REPRESENTING AND SOLVING ASYMMETRIC DECISION PROBLEMS USING VALUATION NETWORKS

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1. Introduction

This paper deals with asymmetric decision problems. An asymmetric decision problem can be defined most easily using its decision tree representation. In a decision tree, a path from the root node to a leaf node is called a *scenario*. We say a decision problem is asymmetric if there exists a decision tree representation of it such that not all scenarios include all variables in the problem. In asymmetric decision problems, some scenarios may exclude either some chance variables, or some decision variables, or both. The main goal of this paper is to describe a valuation network representation and solution of asymmetric decision problems.

Influence diagrams and valuation networks as originally conceived were designed for symmetric decision problems. For asymmetric decision problems, these techniques makes an asymmetric problem symmetric by adding variables and dummy configurations to scenarios. In doing so, we increase the computational burden of solving the problem. For this reason, representing and solving asymmetric problems has been the subject of several studies in recent years.

In the influence diagram literature, four techniques have been proposed by Call and Miller [1990], Smith *et al.* [1993], Fung and Shachter [1990], and Covaliu and Oliver [1992], to deal with asymmetric decision problems. Each of these four techniques is a hybrid of influence diagram and decision tree techniques. In essence, influence diagram representation is used to capture the uncertainty information, and decision tree representation is used to capture the structural asymmetry information.

In this paper, we investigate the use of valuation networks to represent and solve asymmetric decision problems. The structural asymmetry information is represented by indicator valuations. An indicator valuation is a special type of a probability valuation whose values are restricted to either 0 or 1. Indicator valuations enable us to reduce the domain of probability valuations and this contributes greatly to improving the computational efficiency of the solution technique. We use indicator valuations to define effective frames as subsets of frames of variables. All numeric information is specified only for effective frames. The solution technique is mostly the same as in the symmetric case. The main difference is that all computations are done on the effective frames of variables. This contributes to the increased efficiency of the solution technique. Also, when restricted to effective frames, the values of indicator valuations are identically one, and therefore indicator valuations can be handled implicitly and this contributes further to the increased efficiency of the solution technique.

An outline of the remainder of the paper is as follows. In Section 2, we give a verbal statement of the oil wildcatter's problem [Raiffa 1968]. This is an asymmetric decision problem. In Section 3, we describe the valuation network representation method for asymmetric decision problems and illustrate it using the oil wildcatter's problem. In Section 4, we sketch a fusion algorithm for solving valuation network representations. Finally, in Section 5, we summarize and conclude.

2. The Oil Wildcatter's Problem

The oil wildcatter's (OW) problem is reproduced with minor modifications from Raiffa [1968].

An oil wildcatter must decide either to drill (d) or not drill ($\sim d$). He is uncertain whether the hole is dry (dr), wet (we) or soaking (so).

Table I. The utility matrix for the OW problem.

Wildcatter's profit, \$ (υ)			Act: drill (*d*)	Act: not drill (~*d*)	Probability of state
State	Dry	(*dr*)	–70,000	0	0.500
	Wet	(*we*)	50,000	0	0.300
	Soaking	(*so*)	200,000	0	0.200

Table II. Probabilities of seismic test results conditional on the amount of oil.

P(R | O)			Seismic Test Results (R): No Structure (*ns*)	Open Structure (*os*)	Closed Structure (*cs*)
Amount of Oil (O)	Dry	(*dr*)	0.600	0.300	0.100
	Wet	(*we*)	0.300	0.400	0.300
	Soaking	(*so*)	0.100	0.400	0.500

Figure 1. Decision tree representation and solution of the OW problem.

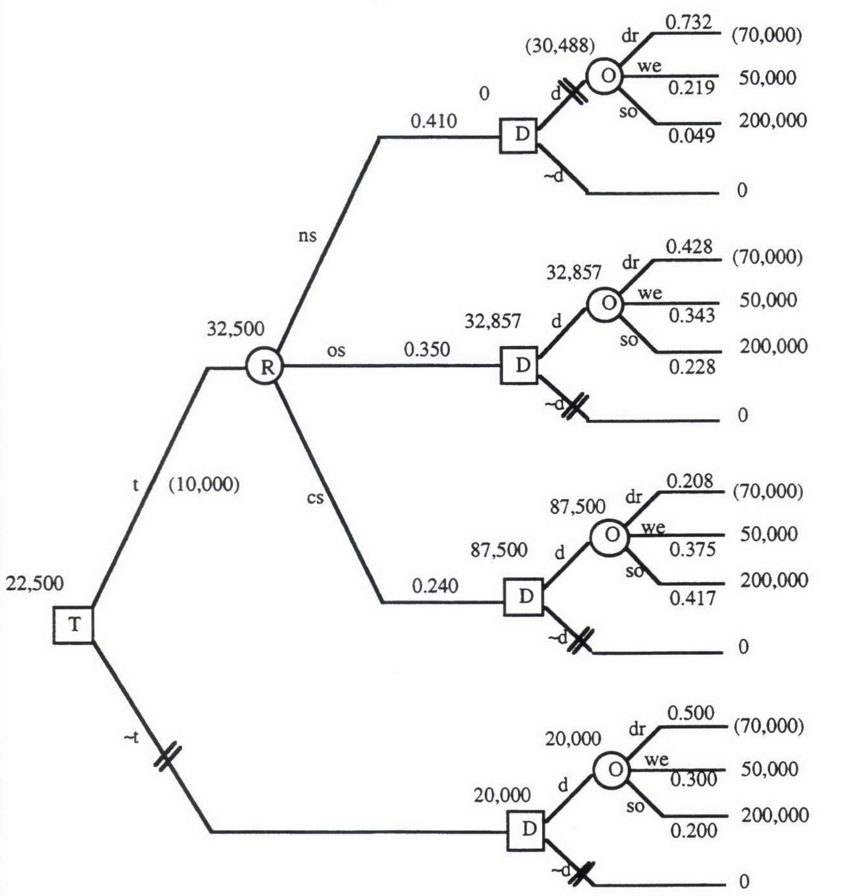


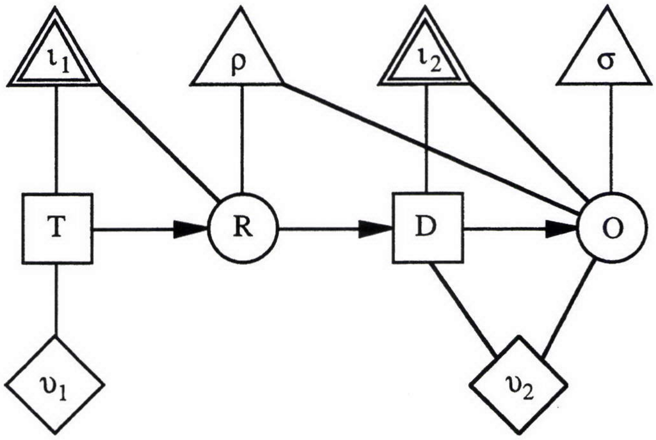
Table I gives his monetary payoffs and his subjective probabilities of the various states. The cost of drilling is \$70,000. The net return associated with the *d-we* pair is \$50,000 which is interpreted as a return of \$120,000 less the \$70,000 cost of drilling. Similarly the \$200,000 associated with the *d-so* pair is a net return (a return of \$270,000 less the \$70,000 cost of drilling).

At a cost of \$10,000, the wildcatter could take seismic soundings which will help determine the geological structure at the site. The soundings will disclose whether the terrain below has no structure (*ns*)—that's bad, or open structure (*os*)—that's so-so, or closed structure (*cs*)—that's really hopeful. The experts have provided us with Table II which shows the probabilities of seismic test results conditioned on the amount of oil.

Figure 1 shows a decision tree representation and solution of this problem. The optimal strategy is to do a seismic test; not drill if seismic test reveals no structure, and drill if the seismic test reveals either open or closed structure. The expected profit associated with this strategy is \$22,500.

Notice that the OW problem is asymmetric. This problem has 16 scenarios. Of these, 9

Figure 2. A valuation network for the OW problem.



scenarios include all four variables, 3 scenarios include only variables T, R, and D, 3 scenarios include only variables T, D, O, and one scenario includes only variables T and D.

3. Valuation Network Representation

In this section, we describe the valuation network representation technique and illustrate it using the oil wildcatter's (OW) problem.

A valuation network representation is specified at three levels—graphical, dependence, and numeric. This is somewhat analogous to Howard and Matheson's [1981] relational, functional, and numerical levels of specification of influence diagrams. The graphical and dependence levels have qualitative (or symbolic) knowledge, whereas the numeric level has quantitative knowledge.

3.1 Graphical Level

At the graphical level, a valuation network representation consists of a graph called a *valuation network*. Figure 2 shows a valuation network for the OW problem. A valuation network consists of two types of nodes — variable and valuation. Variables are further classified as either decision or chance, and valuations are further classified as either indicator, or probability, or utility. Thus in a valuation network, there are in all five different types of nodes — decision, chance, indicator, probability, and utility.

Decision Nodes. Decision nodes correspond to decision variables and are depicted by rectangles. In the OW problem, there are two decision nodes labeled T, and D. T represents the seismic test decision, and D represents the drill decision.

Chance Nodes. Chance nodes correspond to chance variables and are depicted by circles. In the OW problem, there are two chance nodes labeled R and O. R represents the seismic test result, and O represents the amount of oil.

Let $\mathfrak{X}_D$ denote the set of all decision variables, let $\mathfrak{X}_R$ denote the set of all chance variables, and let $\mathfrak{X}$ denote $\mathfrak{X}_D \cup \mathfrak{X}_R$.

Indicator Valuations. Indicator valuations represent qualitative constraints on the joint frames of decision and chance variables and are depicted by double-triangular nodes. The set of variables directly connected to an indicator valuation by undirected edges constitutes the domain of the indicator valuation. In the OW problem, there are two indicator valuations labeled ι_1, and ι_2. ι_1's domain is $\{T, R\}$, and ι_2's domain is $\{D, O\}$. ι_1 represents the constraint that seismic test result is not available if the oil wildcatter decides not to do the seismic test. ι_2 represents the constraint that the amount of oil is only revealed if the oil wildcatter decides to drill.

Utility Valuations. Utility valuations represent factors of the joint utility function and are depicted by diamond-shaped nodes. The set of variables directly connected to a utility valuation constitutes the domain of the utility valuation. Depending on whether the utility function decomposes additively or multiplicatively, the factors are additive or multiplicative (or perhaps some combination of the two). In the OW problem, there are two additive utility valuations labeled υ_1, and υ_2. υ_1's domain is $\{T\}$, and υ_2's domain is $\{D, O\}$. υ_1 represents the profit from the seismic test decision, and υ_2 represents the profit from the drill decision.

Probability Valuations. Probability valuations represent multiplicative factors of the family of joint probability distributions of the chance variables in the problem, and are depicted by triangular nodes. The set of all variables directly connected to a probability valuation constitutes the domain of the probability valuation. In the OW problem, there are two probability valuations labeled σ, and ρ. σ's domain is $\{O\}$, and ρ's domain is $\{R, O\}$.

Information Constraints. The specification of the valuation network at the graphical level includes directed arcs between pairs of distinct variables. These directed arcs represent information constraints. Suppose R is a chance variable and suppose D is a decision variable. An arc R→D means that the true value of R is known to the decision maker (DM) at the time the DM has to choose an alternative from D's frame, and, conversely, an arc from D→R means that the true value of R is not known to the DM at the time the DM has to choose an alternative from D's frame.

3.2. Dependence Level

Next, we specify valuation network representation at the dependence level. Like the graphical level, the dependence level involves only qualitative (or symbolic) knowledge.

Frames. Associated with each variable X is a *frame* $\mathcal{W}_X$. We assume that all variables have finite frames. In the OW problem, $\mathcal{W}_T = \{t, \sim t\}$, where *t* denotes do seismic test, and *~t* denotes not do seismic test; $\mathcal{W}_R = \{ns, os, cs, nr\}$, where *ns* denotes no structure, *os* denotes open structure, *cs* denotes closed structure, and *nr* denotes no result; $\mathcal{W}_D = \{d, \sim d\}$, where *d* denotes drill, and *~d* denotes not drill; $\mathcal{W}_O = \{dr, we, so, uk\}$ where *dr* denotes dry, *we* denotes wet, *so* denotes soaking, and *uk* denotes unknown.

Configurations. We often deal with non-empty subsets of variables in $\mathcal{X}$. Given a non-empty subset h of $\mathcal{X}$, let $\mathcal{W}_h$ denote the Cartesian product of $\mathcal{W}_X$ for X in h, i.e., $\mathcal{W}_h = \times\{\mathcal{W}_X \mid X \in h\}$. We can think of $\mathcal{W}_h$ as the set of possible values of the joint variable h. Accordingly, we call $\mathcal{W}_h$ the *frame for h*. Also, we refer to elements of $\mathcal{W}_h$ as *configurations of h*. We use this terminology even when h consists of a single variable, say X. Thus we refer to elements of $\mathcal{W}_X$ as *configurations of X*.

Indicator Valuations. Suppose s is a subset of variables. An *indicator valuation for s* is a function $\iota: \mathcal{W}_s \rightarrow \{0, 1\}$. The values of indicator valuations represent probabilities. The only values assumed by an indicator valuation are 0 and 1, hence the term indicator valuation. An efficient way of representing an indicator valuation is simply to describe the elements of the frame that have value 1, i.e., we represent ι by Ω_ι where $\Omega_\iota = \{\mathbf{x} \in \mathcal{W}_s \mid \iota(\mathbf{x}) = 1\}$. Obviously, $\Omega_\iota \subseteq \mathcal{W}_s$. To minimize jargon, we also call Ω_ι an indicator valuation for s.

In the OW problem, we have two indicator valuations—ι_1 (or Ω_{ι_1}) with domain $\{T, R\}$, and ι_2 (or Ω_{ι_2}) with domain $\{D, O\}$. These indicator valuations are specified as follows:

$$\Omega_{\iota_1} = \{(t, ns), (t, os), (t, cs), (\sim t, nr)\};$$

and

$$\Omega_{\iota_2} = \{(d, dr), (d, we), (d, so), (\sim d, uk)\}.$$

ι_1 represents the constraint that the seismic test result is not available only if the oil wildcatter decides not to do the seismic test. And ι_2 represents the constraint that the amount of oil remains unknown if the oil wildcatter decides to not drill.

Projection of Configurations. *Projection* of configurations simply means dropping extra coordinates; if *(t, ns, d, dr)* is a configuration of {T, R, D, O}, for example, then the projection of *(t, ns, d, dr)* to {T, R} is simply *(t, ns)*, which is a configuration of {T, R}.

If g and h are sets of variables, $h \subseteq g$, and $\mathbf{x}$ is a configuration of g, then let $\mathbf{x}^{\downarrow h}$ denote the projection of $\mathbf{x}$ to h.

Marginalization of Indicator Valuations. Suppose Ω_{ι_a} is an indicator valuation for a, and suppose $b \subseteq a$. The marginalization of Ω_{ι_a} to b, denoted by $\Omega_{\iota_a}{}^{\downarrow b}$, is an indicator valuation for b given by

$\Omega_{\iota_a}{}^{\downarrow b} = \{\mathbf{x} \in \mathcal{W}_b \mid (\mathbf{x}, \mathbf{y}) \in \Omega_{\iota_a}$ for some $\mathbf{y} \in \mathcal{W}_{a-b}\}$.

To illustrate this definition, consider the indicator valuation Ω_{ι_1} for {T, R} in the OW problem. The marginal of Ω_{ι_1} for {T} is given by the indicator valuation

$$\Omega_{\iota_2}{}^{\downarrow T} = \{t, \sim t\}.$$

Combination of Indicator Valuations. Suppose Ω_{ι_a} is an indicator valuation for a, and suppose Ω_{ι_b} is an indicator valuation for b. The combination of Ω_{ι_a} and Ω_{ι_b}, denoted by $\Omega_{\iota_a} \otimes \Omega_{\iota_b}$, is an indicator valuation for $a \cup b$ given by $\Omega_{\iota_a} \otimes \Omega_{\iota_b} =$

$\{\mathbf{x} \in \mathcal{W}_{a \cup b} \mid \mathbf{x}^{\downarrow a} \in \Omega_{\iota_a}$ and $\mathbf{x}^{\downarrow b} \in \Omega_{\iota_b}\}$.

To illustrate this definition, consider the two indicator valuations Ω_{ι_1} and Ω_{ι_2} in the OW problem. The combination $\Omega_{\iota_1} \otimes \Omega_{\iota_2}$ is an indicator valuation for {T, R, D, O} given as follows:

$\Omega_{\iota_1} \otimes \Omega_{\iota_2} = \{(t, ns, d, dr), (t, ns, d, we), (t, ns, d, so), (t, ns, \sim d, uk), (t, os, d, dr), (t, os, d, we), (t, os, d, so), (t, os, \sim d, uk), (t, cs, d, dr), (t, cs, d, we), (t, cs, d, so), (t, cs, \sim d, uk), (\sim t, nr, d, dr), (\sim t, nr, d, we), (\sim t, nr, d, so), (\sim t, nr, \sim d, uk)\}$.

Effective Frames. Suppose $\{\Omega_{\iota_1}, ..., \Omega_{\iota_p}\}$ is the set of indicator valuations in a given problem such that Ω_{ι_j} is an indicator valuation for s_j, j = 1, ..., p. Without loss of generality, assume that $s_1 \cup ... \cup s_p = \mathcal{X}$. (If a variable, say X, is not included in the domain of some indicator valuation, include the vacuous indicator valuation Ω_ι for {X}, i.e., $\Omega_\iota = \mathcal{W}_X$.) Suppose s is a subset of variables. The effective frame for s, denoted by Ω_s, is given by

$$\Omega_s = \{\otimes\{\Omega_{\iota_k} \mid s_k \cap s \neq \emptyset\}^{\downarrow s}.$$

In words, the effective frame for s is defined in two steps as follows. First we combine indicator valuations whose domains include a variable in s. Second, we marginalize the resulting combination to eliminate variables not in s.

To illustrate this definition, consider the indicator valuations Ω_{ι_1} for {T, R}, and Ω_{ι_2} for {D, O}. Then, for example, the effective frame for {R, O} is given by $\Omega_{\{R, O\}} =$
$(\Omega_{\iota_1} \oplus \Omega_{\iota_2})^{\downarrow\{R, O\}} =$
$\{(ns, dr), (ns, we), (ns, so), (ns, uk), (os, dr), (os, we), (os, so), (os, uk), (cs, dr), (cs, we), (cs, so), (cs, uk), (nr, dr), (nr, we), (nr, so), (nr, uk)\}$.

Notice that the definitions of combination and marginalization of indicator valuations satisfy the three axioms needed for local computation [Shenoy and Shafer 1990]. Thus, we can compute effective frames using local computation. Thus, e.g., to compute the effective frame for {R, O}, by definition, $\Omega_{\{R, O\}} = (\Omega_{\iota_1} \oplus \Omega_{\iota_2})^{\downarrow\{R, O\}}$. However, if we use local computation, we can compute $\Omega_{\{R, O\}} = \Omega_{\iota_1}^{\downarrow R} \oplus \Omega_{\iota_2}^{\downarrow O}$. Notice that the combination in $(\Omega_{\iota_1} \oplus \Omega_{\iota_2})^{\downarrow\{R, O\}}$ is on the frame of {T, R, D, O} whereas the combination in $\Omega_{\iota_1}^{\downarrow R} \oplus \Omega_{\iota_2}^{\downarrow O}$ is only on the frame of {R, O}.

As we will see shortly, all the numeric information in probability and utility valuations are specified on effective frames only. Thus, the definitions of marginalization and combination of indicator valuations allow us to compute effective frames using local computation.

3.3 Numeric Level

Finally, we specify a valuation network at the numeric level. At this level, we specify the details of the utility and probability valuations.

Utility Valuations. Suppose $u \subseteq \mathcal{X}$. A utility valuation υ for u is a function $\upsilon: \Omega_u \rightarrow R$, where R is the set of real numbers. The values of υ are utilities. If υ is a utility valuation for u, we say u is the *domain* of υ.

In the OW problem, there are two utility valuations υ_1 for {T}, and υ_2 for {D, O}. Table III shows the details of these utility valuations.

Probability Valuations. Suppose $p \subseteq \mathcal{X}$. A *probability valuation* π for p is a function $\pi: \Omega_p \rightarrow [0, 1]$. The values of π are probabilities. If π is a valuation for p, then we say p is the *domain* of π.

In the OW problem, there are two probability valuations, σ for {O}, and ρ for {O, R}. σ represents the conditional probability for O given that the values of O are not ruled out by structural constraints, and ρ represents the conditional probability of R given O and the fact that the values of R are not ruled out by structural constraints. Table IV shows the details of these probability valuations.

We have now completely defined a valuation network representation of a decision problem. In summary, a valuation network representation of a decision problem Δ consists of decision variables, chance variables, indicator valuations, probability valuations, utility valuations, and information constraints, $\Delta = \{\mathcal{X}_D, \mathcal{X}_R, \{\iota_1, ..., \iota_p\}, \{\upsilon_1, ..., \upsilon_m\}, \{\rho_1, ..., \rho_n\}, \rightarrow\}$.

Table III. Utility valuations in the OW problem.

$\Omega_{\{D,O\}}$	υ_2
d dr	–70,000
d we	50,000
d so	200,000
~d uk	0

Ω_T	υ_1
t	–10,000
~t	0

Table IV. Probability valuations in the OW problem.

Ω_O	σ
dr	.500
we	.300
so	.200
uk	1

$\Omega_{\{O,R\}}$	ρ
dr ns	.600
dr os	.300
dr cs	.100
we ns	.300
we os	.400
we cs	.300
so ns	.100
so os	.400
so cs	.500
uk ns	.333
uk os	.333
uk cs	.333
dr nr	1
we nr	1
so nr	1
uk nr	1

4. A Fusion Algorithm

In this section, we sketch a fusion algorithm for solving valuation network representations of decision problems.

The fusion algorithm is essentially the same as in the symmetric case [Shenoy 1992]. The main difference is in how indicator valuations are handled. Indicator valuations are treated as probability valuations. However since indicator valuations are identically one on effective frames, there are no computations involved in combining indicator valuations. This contributes to the efficiency of the solution technique. Indicator valuations do contribute domain information and cannot be totally ignored.

Fusion with respect to a decision variable D is defined as follows. All utility valuations that include D in their domain are combined together, and the resulting utility valuation υ is marginalized such that D is eliminated from its domain. A new indicator valuation ζ_D for h corresponding to the decision function for D is created. The utility valuations that do not include D in their domain remain unchanged. All probability valuations that include D in their domain are combined together and the resulting probability valuation ρ is combined with ζ_D and the result is marginalized so that D is eliminated from its domain. The probability valuations that do not include D in their domains remain unchanged.

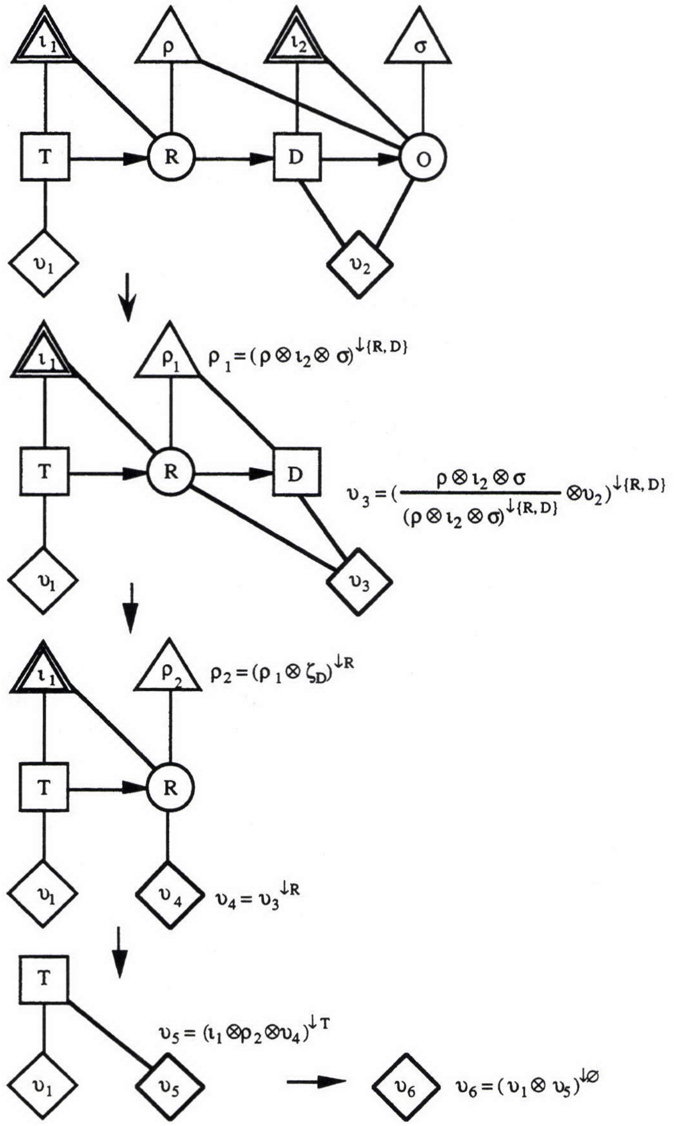
Fusion with respect to a chance variable C is defined as follows. The utility valuations whose domains do not include C, and the probability valuations whose domains do not include C, remain unchanged. A new probability valuation, say ρ, is created by combining all probability valuations whose domain include C and marginalizing C out of the combination. Finally, we combine all probability valuations whose domains include C, divide the resulting probability valuation by the new probability valuation that was created, combine the resulting probability valuation with the utility valuations whose domains include C, and finally marginalize the resulting utility valuation such that C is eliminated from its domain.

The details of the fusion algorithm are given in [Shenoy 1993b]. Figure 3 depicts the fusion algorithm graphically for the OW problem.

5. Summary and Conclusion

The main contribution of this paper is a generalization of the valuation network technique for representing and solving asymmetric decision problems. The structural asymmetry in a decision problem is represented by indicator valuations. An indicator valuation is a special type of a probability valuation. Indicator valuations allow us to reduce the domain of probability valuations. This contributes to the efficiency of the solution technique. Also, indicator valuations are used to define effective frames. An effective frame is a subset of a frame. All computations are done on effective frames, and this contributes also to the efficiency of the solution technique.

Figure 3. The fusion algorithm for the OW problem.



In [Shenoy 1993b], we compare the asymmetric valuation network representation and solution technique with the symmetric valuation network technique described in [Shenoy 1992], and with the influence diagram-based technique of Smith *et al.* [1993]. We note that of all proposed techniques, our technique is the only one that can solve the OW problem using local computation.

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