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Hyperintensional Partial Meet Contractions

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Abstract

Formal frameworks for Epistemology need to have enough logical structure to enable interesting conclusions regarding epistemic phenomena and to be expressive enough to model competing positions in the philosophical and logical literature. While beliefs are commonly accepted as hyperintensional attitudes, i.e., epistemic attitudes, which may differ even towards necessarily equivalent sentences, most work on standard epistemic logic has relied on idealised and intensional agents. This is particularly true in the area of AGM-inspired Belief Change. Although a few recent studies investigate hyperintensional models of belief change, few have been well connected to the AGM framework, the main paradigm in the area. This work investigates hyperintensional notions of belief base contraction and belief set contraction, as studied in the AGM framework, and its connections to partial meet contractions. We also provide suitable representation theorems, characterising the constructions by means of rationality postulates.

1 Introduction

Formal frameworks for Epistemology, i.e., frameworks to reason about beliefs and knowledge, face a double-edged requirement. They need at the same time to be sufficiently structured, in the sense that we can employ them to reason about a broad class of doxastic agents, but flexible and expressive enough to model different theoretical and philosophical positions in the field, as discussed by Berto and Hawke (2021).

Since the work of Hintikka (1962), the epistemic logic tradition has heavily relied on idealised agents, as discussed by Rantala (1982) and Halpern and Puccela (2011). For example, standard epistemic logic has received criticism on the nature of its idealised agents and the intentional nature of beliefs in its models, c. f. the work of Cresswell (1972). It has also been argued in the literature (Halpern and Pucella 2011; Wansing 1990; Jago 2009; Bjerring 2013) that resource-bounded agents are not required to believe all consequences of their currently held beliefs, as an agent can fail to reach the conclusion of a reasoning process due to a lack of cognitive resources.

Similarly, in Belief Change, the area that studies how doxastic agents change their minds after acquiring new information, one of the most influential approaches in the literature, namely the AGM framework (Alchourrón, Gärdenfors, and Makinson 1985), also relies on heavily idealised agents. AGM admits as a representation of an agent's belief state a consequentially-closed set of formulas, requiring thus that an agent believes in all consequences of their beliefs. In fact, Gärdenfors (1988, p. 9) acknowledges that AGM's notion of Belief is but merely an idealisation "judged in relation to the rationality criteria for the epistemological theory".

It is well recognised in the literature, however, as discussed by Özgün and Berto (2020), that beliefs and other mental attitudes are sensitive to hyperintensional distinctions and that these latter are connected to well-studied problems, such as logical omniscience. We call, after Cresswell (1975), *hyperintensional attitudes* those which depend on sentential contents finer-grained than intensions. In other words, these attitudes can draw distinctions between necessarily equivalent contents. For example, while the sentences "*3 is a prime number*" and "*3068 is divisible by 13*" have the same intension as mathematical necessities, they certainly cannot be transparently substituted for the other in the sentence "Alice believes that *3 is a prime number*."

Aiming to obtain a compromise between the logical power of standard epistemic logic, as well as its unrealistic computational demands and the lack of expressiveness to encode current positions in philosophical debate (Berto and Hawke 2021), Levesque (1984) proposes a logic in which we can distinguish the notions of explicit and implicit beliefs. Explicit beliefs represent the notions of Belief that need not be closed under logical consequence and purely intentional, while implicit beliefs represent those that an agent may achieve from their explicit beliefs. In Belief Change, a similar position was advocated by Hansson (1992), who proposes that the representation of an agent's belief state, and its dynamics, depends on the syntactic structure of agents' explicit beliefs.

Despite incorporating some form of hyperintensionality in the representation of beliefs (and belief change) in the area, Hansson's approach does not consider hyperintensional difference in the input, i.e. the received information. For example, let B be a set of logical formulas[1] representing the belief base of an agent, i.e. the representation of their belief state, φ and ψ be logical formulas, and $\star$ be a belief

[1]In this example, we do not require any specific logic. Later, in Section 3, we will define more clearly our notion of logic.

change operation. Hansson's postulate of *uniformity* implies that if $Cn(\varphi) = Cn(\psi)$, then $B \star \varphi = B \star \varphi$, thus belief base change still preserves dependence on intensionality in the dynamics of beliefs.

Recently, work on hyperintensional belief change has risen in the literature to deal with this limitation. Most prominent, Berto (2019) proposes a hyperintensional logic of conditional beliefs and investigates hyperintensional belief revision operations interpreted as conditional beliefs. Other work soon followed, such as Özgün and Berto (2020) who dynamify Berto's conditional logic, proposing a dynamic logic of hyperintensional belief revision. On the other hand, Souza (2020) and Souza and Wassermann (2021) investigate hyperintensional belief change operations using tools similar to that of the AGM framework, based on abstract logics. These authors propose connections between hyperintensional belief change and belief change in non-classical logics, showing that for several logics of interest to AI, we can obtain partial meet-like operations based on a classical logic.

The main drawback of the work of Souza and Wassermann (2021) is that hyperintensional belief change is defined through intensional operators. For example, let C be the consequence operator of Intuitionistic Propositional Calculus (IPC) (Van Dalen 1994), and let Cn be the consequence operator of Classical Propositional Calculus (CPC), Souza and Wassermann show how one can use Cn to compute Belief Change for the logic defined by operator C. However, let p be the proposition that "3068 is divisible by 13". A constructive agent, i.e. an agent with intuitionistic reasoning, with a belief state believing that $\neg p$ is not true, can be described by the set of beliefs $B = \{\neg\neg p\}$. As $\neg\neg p \rightarrow p$ is not valid in IPC, the agent cannot, over the risk of irrationality, state that p is true. However, if the agent comes to the conclusion that there is no support for p, there is no operation defined by Souza and Wassermann in which removing the belief in p from B does not result in a belief state in which the agent believes that $\neg\neg p$, even though the agent did not initially believe p.

In this work, we explore further the notion of hyperintensional belief change, as studied by Souza (2020) and Souza and Wassermann (2021), aiming to overcome their deficiencies. Further, by differentiating between the notions of hyperintensional belief base change and hyperintensional belief set change, we can study the connections between AGM contractions and partial meet contractions in a more general setting, establishing relations to earlier work on the definability of AGM contractions for non-classical logics, such as that of Flouris (2006) on AGM-compatibility and that of Ribeiro (2013) on partial meet contractions for non-classical logics.

Notice that, for Classical Propositional Logic, there is a strong connection between Belief Base Change and Belief Set Change, as demonstrated by Hansson (Hansson 1991; Hansson 1992). However, for non-classical logics this connections seems much more fragile. In fact, as seen in the literature, for some logics the notions of AGM Belief Set Change and Belief Base Change may differ (Ribeiro 2013). More yet, for non-monotonic and hyperintensional logics, the connection between syntactic representations of the agent's epistemic state, e.g. by sets of formulas, and their beliefs is much more complex. As such, for non-classic logics, the notions of Belief Base Change and Belief Set Change may not always coincide and must be investigated separately.

This work is structured as follows: in Section 2 we discuss some of the related literature focusing on hyperintensional phenomena in belief change; in Section 3, we present the basic concepts and notations employed in this work; in Section 4, we study partial meet belief base contractions and their hyperintensional variants; Section 5 discusses partial meet belief set contractions and varying hyperintensional belief set operations, connecting to previous work in the literature; in Section 6, we study the connection between hyperintensional partial meet contractions and AGM contractions, providing negative results regarding the definability of the latter by the former. Finally, in Section 7, we present our final considerations and future directions of our work.

2 Related Work

Work on hyperintensional phenomena in representations of beliefs and other mental attitudes has a long standing tradition on epistemic logic at least since the work of Cresswell (1972; 1975; 1982), c.f. also (Rantala 1982; Vardi 1986; Fagin and Halpern 1987; Wansing 1990).

Regarding hyperintensional phenomena in Belief Change, a smaller, but not insignificant, number of contributions in the literature investigated how belief change can be modelled based on syntactic representations of the agent's belief state. This is the case, for example, of the work of Hansson (1992), Williams (1995), Rott (1998), and Ågnotes and Walicki (2004).

Work on belief change for non-classical logics, such as that of Girard and Tanaka (2016), have investigated dynamic belief change operators, which in principle could be used to model some hyperintensional notions, as classical consequences need not be valid in such logics. Similarly, Santos et al. (2018) investigate pseudo-contractions, which can be used to study a particular kind of hyperintensional belief change, called by Souza and Wassermann (2021) C-dependent contractions. These strategies, however, deal with hyperintensionality in an implicit way, not providing theoretical and philosophical connections that allow us to understand their underlying commitments.

Work on genuinely hyperintensional models for belief change, i.e. that explicitly consider hyperintensional differences between formulas both in the agent's beliefs and in the input, is far more recent in the literature. To our knowledge, the first of such attempts was proposed by Berto (2019), who investigates a topic-sensitive hyperintensional logic of conditional beliefs, in which belief revision is interpreted as conditionalisation. This work was soon followed by that of Özgün and Berto (2020), who dynamify the previously proposed logic. Unlike their work, however, ours investigates how a general notion of hyperintensional belief change can be defined, based on the AGM approach, that can be connected to different semantic frameworks for hyperintensional reasoning.

Similarly, Bozdag (2021) proposes a hyperintensional doxastic logic, based on the HYPE framework (Leitgeb 2019), in which belief base revision can also be thought of as a form of conditionalisation. As before, it is not completely clear how we can compare her proposal with competing notions of belief change in the AGM-inspired literature since, as observed by Lindström and Rabinowicz (1999), Baltag and Smets (2008), or even Souza et al. (2020), modal-based semantic approaches to AGM Belief Change often surpass the expressive power of the original frameworks and, thus, postulates need to be generalised for these settings.

Our work follows directly the line delineated in (Souza 2020) and (Souza and Wassermann 2021), expanding their framework and establishing connections with belief change in non-classical logics and between different notions of hyperintensional belief change investigated in these previous works.

3 Preliminaries

In this work, we employ the tools from Abstract Logic to study classes of logics in which we can define partial meet-like hyperintensional belief change operations. Differently from work such as that of AGM (Alchourrón, Gärdenfors, and Makinson 1985), Flouris (2006), or Ribeiro (2013), we will not require logics to be Tarskian unless explicitly stated.

We will call a logic any pair $\mathcal{L} = \langle L, Cn \rangle$, where L is a non-empty set, called the logical language, and $Cn : 2^L \to 2^L$ is a function called a consequence operator.

A consequence relation may satisfy some important properties defined below. These are common properties of several logics of interest for Philosophy and Artificial Intelligence.

- **inclusion**: $\Gamma \subseteq Cn(\Gamma)$.
- **idempotence**: $Cn(\Gamma) = Cn(Cn(\Gamma))$.
- **monotonicity**: If $\Gamma \subseteq \Gamma'$ then $Cn(\Gamma) \subseteq Cn(\Gamma')$.
- **tarskianicity**: If Cn satisfies inclusion, idempotence and monotonicity.
- **compactness**: for any $\varphi \in Cn(\Gamma)$, there is some finite $\Gamma' \subseteq \Gamma$ s.t. $\varphi \in Cn(\Gamma')$.
- **distributivity**: for any $\Gamma \subseteq L$ and any finitely representable[2] $\Gamma', \Gamma'' \subseteq L$, it holds that $Cn(\Gamma \cup (Cn(\Gamma') \cap Cn(\Gamma''))) = Cn(\Gamma \cup \Gamma') \cap Cn(\Gamma \cup \Gamma'')$.
- **closure under negation**: for any $\Gamma \subseteq L$ finitely representable, there is $\Gamma' \subseteq L$ finitely representable s.t. $Cn(\Gamma \cup \Gamma') = L$ and $Cn(\Gamma \cap \Gamma') = Cn(\varnothing)$.
- **booleanicity**: Cn is tarskian, distributive and closed under negation.

We say a logic $\mathcal{L} = \langle L, Cn \rangle$ satisfies a certain property, say tarskianicity, if its consequence operator Cn does.

The term hyperintensionality describes phenomena in which it is possible to draw distinctions between necessarily equivalent formulas - or those having the same intension. As such, hyperintensionality is commonly explained through the relation between the contents of a sentence and its intension with respect to a standard semantics. In the remainder of this work, we will represent hyperintensional reasoning by means of the relationship between two consequence operators over a given language. Let us define this notion formally.

Definition 1. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a logic, and $C : 2^L \to 2^L$ be a consequence operator. We say that:*

- *C is $\mathcal{L}$-sound, if for every $\Gamma \subseteq 2^L$, $C(\Gamma) \subseteq Cn(\Gamma)$;*
- *C is $\mathcal{L}$-complete, if for every $\Gamma \subseteq 2^L$, $Cn(\Gamma) \subseteq C(\Gamma)$.*

We will denote by $\mathcal{L}_C$ the logic defined by operator C, i.e. $\mathcal{L}_C = \langle L, C \rangle$.

Given a logic $\mathcal{L}$, the AGM authors investigate three basic belief change operators: expansion, contraction and revision. Belief expansion blindly integrates a new piece of information into the agent's beliefs. Belief contraction removes a currently believed sentence from the agent's set of beliefs with minimal alterations. Finally, belief revision is the operation of integrating new information into an agent's beliefs while maintaining consistency. Among these basic operations, only expansion can be univocally defined. The other two are defined by a set of rational constraints or postulates, usually referred to as the AGM postulates. These postulates define a class of suitable change operators representing different rational ways in which an agent can change their beliefs.

In this work, we will focus on hyperintensional contraction operations based on $\mathcal{L}$-sound consequences and characterise them through appropriate postulates, as commonly pursued by the AGM tradition. Contractions have a central role in the study of AGM Belief Change, based on AGM's result known as the Levi Identity (Alchourrón, Gärdenfors, and Makinson 1985), in which revisions may be defined using contraction operations. In the following, we investigate both notions of belief base contractions and belief set contractions, showing their connection.

4 Belief Base Contractions

The first kind of operation we study in this work is known as Belief Base Change, or more generally Belief Representation Change, in which the belief change operation acts on a representation of the agent's belief state to incorporate new information. These kinds of operations describe several belief change operations in the literature, related to ways of representing the agent's belief state, such as Hansson's belief base operations (Hansson 1992), Williams (1995) ensconcements, and Souza et al.'s (2019; 2021) graph transformations.

Hansson (1992) has advocated that belief dynamics should be dependent on explicit doxastic commitments of the agent, in the sense that some beliefs "have no independent standing, but arise only as inferences from our more basic beliefs" (Hansson 1992, p.240) and that "these consequences should be subject only to exactly those changes that follow from the changes of the primary beliefs"(Hansson 1992, p.240). As such, Belief Base Change aims to take

[2] We say a set $\Gamma \subseteq L$ is finitely representable if there is some finite $X \subseteq L$ s.t. $Cn(X) = Cn(\Gamma)$.

into consideration this relation between the agent's beliefs in its dynamics.

In this work, following Hansson (1992), we will focus on a representation of the agent's belief state as a set of formulas $B \subset L$, for a particular logic $\mathcal{L} = \langle L, Cn \rangle$. We call this set a belief base, representing the explicit beliefs held by the agent. As such, we can define the notion of a belief base change operator formally.

Definition 2. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a logic, we call belief base change operator any function $\star : 2^L \times L \to 2^L$*

We begin our exposition with partial meet belief base contractions, as originally studied by Hansson (1992) and Hansson and Wassermann (2002), and introduce our generalisation of such notion taking into consideration hyperintensional reasoning.

4.1 Partial Meet Belief Base Contraction

A partial meet belief base contraction is an operation that preserves a maximal amount of "safe" information from the agent's explicit beliefs, i.e., information that cannot be used to derive what the agent has ceased to believe. To formalise this notion, Alchourrón and Makinson (1982) propose the notion of remainder set.

Definition 3. *(Hansson and Wassermann 2002) Let $B \subseteq L$ be a set of formulas and $\varphi \in L$ be a formula of L, the remainder set $B \perp_{\mathcal{L}} \varphi$ is the set of sets B' satisfying:*

- $B' \subseteq B$
- $\varphi \notin Cn(B')$
- $B' \subset B'' \subseteq B$ *implies* $\varphi \in Cn(B'')$.

When it is clear to which logic $\mathcal{L}$ we are referring, we will denote $B \perp_{\mathcal{L}} A$ by $B \perp A$.

A partial meet contraction $-$ is an operation for which there is a selection function γ, that characterises this operations. By selection function, we mean that the function γ satisfies (i) $\varnothing \neq \gamma(B \perp A) \subseteq B \perp A$ if $B \perp A \neq \varnothing$ and (ii) $\gamma(B \perp A) = \{B\}$ otherwise.

Definition 4. *(Hansson and Wassermann 2002) We say a belief base change operator $-$ is a belief base contraction on a set $B \subseteq L$ if there is a selection function γ, s.t. for any φ*

$$B - \varphi = \bigcap \gamma(B \perp \varphi).$$

Hansson and Wassermann (2002) show that for any monotonic and compact logic, an operation $-$ is a partial meet contraction on B if and only if it satisfies the following postulates:

(**success**) If $\varphi \notin Cn(\varnothing)$, then $\varphi \notin Cn(B - \varphi)$

(**inclusion**) $B - \varphi \subseteq B$

(**uniformity**) If for any $B' \subseteq B$ it holds that $\varphi \in Cn(B')$ iff $\psi \in Cn(B')$, then it holds that $B - \varphi = B - \psi$

(**relevance**) If $\psi \in B \backslash B - \varphi$, then there is some $B' \subseteq B$ s.t. $B - \varphi \subseteq B'$, $\varphi \notin Cn(B')$, and $\varphi \in Cn(B' \cup \{\psi\})$

Let us revisit the example discussed in the introduction of a constructive agent having a belief set based on the information that $\neg p$ is not true and apply this notion to investigate the example of trying to remove the information that p from their beliefs.

Example 5. *Let $\mathcal{L} = \langle L, Cn \rangle$ be the Classical Propositional Logic, C be the consequence operator of Intuitionistic Propositional Logic, and $p \in L$ a propositional symbol of L. Take $B = \{\neg\neg p\}$, then $B \perp p = \{\varnothing\}$ and thus $B - p = \varnothing$, for any belief base partial meet contraction $-$. However, $B \perp_{\mathcal{L}_C} p = \{B\}$.*

Notice in Example 5 that the partial meet belief base contractions defined in $\mathcal{L}$ and in $\mathcal{L}_C$ differ substantially, in the sense that there is no operation for which the result of $B - p$ coincides in both logics. In the next section, we will discuss how we can construct contraction operations for the logic $\mathcal{L}_C$ from partial meet contractions in the logic $\mathcal{L}$, thus removing the requirement for C to be compact in order to obtain such operations.

4.2 Hyperintensional Partial Meet Belief Base Contraction

To deal with the problem of providing an adequate answer to the example illustrated above, we will introduce the notion of a hyperintensional remainder set.

Definition 6. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator, $B \subseteq L$ be a set of formulas and $\varphi \in L$ a formula of $\mathcal{L}$. We define the hyperintensional remainder set of B by φ, relative to C, as the set:*

$$B \perp^C_{\mathcal{L}} \varphi = \{B' \subseteq B | \varphi \notin C(B') \text{ and } \exists B'' \in B \perp \varphi \text{ s.t. } B'' \subseteq B'\}$$

The hyperintensional remainder set of B by φ, relative to C, contains all the subsets of B that do not imply φ, according to the hyperintensional consequence operator C, while maintaining a maximal amount "safe" information in B, i.e. that does not imply φ according to Cn.

Lemma 7. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator. For any $B \subseteq L$ and $\varphi \in L$, $B \perp \varphi \subseteq B \perp^C_{\mathcal{L}} \varphi$.*

We can understand the hyperintensional remainder set of B by φ as a set of approximations of the 'optimal' sets that do not imply φ according to C. If C is monotonic and compact, the notion of 'optimal' commonly adopted in the literature is that of maximality and, indeed, all maximal subsets of B that do not imply φ, according to C, are in $B \perp^C_{\mathcal{L}} \varphi$.

Lemma 8. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a monotonic and compact logic, C be a monotonic and compact $\mathcal{L}$-sound consequence operator. For any $B \subseteq L$ and $\varphi \in L$, $B \perp_{\mathcal{L}_C} \varphi \subseteq B \perp^C_{\mathcal{L}} \varphi$.*

With that, we can define the class of hyperintensional partial meet belief base contractions, as usual.

Definition 9. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator, and $B \subseteq L$ be a set of formulas. We say a belief base change operator $- : 2^L \times L \to 2^L$ is a hyperintensional belief base contraction* *on B, according to C, iff there is a selection function γ s.t. for any $\varphi \in L$:*

$$B - \varphi = \bigcap \gamma(B \perp_{\mathcal{L}}^{C} \varphi).$$

From Lemmas 7 and 8, it is easy to see that every partial meet belief base contraction (on Cn or on C) is a hyperintensional partial meet belief base contraction, in which the selection function always selects elements of $B \perp_{\mathcal{L}} \varphi$ ($B \perp_{\mathcal{L}_C} \varphi$). Thus, our notion is more general than that of Hansson, while maintaining the desideratum of preserving the maximal amount of 'safe' information from the agent's belief state.

Corollary 10. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a monotonic and compact logic and C be any $\mathcal{L}$-sound consequence operator, every partial meet belief base contraction in $\mathcal{L}$ is a hyperintensional belief base contraction. Further, if C is a monotonic and compact $\mathcal{L}$-sound consequence operator, every partial meet belief base contraction in $\mathcal{L}_C$ is also a hyperintensional belief base contraction.*

Let us see that this notion can actually be used to give a proper response to our running example.

Example 11. *Let $\mathcal{L} = \langle L, Cn \rangle$ be the Classical Propositional Logic, C be the consequence operator of Intuitionistic Propositional Logic, and $p \in L$ a propositional symbol of L. Take the constructive agent with beliefs $B = \{\neg\neg p\}$, then $B \perp_{\mathcal{L}}^{C} p = \{\varnothing, \{\neg\neg p\}\}$. Clearly $B \in B \perp_{\mathcal{L}}^{C} p$ and, thus, there is some hyperintensional belief base partial meet contraction $-$ on B s.t. $B - p = B$.*

To characterise this operation, as commonly pursued by the AGM tradition, we will employ the following postulates - a modification of Hansson and Wassermann's (2002) postulates for belief base contraction.

(inclusion) $B - \varphi \subseteq B$

(C-success) If $\varphi \notin C(\varnothing)$, then $\varphi \notin C(B - \varphi)$

(hyperintensional uniformity) If for any $B', B'' \subseteq B$ it holds that

1. $\varphi \in Cn(B')$ iff $\psi \in Cn(B')$
2. $\varphi \notin Cn(B')$ and $\varphi \in C(B' \cup B'')$ implies that there is some $B''' \subseteq B$ s.t. $B' \subseteq B'''$, $\psi \notin Cn(B''')$ and $\psi \in C(B''' \cup B'')$
3. $\psi \notin Cn(B')$ and $\psi \in C(B' \cup B'')$ implies that there is some $B''' \subseteq B$ s.t. $B' \subseteq B'''$, $\varphi \notin Cn(B''')$ and $\varphi \in C(B''' \cup B'')$

then $B - \varphi = B - \psi$

(hyperintensional relevance) If $\psi \in B \backslash B - \varphi$, there is some $B' \subseteq B$ s.t. $B - \varphi \subseteq B'$, $\varphi \notin C(B')$ but $\psi \notin B'$ and $\varphi \in Cn(B' \cup \{\psi\})$. Furthermore, there is some $B'' \subseteq B'$ s.t. $\varphi \notin Cn(B'')$ but $\varphi \in Cn(B'' \cup \{\xi\})$ for any $\xi \in B \backslash B''$.

The postulate of *(inclusion)* states that no information not currently believed by the agent will be included in her belief state after performing a contraction. The postulate of *(C-success)* states that the agent will not believe the information being removed after contraction unless it is a C-validity. The postulate of *(hyperintensional uniformity)* reinforces Hansson's original *(uniformity)* requiring that the contraction operation coincides for any two formulas when they behave similarly with respect to the logic Cn, but also with respect to C in the subsets of a belief base B. Finally, the postulate of *(hyperintensional relevance)* states that if a formula ψ is removed in a contraction, there is a set containing a maximal set with respect to Cn, in which ψ can be used to prove the information to be removed.

With these postulates, we can prove the characterisation of hyperintensional partial meet belief contraction for a monotonic logic C.

Proposition 12. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a monotonic and compact logic, C be a monotonic $\mathcal{L}$-sound consequence operator, and $B \subseteq L$ be a set of formulas. If $-$ is a hyperintensional belief base contraction on B, then $-$ satisfies (inclusion), (C-success), (hyperintensional uniformity) and (hyperintensional relevance).*

Proof. The proofs for *(inclusion)*, *(C-success)* and *(hyperintensional relevance)* are immediate by construction. Let's show that *(hyperintensional uniformity)* holds.

Take $\varphi, \psi \in L$ satisfying the hypothesis, we only need to show that $B \perp_{\mathcal{L}}^{C} \varphi = B \perp_{\mathcal{L}}^{C} \psi$. From $\varphi \in Cn(B')$ iff $\psi \in Cn(B')$ for every $B' \subseteq B$, we conclude that $B \perp \varphi = B \perp \psi$.

Take $B' \in B \perp_{\mathcal{L}}^{C} \varphi$, then $\varphi \notin C(B')$. As $B' \in B \perp_{\mathcal{L}}^{C} \varphi$, then there is some $B'' \in B \perp \varphi$ s.t. $B'' \subseteq B'$. As $B \perp \varphi = B \perp \psi$, we just need to show that $\psi \notin C(B')$. Suppose $\psi \in C(B')$, then $\psi \notin Cn(B'')$ and $\psi \in C(B'' \cup B' \backslash B'')$. By hypothesis, there is some $B''' \subseteq B$ s.t. $B'' \subseteq B'''$, $\varphi \notin B''$ and $\varphi \in C(B''' \cup B' \backslash B''')$. However, $B'' \in B \perp \varphi$, thus $B''' = B''$ and $\varphi \in C(B'' \cup B' \backslash B'') = C(B')$, which is a contradiction. Thus, $\psi \notin C(B')$ and $B' \in B \perp_{\mathcal{L}}^{C} \psi$. □

More yet, any operation satisfying these postulates are hyperintensional partial meet belief base contractions.

Theorem 13. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a monotonic and compact logic, C be a monotonic $\mathcal{L}$-sound consequence operator, and $B \subseteq L$ be a set of formulas. The operator $-$ is a hyperintensional belief base contraction on B iff $-$ satisfies (inclusion), (C-success), (hyperintensional uniformity) and (hyperintensional relevance).*

Proof. Proposition 12 already showed that hyperintensional belief base contractions satisfy the postulates when C is monotonic. Let us show that any belief base change operation satisfying these postulates is a hyperintensional belief base contraction.

Let $\star : 2^L \times L \to 2^L$ be a belief base change operation satisfying the four postulates for a given belief base B. Let us construct a selection function γ that coincides with the result of $B \star \varphi$ for any φ.

Define $\gamma : 2^{2^L} \to 2^{2^L}$ a function s.t. $\gamma(B \perp_{\mathcal{L}}^{C} \varphi) = \{B' \in B \perp_{\mathcal{L}}^{C} \varphi \mid B \star \varphi \subseteq B'\}$ if $B \perp_{\mathcal{L}}^{C} \varphi \neq \varnothing$ and $\gamma(B \perp_{\mathcal{L}}^{C} \varphi) = \{B\}$, otherwise.

We need to show that γ is a selection function and that $B \star \varphi = \bigcap \gamma(B \perp_{\mathcal{L}}^{C} \varphi)$. First, it suffices to see that if $B \perp_{\mathcal{L}}^{C} \varphi = B \perp_{\mathcal{L}}^{C} \psi$, then it holds that for any $B', B'' \subseteq B$: (i) $\varphi \in$

$Cn(B')$ iff $\psi \in Cn(B')$, (ii) if $\varphi \notin Cn(B')$ and $\varphi \in C(B' \cup B'')$, it holds that there is some $B''' \subseteq B$ s.t. $\psi \notin Cn(B''')$ and $\psi \in C(B''' \cup B'')$, and (iii) if $\psi \notin Cn(B')$ and $\psi \in C(B' \cup B'')$, then it holds that there is some $B''' \subseteq B$ s.t. $\varphi \notin Cn(B''')$ and $\varphi \in C(B''' \cup B'')$. By *(hyperintensional uniformity)* of $\star$, $B \star \varphi = B \star \psi$ and, thus, $\gamma(B\perp^C_\mathcal{L}\varphi) = \gamma(B\perp^C_\mathcal{L}\psi)$. Further, $\gamma(B\perp^C_\mathcal{L}\varphi) \neq \varnothing$ if $B\perp^C_\mathcal{L}\varphi \neq \varnothing$, as (i) if $\varphi \notin Cn(B \star \varphi)$, it holds that there is some $B' \in B\perp\varphi \subseteq B\perp^C_\mathcal{L}\varphi$ s.t. $B \star \varphi \subseteq B'$ and (ii) if $\varphi \in Cn(B \star \varphi)$, by *(hyperintensional relevance)* there is a $B' \in B\perp\varphi$ s.t. $\varphi \notin C(B' \cup B\star\varphi)$, thus $B' \cup B\star\varphi \in B\perp^C_\mathcal{L}\varphi$ and, by construction, $B' \cup B \star \varphi \in \gamma(B\perp^C_\mathcal{L}\varphi)$. Additionally, if $B\perp^C_\mathcal{L}\varphi = \varnothing$ then $\gamma(B\perp^C_\mathcal{L}\varphi) = \{B\}$, i.e. γ is a selection function.

Let us show that $B\star\varphi = \bigcap\gamma(B\perp^C_\mathcal{L}\varphi)$. Well, by construction, it is easy to see that $B\star\varphi \subseteq \bigcap\gamma(B\perp^C_\mathcal{L}\varphi)$, then we only need to show the opposite inclusion. Take $\psi \in \bigcap\gamma(B\perp^C_\mathcal{L}\varphi)$, then for any $B' \in B\perp^C_\mathcal{L}\varphi$ s.t. $B \star \varphi \subseteq B'$, $\psi \in B'$. Suppose $\psi \notin B\star\varphi$, then by *(hyperintensional relevance)* of $\star$, there is some $B' \subseteq B$ s.t. $B - \varphi \subseteq B'$, $\psi \notin B'$ and $\varphi \notin C(B')$, and there is $B'' \subseteq B'$ s.t. $\varphi \notin Cn(B'')$ but $\varphi \in Cn(B'' \cup \{\xi\})$ for any $\xi \in B\backslash B''$. Clearly, $B' \in \gamma(B\perp^C_\mathcal{L}\varphi)$, but $\psi \notin B'$, which is a contradiction to $\psi \in \bigcap\gamma(B\perp^C_\mathcal{L}\varphi)$. Thus, $\psi \in B \star \varphi$.

□

5 Belief Set Contractions

In their seminal work, AGM defines a class of operations over the set of theories of a logic $\mathcal{L}$, i.e. sets of formulas satisfying $K = Cn(K)$. These sets of formulas represent the beliefs currently held by an agent. Belief Set Change operations, thus, represent how the currently held beliefs of an agent may be changed due to some new information. As discussed before, changes in belief sets and changes in belief bases may differ in nature for non-classical logics. In fact, the notion of belief set as a consequentially-closed set only applies to Tarskian logics, since for logics not satisfying **idempotence** (c.f. Section 3), there is no guarantee of the existence of such theories. As such, we need to generalise this notion to the context of non-classical logics.

Definition 14. *Let $\mathcal{L} = \langle L, Cn\rangle$ be a logic. The set of theories of $\mathcal{L}$, denoted by $Th(\mathcal{L})$, is the set of all conclusions of some set of hypotheses, i.e.*

$$Th(\mathcal{L}) = \{Cn(B) \mid B \subseteq \mathcal{L}\}.$$

We say a logic $\mathcal{L}$ is closed under intersection if its set of theories is closed under intersections, i.e. for any $T, T' \in Th(\mathcal{L})$, $T \cap T' \in Th(\mathcal{L})$. That is a common property of Tarskian logics, in fact of any monotonic and idempotent logic, but may not hold for all logics considered. Closure under intersection will be an important property of the logics to guarantee the existence of contractions that we investigate in the following.

We call, following AGM, a belief set any element of $Th(\mathcal{L})$, representing the beliefs of the agent obtained from the representation of their belief state as a set of formulas. Belief set contractions are, thus, functions that change the belief set of an agent. Notice that, since different belief bases can give rise to the same belief set and representing the explicit beliefs of an agent is an important feature for hyperintensional modellings of belief change, taking the original belief state of the agent into consideration when computing changes is of great importance. As such, we will employ pseudo-contractions (Santos et al. 2018) to construct belief set contractions in our work. Let us define this notion formally.

Definition 15. *Let $\mathcal{L} = \langle L, Cn\rangle$ be a logic. We say a function $\star : 2^L \times L \to 2^L$ is a belief set change operation if $B \star \varphi \in Th(\mathcal{L})$ for any B and φ.*

Finally, as our logics are not necessarily Tarskian, we need to establish a differentiation among the notions of a remainder set, as defined in Definition 3, used to define partial meet belief base changes, and a different notion, which we call residual beliefs, used to define partial meet belief set changes. The set of residuals describes the possible sets of beliefs after removing an information φ from an epistemic state described by a belief base B. Differently then remainder sets, residual sets do not preserve the structure of the epistemic state, i.e. the agent's explicit doxastic commitments.

Definition 16. *Let $\mathcal{L} = \langle L, Cn\rangle$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator, $B \subseteq L$ be a set of formulas and $\varphi \in L$ a formula of $\mathcal{L}$. The set of belief residuals from B by φ, according to $\mathcal{L}$, is the set*

$$B\triangle_\mathcal{L}\varphi = \{Cn(B') \mid B' \in B\perp\varphi\}.$$

Similarly,

$$B\triangle_{\mathcal{L}_C}\varphi = \{C(B') \mid B' \in B\perp\varphi\}.$$

Similarly to what we did for belief base change, in Definition 6, we will also define a hyperintensional notion of residuals, with which we define different notions of hyperintensional belief set contractions. Analogously to hyperintensional remainders, hyperintensional residuals describe the subsets of the agent's beliefs that do not support the information φ and contain some residual of the agent's beliefs after removing φ, representing the hyperintensional belief (sub)sets in which information φ does not hold, but maintain some maximal amount of "safe" information.

Definition 17. *Let $\mathcal{L} = \langle L, Cn\rangle$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator, $B \subseteq L$ be a set of formulas and $\varphi \in L$ a formula of $\mathcal{L}$. The set of hyperintensional belief residuals from B by φ, according to $\mathcal{L}$, is the set*

$$B\blacktriangle_{\mathcal{L}_C}\varphi = \{C(B') \mid B' \subseteq B, \varphi \notin C(B') \text{ and there is } K \in B\triangle_{\mathcal{L}_C}\varphi \text{ s.t. } K \subseteq C(B') \subseteq C(B)\}.$$

5.1 Partial Meet Belief Set Contraction

As for belief base contractions, let us define partial meet belief set contractions, as defined by AGM.

Definition 18. *Let $\mathcal{L} = \langle L, Cn\rangle$ be a monotonic and compact logic, and $B \subseteq L$ be a set of formulas. We say a belief set change operator $- : Th(\mathcal{L}) \times L \to Th(\mathcal{L})$ is partial meet belief set contraction on B iff there is a selection function γ s.t. for any $\varphi \in L$:*

$$B - \varphi = \bigcap\gamma(B\triangle_\mathcal{L}\varphi) \cap Cn(B).$$

Notice that, as is the case for AGM belief contraction and Hansson's belief base contractions, it is not always the case that partial meet belief set contractions coincide with the consequence of a partial meet base contraction. In fact, take $B = \{p, p \to q\}$, then $B \perp p = \{\{p \to q\}\}$, thus $B \triangle p = \{Cn(\{p \to q\})\}$, while $Cn(B) \perp p = \{Cn(\{p \to q\}), Cn(\{q\})\}$. This means that there is some partial meet belief base contraction $-$ s.t. for any partial meet belief set contraction $\ominus$, $B \ominus p \neq Cn(B) - p$. The reason for this is that $Cn(B) \perp p$ does not preserve the structure of the belief state B, as we intend to do.

Example 19. *Let $\mathcal{L} = \langle L, Cn \rangle$ be the Classical Propositional Logic, C be the consequence operator of Intuitionistic Propositional Logic, and $p \in L$ a propositional symbol of L. Take the constructive agent with beliefs $B = \{\neg\neg p\}$, then $B \triangle p = \{Cn(\varnothing)\}$ and thus $B - p = Cn(\varnothing)$, for any belief set partial meet contraction $-$.*

We will employ the following postulates to characterise partial meet belief set contraction for any monotonic and compact logic closed under intersection.

(logical closure) $B - \varphi = C(\Gamma)$, for some $\Gamma \subseteq L$

(logical inclusion) $B - \varphi \subseteq Cn(B)$

(success) If $\varphi \notin Cn(\varnothing)$, then $\varphi \notin B - \varphi$

(uniformity) If for any $B' \subseteq B$, $\varphi \in Cn(B')$ iff $\psi \in Cn(B')$, then $B - \varphi = B - \psi$

(logical relevance) If $\psi \notin Cn(B) \backslash B - \varphi$, then there is some $B' \subseteq B$ s.t. $B - \varphi \subseteq Cn(B')$, $\varphi \notin Cn(B')$, and $\varphi \in Cn(B' \cup \{\psi\})$

We will omit the proof of this characterisation since it is a simple modification of the one presented by Hansson and Wassermann (2002) to consider the set $B \triangle_{\mathcal{L}} \varphi$ instead of $B \perp \varphi$.

From the same structure of partial meet belief set contraction, we can define different notions hyperintensional belief set contraction, varying the elements used, i.e. substituting $B \triangle_{\mathcal{L}} \varphi$ for $B \triangle_{\mathcal{L}_C} \varphi$ or $B \blacktriangle_{\mathcal{L}_C} \varphi$, substituting $Cn(B)$ for $C(B)$, etc. These variations give rise to different operations previously investigated in the literature by Santos et al. (2018), Souza (2020) and Souza and Wassermann (2021), as well as some interesting new ones. In this work, we will focus on only two of them.

5.2 Belief Set C-base Contraction

Based on the definition of (a subclass of) belief base contractions using AGM contractions by Hansson, Souza and Wassermann (2021) proposed the notion of C-base contractions, a generalisation of the C-dependent contraction previously studied by Santos et al. (2018) and Souza (2020). C-base contractions are hyperintensional contractions in that they consider hyperintensional differences in the belief base to compute a contraction, generalising the reasoning behind Hansson's (1992) belief base contraction to any hyperintensional consequence C.

Definition 20. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator, and $B \subseteq L$ be a set of formulas. We say a belief base change operator $- : 2^L \times L \to 2^L$ is a C-base belief set contraction on B iff there is a selection function γ s.t. for any $\varphi \in L$:*

$$B - \varphi = \bigcap \gamma(B \triangle_{\mathcal{L}} \varphi) \cap C(B).$$

Let us apply this definition to investigate our running example.

Example 21. *Let $\mathcal{L} = \langle L, Cn \rangle$ be the Classical Propositional Logic and C be the consequence operator of Intuitionistic Propositional Logic and $p \in L$ a propositional symbol of L. Take the constructive agent with beliefs $B = \{\neg\neg p\}$, then $B \triangle p = \{Cn(\varnothing)\}$ and thus $B - p = C(\varnothing)$, for any belief set C-base contraction $-$.*

Notice that, in general, $\bigcap \gamma(B \triangle \varphi) \cap C(B)$ is not a C-theory, i.e. $\bigcap \gamma(B \triangle \varphi) \cap C(B) \notin Th(\mathcal{L}_C)$. Thus, to properly characterise this operation as a hyperintensional belief set contraction, we must require a further property of the logic C: upward closure under intersection.

Given a logic $\mathcal{L} = \langle L, Cn \rangle$ and a $\mathcal{L}$-sound consequence operator C. We say $\mathcal{L}_C$ is upwards closed under intersection if it is closed under intersection and for any $T \in Th(\mathcal{L})$ and $T' \in Th(\mathcal{L}_C)$, it holds that $T \cap T' \in Th(\mathcal{L}_C)$.

To characterise belief set C-base contractions, we employ the following postulates - a modified version of those presented previously (Souza and Wassermann 2021).

(logical closure) $B - \varphi = C(\Gamma)$, for some $\Gamma \subseteq L$

(C-logical inclusion) $B - \varphi \subseteq C(B)$.

(success) If $\varphi \notin Cn(\varnothing)$, then $\varphi \notin B - \varphi$

(uniformity) If for all $B' \subseteq Cn(B)$, it holds that $\varphi \in Cn(B')$ iff $\psi \in Cn(B')$, then $B - \varphi = B - \psi$

(C-local logical relevance) If $\psi \in C(B) \backslash B - \varphi$, then there is some $B' \subseteq B$, s.t. $B - \varphi \subseteq Cn(B')$, $\varphi \notin Cn(B')$ and $\varphi \in Cn(B' \cup \{\psi\})$.

Again, the proof of this characterisation, for any logic $\mathcal{L}_C$ upwards closed under intersection, can be obtained by a simple modification of the proof provided previously (Souza and Wassermann 2021) to account for the use of the residual set $B \triangle_{\mathcal{L}} \varphi$.

5.3 Hyperintensional Partial Meet Belief Set Contraction

Finally, analogously to how we defined hyperintensional partial meet belief base contractions, we can define a general notion of hyperintensional belief set contraction that allows us to approximate changes in the agent's beliefs to remove certain information. Differently than Definition 9, however, by using hyperintensional residuals, we are concerned not with the structure of the agent's resulting epistemic state but which beliefs are maintained by the agent after contraction. As a result, hyperintensional belief set contractions can select the hyperintensional subtheories of an agent's beliefs which do not imply the information to be removed.

Definition 22. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator, and $B \subseteq L$ be a set of formulas. We say a belief set change* *operator* $- : 2^L \times L \to Th(\mathcal{L}_C)$ *is a hyperintensional C-dependent belief set contraction on B, according to C, iff there is a selection function* γ *s.t. for any* $\varphi \in L$*:*

$$B - \varphi = \bigcap \gamma(B \blacktriangle_{\mathcal{L}_C} \varphi) \cap Cn(B).$$

As for hyperintensional partial meet belief base contractions, hyperintensional partial meet belief set contractions seek to obtain hyperintensional subtheories of the agent's belief set $Cn(B)$, i.e. elements $K \in Th(\mathcal{L}_C)$ s.t. $K \subseteq Cn(B)$, which do not prove the formula being removed. As for partial meet belief set contractions, it is not always the case that $\bigcap \gamma(B \blacktriangle_{\mathcal{L}_C} \varphi)) = C(\bigcap \gamma(B \perp^C_{\mathcal{L}} \varphi))$, so there is some hyperintensional partial meet belief base contraction $-$ for which there is no hyperintensional partial meet belief set contraction $\ominus$ satisfying $B - \varphi = B \ominus \varphi$.

Again, we can employ this operation to study our running example.

Example 23. *Let* $\mathcal{L} = \langle L, Cn \rangle$ *be the Classical Propositional Logic, C be the consequence operator of Intuitionistic Propositional Logic, and* $p \in L$ *a propositional symbol of L. Take the constructive agent with beliefs* $B = \{\neg\neg p\}$*, then* $B \blacktriangle_{\mathcal{L}_C} p = \{C(\varnothing), C(\{\neg\neg p\}\}$*. Clearly* $C(B) \in B \blacktriangle_{\mathcal{L}_C} p$ *and, thus, there is some hyperintensional belief set partial meet contraction* $-$ *s.t.* $B - p = C(B)$*.*

We can thus see that the hyperintensional residual set $B \blacktriangle_{\mathcal{L}_C} p$ contains subtheories of $C(B)$ that do not prove p. To characterise this operation, we will employ the following postulates:

(logical closure) $B - \varphi = C(\Gamma)$, for some $\Gamma \subseteq L$

(C-logical inclusion) $B - \varphi \subseteq C(B)$

(C-success) If $\varphi \notin C(\varnothing)$, then $\varphi \notin B - \varphi$

(hyperintensional uniformity) If for any $B', B'' \subseteq B$ it holds that

1. $\varphi \in Cn(B')$ iff $\psi \in Cn(B')$
2. $\varphi \notin Cn(B')$ and $\varphi \in C(B' \cup B'')$ implies that there is some $B''' \subseteq B$ s.t. $B' \subseteq B'''$, $\psi \notin Cn(B''')$ and $\psi \in C(B''' \cup B'')$
3. $\psi \notin Cn(B')$ and $\psi \in C(B' \cup B'')$ implies that there is some $B''' \subseteq B$ s.t. $B' \subseteq B'''$, $\varphi \notin Cn(B''')$ and $\varphi \in C(B''' \cup B'')$

then $B - \varphi = B - \psi$

(hyperintensional logical relevance) If $\psi \in Cn(B) \backslash B - \varphi$, there is some $B' \subseteq B$, s.t. $B - \varphi \subseteq C(B')$, $\varphi \notin C(B')$, and $\psi \notin C(B')$, but $\varphi \in Cn(B' \cup \{\psi\})$. Further, there is some $B'' \subseteq B$ s.t. $C(B'') \subseteq C(B')$, $\varphi \notin Cn(B'')$ but $\varphi \in Cn(B'' \cup \{\xi\})$ for any $\xi \in B \backslash B''$.

We can show that these postulates characterise hyperintensional partial meet belief set contraction for all monotonic and upwardly closed under intersection logics $\mathcal{L}_C$.

Theorem 24. *Let* $\mathcal{L} = \langle L, Cn \rangle$ *be a monotonic and compact logic, C be a monotonic* $\mathcal{L}$*-sound consequence operator with* $\mathcal{L}_C$ *an upwardly closed under intersection logic, and* $B \subseteq L$ *be a set of formulas. An operation* $- : 2^L \times L \to Th(\mathcal{L}_C)$ *is a hyperintensional belief set partial meet contraction iff it satisfies (C-logical inclusion), (C-success), (hyperintensional uniformity), and (hyperintensional logical relevance).*

Sketch of the proof. Satisfaction of the postulates is similar to that presented in Proposition 12, with *(logical closure)* following from upward closure by intersection of $\mathcal{L}_C$. To prove the characterisation, construct the selection function $\gamma(B \blacktriangle_{\mathcal{L}_C})\varphi = \{B' \in B \blacktriangle_{\mathcal{L}_C} \varphi \mid B - \varphi \subseteq B'\}$ if $B \blacktriangle_{\mathcal{L}_C} \varphi \neq \varnothing$ and $\gamma(B \blacktriangle_{\mathcal{L}_C} \varphi)\varphi = C(B)$, otherwise.

To prove that γ is selection function, similarly to the proof for hyperintensional partial meet base contraction, it suffices to see that *(hyperintensional logical relevance)* implies that if $B \blacktriangle_{\mathcal{L}_C} \varphi \neq \varnothing$ then $\gamma(B \blacktriangle_{\mathcal{L}_C} \varphi) \neq \varnothing$ and *(hyperintensional uniformity)* implies that if $B \blacktriangle_{\mathcal{L}_C} \varphi = B \blacktriangle_{\mathcal{L}_C} \psi$ then $\gamma(B \blacktriangle_{\mathcal{L}_C} \varphi) = \gamma(B \blacktriangle_{\mathcal{L}_C} \psi)$.

To show that $B - \varphi = \gamma(B \blacktriangle_{\mathcal{L}_C} \varphi) \cap Cn(B)$, it suffices to see that $B - \varphi \subseteq \gamma(B \blacktriangle_{\mathcal{L}_C} \varphi) \cap Cn(B)$ by construction and that for any $\psi \in \gamma(B \blacktriangle_{\mathcal{L}_C} \varphi) \cap Cn(B)$, $\psi \in B - \varphi$ by *(hyperintensional logical relevance)*. □

6 AGM Contractions and Hyperintensional Belief Set Contractions

In this section, we wish to explore whether AGM belief contraction can be understood as a form of partial meet hyperintensional belief contraction, as studied in this work. The motivation for such investigation lies in the fact that the notions of AGM contraction and partial meet contraction coincide in several logics of interest but diverge in some non-classical logics (Ribeiro 2013). Thus, we investigate whether these notions can be reconciled within a broader framework that explains these previous results.

Flouris (2006) studied the definability of AGM contraction operations, i.e. operations satisfying the original AGM postulates, in Tarskian logics, obtaining sufficient and necessary conditions for such definability. Let us introduce these notions in order to compare our hyperintensional belief change operations and AGM contractions.

Definition 25. *(Flouris 2006) A tarskian logic* $\mathcal{L} = \langle L, Cn \rangle$ *is said to be decomposable if for any* $B \subseteq L$ *and* $\varphi \in L$*, with* $\varphi \notin Cn(\varnothing)$*, the set*

$$K^{-\varphi} = \{B' \subseteq B \mid \varphi \notin Cn(B') \wedge Cn(B) = Cn(B' \cup \{\varphi\})\}$$

is not empty.

Flouris introduces the notion of decomposability as the possibility of constructing a contraction, i.e. that there is a set that is an admissible result for a contraction by a formula, for any set B and any formula φ. Further, the author shows that all Tarskian and boolean logics are decomposable.

Theorem 26. *(Flouris 2006) Any Tarskian logic* $\mathcal{L} = \langle L, Cn \rangle$ *that is also boolean is decomposable.*

Flouris (2006) show that decomposability is a necessary and sufficient condition for the definability of AGM-compliant contraction operations.

Proposition 27. *(Flouris 2006) Let* $\mathcal{L} = \langle L, Cn \rangle$ *be a decomposable logic and* $- : Th(\mathcal{L}) \times L \to Th(\mathcal{L})$ *be a belief change operation.* $-$ *is an AGM contraction, i.e. satisfies AGM's six original postulates, iff for any* $K = Cn(K)$ *and* $\varphi \notin Cn(\varnothing)$ *it holds that* $K - \varphi \in K^{-\varphi}$*.*

From AGM's original characterisation of AGM contractions as partial meet contractions, it is easy to obtain the following result.

Proposition 28. *Let $\mathcal{L} = \langle L, Cn\rangle$ be a monotonic and compact logic, and C be a tarskian, boolean and compact $\mathcal{L}$-sound consequence operator. For any set of formulas $K \subseteq L$ s.t. $K = C(K)$ and formula $\varphi \in L$, it holds that any set $K' \subseteq K$ s.t. $\varphi \notin K$ and $K = Cn(K' \cup \{\varphi\})$, $K' \in K\perp_{\mathcal{L}_C}\varphi$.*

From that, we can obtain the simple result that, if the logic $\mathcal{L}_C$ is compact and boolean, every AGM contraction on $\mathcal{L}$ can be obtained as a hyperintensional contraction on $\mathcal{L}$.

Corollary 29. *Let $\mathcal{L} = \langle L, Cn\rangle$ be a monotonic and compact logic, and C be a tarskian, boolean and compact $\mathcal{L}$-sound consequence operator. Let $- : Th(\mathcal{L}_C) \times L \to Th(\mathcal{L}_C)$ be an AGM contraction on C, then there is a hyperintensional C-base contraction $\ominus$ s.t. for any $K \subseteq L$ s.t. $K = C(K)$ and $\varphi \in L$, it holds that $K - \varphi = K \ominus \varphi$.*

This result is not particularly surprising since the connection between partial meet contractions and AGM contractions for boolean and compact logics has been known since the seminal work of AGM. It seems, however, that these connections cannot be extended further if the logic $\mathcal{L}_C$ does not satisfy one of such properties. Consider the following example.

Example 30. *Consider the tarskian logic $\mathcal{L} = \langle L, Cn\rangle$, with $L = \{a, b, p_0, p_1, \cdots, q_0, q_1, \cdots\}$ and Cn with:*

$Cn(L) = L$
$Cn(\varnothing) = \varnothing$
$Cn(a) = Cn(\{p_1\}) = Cn(\{p_2\}) = \cdots = \{a, p_0, p_1, \cdots\}$
$Cn(b) = Cn(\{q_1\}) = Cn(\{q_2\}) = \cdots = \{b, q_0, q_1, \cdots\}$
$Cn(\{p_0\}) = \{p_0\}$
$Cn(\{q_0\}) = \{q_0\}$
$Cn(\{p_i, q_j\}) = L$ *for* $i, j \in \mathbb{N}$

Take also the $\mathcal{L}$-sound consequence C with the following structure:

$C(L) = L$
$C(\varnothing) = \varnothing$
$C(a) = Cn(\{p_0, p_1, \cdots\} = \{a, p_0, p_1, \cdots\}$
$C(b) = Cn(\{q_0, q_1, \cdots\} = \{b, q_0, q_1, \cdots\}$
$C(\{p_i\}) = \{p_0, \cdots p_i\}$ *for* $i \in \mathbb{N}$
$C(\{q_i\}) = \{q_0, \cdots q_i\}$ *for* $i \in \mathbb{N}$
$C(\{p_i, q_j\}) = L$ *for* $i, i \in \mathbb{N}$

Clearly, Cn is boolean and compact and C is boolean. Consider the agent with a belief set $K = C(a)$, there is a set $K' = \varnothing \subseteq K$ s.t. $C(K' \cup \{a\}) = K$ and $K' \notin K\perp^C_{\mathcal{L}} a$. Thus, there is some AGM contraction $-$ with $K - a = \varnothing$ that is not a hyperintensional belief set contraction, for any of such notions investigated in this work.

More yet, Ribeiro (2013) shows a compact and non-distributive logic in which the notions of AGM contraction and partial meet contraction do not coincide. This indicates that these notions may be irreconcilable for non-boolean or non-compact logics.

7 Final Considerations

As Berto and Hawke (2021) point out, a useful framework for epistemic logic needs to be both robust enough, i.e. have enough logical structure, to allow one to draw interesting conclusions regarding epistemic phenomena, and flexible enough as to be able to encode disputing positions in the philosophical and logical literature. We believe our framework for hyperintensional belief change achieves this balance by allowing the construction of a broad class of belief change operations for a wide range of non-classical logics while still guaranteeing minimal criteria (or properties) for minimal change.

Besides the connections with belief change in non-classical logics, established in (Souza and Wassermann 2021), our approach also is similar, in some sense, to the idea of approximate belief change (Chopra, Parikh, and Wassermann 2001). In their work, Chopra et al. (2001) aim to approximate belief contraction operations on a logic $\mathcal{L}$ through belief change operations for a sequence $\mathcal{L}$-sound consequence operators C^i_S. Our approach, however, employs belief change operations in the "well-behaved" logic $\mathcal{L}$ to construct (or approximate) belief change operations in the logic defined by a $\mathcal{L}$-sound consequence operator C.

Notice that, by encoding hyperintensional reasoning using the relation among two different consequence operators C and Cn, our approach is general enough to be connected to different foundational theories of hyperintensionality - such as the structural perspective underlying the structured propositions tradition (Cresswell 1975) and the informational perspective of Berto's (2019) mereological treatment. This allows the investigation of dynamic phenomena in several modellings of hyperintensionality. In fact, the connection of these operations to one such framework, namely impossible-world semantics, is the object of future work developed by the authors.

Our framework also provides a richer setting to study the connection between different competing notions of rational belief change in the literature as partial meet belief change and AGM belief change. Our results indicate, for example, that these two notions may be irreconcilable in a wide variety of logics.

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