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On the Representation of Darwiche and Pearl's Epistemic States for Iterated Belief Revision

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Abstract

The seminal characterization of iterated belief revision was proposed by Darwiche and Pearl, which uses an abstract notion of epistemic states. In this work we look for a canonical representation of these epistemic states. Total preorders are not expressive enough to be used as such a canonical representation. Actually, we show that some operators can even not be represented on a countable epistemic space. Nonetheless, under a very reasonable assumption on the epistemic space, we show that OCFs (Ordinal Conditional Functions) can be considered as a canonical representation.

1 Introduction

Belief revision is an important issue in many domains. In particular, if one wants to conceive a truly autonomous agent, it has to be able to revise incorrect information on its model of its environment.

The standard model for belief revision as been proposed by Alchourron, Gärdenfors and Makinson (Alchourrón, Gärdenfors, and Makinson 1985; Gärdenfors 1988; Fermé and Hansson 2011), and is called the AGM theory. This theory perfectly gives account of a one step revision, as attested by several representation theorems (Alchourrón, Gärdenfors, and Makinson 1985; Alchourrón and Makinson 1985; Katsuno and Mendelzon 1991; Grove 1988; Hansson 1994; del Cerro and Herzig 1996), showing that the AGM rationality postulates correspond exactly to the most sensible way to build revision operators. The close formal links between belief revision and non-monotonic inference relations (Makinson and Gärdenfors 1989; Gärdenfors 1990) on one side and possibilistic logic (Dubois and Prade 1991) on the other side, also prove that this theory adequatly captures this fundamental process.

Nonetheless, one weakness of the AGM theory is that it provides no constraint on sequences of revision, allowing several inadequate behaviors. This may be seen as a real problem, since any autonomous agent is typically expected to perform a large number of revisions during its activity.

To fill the gap, many papers aimed to identify the correct model for *iterated* belief revision, which was a very active topic during the nineties (Nayak 1994; Nayak et al. 1994; Nayak et al. 1996; Rott 2009). What is considered now as the most convincing approach for iterated belief revision was proposed by Darwiche and Pearl (1997), who introduced four additional postulates about successive revisions. Some extensions of this work have also been investigated (Booth and Meyer 2006; Jin and Thielscher 2007; Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010).

While the AGM framework uses simple logical theories to represent an agent's epistemic state, the Darwiche and Pearl's framework requires more complex objects (Darwiche and Pearl 1994; Freund and Lehmann 1994; Lehmann 1995; Friedman and Halpern 1999). Indeed, if one needs not only to represent the current beliefs of an agent but also some information on the relative plausibility of currently disbelieved information, then this additional information has to be represented somewhere to guide the iterated revision process. Yet epistemic states, in Darwiche and Pearl's framework, are very abstract objects: we know very little about them, besides the fact that each epistemic state is associated with a logical formula that represents the current beliefs of an agent. This can be seen both as a strength and a weakness of the approach. One can see it as a weakness, because as opposed to the standard AGM case, the Darwiche and Pearl's representation theorem is not definitional: one can characterize the revised beliefs of the agent (the formula associated with the epistemic state), but not the whole epistemic state. One can also see it as a strength, because it is general enough to encompass all possible ways to practically encode the epistemic states.

The Darwiche and Pearl's representation theorem is an extension of the Katsuno and Mendelzon's one for AGM belief revision (Katsuno and Mendelzon 1991). It states that each epistemic state can be associated with a total preorder on propositional worlds, such that the new beliefs after revision of this epistemic state is obtained by selecting the most plausible models of the new piece of information w.r.t. this total preorder. It may then be tempting to identify the epistemic states and these total preorders on worlds: not only the representation theorem itself seems to suggest total preorders on worlds as a canonical representation, but they are also a very intuitive way to represent the relative plausibility of each formula. In fact, some works explicitly choose this representation for epistemic states (Booth and Meyer 2011; Ramachandran, Nayak, and Orgun 2012; Booth and Chandler 2016; Booth and Chandler 2019), and this is enough for a lot of applications.

But one has to be aware not to make a step further, by considering these total preorders to be *the* canonical representation of Darwiche and Pearl's epistemic states, and to restrict them to this case, as it is done in some works[1]. For instance, Aravanis *et. al* (2019) showed an example of a revision operator that can not be represented as functional transitions between total preorders, would those total preorders be the ones given by the Darwiche and Pearl's representation theorem. In this paper, we show without any further assumption that total preorders are not the canonical representation of Darwiche and Pearl's epistemic states via an example using ordinal conditional functions (OCFs), a well-known representation of epistemic states for belief change. Then, we investigate the extent to which a canonical representation can be found for these epistemic states. To do so, based on the work initiated in (Aravanis, Peppas, and Williams 2019; Schwind and Konieczny 2020), we make precise the condition under which a revision operator defined on a given epistemic space can be "instantiated" into another epistemic space. Would such a target epistemic space exist for every Darwiche and Pearl's revision operator, that epistemic space could be considered as an appropriate canonical representation of Darwiche and Pearl's epistemic states.

We start from the important remark that whatever the chosen representation, every epistemic state can be viewed as a black box associating each *finite* sequence of formulae with a formula representing the beliefs of the agent after the successive revision of the epistemic state by each formula from the sequence. Based on that observation, two epistemic states are *strongly equivalent* according to a revision operator if they cannot be distinguished from each other by any such successive revision steps, which means that these epistemic states have the same behavior for that revision operator. So the question now is whether one can find a canonical representation of the whole *quotient set* of epistemic states under this strong equivalence relation. We show that in general this is not possible. The number of possible (quotient set of) epistemic states is just too large. Nevertheless, we show that under the very natural assumption that every epistemic state is reachable from an initial, "empty", epistemic state, through a finite succession of revisions, OCFs are a possible candidate of canonical representation of epistemic states. This suggests that to encode all such possible epistemic states, what is missing in the total preorder representation is the additional numerical information OCFs accordingly provide.

2 Preliminaries

We consider a propositional language $\mathcal{L}_P$ built up from a finite set of propositional variables P and the usual connectives. The set of consistent formulae is denoted by $\mathcal{L}_P^*$. $\bot$ (resp. $\top$) is the Boolean constant always false (resp. true). An interpretation (or world) is a mapping from P to $\{0, 1\}$. The set of all worlds on $\mathcal{L}_P$ is denoted by Ω. $\models$ denotes logical entailment, $\equiv$ logical equivalence, and $[\varphi]$ denotes the set of models of the formula φ.

[1]See (Meng, Kou, and Li 2015) for instance.

In iterated belief change, it is standard to assume that the current set of beliefs of an agent is represented by an *epistemic state*. An epistemic state allows one to represent the current beliefs of the agent and some conditional information guiding the revision process. In all generality, an epistemic state can be any object Ψ from which the set of beliefs of the agent can be extracted through a mapping Bel, so that $Bel(\Psi)$ is a propositional formula from $\mathcal{L}_P$. Formally:

Definition 1 (Epistemic Space)**.** *An* epistemic space $\mathcal{E}$ *is a tuple* $\langle E, Bel\rangle$*, where* E *is a set and* Bel *is a mapping* $Bel : E \to \mathcal{L}_P^*$.

We illustrate this concept with some basic epistemic spaces:

Example 1. *Let us define the* TPO*-based epistemic space*[2] *The* TPO-based epistemic space *is the epistemic space* $\mathcal{E}_{tpo} = \langle U_{tpo}, B_{tpo}\rangle$ *where:*

- U_{tpo} *is the set of all total preorders over the set of all worlds from* Ω*;*
- B_{tpo} *is the mapping associating each total preorder* $\preceq$ *from* U_{tpo} *with a formula* $\psi \in \mathcal{L}_P^*$ *such that* $[\psi] = \min([\top], \preceq)$.

The second example of epistemic space is built with the ordinal conditional functions (OCFs) (Spohn 1988; Williams 1995). An OCF κ is a function associating each world with a non-negative integer[3] such that there is a world ω such that $\kappa(\omega) = 0$.

Example 2. *The* OCF-based epistemic space *is the epistemic space* $\mathcal{E}_{ocf} = \langle U_{ocf}, B_{ocf}\rangle$ *where:*

- U_{ocf} *is the set of all* OCF*s over* Ω*;*
- B_{ocf} *is the mapping associating each* OCF κ *from* U_{ocf} *with a formula* ψ *such that* $[\psi] = \{\omega \mid \kappa(\omega) = 0\}$.

Given an epistemic space $\mathcal{E} = \langle E, Bel\rangle$, a belief revision operator $\circ$ on $\mathcal{E}$ associates every epistemic state Ψ from E and every consistent formula μ with a new epistemic state from E, denoted by $\Psi \circ \mu$, i.e, $\circ$ is a mapping $\circ : E \times \mathcal{L}_P^* \to E$. In the rest of the paper, when a revision operator $\circ$ will be referred to without the epistemic space on which it is defined, we will implicitly assume that $\circ$ is defined on some epistemic space denoted by $\mathcal{E} = \langle E, Bel\rangle$.

Let us recall the set of postulates which are expected for such operators to have a good iterative behavior (Darwiche and Pearl 1997):

Definition 2 (DP operator (Darwiche and Pearl 1997))**.** *A revision operator* $\circ$ *is a* DP operator *if the following properties are satisfied, for each epistemic state* Ψ *and all formulae* μ, μ'*:*

(R*1) $Bel(\Psi \circ \mu) \models \mu$;
(R*2) *If* $Bel(\Psi) \wedge \mu \not\models \bot$*, then* $Bel(\Psi \circ \mu) \equiv Bel(\Psi) \wedge \mu$*;*
(R*3) *If* $\mu \not\models \bot$*, then* $Bel(\Psi \circ \mu) \not\models \bot$*;*

[2]TPO stands for total preorder.

[3]Note that in the original definition (Spohn 1988) OCFs are defined on ordinals. But here, as in mosts cases, the integer restriction is enough. Note however than using the full power of ordinals can prove useful in some cases (Konieczny 2009).

(R*4) *If* $\mu \equiv \mu'$, *then* $Bel(\Psi \circ \mu) \equiv Bel(\Psi \circ \mu')$;
(R*5) $Bel(\Psi \circ \mu) \wedge \mu' \models Bel(\Psi \circ (\mu \wedge \mu'))$;
(R*6) *If* $Bel(\Psi \circ \mu) \wedge \mu' \not\models \bot$, *then* $Bel(\Psi \circ (\mu \wedge \mu')) \models Bel(\Psi \circ \mu) \wedge \mu'$.
(C1) *If* $\alpha \models \mu$, *then* $Bel((\Psi \circ \mu) \circ \alpha) \equiv Bel(\Psi \circ \alpha)$
(C2) *If* $\alpha \models \neg\mu$, *then* $Bel((\Psi \circ \mu) \circ \alpha) \equiv Bel(\Psi \circ \alpha)$
(C3) *If* $Bel(\Psi \circ \alpha) \models \mu$, *then* $Bel((\Psi \circ \mu) \circ \alpha) \models \mu$
(C4) *If* $Bel(\Psi \circ \alpha) \not\models \neg\mu$, *then* $Bel((\Psi \circ \mu) \circ \alpha) \not\models \neg\mu$

The postulates **(R*1-R*6)** are a direct adaptation of the standard KM postulates to epistemic states. The remaining four postulates, **(C1-C4)**, add constraints w.r.t. iteration.

Darwiche and Pearl also provided a characterization of DP operators in terms of total preorders over worlds:

Definition 3 (Faithful assignment)**.** *Given an epistemic space* $\mathcal{E} = \langle E, Bel \rangle$, *a mapping* $\Psi \mapsto \preceq_\Psi$ *associating each epistemic state* $\Psi \in E$ *with a total preorder*[4] *over worlds* $\preceq_\Psi$ *is a* faithful assignment (on $\mathcal{E}$) *if and only if for all worlds* $\omega, \omega' \in \Omega$:

1. *If* $\omega \models Bel(\Psi)$ *and* $\omega' \models Bel(\Psi)$, *then* $\omega \simeq_\Psi \omega'$
2. *If* $\omega \models Bel(\Psi)$ *and* $\omega' \not\models Bel(\Psi)$, *then* $\omega \prec_\Psi \omega'$

Theorem 1 ((Darwiche and Pearl 1997))**.** *An operator* $\circ$ *is a DP operator if and only if there exists a faithful assignment* $\Psi \mapsto \preceq_\Psi$ *that satisfies the following properties:*

CR1. If $\omega \models \mu$ *and* $\omega' \models \mu$, *then* $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\mu} \omega'$
CR2. If $\omega \not\models \mu$ *and* $\omega' \not\models \mu$, *then* $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\mu} \omega'$
CR3. If $\omega \models \mu$ *and* $\omega' \not\models \mu$, *then* $\omega \prec_\Psi \omega' \Rightarrow \omega \prec_{\Psi\circ\mu} \omega'$
CR4. If $\omega \models \mu$ *and* $\omega' \not\models \mu$, *then* $\omega \preceq_\Psi \omega' \Rightarrow \omega \preceq_{\Psi\circ\mu} \omega'$,

and such that for each epistemic state Ψ *and each formula* μ, $[Bel(\Psi \circ \mu)] = \min([\mu], \preceq_\Psi)$.

When a faithful assignment exists for $\circ$, we will call such a faithful assignment a *DP assignment corresponding to* $\circ$. As a matter of fact, when such assignment exists it is unique[5].

Conditions (CR1-CR4) above correspond to the iteration postulates **(C1-C4)**. They impose constraints on the total preorder $\preceq_{\Psi\circ\mu}$: CR1 and CR2, the "rigidity" conditions, say that the order between models of μ is preserved and the order between models of $\neg\mu$ is also preserved. CR3 and CR4 say that there is no worsening between the models of μ and the models of $\neg\mu$.

Theorem 1 has important applications. One of them is that, for some epistemic spaces, it allows to define some DP operators in a constructive way, for which the verification that they satisfy indeed the DP postulates is done almost trivially. Let us illustrate this through the following examples.

Example 3. *We consider the Boutilier's natural revision operator* $\circ_B$ *defined over the epistemic space* $\mathcal{E}_{tpo}$. *This operator associates each total preorder* $\Psi \in U_{tpo}$ *and each formula* μ *with a total preorder* $\Psi \circ_B \mu$ *that satisfies* $\min(\Psi \circ \mu) = \min([\mu], \Psi)$ *and the following condition:*

B. If $\omega, \omega' \notin \min([\mu], \Psi)$, *then* $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi\circ_B\mu} \omega'$, *where* $\preceq_\Psi$ *denotes* Ψ *and* $\preceq_{\Psi\circ_B\mu}$ *denotes* $\Psi \circ_B \mu$.

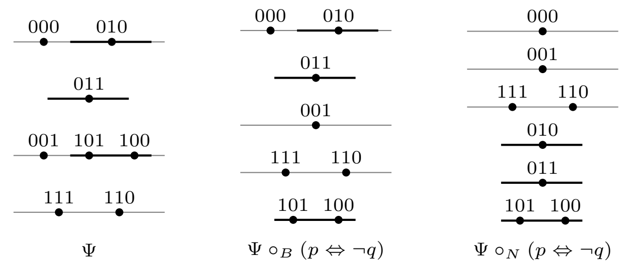

Figure 1: The Boutilier and Nayak operators.

That is, Boutilier's revision operator on $\mathcal{E}_{tpo}$ consists in selecting the set of all models of μ that are minimal according to an input preorder, and defining this set as the first level of the revised preorder while leaving the rest of the preorder unchanged. This is a DP operator: one of the most easy ways to see that is to note that the assignment $\Psi \mapsto \Psi$ is a faithful assignment which satisfies the conditions (CR1-CR4), and that $\min(\Psi \circ \mu) = \min([\mu], \Psi)$. Then by Theorem 1, $\circ_B$ is a DP operator.

Example 4. *Another example is the Nayak's lexicographic operator* $\circ_N$ *defined also in the epistemic space* $\mathcal{E}_{tpo}$. *It is defined by* $\min(\Psi \circ \mu) = \min([\mu], \Psi)$, *conditions (CR1-CR4) and:*

N. If $\omega \models \mu$ *and* $\omega' \not\models \mu$, *then* $\omega \prec_{\Psi\circ_N\mu} \omega'$

Nayak's revision moves all models of μ below all models of $\neg\mu$, and keeps the relationships between worlds of μ (resp. of $\neg\mu$) unchanged. Using the same argument as in the previous example, one can see that this operator is also a DP operator.

Let us illustrate how the behaviors of Boutilier and Nayak revision operators depart from each other.

Example 5. *Let* $P = \{p, q, r\}$. *Figure 1 depicts a total preorder* Ψ *over worlds*[6] *, and the revised total preorders* $\Psi \circ_B (p \Leftrightarrow \neg q)$ *and* $\Psi \circ_N (p \Leftrightarrow \neg q)$. *We have that* $B_{tpo}(\Psi) \equiv p \wedge q$, *and* $B_{tpo}(\Psi \circ_B (p \Leftrightarrow \neg q) \equiv B_{tpo}(\Psi \circ_N (p \Leftrightarrow \neg q)) \equiv p \wedge \neg q$. *Then it is easy to see from the figure that:*

- $B_{tpo}(\Psi \circ_B (p \Leftrightarrow \neg q) \circ_B q \equiv p \wedge q$
- $B_{tpo}(\Psi \circ_N (p \Leftrightarrow \neg q) \circ_N q \equiv \neg p \wedge q \wedge r$,

i.e., $B_{tpo}(\Psi \circ_B (p \Leftrightarrow \neg q) \circ_B q \not\equiv B_{tpo}(\Psi \circ_N (p \Leftrightarrow \neg q) \circ_N q$.

The next example is an operator on the epistemic space of OCFs, that is, it is defined on $\mathcal{E}_{ocf}$.

Example 6. *Given an* OCF κ *and a formula* α, *we define* $\kappa(\alpha) = min\{n : \exists \omega \in [\alpha],\ \kappa(\omega) = n\}$.

Consider an OCF κ, *a formula* μ, *and an integer* x *such that* $x \geq 1$. *The* (μ, x)*-conditionalization (Spohn 1988) of* κ

[4]For each preorder $\preceq$, $\simeq$ denotes the corresponding indifference relation, and $\prec$ the corresponding strict ordering.

[5]That is due to the fact that if an assignment satisfies $[Bel(\Psi \circ \mu)] = \min([\mu], \preceq_\Psi)$, then $\omega \preceq_\Psi \omega'$ iff $\omega \in [\Psi \circ \alpha_{\omega,\omega'}]$ (where $[\alpha_{\omega,\omega'}] = \{\omega, \omega'\}$), from which the unicity follows.

[6]A world ω is at the same or at a lower level than a world ω' iff $\omega \leq_\Psi \omega'$. So minimal (i.e., most plausible) worlds are at the lowest levels.

is the OCF κ' *defined for each world* ω *by:*

$$\kappa'(\omega) = \begin{cases} \kappa(\omega) - \kappa(\mu) & \text{if } \omega \models \mu \\ \kappa(\omega) - \kappa(\neg\mu) + x & \text{if } \omega \not\models \mu \end{cases}$$

The operator $\circ_C$ *on* $\mathcal{E}_{ocf}$ *is defined for each* $\kappa \in U_{ocf}$ *and each* $\mu \in \mathcal{L}_P^*$ *by* $\kappa \circ_C \mu = \kappa'$*, where* κ' *is the* (μ, x)*-conditionalization of* κ*.*

Conditionalization was the first change operator proposed on OCFs (Spohn 1988) (see (Williams 1995) for a generalization and other operators on OCFs). The conditionalization of an OCF performs a (DP) revision under some conditions (if $x > 0$ and $\kappa(\mu) > 0$), but in the other cases we can obtain a contraction or a re-ordering. Please see Definition 7 in Section 4 for an example of a pure DP revision on OCFs.

The most commonly used epistemic spaces are $\mathcal{E}_{tpo}$ and $\mathcal{E}_{ocf}$. Their popularity can be explained by their simplicity and by the fact that the representation theorems for iterated change operators as (Darwiche and Pearl 1997; Booth and Meyer 2006; Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010; Medina Grespan and Pino Pérez 2013) allow to define operators as transitions between states (cf. Example 3, 4 and 6. The choice of such epistemic spaces is for instance used in (Booth and Meyer 2011; Booth and Chandler 2019; Spohn 1988).

Before concluding this section, let us introduce a few notations that will be used in some subsequent proofs. Given a set E, $card(E)$ denotes the cardinality of E. Given any total preorder $\Psi \in U_{tpo}$ and any world ω, $rank(\Psi, \omega)$ denotes the rank of ω in Ψ. More precisely, $rank(\Psi, \omega) = k - 1$ when there is a chain $(\omega_1, \ldots, \omega_k)$ of size k such that $\omega = \omega_k$ and for each i, j such that $i < j$, $\omega_i \prec \omega_j$ ($\prec$ denotes the strict ordering corresponding to Ψ), and such that no such chain of size k' with $k < k'$ exists. Then, given a non-negative integer i, $lvl(\Psi, i)$ denotes the set of models at the i^{th} level in Ψ, i.e., $lvl(\Psi, i) = \{\omega \mid rank(\Psi, \omega) = i\}$, and $max(\Psi) = \max\{i \mid i \in \mathbb{N}, lvl(\Psi, i) \neq \emptyset\}$. Lastly, given an OCF $\kappa \in U_{ocf}$, $tpo(\kappa)$ denotes the total preorder associated with κ, i.e., $tpo(\kappa)$ is defined as the total preorder $\preceq$ where for all worlds ω, ω', $\omega \preceq \omega$ if and only if $\kappa(\omega) \leq \kappa(\omega')$.

3 Instantiability of Revision Operators

Each revision operator is defined on a given epistemic space, but can be equivalently defined on a different epistemic space. Intuitively, given two epistemic spaces $\mathcal{E} = \langle E, Bel \rangle$ and $\mathcal{I} = \langle U, B \rangle$, we say that a revision operator $\circ$ on $\mathcal{E}$ is "safely $\mathcal{I}$-instantiable" whenever one can find a revision operator $\circ_\mathcal{I}$ on $\mathcal{I}$ that has the same "behavior" as $\circ$, i.e., $\circ_\mathcal{I}$ models the same transitions between epistemic states from U as $\circ$ between those from $\mathcal{E}$, modulo the beliefs that can be observed from these epistemic states in the context of belief revision. The set of all epistemic states from E is mapped onto another set U, and B is used to associate with each mapped epistemic state from U its corresponding beliefs. Formally:

Definition 4 ((Safe) translation)**.** *Let* $\circ$ *be a revision operator on* $\mathcal{E} = \langle E, Bel \rangle$*, and let* $\mathcal{I} = \langle U, B \rangle$ *be an epistemic space. A* translation of $\circ$ into $\mathcal{I}$ *is a pair* $(f_\mathcal{E}, \circ_\mathcal{I})$*, where* $f_\mathcal{E}$ *is a mapping from* E *to* U *and* $\circ_\mathcal{I}$ *is a revision operator on* I *such that for each* $\Psi \in \mathcal{E}$*,* $Bel(\Psi) \equiv B(f_\mathcal{E}(\Psi))$*, and for each* $\Psi \in E$ *and each* $\mu \in \mathcal{L}_P^*$*,* $f_\mathcal{E}(\Psi \circ \mu) = f_\mathcal{E}(\Psi) \circ_\mathcal{I} \mu$*. Such a translation is said to be* safe *if there exists a mapping* $h_I : U \to E$ *such that* $(h_I, \circ)$ *is a translation of* $\circ_\mathcal{I}$ *into* $\mathcal{E}$*.*

When a such a (safe) translation exists we say that $\circ$ *is (safely)* $\mathcal{I}$*-instantiable.*

Note that the notion of translation is close to the notions of simulation in (Aravanis, Peppas, and Williams 2019) and of instantiation in (Schwind and Konieczny 2020). We intend now to explain why safe translations are of particular interest. Let $\mathcal{I} = (U, B)$, $\mathcal{I}' = (U', B')$ be two epistemic spaces, let $\circ_\mathcal{I}$ be a revision operator on $\mathcal{I}$ and $\circ_{\mathcal{I}'}$ be a revision operator on $\mathcal{I}'$. Given $\Psi \in U$ and $\Psi' \in U'$, we say that Ψ is *strongly equivalent*[7] to Ψ' w.r.t. $(\circ_\mathcal{I}, \circ_{\mathcal{I}'})$, denoted by $\Psi \equiv_{(\circ_\mathcal{I}, \circ_{\mathcal{I}'})} \Psi'$, if $B(\Psi) \equiv B'(\Psi')$ and for each finite sequence of formulae $\sigma = (\mu_1, \ldots, \mu_k)$ with $k \geq 1$, $B(\Psi \circ_\mathcal{I} \sigma) \equiv B'(\Psi' \circ_{\mathcal{I}'} \sigma)$.

In the particular case where $\mathcal{I} = \mathcal{I}'$ and $\circ_\mathcal{I} = \circ_{\mathcal{I}'} = \circ$, we simply say that Ψ and Ψ' are strongly equivalent w.r.t. $\circ$ and simplify the notation $\Psi \equiv_{(\circ_\mathcal{I}, \circ_{\mathcal{I}'})} \Psi'$ into $\Psi \equiv_\circ \Psi'$. Then given a revision operator $\circ$ on $\mathcal{I} = (U, B)$, we denote by $U_{\equiv_\circ}$ the the quotient of U by the equivalence relation $\equiv_\circ$. That is, if $(\Psi)_{\equiv_\circ}$ denotes the equivalence class[8] of Ψ, then $U_{\equiv_\circ} = \{(\Psi)_{\equiv_\circ} \mid \Psi \in U\}$.

The notion of strong equivalence between epistemic states is very important. Indeed, each epistemic state, whatever the epistemic space to which it belongs, can actually be seen as a black box that associates each finite sequence of formulae with a formula representing the beliefs of the agent. And even if epistemic states may contain more complex information, this additional information does not have any further impact in the iterated revision process. That is, if two epistemic states have the same behavior for any Darwiche and Pearl's iterated revision, then they can not be distinguished from one another in this context, and in that case they belong to the same equivalence class.

Let us now use the notion of strong equivalence between epistemic states to define the notion of equivalence between two operators defined on two different epistemic spaces:

Definition 5. *Let* $\mathcal{I} = (U, B)$ *and* $\mathcal{I}' = (U', B')$ *be two epistemic spaces,* $\circ_\mathcal{I}$ *be a revision operator on* $\mathcal{I}$*, and* $\circ_{\mathcal{I}'}$ *be a revision operator on* $\mathcal{I}'$*. We say that* $\circ_\mathcal{I}$ *and* $\circ_{\mathcal{I}'}$ *are* equivalent *if there is a one-to-one and onto correspondence* g *from* $U_{\equiv_{\circ_\mathcal{I}}}$ *to* $U'_{\equiv_{\circ_{\mathcal{I}'}}}$ *such that for each* $V \in U_{\equiv_{\circ_\mathcal{I}}}$ *and for all* $\Psi \in V$*,* $\Psi' \in g(V)$*, we have that* $\Psi \equiv_{(\circ_\mathcal{I}, \circ_{\mathcal{I}'})} \Psi'$*.*

Let us state an important lemma:

Lemma 1. *Let* $\circ$ *be a revision operator,* $\mathcal{I} = (U, B)$ *be an epistemic space, and* $(f_\mathcal{E}, \circ_\mathcal{I})$ *be a translation of* $\circ$ *into* $\mathcal{I}$*. Then for each* $\Psi \in E$*, we have that* $\Psi \equiv_{(\circ, \circ_\mathcal{I})} f_\mathcal{E}(\Psi)$*.*

[7]In standard DP papers, two epistemic states are said to be equivalent iff $B(\Psi) \equiv B(\Psi')$. This is why we call this notion strong equivalence. Note that a similar distinction was made in (Konieczny and Pino Pérez 2000).

[8]$(\Psi)_{\equiv_\circ} = \{\Psi' \in U \mid \Psi' \equiv_\circ \Psi\}$

Proof. Let $\Psi \in E$. First, by definition of $f_{\mathcal{E}}$ we get that $Bel(\Psi) \equiv B(f_{\mathcal{E}}(\Psi))$. Second, for each sequence of formulae $\sigma = (\mu_1, \ldots, \mu_k)$, $k \geq 1$, we know that $Bel(\Psi \circ \sigma) \equiv B(f_{\mathcal{E}}(\Psi \circ \sigma))$. Since $f_{\mathcal{E}}(\Psi \circ \mu) = f_{\mathcal{E}}(\Psi) \circ_{\mathcal{I}} \mu$ for any formula μ, it is easy to see, by induction, that $f_{\mathcal{E}}(\Psi \circ \sigma) = f_{\mathcal{E}}(\Psi) \circ_{\mathcal{I}} \sigma$, thus $Bel(\Psi \circ \sigma) \equiv B(f_{\mathcal{E}}(\Psi) \circ_{\mathcal{I}} \sigma)$. This shows that $\Psi \equiv_{(\circ, \circ_{\mathcal{I}})} f_{\mathcal{E}}(\Psi)$. □

Taking advantage of Lemma 1, we are ready to make clear why the notion of safe translation is of particular interest:

Proposition 1. *Let $\circ$ be a revision operator and $\mathcal{I} = (U, B)$ be an epistemic space. If $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$ is a safe translation of $\circ$ into $\mathcal{I}$, then $\circ$ and $\circ_{\mathcal{I}}$ are equivalent.*

Proof. Let $\circ$ be a revision operator on $\mathcal{E} = (E, Bel)$, $\mathcal{I} = (U, B)$ be an epistemic space, and $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$ be a safe translation of $\circ$ into $\mathcal{I}$. For each $(\Psi)_{\equiv_\circ} \in E_{\equiv_\circ}$, let g be the mapping $(\Psi)_{\equiv_\circ} \mapsto (f_{\mathcal{E}}(\Psi))_{\equiv_{\circ_{\mathcal{I}}}}$. From Lemma 1, it is easy to see that g is well defined and defines an injective mapping from $E_{\equiv_\circ}$ to $U_{\equiv_{\circ_{\mathcal{I}}}}$, and we directly get for each $(\Psi)_{\equiv_\circ} \in E_{\equiv_\circ}$ and for all $\Psi_1 \in (\Psi)_{\equiv_\circ}$, $\Psi_2 \in (f_{\mathcal{E}}(\Psi))_{\equiv_{\circ_{\mathcal{I}}}}$, that $\Psi_1 \equiv_{(\circ, \circ_{\mathcal{I}})} \Psi_2$. Since $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$ is safe, there is a mapping $h_{\mathcal{I}} : U \to E$ such that $(h_{\mathcal{I}}, \circ)$ is a translation of $\circ_{\mathcal{I}}$ into $\mathcal{E}$. Then, similar to g, let us define the function g', mapping each $(\Psi')_{\equiv_{\circ_{\mathcal{I}}}} \in U_{\equiv_{\circ_{\mathcal{I}}}}$ into $(h_{\mathcal{I}}(\Psi'))_{\equiv_\circ}$. From Lemma 1 again, g' defines an injective mapping from $U_{\equiv_{\circ_{\mathcal{I}}}}$ to $E_{\equiv_\circ}$, and it is easy to verify that for each $(\Psi)_{\equiv_\circ} \in E_{\equiv_\circ}$ and for each$(\Psi')_{\equiv_{\circ_{\mathcal{I}}}} \in U_{\equiv_{\circ_{\mathcal{I}}}}$ we have $g'(g((\Psi)_{\equiv_\circ})) = (\Psi)_{\equiv_\circ}$ and $g(g'((\Psi')_{\equiv_{\circ_{\mathcal{I}}}})) = (\Psi')_{\equiv_{\circ_{\mathcal{I}}}}$. Thus, g is a one-to-one and onto correspondence from $E_{\equiv_\circ}$ to $U_{\equiv_{\circ_{\mathcal{I}}}}$. Therefore, g satisfies the requirements of Definition 5, which means that $\circ$ and $\circ_{\mathcal{I}}$ are equivalent. □

We have also the following result:

Proposition 2. *If a revision operator $\circ$ is not $\mathcal{I}$-instantiable, then there is no revision operator on $\mathcal{I}$ equivalent to $\circ$.*

Proof. We prove the contrapositive of the statement. Let $\circ$ be a revision operator, $\mathcal{I} = (U, B)$ be an epistemic space, and let $\circ_{\mathcal{I}}$ be an operator on $\mathcal{I}$ that is equivalent to $\circ$. So there is a bijection $g : E_{\equiv_\circ} \to U_{\equiv_{\circ_{\mathcal{I}}}}$ such that for each $F \in E_{\equiv_\circ}$, each $\Psi \in F$ and each $\Psi' \in g(F)$, $\Psi \equiv_{(\circ, \circ_{\mathcal{I}})} \Psi'$. First, let $\delta_U : U_{\equiv_{\circ_{\mathcal{I}}}} \to U$ be a mapping associating every equivalence class $F \in U_{\equiv_{\circ_{\mathcal{I}}}}$ with an arbitrary chosen epistemic state from F, in particular, if $\delta_U(V) = \Psi$, then $V = (\Psi)_{\equiv_{\circ_{\mathcal{I}}}}$. Define now the revision operator $\circ'_{\mathcal{I}}$ on $\mathcal{I}$ for each $\Psi \in U$ and each $\mu \in \mathcal{L}^*_P$ by $\Psi \circ'_{\mathcal{I}} \mu = \delta_U((\Psi \circ_{\mathcal{I}} \mu)_{\equiv_{\circ_{\mathcal{I}}}})$. Obviously enough, $\circ'_{\mathcal{I}}$ is equivalent to $\circ_{\mathcal{I}}$, thus $\circ'_{\mathcal{I}}$ is equivalent to $\circ$. Now, define $f_{\mathcal{E}} : E \to U$ for each $\Psi \in E$ as $f_{\mathcal{E}}(\Psi) = \delta_U(g((\Psi)_{\equiv_\circ}))$. It is easy to see for each $\Psi \in E$ that $Bel(\Psi) \equiv B(f_{\mathcal{E}}(\Psi))$. And for each $\Psi \in \mathcal{E}$ and each $\mu \in \mathcal{L}^*_P$, $f_{\mathcal{E}}(\Psi) \circ'_{\mathcal{I}} \mu = \delta_U(g((\Psi)_{\equiv_\circ})) \circ'_{\mathcal{I}} \mu = \delta_U((\delta_U(g((\Psi)_{\equiv_\circ})) \circ_{\mathcal{I}} \mu)_{\equiv_{\circ_{\mathcal{I}}}}) = \delta_U(g((\Psi \circ \mu)_{\equiv_\circ})) = f_{\mathcal{E}}(\Psi \circ \mu)$. This shows that $(f_{\mathcal{E}}, \circ'_{\mathcal{I}})$ is a translation of $\circ$ into $\mathcal{I}$, i.e., $\circ$ is $\mathcal{I}$-instantiable. □

Propositions 1 and 2 show why the notion of (safe) instantiability is a key concept: (safe) $\mathcal{I}$-instantiability it is a (sufficient and) necessary condition for defining an operator equivalent to $\circ$ in the space $\mathcal{I}$.

Another important consequence of Lemma 1 is the following:

Proposition 3. *Let $\circ$ be a revision operator and $\mathcal{I} = (U, B)$ be an epistemic space. If $\circ$ is $\mathcal{I}$-instantiable, then $card(E_{\equiv_\circ}) \leq card(U)$.*

Proof. Let $\circ$ be a revision operator and $\mathcal{I} = (U, B)$ be an epistemic space. Assume that $\circ$ is $\mathcal{I}$-instantiable, and let $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$ be a translation of $\circ$ into $\mathcal{I}$. So let $\Psi, \Psi' \in E$ and assume that $\Psi \not\equiv_\circ \Psi'$. From Lemma 1, we know that $f_{\mathcal{E}}(\Psi) \not\equiv f_{\mathcal{E}}(\Psi')$, thus $f_{\mathcal{E}}(\Psi) \neq f_{\mathcal{E}}(\Psi')$. This means that there is an injective mapping from $\mathcal{E}_{\equiv_\circ}$ to U. Hence, $card(E_{\equiv_\circ}) \leq card(U)$. □

This result is quite intuitive: in order to be used as an appropriate space for defining an operator equivalent to a given operator $\circ$, an epistemic space has to be "large enough" to encode all the equivalence classes (w.r.t. $\circ$) of epistemic states in the initial epistemic space. This point is important for our subsequent results.

4 TPO vs OCF

In this section, we investigate the links between the TPO and OCF epistemic spaces. We start by defining the notion of "structure preservation", in order to work with epistemic states that are closely related to the total preorders obtained by the representation theorem. Then we show how to find for every operator on the TPO epistemic space an equivalent operator on the OCF epistemic space. Lastly, we show that a converse translation is not possible.

A very interesting feature of operators defined in the spaces $\mathcal{E}_{tpo}$ and $\mathcal{E}_{ocf}$ is that they are typically built using the ordered structure of the epistemic states in a natural way. For instance, the DP assignment corresponding to the operators in Examples 3 and 4 is the identity, and the one corresponding to the operator on $\mathcal{E}_{ocf}$ introduced later in Definition 7 is "almost" the identity. An appealing property of an operator defined on $\mathcal{E}_{tpo}$ or $\mathcal{E}_{ocf}$ is thus to preserve the internal ordered structure of their epistemic states according to its corresponding assignment. Let us formalize this property:

Definition 6. *Let $\circ$ be a DP operator defined on $\mathcal{E}_{tpo}$ and $\Psi \mapsto \preceq_\Psi$ be the DP assignment corresponding to $\circ$. We say that $\circ$ is* structure preserving *if for each $\Psi \in U_{tpo}$, $\Psi = \preceq_\Psi$. Likewise, let $\circ$ be a DP operator defined on $\mathcal{E}_{ocf}$ and $\kappa \mapsto \preceq_\kappa$ be the DP assignment corresponding to $\circ$. We say that $\circ$ is* structure preserving *if for each $\kappa \in U_{ocf}$, $tpo(\kappa) = \preceq_\kappa$.*

One can wonder if there exist DP operators that are not structure preserving. The following example shows an operator defined on $\mathcal{E}_{tpo}$ which is not structuring preserving.

Example 7. *Consider a propositional language with two propositional variables. Thus, there are four interpretations ω_1, ω_2, ω_3 and ω_4. Let $\circ_N$ be the Nayak operator on $\mathcal{E}_{tpo}$. Consider the two following total preorders: Ψ_1 defined by $\omega_1 \sim \omega_2 \prec \omega_3 \prec \omega_4$ and Ψ_2 defined by $\omega_1 \sim \omega_2 \prec \omega_4 \prec$*

ω_3. *Clearly* $\Psi_1 \neq \Psi_2$ *and* $B_{tpo}(\Psi_1) \equiv B_{tpo}(\Psi_2)$. *Now, define a new operator* $\star$ *as follows:*

$$\Psi \star \mu = \begin{cases} \Psi_2 \circ_N \mu & \text{if } \Psi = \Psi_1 \\ \Psi_1 \circ_N \mu & \text{if } \Psi = \Psi_2 \\ \Psi \circ_N \mu, & \text{otherwise} \end{cases}$$

It is easy to see that $\star$ *is a DP operator and, moreover,* $\Psi_1 = \preceq^\star_{\Psi_2}$. *Therefore* $\star$ *is not structure preserving.*

However, we can show that:

Proposition 4. *An operator* $\circ$ *on* $\mathcal{E}_{tpo}$ *is a DP operator if and only if there is a structure preserving DP operator* $\circ'$ *on* $\mathcal{E}_{tpo}$ *that is equivalent to* $\circ$.

Proof. The (if) part of the proof is direct: since $\circ'$ is equivalent to $\circ$ and $\circ'$ is a DP operator, $\circ$ is a DP operator as well. Let us prove the (only if) part of the proof. Let $\circ$ be a DP operator. Let τ denote the faithful assignment corresponding to $\circ$ (recall that it is unique), and let us first show that τ is a one-to-one-correspondence on U_{tpo}. It is enough to prove that τ is surjective. In the following, $\tau(\Psi)$ will sometimes be denoted by $\preceq_\Psi$.

Let us start with a useful observation. Let $\Psi \in U_{tpo}$ and α be a formula such that $[\alpha] \subseteq lvl(\preceq_\Psi, max(\preceq_\Psi))$, i.e., α is a formula whose models lie at the last level of $\preceq_\Psi$. Then from the representation theorem (Theorem 1) and condition CR2, the total preorder $\preceq_{\Psi\circ\alpha}$ is uniquely defined by:

- if $\omega, \omega' \models \alpha$, then $\omega \preceq_{\Psi\circ\alpha} \omega'$
- if $\omega \models \alpha, \omega' \not\models \alpha$, then $\omega \prec_{\Psi\circ\alpha} \omega'$
- if $\omega, \omega' \not\models \alpha$, then $\omega \preceq_{\Psi\circ\alpha} \omega'$ iff $\omega \preceq_\Psi \omega'$

From this observation, we can show that every total preorder $\preceq$ corresponds to $\tau(\Psi)$ from some $\Psi \in U_{tpo}$. Indeed, let $\preceq$ be any total preorder from U_{tpo}, let $\Psi_\top \in U_{tpo}$ be such that $B_{tpo}(\Psi_\top) \equiv \top$ (note that $\preceq_{\Psi_\top}$ is always the "flat" total preorder defined for all worlds ω, ω' by $\omega \simeq_{\Psi_\top} \omega'$), and let $\sigma = (\alpha_1, \ldots, \alpha_{max(\preceq)})$ be a sequence of formulae such that for each α_i, $[\alpha_i] = lvl(\preceq, max(\preceq) - i)$. It is easy to see from the observation above that $\tau(\Psi_\top \circ \sigma) = \preceq$. This shows that τ is surjective, so it is a one-to-one correspondence.

Now, let us consider the operator $\circ'$ on $\mathcal{E}_{tpo}$ defined for each $\Psi \in U_{tpo}$ and each $\mu \in \mathcal{L}^*_P$ by $\Psi \circ' \mu = \tau(\tau^{-1}(\Psi) \circ \mu)$, and let us show that $(\tau, \circ')$ is a safe translation of $\circ$ into $\mathcal{E}_{tpo}$. Let $\Psi \in U_{tpo}$ and $\mu \in \mathcal{L}^*_P$. The fact that $B_{tpo}(\Psi) \equiv B_{tpo}(\tau(\Psi))$ is direct from the representation theorem (cf. Theorem 1) and conditions 1 and 2 of a faithful assignment. Yet we also have that $\tau(\Psi) \circ' \mu = \tau(\tau^{-1}(\tau(\Psi)) \circ \mu) = \tau(\Psi \circ \mu)$. This shows that $(\tau, \circ')$ is a translation of $\circ$ into $\mathcal{E}_{tpo}$. To show that $(\tau, \circ')$ is safe, it is enough to remark that $(\tau^{-1}, \circ)$ is a translation of $\circ'$ into $\mathcal{E}_{tpo}$, since $B_{tpo}(\Psi) \equiv B_{tpo}(\tau^{-1}(\Psi))$ and $\tau^{-1}(\Psi \circ' \mu) \equiv \tau^{-1}(\tau(\tau^{-1}(\Psi) \circ \mu)) = \tau^{-1}(\Psi) \circ \mu$.

Since $(\tau, \circ')$ is a safe translation of $\circ$ into $\mathcal{E}_{tpo}$, from Proposition 1 we get that $\circ$ and $\circ'$ are equivalent, which concludes the proof. □

This result is very important, since it means that we can always assume to work with structure preserving operators for any DP operator on $\mathcal{E}_{tpo}$.

Now, let us show that every revision operator on $\mathcal{E}_{tpo}$ is safely $\mathcal{E}_{ocf}$-instantiable. Let $\circ$ be any revision operator on $\mathcal{E}_{tpo}$. Let $f_{\mathcal{E}_{tpo}}$ be the mapping from U_{tpo} to U_{ocf} defined for each $\Psi \in U_{tpo}$ by $f_{\mathcal{E}_{tpo}}(\Psi) = \kappa_\Psi$, where κ_Ψ is the OCF defined for each world ω by $\kappa_\Psi(\omega) = rank(\Psi, \omega)$. Now, let us define a revision operator $\circ_{\mathcal{E}_{ocf}}$ on $\mathcal{E}_{ocf}$ for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}^*_P$ as $\kappa \circ_{\mathcal{E}_{ocf}} \mu = f_{\mathcal{E}_{tpo}}(tpo(\kappa) \circ \mu)$. Then:

Proposition 5. $(f_{\mathcal{E}_{tpo}}, \circ_{\mathcal{E}_{ocf}})$ *is a safe translation of* $\circ$ *into* $\mathcal{E}_{ocf}$.

Proof. First, it is easy to see that for each $\Psi \in B_{tpo}$, we have that $B_{tpo}(\Psi) = B_{ocf}(\kappa_\Psi)$. Then, remark that for each $\Psi \in U_{tpo}$, by construction of κ_Ψ we get that $tpo(\kappa_\Psi) = \Psi$. Thus for each $\Psi \in U_{tpo}$ and each $\mu \in \mathcal{L}^*_P$ we get that $f_{\mathcal{E}_{tpo}}(\Psi) \circ_{\mathcal{E}_{ocf}} \mu = \kappa_\Psi \circ_{\mathcal{E}_{ocf}} \mu = f_{\mathcal{E}_{tpo}}(tpo(\kappa_\Psi) \circ \mu) = f_{\mathcal{E}_{tpo}}(\Psi \circ \mu)$. This shows that $(f_{\mathcal{E}_{tpo}}, \circ_{\mathcal{E}_{ocf}})$ is a translation of $\circ$ into $\mathcal{E}_{ocf}$.

To see that this is also a safe translation, simply consider the mapping $h_{\mathcal{E}_{ocf}} : U_{ocf} \to U_{tpo}$ defined for each $\kappa \in U_{ocf}$ as $h_{\mathcal{E}_{ocf}}(\kappa) = tpo(\kappa)$. On the one hand, it is easy to see that $B_{ocf}(\kappa) \equiv B_{tpo}(h_{\mathcal{E}_{ocf}}(\kappa))$. On the other hand, remark that for each $\Psi \in U_{tpo}$, $h_{\mathcal{E}_{ocf}}(f_{\mathcal{E}_{tpo}}(\Psi)) = tpo(\kappa_\Psi) = \Psi$. Hence, we get for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}^*_P$ that $h_{\mathcal{E}_{ocf}}(\kappa \circ_{\mathcal{E}_{ocf}} \mu) = h_{\mathcal{E}_{ocf}}(f_{\mathcal{E}_{tpo}}(tpo(\kappa) \circ \mu)) = tpo(\kappa) \circ \mu = h_{\mathcal{E}_{ocf}}(\kappa) \circ \mu$. This shows that $(h_{\mathcal{E}_{ocf}}, \circ)$ is a translation of $\circ_{\mathcal{E}_{ocf}}$ into U_{tpo}, which proves that $(f_{\mathcal{E}_{tpo}}, \circ_{\mathcal{E}_{ocf}})$ is a safe translation of $\circ$ into $\mathcal{E}_{ocf}$. □

We call the pair $(f_{\mathcal{E}_{tpo}}, \circ_{\mathcal{E}_{ocf}})$ the *canonical* translation of $\circ$ into $\mathcal{E}_{ocf}$.

A particular consequence of Proposition 5 is that every revision operator $\circ$ on $\mathcal{E}_{tpo}$ is safely $\mathcal{E}_{ocf}$-instantiable. Moreover, it is easy to see by construction of $\circ_{\mathcal{E}_{ocf}}$ that if $\circ$ is a structure preserving DP operator, then $\circ_{\mathcal{E}_{ocf}}$ is also structure preserving. Hence, a direct consequence of Propositions 4 and 5 is that:

Corollary 1. *For every DP operator* $\circ$ *on* $\mathcal{E}_{tpo}$*, there is an operator on* $\mathcal{E}_{ocf}$ *equivalent to* $\circ$ *that is structure preserving.*

So we can encode any DP revision defined on TPOs into the OCF epistemic space by a structure preserving operator. Let us now show that the converse translation is not possible, by providing an example of operator that is defined on OCFs and that is not $\mathcal{E}_{tpo}$-instantiable.

Definition 7. *Let* $\bullet_+$ *be the revision operator on* $\mathcal{E}_{ocf}$ *defined for each* $\kappa \in U_{ocf}$ *and each* $\mu \in \mathcal{L}^*_P$ *by* $\kappa' = \kappa \bullet_+ \mu$*, where for each world* $\omega \in \Omega$, $\kappa'(\omega)$ *is defined by:*

$$\kappa'(\omega) = \begin{cases} 0 & \text{if } \omega \models \mu \text{ and } \kappa(\omega) = \kappa(\mu) \\ \kappa(\omega) & \text{if } \omega \models \mu \text{ and } \kappa(\omega) > \kappa(\mu) \\ \kappa(\omega) + 1 & \text{if } \omega \not\models \mu \end{cases}$$

This operator can be related with the $\bullet$ operator proposed by Darwiche and Pearl to illustrate their DP postulates (Darwiche and Pearl 1997), since $\bullet$ and $\bullet_+$ deal with the countermodels of μ in a similar way. However, $\bullet_+$ depart from $\bullet$ in its treatment of the models of μ that do not move (except in the case where they are the minimal ones).

First, please observe that :

Proposition 6. *$\bullet_+$ is a DP revision operator.*

Proof. Let τ be the mapping from U_{ocf} to U_{tpo} defined for each $\kappa \in U_{ocf}$ by $\tau(\kappa) = tpo(\kappa)$. The fact that $\bullet_+$ is a DP operator is shown as follows: one can easily observe by definition of $\bullet_+$ that τ satisfies conditions 1 and 2 of a faithful assignment and conditions (CR1-CR4) of Theorem 1, and for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$, $[B_{ocf}(\kappa \bullet_+ \mu)] = \min([\mu], tpo(\kappa))$. That is, τ defines a DP assignment corresponding to $\bullet_+$, which from Theorem 1 shows that $\bullet_+$ is a DP operator. □

Let us illustrate the behavior of this operator through the following example:

Example 8. *Let $P = \{p, q\}$, and consider the two* OCF*s κ and π defined by:*[9]
$\kappa(00) = 0$ $\kappa(01) = 1$ $\kappa(10) = 1$ $\kappa(11) = 2$
$\pi(00) = 0$ $\pi(01) = 1$ $\pi(10) = 1$ $\pi(11) = 4$
Then $\kappa \bullet_+ p = \kappa_1$, $\pi \bullet_+ p = \pi_1$, $\pi_1 \bullet_+ p = \pi_2$, $\pi_2 \bullet_+ p = \pi_3$, as illustrated in Figure 2.[10]

Note that $\bullet_+$ is structure preserving, so its corresponding faithful assignment associates κ and π with the same total preorder $\preceq$. Would $\bullet_+$ be safely $\mathcal{E}_{tpo}$-instantiable, Propositions 1 and 4 together show that there would be a structure preserving operator on $\mathcal{E}_{tpo}$ that is equivalent to $\bullet_+$, and so κ and π would be translated to the same total preorder. But this example shows that this can not be the case, since $B_{ocf}((\kappa \bullet_+ p) \bullet_+ q) \not\equiv B_{ocf}((\pi \bullet_+ p) \bullet_+ q)$, and so κ and π are not strongly equivalent. Similarly, one can remark that the TPO translated from π_1 and π_2 should be exactly the same, but $B_{ocf}((\pi_1 \bullet_+ p) \bullet_+ q) \not\equiv B_{ocf}((\pi_2 \bullet_+ p) \bullet_+ q)$.

This example is enough to conclude that $\bullet_+$ is not safely $\mathcal{E}_{tpo}$-instantiable, but let us give a more general result:

Proposition 7. *$U_{ocf,\equiv_{\bullet_+}}$ is countably infinite.*

Proof. Without loss of generality, assume that $\mathcal{L}_P$ is such that $|P| \geq 2$. Let $\omega_1, \omega_2, \omega_3$ be three distinct worlds from Ω. Since U_{ocf} is a countable set, $U_{ocf,\equiv_{\bullet_+}}$ is a countable set. Let $\mathbb{N}_*$ denotes the set of positive integers. To conclude the proof that $U_{ocf,\equiv_{\bullet_+}}$ is countably *infinite*, since $\mathbb{N}_*$ is infinite it is enough to build an injective mapping τ from $\mathbb{N}_*$ to $U_{ocf,\equiv_{\bullet_+}}$. So let us define $\tau : \mathbb{N}_* \rightarrow U_{ocf,\equiv_{\bullet_+}}$ for each $i \in \mathbb{N}_*$ as $\tau(i) = \kappa^i$, where κ^i is any arbitrary OCF that satisfies $\kappa^i(\omega_1) = 0$, $\kappa^i(\omega_2) = 1$, and $\kappa^i(\omega_3) = i + 2$. For each $k \in \mathbb{N}_*$, let σ^k be the sequence of formulae $(\alpha_{13}, \ldots, \alpha_{13})$ where $[\alpha_{13}] = \{\omega_1, \omega_3\}$ and α_{13} appears exactly k times in the sequence. Let us denote by κ_k^i the OCF $\kappa^i \bullet_+ \sigma^k$. By definition of $\bullet_+$, we get for all $i, k \in \mathbb{N}_*$ that $\kappa_k^i(\omega_1) = \kappa^i(\omega_1) = 0$, $\kappa_k^i(\omega_2) = \kappa^i(\omega_2) + k = k + 1$, and $\kappa_k^i(\omega_3) = \kappa^i(\omega_3) = i + 2$. Hence,

- $\kappa_{i+1}^i(\omega_2) = \kappa_{i+1}^i(\omega_3)$
- if $k < i + 1$, then $\kappa_k^i(\omega_2) < \kappa_k^i(\omega_3)$
- otherwise, i.e., if $k > i + 1$, then $\kappa_k^i(\omega_2) > \kappa_k^i(\omega_3)$

[9]A world is denoted by an ordered pair of two symbols from $\{0, 1\}$ according to the ordering $p < q$.

[10]$\kappa(\omega) = i$ iff ω appears at the level associated to the integer i.

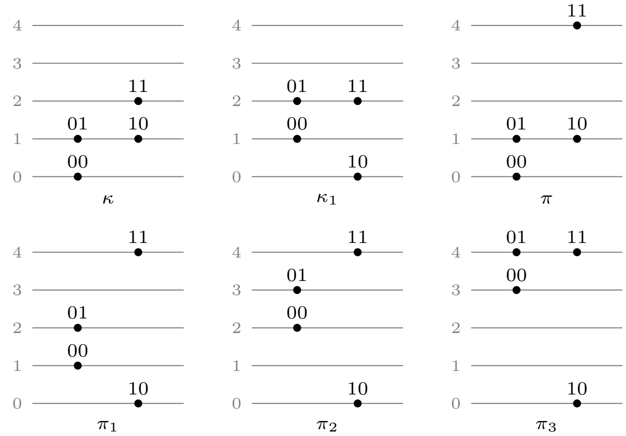

Figure 2: The DP operator $\bullet_+$

This shows that $B_{ocf}(\kappa_k^i \bullet_+ \alpha_{23}) \equiv \alpha_{23}$ if and only if $k = i + 1$, where $[\alpha_{23}] = \{\omega_2, \omega_3\}$. Stated otherwise, for all $i, j \in \mathbb{N}_*$, if $i \neq j$ then the sequence $\sigma = \sigma^{i+1} \cdot \alpha_{23}$ is such that $B_{ocf}(\kappa^i \bullet_+ \sigma) \not\equiv B_{ocf}(\kappa^j \bullet_+ \sigma)$. This means that if $i \neq j$ then $\kappa_i \not\equiv_{\bullet_+} \kappa_j$, which shows that τ is an injective mapping from $\mathbb{N}_*$ to $U_{ocf,\equiv_{\bullet_+}}$, and concludes the proof that $U_{ocf,\equiv_{\bullet_+}}$ is countably infinite. □

Then from Propositions 3 and 7 and from the fact that U_{tpo} is a finite set, we can conclude that:

Corollary 2. *$\bullet_+$ is not $\mathcal{E}_{tpo}$-instantiable.*

Proposition 7 is more interesting than the counter-example, since the fact that $U_{ocf,\equiv_{\bullet_+}}$ is countably infinite means not only that an operator on $\mathcal{E}_{tpo}$ that is equivalent to $\bullet_+$ can not be found on the same propositional language, but also that trying to use a larger language in order to gain in expressivity (i.e., artifically adding new atoms to have more TPOs) can not work either.

5 An Impossibility Result

Let us start this section by the definition of an epistemic space $\mathcal{I}$ being C-complete where C is a class of revision operators, meaning that $\mathcal{I}$ is rich enough to endorse any operator of a given class C (e.g., satisfying a set of postulates).

Definition 8. *Given a class C of revision operators, we say that an epistemic space $\mathcal{I} = (U, B)$ is C-*complete *if every operator from C is safely $\mathcal{I}$-instantiable.*

When C is the class of DP operators and $\mathcal{I}$ is C-complete, we simply say that $\mathcal{I}$ is DP-complete.

In this section, we intend to introduce an impossibility result, that is, no epistemic space $\mathcal{I} = (U, B)$ where U is countable is DP-complete. To this end, we intend to introduce a specific DP revision operator that is not $\mathcal{I}$-instantiable, for any $\mathcal{I} = (U, B)$ where U is a countable set.

Let us first define our revision operator, denoted by $\bullet^*$. Without loss of generality, we can assume that $\mathcal{L}_P$ is such that $|P| \geq 2$. Let $\omega_1, \omega_2, \omega_3$ be three distinct worlds from Ω. Let $\mathbb{B}^{\mathbb{N}}$ be the set of all (countably infinite) binary sequences of values from $\{0, 1\}$. For each $\delta \in \mathbb{B}^{\mathbb{N}}$ and for each integer $i \geq 1$, $\delta(i)$ denotes the i^{th} value of δ in the sequence, and $(\delta + 1)$ denotes the sequence from $\mathbb{B}^{\mathbb{N}}$ defined for each integer $i \geq 1$ as $(\delta + 1)(i) = \delta(i + 1)$ (e.g., if $\delta = (00101 \cdots)$, then $(\delta + 1) = (0101 \cdots)$). Moreover, let $\circ_B$ and $\circ_N$ be respectively the Boutilier revision operator and the Nayak lexicographic operator on $\mathcal{E}_{tpo}$ (recall that both operators are DP operators). Lastly, let Φ_{123} denote the subset of total preorders from U_{tpo} that satisfy $\omega_1 < \omega_2 < \omega_3$ and for each $\omega \notin \{\omega_1, \omega_2, \omega_3\}$, $\omega_3 < \omega$.

Let us now define an epistemic space $\mathcal{E}^* = \langle E^*, Bel^* \rangle$ as follows. Let $E^* = U_{tpo} \times \mathbb{B}^{\mathbb{N}}$, and let Bel^* be defined for each $(\Psi, \delta) \in E^*$ as $Bel^*((\Psi, \delta)) = B_{tpo}(\Psi)$. We now define the revision operator $\bullet^*$ on $\mathcal{E}^*$ as follows, for each $(\Psi, \delta) \in E^*$:

- $(\Psi, \delta) \bullet^* \mu = (\Psi \circ_N \mu, (\delta + 1))$, if $\Psi \in \Phi_{123}, [\mu] = \{\omega_1, \omega_3\}$, and $\delta(1) = 1$,
- $(\Psi, \delta) \bullet^* \mu = (\Psi \circ_B \mu, (\delta + 1))$, if $\Psi \in \Phi_{123}, [\mu] = \{\omega_1, \omega_3\}$, and $\delta(1) = 0$,
- $(\Psi, \delta) \bullet^* \mu = (\Psi \circ_B \mu, \delta)$, in the remaining cases.

Let us first show that:

Proposition 8. $\bullet^*$ *is a DP operator.*

Proof. Let τ be the mapping associating each epistemic state $(\Psi, \delta) \in E^*$ with the total preorder Ψ. On the one hand, since $Bel^*((\Psi, \delta)) = B_{tpo}(\Psi)$ for every $(\Psi, \delta) \in E^*$, by definition of $B_{tpo}(\Psi)$ we directly get that τ satisfies conditions 1 and 2 of a faithful assignment. On the other hand, by construction of $\bullet^*$, for every $(\Psi, \delta) \in E^*$, we have that $\tau((\Psi, \delta) \bullet^* \mu) \in \{\Psi \circ_N \mu, \Psi \circ_B \mu\}$. Yet $\circ_B$ and $\circ_N$ are both structure preserving, so the identity function on U_{tpo} defines a DP assignment corresponding to both $\circ_B$ and $\circ_N$. Then τ satisfies conditions (CR1-CR4) as well, and for each epistemic state (Ψ, δ) and each formula μ, $[Bel((\Psi, \delta) \bullet^* \mu)] = \min([\mu], \Psi)$. This means that τ is the DP assignment corresponding to $\bullet^*$. Then, by Theorem 1, $\bullet^*$ is a DP revision operator. □

Our impossibility result is based on the following statement, which shows $E^*_{\equiv_{\bullet^*}}$ is uncountable:

Proposition 9. $card(\mathbb{B}^{\mathbb{N}}) \leq card(E^*_{\equiv_{\bullet^*}})$.

Proof. Let σ_{321} be the sequence of formulae $(\alpha_3, \alpha_2, \alpha_1)$ with $[\alpha_3] = \{\omega_3\}$, $[\alpha_2] = \{\omega_2\}$, and $[\alpha_1] = \{\omega_1\}$. Let Ψ be any total preorder from U_{tpo}, and let us note $\Psi' = \Psi \circ_B \sigma_{123}$. It is easy to verify by construction of $\circ_B$ that $\Psi' \in \Phi_{123}$. Yet by definition of $\circ_*$, we know that for each epistemic state $(\Psi, \delta) \in E^*$, $(\Psi, \delta) \bullet^* \sigma_{123} = (\Psi \circ_B \sigma_{123}, \delta)$. Hence, for each epistemic state $(\Psi, \delta) \in E^*$, $(\Psi, \delta) \bullet^* \sigma_{123}$ is an epistemic state (Ψ', δ) such that $\Psi' \in \Phi_{123}$.

Now, given any such epistemic state $(\Psi', \delta) \in \mathcal{E}^*$ such that $\Psi' \in \Phi_{123}$, when α_{13}, α_{23} are formulae such that $[\alpha_{13}] = \{\omega_1, \omega_3\}$ and $[\alpha_{23}] = \{\omega_2, \omega_3\}$, we have that:

- if $\delta(1) = 1$, then $(\Psi', \delta) \bullet^* \alpha_{13} \bullet^* \alpha_{23} = (\Psi' \circ_N \alpha_{13}, (\delta + 1)) \bullet^* \alpha_{23} = (\Psi' \circ_N \alpha_{13} \circ_B \alpha_{23}, (\delta + 1))$, which we denote by $(\Psi_1, (\delta + 1))$; and
- if $\delta(1) = 0$, then $(\Psi', \delta) \bullet^* \alpha_{13} \bullet^* \alpha_{23} = (\Psi' \circ_B \alpha_{13}, (\delta + 1)) \bullet^* \alpha_{23} = (\Psi' \circ_B \alpha_{13} \circ_B \alpha_{23}, (\delta + 1))$, which we denote by $(\Psi_2, (\delta + 1))$.

And it is easy to verify by construction of $\circ_B$ and $\circ_N$ that $B_{tpo}(\Psi_1) \equiv \alpha_3$ and $B_{tpo}(\Psi_2) \equiv \alpha_2$, thus $Bel((\Psi_1, (\delta + 1)) \equiv \alpha_3$ and $Bel((\Psi_2, (\delta + 1)) \equiv \alpha_2$.

Overall, we have shown that for each epistemic state $(\Psi, \delta) \in E^*$, when σ is the sequence of formulae $\sigma = \sigma_{123} \cdot \alpha_{13} \cdot \alpha_{23}$,

- if $\delta(1) = 1$, then $(\Psi, \delta) \bullet^* \sigma = (\Psi_1, (\delta + 1))$ and $Bel((\Psi_1, (\delta + 1))) \equiv \alpha_3$, and
- if $\delta(1) = 0$, then $(\Psi, \delta) \bullet^* \sigma = (\Psi_2, (\delta + 1))$ and $Bel((\Psi_2, (\delta + 1))) \equiv \alpha_2$.

That is, for all epistemic states $(\Psi, \delta), (\Psi, \delta') \in E^*$,

$$\delta(1) \neq \delta'(1) \Rightarrow Bel((\Psi, \delta) \bullet^* \sigma) \not\equiv Bel((\Psi, \delta') \bullet^* \sigma) \quad (1)$$

We intend now to show that if $(\Psi, \delta), (\Psi, \delta')$ are two epistemic states from E^* such that $\delta \neq \delta'$, then $(\Psi, \delta) \not\equiv_{\bullet^*} (\Psi, \delta')$. Given a vector $\delta \in \mathbb{B}^{\mathbb{N}}$ and an integer $k \geq 1$, δ can be rewritten as the concatenation of two vectors δ_{-k} and δ_{+k}, i.e., $\delta = \delta_{-k} \cdot \delta_{+k}$, where $\delta_{-k} = (\delta(1), \ldots, \delta(k-1))$ (δ_{-k} is possibly empty) and $\delta_{+k} = (\delta(k), \delta(k+1), \ldots)$. Now, let $\delta \neq \delta'$, then let i be the smallest integer such that $\delta(i) \neq \delta'(i)$, formally $i = \min(\{j \in \mathbb{N} \mid \delta(j) \neq \delta'(j)\})$. We have that $\delta_{-i} = \delta'_{-i}$ and $\delta_{+i}(1) \neq \delta'_{+i}(1)$. On the one hand, by definition of $\bullet^*$, we get that $(\Psi, \delta) \bullet^* \sigma^{(i)} = (\Psi_i, \delta_{+i})$ and $(\Psi, \delta') \bullet^* \sigma^{(i)} = (\Psi_i, \delta'_{+i})$ for some unique total preorder Ψ_i. On the other hand, we have shown in Equation 1 above that since $\delta_{+i}(1) \neq \delta'_{+i}(1)$, $Bel(\Psi_i, \delta_{+i}) \bullet^* \sigma \not\equiv Bel(\Psi_i, \delta'_{+i}) \bullet^* \sigma$. That is, the sequence of formulae $\sigma^{(i+1)}$ is such that $Bel((\Psi, \delta) \bullet^* \sigma^{(i+1)}) \not\equiv Bel((\Psi, \delta') \bullet^* \sigma^{(i+1)})$. This shows that $(\Psi, \delta) \not\equiv_{\bullet^*} (\Psi, \delta')$.

We have shown that each total preorder $\Psi \in U_{tpo}$ and for all $\delta, \delta' \in \mathbb{B}^{\mathbb{N}}$, if $\delta \neq \delta'$ then $(\Psi, \delta) \not\equiv_{\bullet^*} (\Psi, \delta')$. This shows that there is an injective mapping from $\mathbb{B}^{\mathbb{N}}$ to $E^*_{\equiv_{\bullet^*}}$, from which we can conclude that $card(\mathbb{B}^{\mathbb{N}}) \leq card(E^*_{\equiv_{\bullet^*}})$. □

And so, as a consequence of Propositions 3 and 9 and from the fact that the set $\mathbb{B}^{\mathbb{N}}$ is uncountable, we get that:

Corollary 3. *Let $\mathcal{I} = (U, B)$ be an epistemic space such that U is a countable set. Then $\bullet^*$ is not $\mathcal{I}$-instantiable.*

We are ready to state our impossibility result. Since $\bullet^*$ is a DP operator (cf. Proposition 8), we get as a direct consequence of Corollary 3 that:

Corollary 4. *There is no DP-complete epistemic space $\mathcal{I} = (U, B)$ where U is a countable set.*

In particular, $\mathcal{E}_{ocf}$ and $\mathcal{E}_{tpo}$ are not DP-complete, and thus these epistemic spaces cannot be used to define *every* DP operator. This can be seen as a very negative result. But this is a simple consequence of the generality of Darwiche and Pearl's definition of epistemic states. Nevertheless, one can be more positive if one makes a very natural assumption on the operator and the epistemic space on which it is defined. This is the topic of the next section.

6 A Possibility Result

Let us now investigate a particular, yet very natural, case, where all epistemic states can be finitely generated from an initial empty epistemic state and a sequence of revisions.

Let $\mathcal{E} = \langle E, Bel \rangle$ be an epistemic space and $\circ$ be a revision operator on $\mathcal{E}$. We consider the following property:

(G) There exists a unique $\Psi_\top \in \mathcal{E}$ such that $Bel(\Psi_\top) \equiv \top$, and for each $\Psi \in \mathcal{E}$, there exists a finite sequence of formulae $\sigma = (\mu_1, \ldots, \mu_k)$ such that $\Psi = \Psi_\top \circ \sigma$

A DP operator satisfying **(G)** is said to be a DP+G operator. And when C is the class of DP operators satisfying **(G)** and $\mathcal{I}$ is C-complete, we simply say that $\mathcal{I}$ is DP+G-complete.

We intend to show that every DP+G operator is safely $\mathcal{E}_{ocf}$-instantiable. For this purpose, we intend to provide a canonical, safe translation of any DP+G operator $\circ$ into an equivalent operator on OCFs, denoted by $\circ_{\mathcal{E}_{ocf}}$. So let $\circ$ be any DP+G operator on $\mathcal{E} = (E, Bel)$. By condition **(G)**, every epistemic state from E can be identified by a finite sequence of formulae. Since the set of all finite sequence of formulae is a countable set, E is also countable. So one can denote by $(\Psi_1, \Psi_2, \ldots)$ the list of all epistemic states from E. For each $\Psi_i \in E$, let $\preceq_{\Psi_i}$ be the total preorder associated with Ψ_i by the DP assignment corresponding to $\circ$. Now, let $f_\mathcal{E}$ be the mapping from E to U_{ocf} defined for each $\Psi_i \in \mathcal{E}$ by $f_\mathcal{E}(\Psi_\top) = \kappa_\top$, where $\kappa_\top$ is the OCF defined for each world ω by $\kappa_\top(\omega) = 0$; and if $\Psi_i \neq \Psi_\top$, then $f_\mathcal{E}(\Psi_i) = \kappa_i$, where κ_i is the OCF defined for each world ω by $\kappa_i(\omega) = 0$ if ω is at the first level of $\preceq_{\Psi_i}$ otherwise $\kappa_i(\omega) = |\Omega| + i$ if $\omega \in lvl(\preceq_{\Psi_i}, max(\preceq_{\Psi_i}))$ (i.e., if ω lies at the last level of $\preceq_{\Psi_i}$), otherwise $\kappa_i(\omega) = rank(\preceq_{\Psi_i}, \omega)$. By construction, $f_\mathcal{E}$ is injective. Let $\mathsf{Im}(f_\mathcal{E})$ be the subset of OCFs from U_{ocf} that is the image of $f_\mathcal{E}$, i.e., $\mathsf{Im}(f_\mathcal{E}) = \{f_\mathcal{E}(\Psi_i) \mid \Psi_i \in \mathcal{E}\}$. The fact that $f_\mathcal{E}$ is injective allows one to define another mapping $h_{\mathcal{E}_{ocf}} : U_{ocf} \rightarrow E$ defined for each OCF $\kappa \in U_{ocf}$ by $h_{\mathcal{E}_{ocf}}(\kappa) = f_\mathcal{E}^{-1}(\kappa)$ if $\kappa \in \mathsf{Im}(f_\mathcal{E})$, otherwise $h_{\mathcal{E}_{ocf}}(\kappa) = \Psi_i$, where Ψ_i is any arbitrary chosen epistemic state from E such that $tpo(\kappa) = \preceq_{\Psi_i}$. (Note that such an epistemic state $\Psi_i \in E$ exists, since the DP assignment corresponding to $\circ$ is a surjective mapping from E to U_{tpo}: similar to the proof of Proposition 4 and by condition **(G)**, we can prove that for any total preorder $\preceq$, there exists a finite sequence σ such that $\Psi_i = \Psi_\top \circ \sigma$ and $\preceq = \preceq_{\Psi_i}$.) Then we define the revision operator $\circ_{\mathcal{E}_{ocf}}$ on $\mathcal{E}_{ocf}$ defined for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$ by $\kappa \circ_{\mathcal{E}_{ocf}} \mu = f_\mathcal{E}(h_{\mathcal{E}_{ocf}}(\kappa) \circ \mu)$.

Using this translation, we get that:

Proposition 10. *$(f_\mathcal{E}, \circ_{\mathcal{E}_{ocf}})$ is a safe translation of $\circ$ into $\mathcal{E}_{ocf}$.*

Proof. First, for each $\Psi_i \in E$, let $\preceq_{\Psi_i}$ be the total preorder associated with Ψ_i by the DP assignment corresponding to $\circ$. Then we know by conditions 1 and 2 of a faithful assignment that $[Bel(\Psi_i)] = \min(\Omega, \preceq_{\Psi_i})$. And by construction of $f_\mathcal{E}(\Psi_i) = \kappa_i$, we have that $\kappa_i(\omega) = 0$ if and only if $rank(\preceq_{\Psi_i}, \omega) = 0$ if and only if $\omega \in \min(\Omega, \preceq_{\Psi_i})$. Hence, for each $\Psi_i \in E$, we have that $Bel(\Psi_i) = B_{ocf}(f_\mathcal{E}(\Psi_i))$. Second, for each $\Psi_i \in E$ and each $\mu \in \mathcal{L}_P^*$ we get that $f_\mathcal{E}(\Psi_i) \circ_{\mathcal{E}_{ocf}} \mu = \kappa_i \circ_{\mathcal{E}_{ocf}} \mu = f_\mathcal{E}(h_{\mathcal{E}_{ocf}}(\kappa_i) \circ \mu) = f_\mathcal{E}(f_\mathcal{E}^{-}1(\kappa_i) \circ \mu) = f_\mathcal{E}(\Psi_i \circ \mu)$. This shows that $(f_\mathcal{E}, \circ_{\mathcal{E}_{ocf}})$ is a translation of $\circ$ into $\mathcal{E}_{ocf}$.

Yet by definition of $h_{\mathcal{E}_{ocf}}$, on the one hand we get for each $\kappa \in U_{ocf}$ that $B_{ocf}(\kappa) \equiv Bel(h_{\mathcal{E}_{ocf}}(\kappa))$ (since by denoting $\Psi_i = h_{\mathcal{E}_{ocf}}(\kappa)$, we have that $tpo(\kappa) = \preceq_{\Psi_i}$). On the other hand, for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$ we get that $h_{\mathcal{E}_{ocf}}(\kappa \circ_{\mathcal{E}_{ocf}} \mu) = h_{\mathcal{E}_{ocf}}(f_\mathcal{E}(h_{\mathcal{E}_{ocf}}(\kappa) \circ \mu)) = f_\mathcal{E}^{-}1(f_\mathcal{E}(\Psi_i \circ \mu)) = \Psi_i \circ \mu = h_{\mathcal{E}_{ocf}}(\kappa_i) \circ \mu$. This shows that $(h_{\mathcal{E}_{ocf}}, \circ)$ is a translation of $\circ_{\mathcal{E}_{ocf}}$ into $\mathcal{E}$. Hence, $(f_\mathcal{E}, \circ_{\mathcal{E}_{ocf}})$ is a safe translation of $\circ$ into $\mathcal{E}_{ocf}$. □

We know now that every DP+G operator $\circ$ is safely $\mathcal{E}_{ocf}$-instantiable. Moreover, by construction, $\circ_{\mathcal{E}_{ocf}}$ is structure preserving. Therefore:

Corollary 5. *For every DP+G operator $\circ$, the operator $\circ_{\mathcal{E}_{ocf}}$ is equivalent to $\circ$ and is structure preserving.*

And we can conclude that:

Corollary 6. *$\mathcal{E}_{ocf}$ is DP+G-complete.*

This means that every DP operator defined on any epistemic space and satisfying the property **(G)** (that simply states that there is an initial, empty, epistemic state and that each epistemic state is reachable from it through a finite sequence of revisions) can be safely translated to the OCF epistemic space. As a consequence, all these operators can be defined equivalently in the OCF epistemic space.

7 Conclusion

In this work, we focused on Darwiche and Pearl's epistemic states and iterated revision operators from the representational perspective. We defined epistemic spaces as the structures in which these epistemic states and revision operators are defined. We highlighted the fact that epistemic states can be distinguished from each other by revision operators only by their behavior under any finite sequence of successive revisions. This led us to focus on the notion of strong equivalence between epistemic states, and to define a notion of equivalence between revision operators defined on different epistemic spaces. Based on these notions, we made precise the condition under which an operator is "instantiable" into another epistemic space.

Then, we looked for a canonical representation for the whole quotient set of epistemic states under this strong equivalence relation. But we showed that in general, such a canonical representation can not exist. The number of possible (quotient set of) epistemic states is just too large.

But then, if we restrict ourselves to epistemic states finitely generated from an initial, "empty" epistemic state, that is a quite reasonable assumption, then we showed that we can find such a canonical representation in terms of OCFs (Ordinal Conditional Functions). OCFs are a well-known representation choice for belief change operators, and are really close to total preorders. But this paper shows that the numerical information they add compared to total preorders is what is missing in order to encode all possible finitely generated epistemic state spaces.

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