ implies that $w \ R^* \ u$ by transitivity,

$$\vdash \chi^+(v) \Rightarrow \bigcirc \bigvee_{wR^*u} \chi^+(u).$$

Since v was arbitrary, we obtain

$$\vdash \bigvee_{wR^*v} \chi^+(v) \Rightarrow \bigcirc \bigvee_{wR^*u} \chi^+(u),$$

which by a change of variables yields the original claim.
Item 2 is similar, but uses Proposition 28(3). $\square$

In order to complete our proof that $\mathfrak{C}/\Sigma$ is ω-sensible, it suffices to apply induction to the formulas of Lemma 29.

Proposition 30.

1. *If $w \in |\mathfrak{C}/\Sigma|$ and $\Diamond\psi \in \ell^+(w)$, then there is $v \in R^*(w)$ such that $\psi \in \ell^+(v)$.*
2. *If $w \in |\mathfrak{C}/\Sigma|$ and $\Box\psi \in \ell^-(w)$, then there is $v \in R^*(w)$ such that $\psi \in \ell^-(v)$.*

Proof. **1.** Towards a contradiction, assume that $w \in |\mathfrak{C}/\Sigma|$ and $\Diamond\psi \in \ell^+(w)$ but, for all $v \in R^*(w)$, $\psi \in \ell^-(w)$. By Lemma 29, $\vdash \bigcirc \bigwedge_{wR^*v} \chi^-(v) \Rightarrow \bigwedge_{wR^*v} \chi^-(v)$. By the $\Diamond$-induction axiom IV.J and standard modal reasoning, $\vdash \Diamond \bigwedge_{wR^*v} \chi^-(v) \Rightarrow \bigwedge_{wR^*v} \chi^-(v)$; in particular,

$$\vdash \Diamond \bigwedge_{wR*v} \chi^-(v) \Rightarrow \chi^-(w). \tag{1}$$

Now let $v \in R^*(w)$. By Proposition 28(1) and the assumption that $\psi \in \ell^-(v)$ we have that $\vdash \psi \Rightarrow \chi^-(v)$, and since v was arbitrary, $\vdash \psi \Rightarrow \bigwedge_{wR^*v} \chi^-(v)$. Using distributivity IV.F we further have that $\vdash \Diamond\psi \Rightarrow \Diamond \bigwedge_{wR^*v} \chi^-(v)$. This, along with (1), shows that $\vdash \Diamond\psi \Rightarrow \chi^-(w)$. However, by Proposition 27(2) and our assumption that $\Diamond\psi \in \ell^+(w)$ we have that $\vdash \big(\Diamond\psi \Rightarrow \chi^-(w)\big) \Rightarrow \chi^-(w)$. Hence by modus ponens we obtain $\vdash \chi^-(w)$. Writing $w = [\Gamma]$, Proposition 25 yields $\chi^-(w) \notin \Gamma^+$, but this contradicts $\vdash \chi^-(w)$. We conclude that there is $v \in R^*(w)$ with $\psi \in \ell^+(v)$, as needed.

2. This is similar to the first item, but dualised. Towards a contradiction, assume that $w \in |\mathfrak{C}/\Sigma|$ and $\Box\psi \in \ell^-(w)$ but, for all $v \in R^*(w)$, $\psi \in \ell^+(w)$. By Lemma 29, $\vdash \bigvee_{wR^*v} \chi^+(v) \Rightarrow \bigcirc \bigvee_{wR^*v} \chi^+(v)$. By the $\Box$-induction axiom IV.I, $\vdash \bigvee_{wR^*v} \chi^+(v) \Rightarrow \Box \bigvee_{wR^*v} \chi^+(v)$; in particular,

$$\vdash \chi^+(w) \Rightarrow \Box \bigvee_{wR^*v} \chi^+(v). \tag{2}$$

Now let $v \in R^*(w)$. By Proposition 27(2) and the assumption that $\psi \in \ell^+(v)$, we have that $\vdash \chi^+(v) \Rightarrow \psi$, and since v was arbitrary, $\vdash \bigvee_{wR^*v} \chi^+(v) \Rightarrow \psi$. Using distributivity IV.E we further have that $\vdash \Box \bigvee_{wR^*v} \chi^+(v) \Rightarrow \Box\psi$. This, along with (2), shows that

$$\vdash \chi^+(w) \Rightarrow \Box\psi. \tag{3}$$

By Proposition 27(1) and our assumption that $\Box\psi \in \ell^-(w)$ we have that $\vdash \chi^+(w) \Rightarrow \big(\chi^+(w) \Leftarrow \Box\psi\big)$, hence by (3) and Rule II.B we obtain $\vdash \chi^+(w) \Rightarrow (\Box\psi \Leftarrow \Box\psi)$. In view of Lemma 7.5, this implies that $\chi^+(w)$ is contradictory. Writing $w = [\Gamma]$, Proposition 25 yields $\chi^+(w) \in \Gamma$, which once again is impossible. We conclude that there is $v \in R^*(w)$ with $\psi \in \ell^-(v)$. $\square$

Corollary 31. *If $\Sigma \Subset \mathcal{L}$, then $\mathfrak{C}/\Sigma$ is a quasimodel.*

Proof. By Lemma 22, $\mathfrak{C}/\Sigma$ is based on a labelled system, while by Proposition 30, R is ω-sensible. By definition, $\mathfrak{C}/\Sigma$ is a quasimodel. $\square$

We are now ready to prove that our calculus is complete.

Theorem 32. *If $\varphi \in \mathcal{L}$ is valid, then $\vdash \varphi$.*

Proof. We prove the contrapositive. Suppose φ is an unprovable formula and let Σ be the set of subformulas of φ. Since φ is unprovable, there is $\Gamma \in |\mathfrak{C}|$ with $\varphi \in \Gamma^-$. Hence $[\Gamma] \in |\mathfrak{C}/\Sigma|$ is a point in a quasimodel falsifying φ, so that by Theorem 11, φ is not valid. $\square$

9 Concluding Remarks

We have provided a sound and complete calculus for the Gödel temporal logic GTL. These results further cement GTL as a privileged logic for fuzzy temporal reasoning and pave the way for a proof-theoretic treatment of these logics. Among the challenges in this direction is the design of cut-free or cyclic calculi.

In proving our main results, we have developed tools for the treatment of superintuitionistic temporal logics, specifically identifying the usefulness of combining ‘henceforth’ with co-implication. We believe that this insight will lead to completeness proofs for related logics, including intuitionistic LTL, where complete calculi for ‘eventually’ are available, but not so for ‘henceforth’. Along these lines, it should be remarked that the techniques of (Diéguez and Fernández-Duque 2018) should lead to a sound and complete calculus for the logic with $\Rightarrow$, $\bigcirc$ and $\Diamond$ (but no co-implication or henceforth), although such a result does not follow immediately from the present work.

Another subject that would be worth studying in the near future is bisimulation in Gödel temporal logic. This tool has been used to determine that temporal operators are not interdefinable in the intuitionistic temporal setting (Balbiani et al. 2018; Balbiani et al. 2020). For the class of temporal here-and-there models, ‘henceforth’ is a basic operator that cannot be defined, while ‘eventually’ becomes definable in terms of ‘henceforth’, ‘next’, and implication (Balbiani et al. 2018; Balbiani et al. 2020). When introducing co-implication, results on definability exist in the literature: for a combination of the logic of here-and-there, co-implication and the basic modal logic K, it has been proven by (Balbiani and Diéguez 2018) that modal operators become interdefinable. We do not know if co-implication has the same effect to our Gödel temporal logic; a negative answer would require a suitable notion of bisimulation preserving both implications, as well as the temporal operators.

In a different direction, logics such as PDL or CTL may also enjoy naturally axiomatizable Gödel counterparts. The techniques developed here and by (Aguilera et al. 2022) could very well be applicable in these settings.

Acknowledgements

This work has been partially supported by FWO-FWF grant G030620N/I4513N (J.P.A. and D.F.D.), FWO grant 3E017319 (J.P.A.), the projects EL4HC and étoiles montantes CTASP at Région Pays de la Loire, France (M.D.), the COST action CA-17124 (M.D. and D.F.D.), and SNSF–FWO Lead Agency Grant 200021L_196176/G0E2121N (B.M. and D.F.D.).

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