Epistemic Actions: Comparing Multi-agent Belief Bases with Action Models

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Abstract

We compare the syntactic multi-agent belief base approach, and the dynamic epistemic logic possible world semantic approach. In the belief base approach, the language provides an implicit and an explicit belief operators, plus a dynamic modality for actions consisting in adding formulae to bases. For the semantic approach, we rely on action models of Dynamic Epistemic Logic (DEL). We first show how to translate a formula of the belief base approach into DEL: in particular, we provide a specific action model scheme corresponding to the addition of a formula in a belief base. Conversely, we identify a fragment of DEL that can be translated in the multi-agent belief base language.

1 Introduction

The formalization of epistemic states and their dynamics is one of the key topics in the area of knowledge representation and reasoning. There are two traditions in this area. On the one hand, there is epistemic logic which started with the seminal work of Hintikka (Hintikka 1962) on the logics of knowledge and belief. It was extended to the multi-agent setting at a later stage (Fagin et al. 1995; Meyer and van der Hoek 1995) and, more recently, to modeling knowledge and belief change with growing research on dynamic epistemic logic (DEL) (Baltag, Moss, and Solecki 1998; Baltag and Moss 2004; van Ditmarsch, van der Hoek, and Kooi 2007a). The standard approach to epistemic logic is extensional. Its formal semantics exploits the so-called multi-agent *Kripke models*, namely, multi-relational structures equipped with valuation functions for the interpretation of atomic formulas. Binary relations in a multi-agent Kripke model are called *epistemic (or doxastic) accessibility relations* and are used to describe the agents' epistemic states and uncertainty.

On the other hand, we have the so-called syntactic approach. It includes, for instance, work on belief base and knowledge base revision (Hansson 1993; 1999; Benferhat et al. 2002), belief base merging (Konieczny and Pérez 2002) and input-output logic (Makinson and van der Torre 2000). The syntactic approach typically leverages belief bases, or more generally knowledge bases, for representing what an agent knows or believes. A natural distinction in this approach is between explicit and implicit belief. An agent's explicit belief is seen as a piece of information in the agent's belief base, while an implicit belief corresponds to a piece of information that is derivable from the agent's belief base (i.e., included in the deductive closure of the agent's belief base). The syntactic approach was put forward among other things as a solution to the logical omniscience problem in virtue of the fact that an agent's beliefs are described either by a set of formulas which is not necessarily closed under deduction (Eberle 1974; Moore and Hendrix 1982) or by a set of formulas obtained by the application of an incomplete set of deduction rules (Konolige 1986; Jago 2009).

The two approaches have been recently reconciled in (Lorini 2018; 2020) in which a semantics for multi-agent epistemic logic using belief bases was proposed. The central idea of this semantics is that an agent's epistemic indistinguishability relation should be computed from the agent's belief base by stipulating that *a state is considered possible by the agent if and only if it satisfies all information in the agent's belief base*. Moreover, at a dynamic level, the dynamics of the agents' explicit and implicit beliefs are supposed to depend on how their belief bases change over time. For instance, by privately expanding its belief base with a new information α, an agent will start to explicitly believe α and, consequently, it will be able to deduce new facts from its expanded belief base.

In this paper we push forward the comparison between the two approaches by investigating the connection between the update semantics for DEL using so-called action models (Baltag, Moss, and Solecki 1998) and the update semantics for epistemic logic using belief bases introduced in (Lorini 2020). We believe comparing the two approaches is useful since each of them has its own advantages and disadvantages and it is important to have a clear understanding of their relationship. For instance, the extensional approach using multi-relational Kripke models and action models is general, as it allows us to model a large variety of multi-agent information dynamics, but it is not compact: even simple situations require big models. On the contrary, the syntactic approach is less general, but it represents information in a more compact way with the help of belief bases. We will answer the following questions: what is the extensional counterpart in terms of action model of the notion of belief base expansion? Which class of multi-agent information dynamics can be represented through update operations on belief bases? Both directions will be explored: from multi-agent belief bases to action models and back from action models to multi-agent belief bases. On the one hand, we will show how to translate the notion of private belief base expansion into a specific class of action models representing private information change. On the other hand, we will identify a specific class of "ruby" actions models that can be translated into the belief base semantics. In a "ruby" action model each agent privately learns a new fact and updates its beliefs accordingly assuming that the others do not learn anything. The non-trivial aspect of the construction from the DEL semantics to the belief base semantics lies in syntactically representing action models with belief bases.

The paper is organized as follows.[1] In Section 2 we present the background material on the language and the semantics for epistemic logic exploiting belief bases introduced in (Lorini 2020). Section 3 presents the background material on DEL: the notions of action model and product update between an epistemic model and an action model. In Section 4, we study the connection between the update semantics for epistemic logic using multi-agent belief bases and the DEL semantics using action models. We provide a polynomial embedding of the former into the latter and then we show how to represent "ruby" action models through the notion of private belief base expansion. In Section 5, we discuss the two semantics under the assumption of epistemic introspection. In Section 6, we discuss some related work.

2 Background on Multi-agent Belief Bases

This section presents the dynamic epistemic language for beliefs of both explicit and implicit types (Lorini 2020).

2.1 Language

Assume a countably infinite set of atomic propositions $Ap = \{p, q, \dots\}$ and a finite set of agents $Ag = \{1, \dots, n\}$. Let $2^{Ag*} = 2^{Ag} \setminus \{\emptyset\}$ be the set of non-empty coalitions.

We define the language in two steps. First define the language $\mathcal{L}_0$ by:

$$\mathcal{L}_0 \stackrel{\text{def}}{=} \quad \alpha \quad ::= \quad p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \triangle_i \alpha,$$

where p ranges over Ap and i ranges over Ag. $\mathcal{L}_0$ is the language used to represent *explicit* beliefs. The formula $\triangle_i \alpha$ reads "agent i has the explicit belief that α". The language $\mathcal{L}_B$ extends $\mathcal{L}_0$ and is defined by:

$$\mathcal{L}_B \stackrel{\text{def}}{=} \quad \varphi \quad ::= \quad \alpha \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi \mid [+_J \alpha]\varphi,$$

where α ranges over $\mathcal{L}_0$, i over Ag and J over 2^{Ag*}.

The other Boolean constructions $\top$, $\bot$, $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined from p, $\neg$ and $\wedge$ in the standard way. The formula $\Box_i \varphi$ reads "agent i implicitly believes that φ". The abbreviation $\Diamond_i \varphi \stackrel{\text{def}}{=} \neg\Box_i\neg\varphi$ defines the concept of belief compatibility. The formula $\Diamond_i \varphi$ reads "φ is compatible with agent i's explicit beliefs". The operator $[+_J \alpha]$ is used to model private belief expansion. Specifically, the formula $[+_J \alpha]\varphi$ reads "φ holds after every agent in the coalition J has expanded its belief base with α".

[1]The supplementary material contains the long version of this submission.

2.2 Belief Base Semantics

The formal semantics for the language $\mathcal{L}_B$ exploits belief bases. Unlike the standard Kripke semantics in which possible worlds and epistemic alternatives are primitive, they are here defined from the primitive concept of belief base.

Definition 1 (State) *A state is a tuple $B = \big((B_i)_{i \in Ag}, V\big)$ where $B_i \subseteq \mathcal{L}_0$ is agent i's belief base, and $V \subseteq Ap$ is the actual environment. The set of all states is noted $\mathbf{S}$.*

The following definition specifies truth conditions for formulas in the sublanguage $\mathcal{L}_0$.

Definition 2 (Satisfaction relation) *For any state $B = \big((B_i)_{i \in Ag}, V\big) \in \mathbf{S}$:*

$$\begin{aligned} B \models p &\iff p \in V, \\ B \models \neg\alpha &\iff B \not\models \alpha, \\ B \models \alpha_1 \wedge \alpha_2 &\iff B \models \alpha_1 \textit{ and } B \models \alpha_2, \\ B \models \triangle_i \alpha &\iff \alpha \in B_i. \end{aligned}$$

Observe the set-theoretic interpretation of the explicit belief operators in the previous definition: agent i has the explicit belief that α if and only if α is included in its belief base. The following definition introduces the notion of epistemic alternative.

Definition 3 (Epistemic alternatives) *Let $i \in Ag$. Then, $\mathcal{R}_i$ is the binary relation on the set $\mathbf{S}$ such that, for all $B = \big((B_i)_{i \in Ag}, V\big), B' = \big((B'_i)_{i \in Ag}, V'\big) \in \mathbf{S}$:*

$$B \mathcal{R}_i B' \textit{ if and only if } \forall \alpha \in B_i : B' \models \alpha.$$

$B \mathcal{R}_i B'$ means that B' is an epistemic alternative for agent i at B, that is to say, B' is a state that agent i considers possible at B. The idea of the previous definition is that B' is an epistemic alternative for agent i at B if and only if, B' satisfies all facts that agent i explicitly believes at B.

A multi-agent belief model (MAB), or simply model, is defined to be a state supplemented with a set of states, called *context*. The context Cxt is not necessarily equal to the set of all states $\mathbf{S}$, since there could be states in $\mathbf{S}$ incompatible with the general "laws of the domain" and, consequently, with the agents' epistemic states. For example, we might want to exclude from the context Cxt all states in which the propositions "1+1=2" and "1+1=3" are true concomitantly.

Definition 4 (Multi-agent belief model) *A multi-agent belief model (MAB) is a pair (B, Cxt), where $B \in \mathbf{S}$ and $Cxt \subseteq \mathbf{S}$. The class of MABs is noted $\mathbf{M}$.*

Note that in Definition 4 we do not require $B \in Cxt$. The following definition extends Definition 2 to the full language $\mathcal{L}_B$. Its formulas are interpreted with respect to MABs. (We omit Boolean cases, as they are defined in the usual way.)

Definition 5 (Satisfaction relation (cont.)) *Let $(B, Cxt) \in \mathbf{M}$. Then:*

$$\begin{aligned} (B, Cxt) \models \alpha &\iff B \models \alpha, \\ (B, Cxt) \models \Box_i \varphi &\iff \forall B' \in Cxt, \textit{if } B \mathcal{R}_i B' \textit{ then} \\ &\qquad (B', Cxt) \models \varphi, \\ (B, Cxt) \models [+_J \alpha]\varphi &\iff (B^{+_J \alpha}, Cxt) \models \varphi, \end{aligned}$$

with $V^{+_J \alpha} = V$, $B_i^{+_J \alpha} = B_i \cup \{\alpha\}$ for all $i \in J$, and $B_j^{+_J \alpha} = B_j$ for all $j \notin J$.

According to the previous definition, agent i implicitly believes that φ if and only if φ is true at all states in the context that i considers possible. Moreover, every agent in coalition J privately expands its belief base with α if every agent in J adds the information α to its belief base, while all agents outside of J keep their beliefs unchanged.

Let $\varphi \in \mathcal{L}_B$. Formula φ is said to be valid relative to the class $\mathbf{M}$, noted $\models_{\mathbf{M}} \varphi$, if and only if, for every $(B, Cxt) \in \mathbf{M}$, we have $(B, Cxt) \models \varphi$. Formula φ is said to be satisfiable for the class $\mathbf{M}$ if and only if $\neg\varphi$ is not valid for the class $\mathbf{M}$.

3 Background on DEL

An epistemic model is a Kripke structure (i.e. a graph) in which nodes are possible worlds and edges are labelled by agents.

Definition 6 *An* epistemic model $\mathcal{M} = (W, (R_i)_{i\in Ag}, V)$ *is a tuple where:*

- W *is a non-empty finite set of possible* worlds*,*
- $R_i \subseteq W \times W$ *is an* accessibility relation *for agent* i*,*
- $V : W \longrightarrow 2^{Ap}$ *is a* valuation function*.*

A static epistemic situation is then classically represented by a so-called pointed epistemic model, which is a pair $\mathcal{M}, w$ where w is the current world.

Example 1 *In Figure 1 (on the left), the epistemic model $\mathcal{M}$ contains two worlds w and u that are indistinguishable for both agents i and j. Moreover, the current world is w.*

To model epistemic actions (public/private announcements, etc.), DEL provides the notion of action model (Baltag, Moss, and Solecki 1998). An action model is also a Kripke structure. Nodes are atomic actions, also called events, labelled by a pair of pre- and post-conditions. The precondition $pre(a)$ of an atomic action a is the epistemic formula that should be true before the execution of a, and the postcondition function $post(a)$ assigns to each proposition p a new truth value obtained by the evaluation of the formula $post(a)(p)$.

Definition 7 *An* action model *is a tuple* $\mathcal{A} = (\mathsf{A}, (R_i^{\mathcal{A}})_{i\in Ag}, pre, post)$ *where:*

- A *is a non-empty finite set of possible* atomic actions*,*
- $R_i^{\mathcal{A}} \subseteq \mathsf{A} \times \mathsf{A}$ *is the* accessibility relation *on* A *for* i*,*
- $pre : \mathsf{A} \longrightarrow \mathcal{L}_{\mathbf{EL}}$ *is a* precondition function*,*
- $post : \mathsf{A} \times Ap \longrightarrow \mathcal{L}_{\mathbf{EL}}$ *is a* postcondition function*.*

Example 2 *In Figure 1 (top), the action model $\mathcal{A}$ contains two atomic actions a and a'. For a to be executed, its precondition p should be true. Then, action a assigns p to false. Action a' is always executable since its precondition is $\top$, and it does not change the value of any proposition. Action a is the current event and the sole possible action for agent i, while agent j considers that the trivial action a' happens. The pointed action model $\mathcal{A}, a$ corresponds to the private announcement of p to agent i and the private assignment of p to false for agent i (that is, only i sees that p has been assigned to false).*

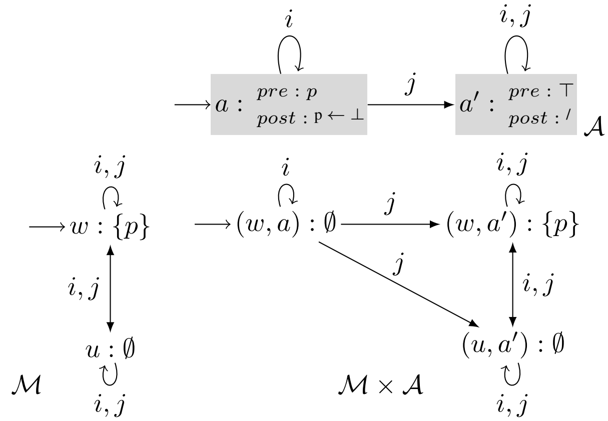


Figure 1: Example of product $\mathcal{M} \times \mathcal{A}$.

Now we recall the definition of the product $\mathcal{M} \times \mathcal{A}$ which corresponds to the epistemic model obtained by executing the action model $\mathcal{A}$ in the initial epistemic model $\mathcal{M}$.

Definition 8 *Let* $\mathcal{M} = (W, (R_i)_{i\in Ag}, V)$ *be a Kripke model. Let* $\mathcal{A} = (\mathsf{A}, (R_i^{\mathcal{A}})_{i\in Ag}, pre, post)$ *be an action model. The* product *of $\mathcal{M}$ and $\mathcal{A}$ is* $\mathcal{M} \times \mathcal{A} = (W', (R'_i)_{i\in Ag}, V')$ *where:*

1. $W' = \{(w, a) \in W \times \mathsf{A} \mid \mathcal{M}, w \models pre(a)\}$*;*
2. $(w, a)R'_i(w', a')$ *iff* wR_iw' *and* $aR_i^{\mathcal{A}}a'$*;*
3. $V'((w, a)) = \{p \in Ap \mid \mathcal{M}, w \models post(a, p)\}$*.*

Worlds in $\mathcal{M} \times \mathcal{A}$ are pairs (w, a) in which the precondition of a holds in $\mathcal{M}, w$.

Example 3 *Figure 1 shows the resulting product $\mathcal{M} \times \mathcal{A}$ by executing $\mathcal{A}$ from $\mathcal{M}$. Note that the pair (u, a') is not present because the precondition p is not true in u. Note also that p is false in (w, a), due to the postcondition of a.*

The product of a pointed epistemic model $(\mathcal{M}, w)$ with a pointed event model $(\mathcal{A}, a)$ is defined as $(\mathcal{M}, w)\times(\mathcal{A}, a) := (\mathcal{M} \times \mathcal{A}, (w, a))$. It is defined only if $\mathcal{M}, w \models pre(a)$.We recall the language of DEL, noted $\mathcal{L}_{DEL}$, by extending standard epistemic logic with dynamic modalities $\langle \mathcal{A}, a\rangle$:

$$\varphi \quad ::= \ \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{K}_i\varphi \mid \langle \mathcal{A}, a\rangle\varphi$$

Formula $\mathsf{K}_i\varphi$ is read "agent i knows that φ". Formula $\langle \mathcal{A}, a\rangle\varphi$ is read "action $\mathcal{A}, a$ is executable, and after having executed it, φ holds".

Definition 9 *The truth conditions $\mathcal{M}, w \models \varphi$ are defined as follows (Boolean cases are omitted):*

- $\mathcal{M}, w \models p$ *if* $p \in V(w)$*;*
- $\mathcal{M}, w \models \mathsf{K}_i\varphi$ *if for all u such that wR_iu,* $\mathcal{M}, u \models \varphi$*.*
- $\mathcal{M}, w \models \langle \mathcal{A}, a\rangle\varphi$ *if* $\mathcal{M}, w \models pre(a)$ *and* $\mathcal{M} \times \mathcal{A}, (w, a) \models \varphi$*.*

We finish this section by recalling the notion of modal depth $md(\varphi)$ of a formula φ which is the maximum of the number of nested knowledge operators. It is defined by induction: $md(p) := 0$; $md(\neg\varphi) := md(\varphi)$; $md(\varphi \vee \psi) := \max(md(\varphi), md(\psi))$; $md(\mathsf{K}_i\varphi) = 1 + md(\varphi)$ and finally

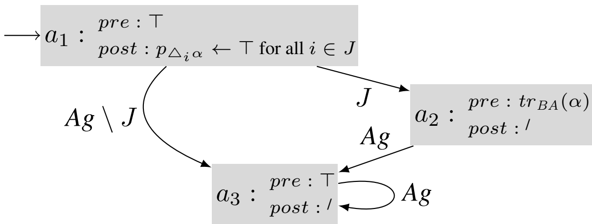


Figure 2: Action model $\mathcal{A}_{+_J\alpha}$ that simulates adding α in bases of agents in J.

$md(\langle \mathcal{A}, a\rangle)\varphi$ is the maximum of the modal depths of φ and the formulas written in $\mathcal{A}$.

The notion of bisimulation in modal logic has been refined to take the modal depth of formulas into account: the notion of n-bimilation noted $\leftrightarroweq_n$. We have that $\mathcal{M}, w \leftrightarroweq_n \mathcal{M}', w'$ iff $\mathcal{M}, w$ and $\mathcal{M}', w'$ agree on formulas of modal depth at most n (see Prop. 2.31 in (Blackburn, de Rijke, and Venema 2001) and Section 6.5 in (Van Ditmarsch, van Der Hoek, and Kooi 2007b)).

4 The Two Translations

In this section, we explore the connection between the DEL semantics using action models and the semantics using belief bases. We first investigate the direction from multi-agent belief bases to action models. Then, we investigate the other direction, from action models to multi-agent belief bases.

4.1 From Belief Base Operations to Action Models

We show how to embed the belief base semantics into the DEL action model semantics. In particular, we provide a polynomial satisfiability preserving translation tr_{BA} of the language $\mathcal{L}_B$ into $\mathcal{L}_{DEL}$. It is defined as follows:

$$
\begin{aligned}
&tr_{BA}(p) = p,\\
&tr_{BA}(\neg\varphi) = \neg tr_{BA}(\varphi),\\
&tr_{BA}(\varphi_1 \wedge \varphi_2) = tr_{BA}(\varphi_1) \wedge tr_{BA}(\varphi_2),\\
&tr_{BA}(\triangle_i\alpha) = p_{\triangle_i\alpha} \wedge \mathsf{K}_i tr_{BA}(\alpha),\\
&tr_{BA}(\Box_i\varphi) = \mathsf{K}_i tr_{BA}(\varphi),\\
&tr_{BA}([+_J\alpha]\varphi) = [\mathcal{A}_{+_J\alpha}, a_1] tr_{BA}(\varphi),
\end{aligned}
$$

where $\mathcal{A}_{+_J\alpha} = (\mathsf{A}, (R_i^{\mathcal{A}})_{i\in Ag}, pre, post)$ is the action model (see Figure 2) such that:

- $\mathsf{A} = \{a_1, a_2, a_3\}$;
- $\forall i \in J$, $R_i^{\mathcal{A}} = \{(a_1, a_2), (a_2, a_3), (a_3, a_3)\}$;
- $\forall i \in Ag \setminus J$, $R_i^{\mathcal{A}} = \{(a_1, a_3), (a_2, a_3), (a_3, a_3)\}$;
- $pre(a_1) = \top, pre(a_2) = tr_{BA}(\alpha), pre(a_3) = \top$;
- $post(a_2, p) = post(a_3, p) = p$ for all $p \in Ap$, $post(a_1, p) = p$ for all $p \notin \bigcup_{i\in J}\{p_{\triangle_i\alpha}\}$, and $post(a_1, p_{\triangle_i\alpha}) = \top$ if $i \in J$.

The following is the core result of this section highlighting the correctness of our polynomial embedding.

Theorem 1 *Let $\varphi \in \mathcal{L}_B$. Then, φ is satisfiable for the class $\mathbf{M}$ if and only if $tr_{BA}(\varphi)$ is DEL-satisfiable.*

PROOF.

The proof relies on the fact the language $\mathcal{L}_B$ can be equivalently interpreted relative to models of the form $\Omega = (S, \mathcal{B}, (\Rightarrow_i)_{i\in Ag}, \tau)$ where S is a non-empty set of states, $\mathcal{B} : Ag \times S \longrightarrow 2^{\mathcal{L}_0}$ is a belief base function, $\tau : Ap \longrightarrow 2^S$ is valuation function, $\Rightarrow_i \subseteq S \times S$ is agent i's epistemic accessibility relation. $\mathcal{L}_B$-formulas are interpreted with respect to pointed models (Ω, s) with $s \in S$ as follows (boolean cases are omitted for simplicity): (i) $(\Omega, s) \models p$ iff $s \in \tau(p)$, (ii) $(\Omega, s) \models \triangle_i\alpha$ iff $\alpha \in \mathcal{B}(i, s)$, (iii) $(\Omega, s) \models \Box_i\varphi$ iff $\forall s' \in S$, if $s \Rightarrow_i s'$ then $(\Omega, s') \models \varphi$, (iv) $(\Omega, s) \models [+_J\alpha]\varphi$ iff $(\Omega^{+_J\alpha}, s^*) \models \varphi$, where $\Omega^{+_J\alpha} = (S^{+_J\alpha}, \mathcal{B}^{+_J\alpha}, (\Rightarrow_i^{+_J\alpha})_{i\in Ag}, \tau^{+_J\alpha})$ with $s^* \notin S$ and:

- $S^{+_J\alpha} = S \cup \{s^*\}$,
- $\mathcal{B}^{+_J\alpha}(i, s') = \mathcal{B}(i, s)$ if $s' \neq s^*$,
- $\mathcal{B}^{+_J\alpha}(i, s^*) = \mathcal{B}(i, s)$ if $i \notin J$,
- $\mathcal{B}^{+_J\alpha}(i, s^*) = \mathcal{B}(i, s) \cup \{\alpha\}$ if $i \in J$,
- $\Rightarrow_i^{+_J\alpha} = \Rightarrow_i \cup \{(s^*, s') \mid s \Rightarrow_i s'\}$ if $i \notin J$,
- $\Rightarrow_i^{+_J\alpha} = \Rightarrow_i \cup \{(s^*, s') \mid s \Rightarrow_i s'$ and $(\Omega, s') \models \alpha\}$ if $i \in J$,
- $\tau^{+_J\alpha}(p) = \tau(p) \cup \{s^*\}$ if $s \in \tau(p)$,
- $\tau^{+_J\alpha}(p) = \tau(p)$ if $s \notin \tau(p)$.

In particular, for every $\varphi \in \mathcal{L}_B$, it is shown in (Lorini 2020) that φ is satisfiable for the class of multi-agent belief models $\mathbf{M}$ iff φ is satisfiable for the subclass of the previous models such that, for every $i \in Ag$ and for every $s \in S$, $\Rightarrow_i(s) \subseteq \bigcap_{\alpha\in\mathcal{B}(i,s)} ||\alpha||_{(\Omega,s)}$ with $||\alpha||_{(\Omega,s)} = \{s' \in S \mid (\Omega, s') \models \alpha\}$. Such a subclass is called the class of "notional" models and is noted $\mathbf{N}$. Given the previous semantic equivalence between the two classes $\mathbf{M}$ and $\mathbf{N}$ wrt language $\mathcal{L}_B$, in the rest of the proof we simply need work with the latter class.

We just need to show that φ is satisfiable for the class $\mathbf{N}$ if and only if $tr_{BA}(\varphi)$ is DEL-satisfiable. The left-to-right direction of the proof is by induction on the structure of φ and relies on a two-step transformation. The non-trivial case is $\psi = [+_J\alpha]\varphi$. Given a pointed notional model (Ω, s) satisfying φ we transform it into an isomorphic pointed epistemic model $\mathcal{M}, w$. Then, we compute the pointed notional model $(\Omega^{+_J\alpha}, s^*)$ on the one hand, and the epistemic model $\mathcal{M} \times \mathcal{A}_{+_J\alpha}, (w, a_1)$ on the other hand. We transform $(\Omega^{+_J\alpha}, s^*)$ into an isomorphic pointed epistemic model $(\mathcal{M}', w')$. We show that $\mathcal{M}', w'$ and $\mathcal{M} \times \mathcal{A}_{+_J\alpha}, (w, a_1)$ are bisimilar. The right-to-left direction is proved in an analogous way. ■

4.2 From Action Models to Belief Base Operations

In this section, we show how to polynomially translate a DEL fragment into $\mathcal{L}_B$. We focus on the DEL fragment, based on so-called *ruby action models*, that encompasses[2]

[2]if we ignore postconditions

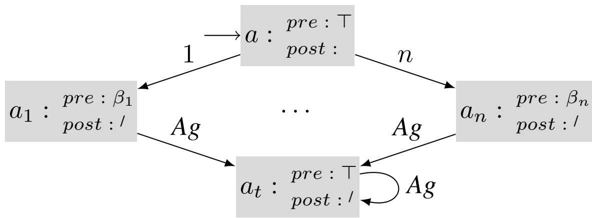


Figure 3: Pointed ruby action model $(\mathcal{A}, a)$.

the action models $\mathcal{A}_{+_J\alpha}$ used in the translation tr_{BA} (see Figure 2).

Ruby Action Models

Definition 10 *A ruby action model $\mathcal{A}, a$ is a pointed action model of the form given in Figure 3 where $\beta_1, \ldots, \beta_n$ are propositional formulas (that is, boolean combinations of propositional variables).*

A ruby action model corresponds to the concurrent private announcement of β_1 to agent $1, \ldots,$ of β_n to agent n. Interestingly, the action model $\mathcal{A}_{+_J\alpha}$ of Figure 2 in which we ignore the postcondition is equivalent to the ruby action model with $\beta_i = tr_{BA}(\alpha)$ for all $i \in J$ and $\beta_i = \top$ for all $i \notin J$.

In the definition of ruby action models, we suppose that the root a_0 is linked to a unique action a_i for agent i. This definition can be leveraged also if this uniqueness is not assumed: several i-successors can be replaced by a single i-successor whose precondition is the disjunction of the preconditions. While, the absence of i-successors can be replaced by a single i-successor whose precondition is $\perp$. The obtained ruby action model is equivalent to the original action model[3].

Note that ruby action models commute ($\mathcal{M} \times \mathcal{A} \times \mathcal{A}' = \mathcal{M} \times \mathcal{A}' \times \mathcal{A}$ when $\mathcal{A}$ and $\mathcal{A}'$ are ruby action models).

Translation We define a translation tr_{AB} from the DEL language with only ruby action models into the language $\mathcal{L}_B$ as follows:

$$\begin{aligned}
tr_{AB}(p) &:= p \\
tr_{AB}(\neg\varphi) &:= \neg tr_{AB}(\varphi) \\
tr_{AB}(\varphi \wedge \psi) &:= tr_{AB}(\varphi) \wedge tr_{AB}(\psi) \\
tr_{AB}(\mathsf{K}_i\varphi) &:= \Box_i\varphi \\
tr_{AB}([\mathcal{A}, a]\varphi) &:= [tr_{Act}(\mathcal{A}, a)]tr_{AB}(\varphi)
\end{aligned}$$

where $tr_{Act}(\mathcal{A}, a)$ is the sequence of operations that $[+_1\beta_1; \ldots; +_n\beta_n]$; when $\mathcal{A}, a$ is the action model depicted in Figure 3. Note that tr_{AB} is computable in polynomial time.

Theorem 2 *Let φ be a DEL formula only containing ruby action models. Then: φ is DEL satisfiable iff $tr_{AB}(\varphi)$ is satisfiable for the class* $\mathbf{M}$.

[3] This is true because there is an emulation (see (van Eijck, Ruan, and Sadzik 2012)) between the original action model and the obtained ruby action model.

Before proving Theoreom 2, we need to make a detour and explain how to encode tree-like epistemic models. For the rest of the section, we suppose that Ap is finite and contains the atomic propositions in φ.

Tree-like Epistemic Models When a DEL formula is satisfiable, it is in a tree. This comes from the fact that epistemic logic has the tree-like model property (see Th. 5.2 in (Vardi 1996), or Proposition 2.15 in (Blackburn, de Rijke, and Venema 2001)), and that any DEL formula is equivalent to a formula in standard epistemic logic (van Ditmarsch, van der Hoek, and Kooi 2007a). We consider here tree-like epistemic models, in which the actual world is the root of the tree. Given a tree-like epistemic model τ with root w and a ruby action model $(\mathcal{A}, a)$, we call $\tau \times (\mathcal{A}, a)$ the connected component of $\tau \times \mathcal{A}$ containing the world (w, a). This component is still tree-like, and $(\tau \times (\mathcal{A}, a), (w, a))$ is bisimilar to $(\tau \times \mathcal{A}, (w, a))$.

We first give a function describing any tree-like epistemic model. We define $desc(\tau)$ by induction on τ:

$$desc(w) = \mathsf{Prop}(w) \wedge \bigwedge_{i \in Ag} \Box_i \perp$$

$$desc\left(\begin{array}{ccc} & w & \\ J_1 \swarrow & & \searrow J_m \\ \tau_1 & \cdots & \tau_m \end{array}\right) = \mathsf{Prop}(w) \wedge \bigwedge_{i \in Ag} \Box_i \bigvee_{k \mid i \in J_k} desc(\tau_k) \wedge \bigwedge_{i,k \mid i \in J_k} \Diamond_i desc(\tau_k)$$

where $\mathsf{Prop}(w)$ is $\bigwedge_{p \in Ap \mid p \text{ is true in } w} p \wedge \bigwedge_{p \in Ap \mid p \text{ is false in } w} \neg p$ (we restrict to relevant atomic propositions, so Ap is finite here). The formula $desc(w)$ intuitively describes the valuation in w and that there are no successors in w. The formula $desc\left(\begin{array}{ccc} & w & \\ J_1 \swarrow & & \searrow J_m \\ \tau_1 & \cdots & \tau_m \end{array}\right)$ intuitively describes the valuation w, says that successors are among trees $\tau_1, \ldots, \tau_k$ and finally says that each tree $\tau_1, \ldots, \tau_k$ is present as a successor.

It is known that the DEL version of this function, which for any τ gives the formula $tr_{BA}(desc(\tau))$, describes epistemic models up to $depth(\tau)$-bisimulation (Moss 2007). We first show that this description function similarly describes belief bases up to bisimulation.

Proposition 1 *For any DEL formula φ in which there are no dynamic operators, for any state B, context Cxt and tree-like epistemic model τ with root w, if $B, Cxt \models desc(\tau)$ then $B, Cxt \models tr_{AB}(\varphi)$ iff $\tau, w \models \varphi$.*

PROOF.
This is shown by induction on φ. Suppose τ is of the form $\begin{array}{ccc} & w & \\ J_1 \swarrow & & \searrow J_m \\ \tau_1 & \cdots & \tau_m \end{array}$.

The case of propositional variables is straightforward as $B, Cxt \models \mathsf{Prop}(w)$. The cases of boolean operators are standard.

If $B, Cxt \models \Box_i tr_{AB}(\varphi)$, consider w' such that wR_iw'. Then w' is the root of τ_k for some k such that $i \in J_k$.

As $B, Cxt \models desc(\tau)$, we have in particular $B, Cxt \models \Diamond_i desc(\tau_k)$. Hence there exists B' such that $B\mathcal{R}_i B'$ and $B', Cxt \models desc(\tau_k)$. But then by the induction hypothesis, $B', Cxt \models tr_{AB}(\varphi)$ iff $\tau_k, w' \models \varphi$, and $B', Cxt \models tr_{AB}(\varphi)$ as $B\mathcal{R}_i B'$. Hence $\tau, w' \models \varphi$ and $\tau, w \models \mathsf{K}_i\varphi$.

If $\tau, w \models \mathsf{K}_i\varphi$, consider B' such that $B\mathcal{R}_i B'$. Then there exists k such that $i \in J_k$ and $B', Cxt \models desc(\tau_k)$. Call w' the root of τ_k in τ. Then $wR_i w'$ so $\tau, w' \models \varphi$, and by the induction hypothesis $B', Cxt \models tr_{AB}(\varphi)$ iff $\tau_k, w' \models \varphi$, the latter being equivalent to $\tau, w' \models \varphi$. Hence $B', Cxt \models tr_{AB}(\varphi)$ and $B, Cxt \models \Box_i tr_{AB}(\varphi)$. ■

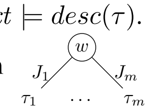
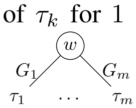
In order to generalize this correspondence to formulas which *do* include dynamic operators, we must first establish that the translation of ruby action models effectively simulates the effects of these actions in belief bases. More precisely, the formula $tr_{BA}(desc(\tau)) \rightarrow [(\mathcal{A}, a)]tr_{BA}(desc(\tau \times (\mathcal{A}, a)))$ is valid in DEL for any tree-like epistemic model τ and pointed ruby action model $(\mathcal{A}, a)$; we show a corresponding property for belief bases.

Proposition 2 *For any tree-like epistemic model τ, for any ruby action model $(\mathcal{A}, a)$, state B and context Cxt, if $B \models desc(\tau)$ then $B^{tr(\mathcal{A},a)}, Cxt \models desc(\tau \times (\mathcal{A}, a))$.*

PROOF.

We show this by induction on the height on τ. Consider a ruby-like action model such as the one in Figure 3.

If $\tau = w$, then $\tau \times (\mathcal{A}, a)$ is equal to τ and $desc(\tau) = desc(\tau \times (\mathcal{A}, a)) = \mathsf{Prop}(w) \wedge \bigwedge_{i\in Ag} \Box_i \bot$. Consider then B and Cxt such that $B, Cxt \models desc(\tau)$. As $B^{tr_{Act}(\mathcal{A},a)}$ and B agree on all propositional variables, we have $B^{tr_{Act}(\mathcal{A},a)}, Cxt \models \mathsf{Prop}(w)$. Moreover, suppose that $B^{tr_{Act}(\mathcal{A},a)}\mathcal{R}_i B'$ for some agent i and some B' in Cxt: then $B\mathcal{R}_i B'$ as $B_i \subseteq B_i^{tr_{Act}(\mathcal{A},a)}$. But $B, Cxt \models \Box_i\bot$, so there can be no such state B', and therefore $B^{tr_{Act}(\mathcal{A},a)}, Cxt \models \Box_i\bot$ for all i. Hence $B^{tr_{Act}(\mathcal{A},a)}, Cxt \models desc(\tau)$.

Suppose now that τ is of the form , and call w_k the root of τ_k for $1 \leqslant k \leqslant m$. Then $\tau \times (\mathcal{A}, a)$ is bisimilar to  where for every $k \leqslant m$, $G_k = \{i \in J_k \mid \tau, w_k \models \beta_i\}$; and

$$desc(\tau \times (\mathcal{A}, a)) = \mathsf{Prop}(w) \wedge \bigwedge_{i\in Ag} \Box_i \bigvee_{k \mid i\in G_k} desc(\tau_k)$$
$$\wedge \bigwedge_{i,k \mid i\in G_k} \Diamond_i desc(\tau_k).$$

Consider B and Cxt such that $B, Cxt \models desc(\tau)$. Once again, it is clear that $B^{tr_{Act}(\mathcal{A},a)}, Cxt \models \mathsf{Prop}(w)$. Consider now a state $B' \in Cxt$ such that $B^{tr_{Act}(\mathcal{A},a)}\mathcal{R}_i B'$. Then as before, $B\mathcal{R}_i B'$,and therefore $B', Cxt \models \bigvee_{k \mid i\in J_k} desc(\tau_k)$, that is, there exists a $k \leqslant m$ such that $i \in J_k$ and $B', Cxt \models desc(\tau_k)$. Moreover, as $\beta_i \in B^{tr_{Act}(\mathcal{A},a)}$, it must be the case that $B', Cxt \models \beta_i$. Then by Proposition 1, we have $\tau_k, w_k \models \beta_i$, hence $\tau, w_k \models \beta_i$ and $i \in G_k$. Therefore $B^{tr_{Act}(\mathcal{A},a)}, Cxt \models \Box_i \bigvee_{k \mid i\in G_k} desc(\tau_k)$ for all i in Ag.

It remains to show that $B^{tr_{Act}(\mathcal{A},a)}, Cxt \models \Diamond_i desc(\tau_k)$ for all i and k such that $i \in G_k$. Consider such a i and k: then $i \in J_k$ and $\tau, w_k \models \beta_i$. As $i \in J_k$ and $B, Cxt \models desc(\tau)$, we know that $B, Cxt \models \Diamond_i desc(\tau_k)$, that is, there exists some B' such that $B\mathcal{R}_i B'$ and $B', Cxt \models desc(\tau_k)$. But then by Proposition 1, $B', Cxt \models \beta_i$ as $\tau, w_k \models \beta_i$. Hence $B^{tr_{Act}(\mathcal{A},a)}\mathcal{R}_i B'$, and $B^{tr_{Act}(\mathcal{A},a)}, Cxt \models \Diamond_i desc(\tau_k)$. We conclude that $B^{tr_{Act}(\mathcal{A},a)}, Cxt \models desc(\tau \times (\mathcal{A}, a))$. ■

We can now extend the result of Proposition 1 to any formulas of the DEL language in which all actions are ruby action models.

Proposition 3 *For any formula φ of the DEL language in which all action models are ruby action models, for any state B, context Cxt and tree-like epistemic model τ with root w, if $B, Cxt \models desc(\tau)$ then $B, Cxt \models tr_{AB}(\varphi)$ iff $\tau, w \models \varphi$.*

PROOF.

We show this by induction on φ. We have shown all cases in the proof of Proposition 1 except for that of dynamic operators.

If $B, Cxt \models desc(\tau)$, then:

$$B, Cxt \models [tr_{Act}(\mathcal{A}, a)]tr_{AB}(\varphi)$$
iff $B^{tr_{Act}(\mathcal{A},a)}, Cxt \models tr_{AB}(\varphi)$
iff $\tau \times \mathcal{A}, (w, a) \models \varphi$ (by IH with Proposition 2)
iff $\tau, w \models [\mathcal{A}, a]\varphi$.

This ends the proof. ■

Correctness of the Translation We are now ready to prove Theorem 2. We show both directions of the equivalence separately.

Proposition 4 *For any formula φ containing only ruby action models, if φ is DEL-satisfiable then $tr_{AB}(\varphi)$ is B-satisfiable.*

PROOF.

Suppose that φ is DEL-satisfiable. Then by the tree-like model property, φ is satisfied in a tree-like epistemic model $\tau = (W, (R_i)_{i\in Ag}, V)$ with root w: $\tau, w \models \varphi$. For each $v \in W$, consider a fresh variable p_v (appearing neither in φ not in $V(w)$ for $w \in W$) and define the model $\tau' = (W, (R_i)_{i\in Ag}, V')$ such that for all $v \in W$, $V'(v) = V(v) \cup \{p_v\}$. Then we also have $\tau', w \models \varphi$. Now define the state $B^v = \big((B_i^v)_{i\in Ag}, V^v\big)$ as follows:

$$B_i^v = \{ \bigvee_{u \mid vR_i u} p_u\}$$
$$V^v = V'(v)$$

Consider the context $Cxt^W = \{B^v \mid v \in W\}$. Then $B^w, Cxt^W \models desc(\tau')$. Hence $B^w, Cxt^W \models tr_{AB}(\varphi)$? by Proposition 3. ■

Proposition 5 *For any formula φ containing only ruby action models, if $tr_{AB}(\varphi)$ is B-satisfiable then φ is DEL satisfiable.*

PROOF.
For any state B and context Cxt, define the epistemic model $\mathcal{M}^{B,Cxt} = (W^{B,Cxt}, (R_i^{B,Cxt})_{i\in Ag}, V^{B,Cxt})$, where:

$$\begin{aligned} W^{B,Cxt} =& \{B\} \cup Cxt;\\ R_i^{B,Cxt} =& \{(B,B') \mid B' \in Cxt \text{ and } B\mathcal{R}_i B'\}\\ &\cup \{(B',B'') \mid B',B'' \in Cxt \text{ and } B'\mathcal{R}_i B''\}\\ &\text{for all } i \in Ag;\\ V^{B,Cxt}(B') =& \{p \mid B' \models p\} \text{ for all } B' \in W. \end{aligned}$$

An induction on φ proves that for any formula φ of the DEL language containing no dynamic operators, for any B and Cxt, $B, Cxt \models tr_{AB}(\varphi)$ iff $\mathcal{M}^{B,Cxt}, B \models \varphi$. (For the proof, remark that given B, Cxt, and any $B' \in Cxt$, $(M^{B,Cxt}, B')$ and $(M^{B',Cxt}, B')$ are equal when $B \in Cxt$ and bisimilar when $B \notin Cxt$.)

Consider now a formula φ of the DEL language only containing ruby action models, and suppose that $B, Cxt \models \varphi$ for some B and Cxt. We know that $M^{B,Cxt}, B$ is $md(\varphi)$-bisimilar to some tree-like epistemic model (τ, w) (consider the unravelling of $M^{B,Cxt}, B$ up to depth $md(\varphi)$), where w is the root of τ. In particular, $M^{B,Cxt}, B \models tr_{BA}(desc(\tau))$, hence $B, Cxt \models desc(\tau)$ as $tr_{BA}(desc(\tau))$ contains no dynamic operators. By Proposition 3, this implies that for any formula ψ of the DEL language only containing ruby action models, $B, Cxt \models tr_{AB}(\psi)$ iff $\tau, w \models \psi$. In particular, $\tau, w \models \varphi$, hence φ is DEL satisfiable. ■

5 Introspective Variant

In this section, we explain the changes to make to keep translations from $\mathcal{L}_B$ into $\mathcal{L}_{DEL}$, and from $\mathcal{L}_{DEL}$ into $\mathcal{L}_B$ wrt. to the introspective variant. The latter translation will however require some technical restrictions. The underlying epistemic logic is K45: the epistemic relations are transitive and Euclidean (see Def. 2.13 of (Van Ditmarsch, van Der Hoek, and Kooi 2007b)).

5.1 Definition

In (Lorini 2020) an alternative definition of the epistemic indistinguishability relation is given. It works for introspective agents that have perfect knowledge of their belief bases. Specifically, for a state to be considered possible by an introspective agent, (i) it must satisfy all information in the agent's actual belief base, and (ii) the agent should have the same belief base in the actual state and in the epistemically accessible state.

Definition 11 (Epistemic alternatives) *The introspective variant of the epistemic indistinguishability relation is the binary relation $\mathcal{R}_i \subseteq \mathbf{S} \times \mathbf{S}$ such that, for all $B = \big((B_i)_{i\in Ag}, V\big), B' = \big((B'_i)_{i\in Ag}, V'\big) \in \mathbf{S}$:*

$$B\mathcal{R}_i^{Int} B' \text{ if and only if } (i)\forall\alpha \in B_i : B' \models \alpha,\quad (ii) B_i = B'_i.$$

In other words, the previous definition adds to Definition 3 the introspection condition (ii). It is easy to verify that the relation $\mathcal{R}_i^{Int}$ so defined is transitive and Euclidean.

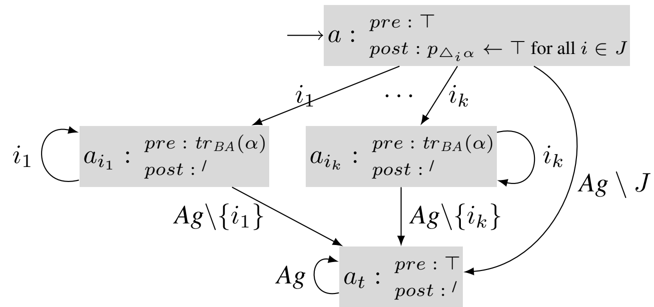


Figure 4: Action model $\mathcal{A}^{intr}_{+_J\alpha}$ that simulates adding α in bases of agents in $J = \{i_1, \dots, i_k\}$ in the introspective variant.

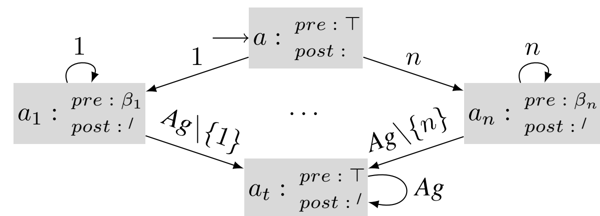


Figure 5: Introspective ruby action model.

5.2 From Belief Base Operations to Action Models

When considering introspective agents (i.e., when the epistemic indistinguishability relations are defined according to Definition 11), we need to slightly redefine the translation from the language $\mathcal{L}_B$ to the language $\mathcal{L}_{DEL}$ given in Section 4.1. In particular, we need to define a new translation tr^{intr}_{BA} whose lines 1-5 are identical to lines 1-5 of translation tr_{BA} and whose line 6 is:

$$tr^{intr}_{BA}([+_J\alpha]\varphi) = [\mathcal{A}^{intr}_{+_J\alpha}, a]tr^{intr}_{BA}(\varphi),$$

where $\mathcal{A}^{intr}_{+_J\alpha}$ is the action model for introspective agents depicted in Figure 4. The following theorem is the analog of Theorem 1 for introspective agents. We omit its proof since it is similar to that of Theorem 1.

Theorem 3 *Let the epistemic indistinguishability relation $\mathcal{R}_i$ be defined according to Definition 11 and let $\varphi \in \mathcal{L}_B$. Then, φ is satisfiable for the class $\mathbf{M}$ if and only if $tr^{intr}_{BA}(\varphi)$ is DEL-satisfiable.*

5.3 From Action Models to Belief Base Operations

In this section, we discuss the natural generalisation of tr_{AB} to the introspective case.

Introspective Ruby Action Models

Definition 12 *An introspective ruby action model $\mathcal{A}, a$ is a pointed action model of the form given in Figure 5 where $\beta_1, \dots, \beta_n$ are propositional formulas.*

The translation tr^{intr}_{AB} differs from tr_{AB} given in Subsection 4.2 in two aspects. First the language is the DEL one with *introspective* ruby action models. Second, we need to add a special variable for each action model $\mathcal{A}$. We define:

$$tr_{AB}^{intr}([\mathcal{A},a]\varphi) := [tr_{Act}(\mathcal{A},a)]tr_{AB}^{intr}(\varphi)$$

where $tr_{Act}(\mathcal{A},a)$ is the sequence of operations that $[+_1\beta_1;\ldots;+_n\beta_n;+_1p_{\mathcal{A}};\ldots;+_np_{\mathcal{A}};+_1p^1_{\mathcal{A}};\ldots;+_np^n_{\mathcal{A}}]$; when $\mathcal{A},a$ is the action model depicted in Figure 5, and where $p_{\mathcal{A}}$ and all $p^i_{\mathcal{A}}$ are fresh atomic propositions. Note that tr_{AB}^{intr} is still computable in polynomial time.

Just like the ruby action models of Definition 10, introspective ruby action models commute. Thus, the sole information of action model $\mathcal{A}$ being executed is sufficient; the order of the executions is not important. In other words, the propositions $p_{\mathcal{A}}$ are sufficient to represent the list of already executed actions. Also, as the variables $p_{\mathcal{A}}$ are fresh, they are supposed to be initially false.

Theorem 4 *Let φ be a DEL formula only containing introspective ruby action models. Then: φ is DEL K45-satisfiable implies $tr_{AB}^{intr}(\varphi)$ is satisfiable for the class $\mathbf{M}$ with the introspective relations.*

The proof of Theorem 4 is involved. We start by giving some definitions and notations surrounding introspective ruby action models. For any formula φ containing only introspective ruby action models, we call $\mathbf{Seq}(\varphi)$ the set of sequences of pointed action models corresponding to sequences of actions found in φ, defined inductively as:

$$\begin{aligned}
\mathbf{Seq}(p) &= \{\epsilon\} \qquad (\epsilon \text{ is the empty sequence})\\
\mathbf{Seq}(\varphi\wedge\psi) &= \mathbf{Seq}(\varphi)\cup\mathbf{Seq}(\varphi)\\
\mathbf{Seq}(\neg\varphi) &= \mathbf{Seq}(\varphi)\\
\mathbf{Seq}([\mathcal{A},a]\varphi) &= \{(\mathcal{A},e)\sigma \mid \sigma\in\mathbf{Seq}(\varphi)\\
&\qquad \text{and } e \text{ is an event in } \mathcal{A}\}
\end{aligned}$$

Example 4 *For example, given two pointed action models $(\mathcal{A},a)$ and $(\mathcal{A}',a')$, we have $\mathbf{Seq}([\mathcal{A},a]p \wedge [\mathcal{A}',a'](q\wedge\neg[\mathcal{A},a]p)) = \{(\mathcal{A},e),(\mathcal{A}',e'),(\mathcal{A}',e')(\mathcal{A},e) \mid$ e is an event of $\mathcal{A}$ and e' is an event of $\mathcal{A}'\}$.*

If w is a world in an epistemic model $\mathcal{M}$ and σ is a sequence $(\mathcal{A}_1,e_1)\ldots(\mathcal{A}_m,e_m)$ of pointed action models, we abbreviate the model $(\mathcal{M}\times\mathcal{A}_1)\times\cdots\times\mathcal{A}_m$ to $\mathcal{M}\times\sigma$ and the world $((w,e_1),\ldots,e_m)$ in $\mathcal{M}\times\sigma$ to (w,σ), when that world is defined in $\mathcal{M}\times\sigma$.

As ruby action models commute, the order of the ruby action models of a sequence σ does not matter when constructing $\mathcal{M}\times\sigma$; we will therefore be identifying any sequence σ of actions with the set of actions found in σ. In particular, two sequences σ and σ' are considered equal when they consist of the same actions.

Finally, given a sequence σ of introspective ruby action models, we call $Add_i(\sigma)$ the set of formulas β such that $+_i\beta$ appears in $tr_{Act}(\mathcal{A},a)$ for some action $(\mathcal{A},a)$ or $(\mathcal{A},a_i)$ in σ. Formally:

$$\begin{aligned}
Add_i(\sigma) = \{\beta \mid \exists(\mathcal{A},e)\in\sigma, e\in\{a,a_i\}\\
\text{and } pre_A(a_i)=\beta\}
\end{aligned}$$

From Epistemic Models to Contexts Remark that introspective ruby action models preserve the K45 properties: the product of a K45 epistemic model with an introspective ruby action model is again a K45 epistemic model.

We now define contexts simulating the effects of actions in an epistemic model. Given a formula φ containing only introspective ruby action models and a K45 epistemic model $(\mathcal{M},r)$ with $\mathcal{M}=(W,(R_i)_{i\in Ag},V)$, we take for every action model $\mathcal{A}$ in φ three fresh variable $p_{\mathcal{A}}$, $p^i_{\mathcal{A}}$ and $p^t_{\mathcal{A}}$ appearing neither in φ nor in $V(w)$ for any $w\in W$, and we similarly take for every $w\in W$ a fresh variable p_w. We define the following context:

$$\begin{aligned}
Cxt^{\varphi,\mathcal{M}} = \{B^{w,\sigma} \mid \sigma\in\mathbf{Seq}(\varphi) \text{ and}\\
(w,\sigma) \text{ is a world of } \mathcal{M}\times\sigma\}
\end{aligned}$$

where for all $w\in W$ and $\sigma\in\mathbf{Seq}(\varphi)$, $B^{w,\sigma} = \big((B_i^{w,\sigma})_{i\in Ag},, V^{w,\sigma}\big)$ with:

$$\begin{aligned}
B_i^{w,\sigma} =& \{\bigvee_{v\mid wR_iv} p_v\}\cup\{p_{\mathcal{A}} \mid (\mathcal{A},e)\in\sigma\}\cup Add_i(\sigma)\\
&\cup\{p^i_{\mathcal{A}} \mid (\mathcal{A},e)\in\sigma, e\in\{a,a_i\}\}\\
&\cup\{p^t_{\mathcal{A}} \mid (\mathcal{A},e)\in\sigma, e\notin\{a,a_i\}\}\\
V^{w,\sigma} =& V(w)\cup\{p_w\}\cup\{p_{\mathcal{A}} \mid (\mathcal{A},e)\in\sigma\}\\
&\cup\{p^i_{\mathcal{A}} \mid i\in Ag, (\mathcal{A},a_i)\in\sigma\}\cup\{p^t_{\mathcal{A}} \mid (\mathcal{A},a_t)\in\sigma\}.
\end{aligned}$$

Intuitively, we want every state $B^{w,\sigma}$ to correspond to the world (w,σ). The fresh variables identify the worlds of w, the actions models in σ, as well as instances of events a_i and a_t. The constraints in $B_i^{w,\sigma}$ correspond to the following properties:

- from the definition of product updates, it follows that is $(w,\sigma)R_i(w',\sigma')$ then wR_iw';
- the inclusion of $p_{\mathcal{A}}$ for all $(\mathcal{A},e)\in\sigma$ identify the world as being part of the model $\mathcal{M}\times\sigma$;
- $Add_i(\sigma)$ corresponds to the new beliefs introduced for i by σ in (w,σ);
- when $e\in\{a,a_i\}$ for some $(\mathcal{A},e)\in\sigma$ then eR_ie' iff $e'=a_i$;
- when $e\notin\{a,a_i\}$ for some $(\mathcal{A},e)\in\sigma$ then eR_ie' iff $e'=a_t$.

We are now going to show that this context corresponds to the collection of all models $\mathcal{M}\times\sigma$ for $\sigma\in\mathbf{Seq}(\varphi)$, which will then allow us to simulate model checking of φ in $\mathcal{M}$.

Proposition 6 *Let φ be a DEL formula containing only introspective ruby action models. Let $(\mathcal{M},r)$ be a K45 epistemic model , and $Cxt^{\varphi,\mathcal{M}}$ be the corresponding context. Then $\mathcal{M},r\models\varphi$ iff $B^r, Cxt^{\varphi,\mathcal{M}}\models tr_{AB}^{intr}(\varphi)$.*

Proof of Theorem 4 Suppose that φ is K45-satisfiable. Then φ is satisfiable in a K45 epistemic model. Consider a K45 epistemic model $(\mathcal{M},r)$ satisfying φ. By Proposition 6, we have $B^{r,\epsilon}, Cxt^{\varphi,\mathcal{M}}\models tr_{AB}^{intr}(\varphi)$.

Other Direction Suppose that $tr_{AB}^{intr}(\varphi)$ is satisfiable. To adapt the proof of Proposition 5, we need to construct a K45 epistemic model for φ from the unravelling of the belief base inferred epistemic model for $tr_{AB}^{intr}(\varphi)$. The issue will be that the application of a_i *must* exactly correspond to the addition of the β_i in the belief bases. And the obtained state *must* be in the context. To cope with that, we simply suppose that the context is the set $\mathbf{S}$ of *all* states.

Theorem 5 *$tr^{intr}_{AB}(\varphi)$ is satisfiable for the class of states whose context is $\mathbf{S}$ implies φ is DEL K45-satisfiable.*

6 Related Work

Let us mention syntactic approaches for describing epistemic actions. Action models of DEL are captured up to n-bisimilarity by a language of programs introduced in (French, Hales, and Tay 2014). A logical language for capturing specific kinds of private/public announcements in which propositions expressing which agents are listening to the source is introduced in (Bolander et al. 2016). However in all of these approaches only actions are syntactic, while the underlying models remain Kripke structures.

As emphasized in the introduction, the multi-agent semantics for epistemic logic using belief bases was first introduced in (Lorini 2018; 2020). It allows us to represent both agents' beliefs about propositional facts and their higher-order beliefs (i.e., an agent i's belief about an agent j's belief).[4] It was applied to the formalization of a large variety of epistemic concepts and of aspects of both individual and collective epistemic reasoning including graded belief (Lorini and Schwarzentruber 2021a), distributed belief (Herzig et al. 2020), common belief (Lorini and Rapion 2022), multi-agent belief revision and planning (Lorini and Schwarzentruber 2021b; Davila et al. 2021). A preliminary comparison of this semantics and the DEL update semantics was given in (Lorini 2020) in which it was shown that private belief base expansion corresponds to a specific kind of DEL private announcement represented through arrow models (Kooi and Renne 2011). In particular, it was shown that the epistemic model obtained via private belief expansion is bisimilar to the epistemic model obtained via private announcement relative to the language of standard of epistemic logic.

The semantics for epistemic logic using belief bases can be seen as a way of defining a single 'canonical' model from the description of a state. This point was explored in (Lorini 2019) in the static setting of epistemic logic. In this paper, we extend the analysis to the dynamic setting (Theorem 2).

In the belief base approach, as seen in Definition 1, the state contains a set of formulas for each agent, and then a 'canonical' model is inferred by the relations of Definition 3. There are other approaches that define a 'canonical' model based on what agents see. The description of what agents see is either given by positions of agents in a geometrical environment (Balbiani, Gasquet, and Schwarzentruber 2013; Gasquet, Goranko, and Schwarzentruber 2016), or given by an abstract specification (Cooper et al. 2021). A similar idea is at the center of symbolic model checking for epistemic logic (van Benthem et al. 2015). Techniques described in Section 4 and 5 may also be applied to the comparison of these frameworks with action models.

[4]Related work on the connection between the syntactic representation of preferences based on priority graphs and Kripke models for preference representation can be found in (de Jongh and Liu 2009; Liu 2011; Souza and Moreira 2021). It was extended to deontic logic in (van Benthem, Grossi, and Liu 2014). This work is focused on the single-agent case and does not consider reasoning about higher-order beliefs or preferences.

7 Conclusion

This paper is about a seminal bridge between two opposite approaches for dynamic knowledge change: the syntactic one with belief base operations, and the semantic one with DEL action models (see Theorems 1 and 2). We also discussed the introspective case (see Theorems 3, 4, 5).

For going from belief bases to action models, we proposed action models for simulating the operation of privately expanding belief bases of some agents by an input α (see Figures 2 and 4). For the other direction, we restricted to so-called *ruby* action models (see Figures 3 and 5). For going from (ruby) action models to belief base operations, we adopted two different proof techniques. For the non-introspective case, we simply add each formula β_j to the base of agent j for simulating a ruby action model. We were also able to fully describe (tree-like) epistemic models obtained by product as a formula of $\mathcal{L}_B$. For the introspective case, the simulation of a ruby action model is more involved and requires the use of fresh atomic propositions.

Our work leaves many open questions for future work.

Modal Formulas. Currently, formulas in ruby action models are Boolean. We plan to generalize to any modal formulas. This is challenging because formulas in action models can only be about implicit beliefs, and the translation from implicit to explicit beliefs is not straightforward.

Belief Revision. Belief bases form an elegant formalism for bringing belief revision in a multi-agent context (Lorini and Schwarzentruber 2021b). We aim at studying the link with the plausibility models for multi-agent belief revision (Baltag and Smets 2006).

Epistemic Planning. On the one hand, multi-agent epistemic planning with belief bases is decidable, even with belief revision (Lorini and Schwarzentruber 2021b). On the other hand, epistemic planning in DEL when models are trees and propositional is also decidable (Bolander, Holm Jensen, and Schwarzentruber 2015). Future work will be devoted to investigate the connection between the two approaches to epistemic planning.

Public Announcement. Recently, the belief base approach was extended to capture a notion of public announcement (Lorini and Rapion 2022). This paper paves the way for investigating the connection between DEL-notions of public and semi-private announcement and corresponding notions expressed the belief base semantics.

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