A New Notion of Individually Fair Clustering: α-Equitable k-Center

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Abstract

Clustering is a fundamental problem in unsupervised machine learning, and due to its numerous societal implications fair variants of it have recently received significant attention. In this work we introduce a novel definition of individual fairness for clustering problems. Specifically, in our model, each point j has a set of other points $\mathcal{S}_j$ that it perceives as similar to itself, and it feels that it is being fairly treated if the quality of service it receives in the solution is α-close (in a multiplicative sense, for some given $\alpha \geq 1$) to that of the points in $\mathcal{S}_j$. We begin our study by answering questions regarding the combinatorial structure of the problem, namely for what values of α the problem is well-defined, and what the behavior of the *Price of Fairness (PoF)* for it is. For the well-defined region of α, we provide efficient and easily-implementable approximation algorithms for the k-center objective, which in certain cases also enjoy bounded-PoF guarantees. We finally complement our analysis by an extensive suite of experiments that validates the effectiveness of our theoretical results.

1 INTRODUCTION

In a typical clustering problem, there is a set of points $\mathcal{C}$ in a metric space characterized by a distance function $d : \mathcal{C}^2 \mapsto \mathbb{R}_{\geq 0}$, where d is some non-increasing function of similarity or proximity. The goal is to choose a set $S \subseteq \mathcal{C}$ of at most k representative centers, and subsequently construct an assignment $\phi : \mathcal{C} \mapsto S$ that maps each point to one of the chosen centers, thus creating a collection of at most k clusters. In addition, the quantity that really matters for each $j \in \mathcal{C}$, is the distance $d(j, \phi(j))$ to its corresponding cluster center $\phi(j)$. This distance *represents the quality of service j receives*. In classical clustering applications $d(j, \phi(j))$ would correspond to how similar $\phi(j)$ is to j, and in facility-location applications to the distance j needs to travel in order to reach its service-provider. Hence, from an individual perspective, each j requires $d(j, \phi(j))$ to be as small as possible. The most popular objectives in the literature (k-center, k-median, k-means) "boil down" this large collection of values $d(j, \phi(j))$, into an increasing function they minimize.

In scenarios where the points correspond to selfish agents, it is natural to assume that they will be mindful of the quality of service other points receive. Specifically, a point j may feel that it is being handled unfairly by a solution (S, ϕ), if $d(j, \phi(j))$ is much larger than the assignment distances a group $\mathcal{S}_j$ of other points obtains. In this context, the points of $\mathcal{S}_j$ are exactly those which j perceives as similar to itself, hence it arguably believes that it should obtain similar treatment as them. As a practical example, consider the following application in an e-commerce site, where the points of $\mathcal{C}$ correspond to its users and $d(j, j')$ measures how similar the profiles of j and j' are. In order to provide relevant recommendations, the website needs to choose a set S of k representative users, and then assign each point to one of those based on a mapping $\phi : \mathcal{C} \mapsto S$. The recommendations j gets will be based on $\phi(j)$'s profile, and in this case the quantity $d(j, \phi(j))$ corresponds to how representative $\phi(j)$ is for j, and hence how suitable j's recommendations are. In this scenario, a point j may feel unfairly treated, if points that are similar to it (points j' with small $d(j, j')$) get better recommendations and consequently better service (see, e.g., the work of Datta et al. (2015) for studies on similar users receiving different types of

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job recommendations).

In addition, this sort of fairness considerations are applicable when seeking equity in healthcare provision, such as in vaccine allocation: the clusters could represent groups of people who would be given health-related resources such as treatment from a facility, and we aim for similar people to get similar commute-times to their resource provider.

Here we formalize this abstract notion of fairness via two rigorous and related constraints, which we incorporate into the k-center problem. We focus on k-center due to its numerous practical applications, but mostly because of its theoretical simplicity, which allows us to explore in depth the combinatorial structure of this novel notion of individually-fair clustering.

1.1 Formal Problem Definitions

We are given a set of points $\mathcal{C}$ in a metric space characterized by the distance function $d : \mathcal{C}^2 \mapsto \mathbb{R}_{\geq 0}$. Moreover, the input includes a positive integer k and a value $\alpha \geq 1$. Finally, for every $j \in \mathcal{C}$ we have a *similarity set* $\mathcal{S}_j \subseteq \mathcal{C}$, denoting the group of points that are deemed similar to j.

Our goal is to choose a set $S \subseteq \mathcal{C}$ of at most k centers, and then find an assignment $\phi : \mathcal{C} \mapsto S$, such that the k-center objective, i.e., $\max_{j\in\mathcal{C}} d(j, \phi(j))$, is minimized. Further, we use two different constraints to capture the notion of fairness we aim to study.

Per-Point Fairness (PP): When we study the problem under this constraint, we want to make sure that for all $j \in \mathcal{C}$ with $\mathcal{S}_j \neq \emptyset$, we have:

$$d(j, \phi(j)) \leq \alpha \cdot \min_{j'\in\mathcal{S}_j} d(j', \phi(j')) \tag{1}$$

Here j is satisfied if its quality of service is at most α times the "best" quality found in $\mathcal{S}_j$. Equivalently, we should guarantee that $d(j, \phi(j)) \leq \alpha \cdot d(j', \phi(j'))$ for all $j \in \mathcal{C}$ and $j' \in \mathcal{S}_j$.

Aggregate Fairness (AG): Here for each $j \in \mathcal{C}$ with $\mathcal{S}_j \neq \emptyset$, we want to guarantee that:

$$d(j, \phi(j)) \leq \alpha \frac{\sum_{j'\in\mathcal{S}_j} d(j', \phi(j'))}{|\mathcal{S}_j|} \tag{2}$$

Hence, here j feels fairly treated if $d(j, \phi(j))$ is at most α times the average quality of $\mathcal{S}_j$.

We call our problem **α-Equitable k-Center**, and denote it by EqCenter. Moreover, we consider it either under constraint (1) or under constraint (2). When we study it under (1) we refer to it as EqCenter-PP, and similarly when we use constraint (2) we denote it by EqCenter-AG. Both variants are NP-hard, since they generalize the already NP-hard k-center.

Constraint (1) provides a stronger notion of fairness, in that each point j cares explicitly about every point $j' \in \mathcal{S}_j$. Constraint (2) is weaker, in the sense that the points now compromise to comparing their quality of service to the *average* quality obtained by their similarity set. Due to this, a solution for (1) also constitutes a solution for (2), and hence for the same instance the optimal value of EqCenter-AG must be no larger than that of EqCenter-PP. *This observation reveals an intriguing trade-off between how strict we want to be in our fairness constraints, and how much we care about the overall objective cost.* We further explore this issue in Section 3.

1.1.1 The Structure of the Similarity Sets $\mathcal{S}_j$

In our work we do not consider an arbitrary model of similarity, but we rather focus on distance based similarity. On a high-level, this means that points which are far apart in the metric space, cannot really be similar. Such an approach for instantiating similarity is extensively utilized for fair clustering (Brubach et al., 2020, 2021; Anderson et al., 2020), with elements of it appearing in (Dwork et al., 2012; Jung et al., 2019) as well. Moreover, this concept is highly realistic, since in many conceivable applications the function d already captures a notion of resemblance. For instance, in the previously mentioned use-case of a recommendation system, two users that are close under d have comparable profiles, and thus can be seen as similar.

The way we capture distance-based similarity here, is by considering sets $\mathcal{S}_j$ that satisfy a well-established assumption from (Brubach et al., 2020), which was used to define similarity between points in a different individually-fair clustering problem. Specifically, suppose that we have an instance $(\mathcal{C}, k)$ of vanilla/"unfair" k-center, whose optimal value is R^*_{unf}. In other words, this is merely an instance of the standard k-center problem, where we want to choose (S, ϕ) with $|S| \leq k$ such that $\max_{j\in\mathcal{C}} d(j, \phi(j))$ is minimized, and no fairness constraints are imposed. Further, assume that this instance is extended to an instance of either EqCenter-PP or EqCenter-AG, by choosing an arbitrary α value and sets $\mathcal{S}_j$. Then the following holds.

Assumption 1.1. For every $j \in \mathcal{C}$ we have $\mathcal{S}_j \subseteq \{j' \in \mathcal{C} : d(j, j') \leq \psi R^*_{unf}\}$, for some $\psi = O(1)$.

Therefore, for instances of EqCenter-PP and EqCenter-AG, two points can be similar if their distance is at most ψR^*_{unf}, where ψ is some small constant and R^*_{unf} is the optimal value of the underlying unfair k-center instance.

Although Assumption 1.1 is adequately justified in (Brubach et al., 2020), we also give some intuition for it. Consider the optimal solution for the unfair problem on $(\mathcal{C}, k)$. Then, the triangle inequality implies that a point j will never be placed in the same cluster as some other j' with $d(j,j') > 2R^*_{unf}$. Hence, the optimal unconstrained/unfair solution that can be thought of as an expert when it comes to determining similarity (it constructs the most intra-similar clusters), does not deem the two points comparable enough to place them in the same cluster. Therefore, following the "advice" of the optimal unconstrained solution yields $\psi = 2$ in Assumption 1.1, and due to the previous explanation, this value can be actually interpreted as the *canonical case* for ψ.

For scenarios where we are not certain of whether Assumption 1.1 holds, or the points have a fuzzy understanding of similarity that does not allow them to meaningfully define their sets $\mathcal{S}_j$, see Appendix C for an explainable way of enforcing $\mathcal{S}_j \subseteq \{j' \in \mathcal{C} : d(j,j') \leq \psi R^*_{unf}\}$ for all j.

To conclude, we need to define some more notation. Given similarity sets $\mathcal{S}_j$ for every $j \in \mathcal{C}$, we define $R_j = \max_{j' \in \mathcal{S}_j} d(j,j')$ and $R_m = \max_{j \in \mathcal{C}} R_j$.

1.2 Discussion of Our Results

We first investigate the combinatorial structure of our newly introduced fairness constraints. At first, a question that naturally arises is for what values of α are our problems well-defined. We call a problem well-defined if it always admits a feasible solution (S, ϕ), i.e., $|S| \leq k$ and ϕ satisfies the corresponding fairness constraint for all j. Ideally, we would like our problems to admit feasible solutions for any possible value of α. However we provide the following result which indicates that absolute equity is not achievable.

Theorem 1.2. *There exist instances with $\alpha < 2$ for both* EqCenter-*PP and* EqCenter-*AG, that do not admit any feasible solution.*

We then proceed by showing that for $\alpha \geq 2$ there is always a feasible solution, thus settling the crucial question about the regime of α for which our problems are well-defined.

Theorem 1.3. *Every instance with $\alpha \geq 2$ for both* EqCenter-*PP and* EqCenter-*AG, always admits a feasible solution.*

Given that $\alpha \geq 2$ is the range we should focus on, we proceed by studying another vital concept, and that is the *Price of Fairness (PoF)* (Bertsimas et al., 2011; Caragiannis et al., 2009). This notion is just a measure of relative loss in system efficiency, when fairness constraints are introduced. Specifically, for a given instance of either EqCenter-PP or EqCenter-AG, PoF is defined as the value of the optimal solution to our fair problem, over the value of the optimal solution to the underlying k-center instance, where we drop the fairness constraints from the problem's requirements. In other words, PoF = (optimal fair value)/(optimal unfair value). In the vast majority of fair clustering problems it is known that there exist instances with unbounded PoF. In line with those results, we show the following.

Theorem 1.4. *There exist instances of* EqCenter-*PP and* EqCenter-*AG with unbounded PoF.*

All previous results are proven for $k \geq 2$. See that the $k = 1$ case is trivial, since one can efficiently try each point as a center, see if any yields a feasible solution, and also find the optimal solution among the computed feasible ones. On the other hand, even when $k = 2$ and we only have $\binom{|\mathcal{C}|}{2} + |\mathcal{C}|$ center sets to check, the number of possible assignments for each set of size 2 is $2^{|\mathcal{C}|}$.

Due to space constraints, we present the proofs of Theorems 1.2, 1.3 and 1.4 in the supplementary material.

In Section 2 we provide an approximation algorithm that covers instances with $\alpha \geq 2$ for EqCenter-PP and EqCenter-AG. The main body of the algorithm remains the same for the two problems, with minor differences to capture each unique case. Our process of choosing centers constitutes an extension of a result by Khuller and Sussmann (2000). Our procedure gives useful guarantees regarding the distances between chosen centers, a feature that is crucially exploited in the assignment phase of the algorithm, where we carefully construct the mapping ϕ. Our result is:

Theorem 1.5. *Suppose we are given an instance with $\alpha \geq 2$ for either* EqCenter-*PP or* EqCenter-*AG, whose optimal value is R^*. Our algorithm provides a feasible solution (S, ϕ) to either problem, for which $\max_{j \in \mathcal{C}} d(j, \phi(j)) \leq 5 \max\{R^*, R_m\}$.*

Due to Assumption 1.1, we have $R_m \leq \psi R^*_{unf}$ with $\psi = O(1)$. Moreover, because R^*_{unf} is an obvious lower bound for R^*, our algorithm produces constant-factor approximate solutions. For example, in the canonical case of $\psi = 2$ it gives a 10-approximate solution.

Even though $R_m = O(R^*_{unf})$, notice that because we might have $R^* \geq R_m$, the algorithm of Theorem 1.5 does not provide bounded PoF guarantees. Nonetheless, in Section 2 we also study the PoF behavior of our algorithms, and specifically we prove the following.

Theorem 1.6. *A small modification to our main algorithm yields (S, ϕ), with: **(i)** $|S| \leq 2k$, **(ii)** both constraints (1) and (2) satisfied by ϕ, and **(iii)** $\max_{j \in \mathcal{C}} d(j, \phi(j)) \leq 5 \max\{\psi R^*_{unf}, R^*_{unf}\}$.*

Theorem 1.7. *When $R_j = R_d$ for all $j \in \mathcal{C}$ and some R_d, our algorithm for* EqCenter-*AG provides a feasible solution with cost at most $5 \max\{\psi R^*_{unf}, R^*_{unf}\}$.*

The result of Theorem 1.6 says that there is an easy way to get an algorithm with bounded PoF guarantees, if we are willing to sacrifice the cardinality constraint on the set of chosen centers. On the other hand, Theorem 1.7 says that when the value R_j is the same for all points, then our main result yields a true approximation with bounded PoF for EqCenter-AG.

Furthermore, we mention that *all algorithms of Section 2 are purely combinatorial, and hence very efficient and easily implementable.*

In Appendix B we study the assignment problem for EqCenter-PP and EqCenter-AG. To be more precise, if we are given the optimal set of centers S^*, can we find the corresponding optimal assignment ϕ^*? In a vanilla clustering setting this is trivial, since assigning points to their closest center is easily seen to yield the necessary results. *However, as is the case in almost all literature on fair clustering, in the presence of fairness constraints like (1) or (2), such an assignment is not necessarily correct.* This was actually among the first observations made in the seminal work of Chierichetti et al. (2017), which initiated the research area of fair clustering. As a side note, the aforementioned observation implies that for a $c \in S^*$, we might end up having $\phi^*(c) \neq c$. Nonetheless, this does not constitute a modeling issue. Recalling the motivational example of a recommendation system for a website, we see that for a client j chosen as a representative, assigning j to a different representative j' is an acceptable outcome, as long as all individuals feel fairly treated.

Therefore, since from a theoretical perspective the assignment problem is fundamental and because in our case it appears non-trivial, we choose to address it explicitly. In the end, we show that with a slightly intricate iterative algorithm, we can indeed compute the optimal assignment ϕ^* in polynomial time.

Finally, Section 3 contains an extensive experimental evaluation, that validates the effectiveness and the efficiency of our proposed algorithms.

1.3 Related Work

The most well-studied notion of fairness in clustering is the demographic one. Herein, the points belong to demographic groups, and what is required is a fair treatment or a proportional representation of these groups. This area started with the groundbreaking work of Chierichetti et al. (2017). Further work on demographic fairness includes (Bercea et al., 2019; Bera et al., 2019; Esmaeili et al., 2020; Huang et al., 2019; Backurs et al., 2019; Ahmadian et al., 2019; Kleindessner et al., 2019; Chen et al., 2019; Abbasi et al., 2021).

The concept of fairness we consider here falls under the broader umbrella of individual fairness. The fundamentals of individual fairness were introduced in the seminal work of Dwork et al. (2012) in the context of classification. In addition, Dwork et al. (2012) demonstrated a series of shortcomings for demographic fairness, making the case for individual fairness stronger. The high-level idea proposed in that work was that *similar individuals should be treated similarly.* Our model follows this paradigm by modeling similarity through the sets $\mathcal{S}_j$, and requiring similar treatment through constraints (2) and (1).

Previous work on individually-fair clustering that adheres to the notion of Dwork et al. (2012) includes (Anderson et al., 2020; Brubach et al., 2020, 2021; Kar et al., 2021). However, these papers interpret similar treatment differently. Specifically, two points j, j' that are similar should be placed in the same cluster (under some stochastic or lower-bounding sense). Hence, similar treatment is defined as guaranteeing $\phi(j) = \phi(j')$. Unlike our model, these papers provide no guarantee on the gap between $d(j, \phi(j))$ and $d(j', \phi(j'))$.

There are also individually-fair problems that do not follow the concept of "similar points should be treated similarly". Mahabadi and Vakilian (2020); Jung et al. (2019) define individual fairness as ensuring that for each j there will be a chosen center within distance r_j from it, where r_j is the minimum radius such that $|\{j' \in \mathcal{C} \mid d(j, j') \leq r_j\}| \geq |\mathcal{C}|/k$. Finally, Kleindessner et al. (2020) view individual fairness as ensuring that each point is on average closer to the points in its own cluster than to the points in any other cluster.

Another work that is closely related to ours is that of Balcan et al. (2019). In that paper the authors study a classification problem where there is a set of already-known labels, and the points need to be assigned to those via some stochastic classifier. The points have preferences over the labels, given by some utility function, and the final classification should be envy-free in the standard sense. Our model differs from that of Balcan et al. (2019) for two reasons. First, our focus is on a clustering problem, where the labels are not known, a metric objective needs to be minimized, and also the assignment has to be deterministic. Secondly, although the concept of envy-freeness is related to constraint (1), there is the crucial difference of points in our case not envying the resources allocated to other individuals, but rather their final utility. In other words, in the language of *Fair Division of Goods*, our model is closer to the notion of an *equitable allocation* (Varian, 1974) rather than an *envy-free* one.

As for "unfair" k-center, the best approximation ratio for it is 2 and this is best-possible unless P=NP (Hochbaum and Shmoys, 1985; Gonzalez, 1985).

2 OUR ALGORITHMS

Suppose that we are given an instance of EqCenter with $\alpha, k \geq 2$, and we are either solving EqCenter-PP or EqCenter-AG. In addition, recall that $R_j = \max_{j' \in \mathcal{S}_j} d(j, j')$, $R_m = \max_{j \in \mathcal{C}} R_j$ and R^* denotes the value of the optimal solution for the corresponding problem.

In this section we demonstrate a procedure that works under an explicitly given value R, with $R \geq R_m$. This process will either return a feasible solution (S_R, ϕ_R) with $\max_{j \in \mathcal{C}} d(j, \phi_R(j)) \leq 5R$, or an infeasibility message. The latter message indicates with absolute certainty that $R < R^*$.

The aforementioned procedure suffices to yield the result of Theorem 1.5. Because R^* is always the distance between two points in $\mathcal{C}$, the total number of possible values for it is only polynomial, specifically at most $\binom{|\mathcal{C}|}{2}$. Hence, we can run the procedure for all such distances that are at least R_m, and in the end keep (S_R, ϕ_R) for the minimum guess R for which we did not receive an infeasibility message. If $R_m \leq R^*$, then our returned solution is guaranteed to have value at most $5R^*$, because R^* is one of the target values we tested. On the other hand, when $R_m > R^*$, the iteration with R_m as the guess cannot return an infeasibility message, and thus it will provide a solution of value at most $5R_m$. As a side note, we mention that we can speed up the runtime of this approach by using binary search over the guesses R, instead of a naive brute-force search.

Therefore, apart from the input instance, assume that we are also given a target value R with $R \geq R_m$. Our framework begins by choosing an initial set of centers S. The full details of this step are presented in Algorithm 1. Besides choosing this set S, Algorithm 1 also creates a partition $P_1, P_2, \dots P_T$ of S for some $T \leq |\mathcal{C}|$, and returns sets $G_c \subseteq \mathcal{C}$ for every $c \in S$.

Initially, all point of $\mathcal{C}$ are considered uncovered ($U = \mathcal{C}$). The algorithm works by trying to expand the current set of centers P_t as much as possible, via finding a new center that is currently uncovered and is within distance $3R$ from some center already placed in P_t. If no such point exists, then we never deal with P_t again, and we move on to create P_{t+1} by choosing an arbitrary uncovered point as the first center for it. In additional, every time a center c is chosen, it covers all uncovered points that are within distance $2R$ from it, and these points constitute the set G_c. This process is repeated until all points get covered, i.e., until the set U becomes empty.

For every $c \in S$, let $t(c)$ be the index of the partition set c belongs to, i.e., $c \in P_{t(c)}$. We also define $S_I = \{c \in S : |P_{t(c)}| = 1\}$ and $S_N = S \setminus S_I$. We interpret the centers of S_I as being *isolated*, since for each $c \in S_I$ its corresponding partition set contains only c, i.e., $P_{t(c)} = \{c\}$. On the other hand, the centers of S_N are *non-isolated*, in the sense of having $|P_{t(c)}| > 1$ for each $c \in S_N$. In addition, for every point $j \in \mathcal{C}$, let $\rho(j)$ the center of S that covered j, i.e., $j \in G_{\rho(j)}$. Note that $d(j, \rho(j)) \leq 2R$. Finally, let $\mathcal{C}_I = \{j \in \mathcal{C} : \rho(j) \in S_I\}$ and $\mathcal{C}_N = \mathcal{C} \setminus \mathcal{C}_I$, where $\mathcal{C}_I$ are the points that got covered by isolated centers, and $\mathcal{C}_N$ the points that got covered by non-isolated centers.

Algorithm 1: Choosing an initial set of centers

```
S ← ∅, U ← C, P_0 ← ∅, t ← 0;
while U ≠ ∅ do
    Q ← {c ∈ U | ∃c' ∈ P_t with d(c, c') ≤ 3R};
    if Q ≠ ∅ then
        Choose a point c ∈ Q;
        P_t ← P_t ∪ {c}, S ← S ∪ {c},
          G_c ← {j ∈ U | d(j, c) ≤ 2R}, U ← U \ G_c;
    else
        Choose an arbitrary c ∈ U;
        t ← t + 1;
        P_t ← {c}, S ← S ∪ {c},
          G_c ← {j ∈ U | d(j, c) ≤ 2R}, U ← U \ G_c;
    end
end
Return the set S, the partition P_1, P_2, ... P_t of S,
  and the sets G_c for every c ∈ S;
```

Observation 2.1. For every distinct $c, c' \in S$ we have $d(c, c') > 2R$.

Observation 2.2. For every $c \in S_N$, there exists a different $c' \in S_N$ such that $d(c, c') \leq 3R$.

Observation 2.3. The sets G_c for all $c \in S$, induce a partition of $\mathcal{C}$.

The three previous observations follow trivially from the definition of Algorithm 1. However, Observation 2.2 is of particular importance, since it will allow us to carefully control the assignment distances of points later on, in a way that would satisfy the underlying fairness constraints.

Lemma 2.4. *For any $c \in S_I$, we have $d(j, j') > R$ for all $j \in G_c$ and all $j' \in \mathcal{C} \setminus G_c$.*

By using Lemma 2.4 and the fact that $R \geq R_m$, we immediately get the following.

Corollary 2.5. *For every $c \in S_I$, we have $\mathcal{S}_j \subseteq G_c \subseteq \mathcal{C}_I$ for all $j \in G_c$.*

Corollary 2.6. *For every $j \in \mathcal{C}_N$, we have $\mathcal{S}_j \cap \mathcal{C}_I = \emptyset$ and hence $\mathcal{S}_j \subseteq \mathcal{C}_N$.*

In words, Corollary 2.5 says that the similarity set of a

Algorithm 2: Assignment for the points of $\mathcal{C}_I$

$S'_I \leftarrow \emptyset$;
for *every* $c \in S_I$ **do**
  Check if there exists any $j \in G_c$, such that assigning all points of G_c to j would result in the appropriate fairness constraint being satisfied for each $j' \in G_c$. Note that checking this feasibility condition is well-defined, since Corollary 2.5 gives $\mathcal{S}_{j'} \subseteq G_c$ for all $j' \in G_c$. If such a j exists, set $S'_I \leftarrow S'_I \cup \{j\}$ and $\phi_I(j') \leftarrow j$ for all $j' \in G_c$;
  If you could not find such a j, use the algorithm of Lemma A.1 on the points of G_c. This will return two points $c_1, c_2 \in G_c$ and an assignment $\phi : G_c \mapsto \{c_1, c_2\}$. Then, set $S'_I \leftarrow S'_I \cup \{c_1, c_2\}$, $\phi_I(j') \leftarrow \phi(j')$ for all $j' \in G_c$;
end
Return S'_I and ϕ_I;

point $j \in \mathcal{C}_I$ is completely contained in $G_{\rho(j)}$, where of course $\rho(j) \in S_I$ and $G_{\rho(j)} \subseteq \mathcal{C}_I$. Similarly, Corollary 2.6 says that the similarity set of a point $j \in \mathcal{C}_N$ is completely contained in $\mathcal{C}_N$.

After computing the set of centers S, our approach proceeds by constructing the appropriate assignment function. This will occur in two steps. The first step takes care of the points in $\mathcal{C}_I$, by choosing a new set of centers $S'_I \subseteq \mathcal{C}_I$, and by constructing an assignment $\phi_I : \mathcal{C}_I \mapsto S'_I$. The second step handles the points of $\mathcal{C}_N$ via a mapping $\phi_N : \mathcal{C}_N \mapsto S_N$. This is well-defined, since $\mathcal{C}_I \cap \mathcal{C}_N = \emptyset$. Note now that due to Corollary 2.5, the fairness constraint of a point $j \in \mathcal{C}_I$ is only affected by ϕ_I, since $\mathcal{S}_j \subseteq G_{\rho(j)} \subseteq \mathcal{C}_I$ and $\mathcal{C}_I \cap \mathcal{C}_N = \emptyset$. Similarly, due to Corollary 2.6, the fairness constraint of a $j \in \mathcal{C}_N$ is only affected by ϕ_N, since $\mathcal{S}_j \subseteq \mathcal{C}_N$ and $\mathcal{C}_I \cap \mathcal{C}_N = \emptyset$. Hence, we can study the satisfaction of fairness constraints separately on $\mathcal{C}_I$ for ϕ_I, and on $\mathcal{C}_N$ for ϕ_N.

Algorithm 2 demonstrates the details of the first assignment step. The algorithm operates by trying to "guess" if the optimal solution uses exactly one center inside each G_c for $c \in S_I$. If it does, so will our algorithm. If not, then our approach will open exactly two centers, and will subsequently construct an assignment that will satisfy the appropriate fairness constraint.

Lemma 2.7. *After the execution of Algorithm 2, for every $j \in \mathcal{C}_I$ we have that the constructed assignment ϕ_I will **1)** satisfy j's fairness constraint, and **2)** guarantee $d(j, \phi_I(j)) \leq 4R$.*

Lemma 2.8. *If $R \geq R^*$, then after the execution of Algorithm 2 we will have $|S'_I| + |S_N| \leq k$.*

Using the contrapositive of Lemma 2.8, we see that if $|S'_I| + |S_N| > k$ then $R < R^*$, and hence we can safely return as our answer an infeasibility message.

Before we proceed to the second step of our assignment process, we need some extra notation. For each $c \in S_N$ define $H^1_c = \{j \in \mathcal{C}_N \mid d(j,c) \leq R\}$ and $H^2_c = G_c \setminus \big(\bigcup_{c' \in S_N} H^1_{c'}\big)$. Combining Observation 2.1, Observation 2.3 and the way we constructed the sets H^1_c, H^2_c, it is easy to see that for each $j \in \mathcal{C}_N$ exactly one of the following two cases will hold.

I) The point j belongs to *exactly one* H^1_c for some $c \in S_N$. In addition, j clearly does not belong to any set $H^2_{c'}$ for $c' \in S_N$. In this case, we call j a *type-1* point, and we set $\pi(j) = c$.

II) The point j belongs to $H^2_{\rho(j)}$. In addition, j does not belong to any H^1_c for $c \in S_N$, and also it does not belong to any H^2_c with $c \neq \rho(j)$. Here we call j a *type-2* point, and set $\pi(j) = \rho(j)$.

Further, let $\mathcal{C}^1_N = \{j \in \mathcal{C}_N \mid j \text{ is a type-1 point}\}$ and $\mathcal{C}^2_N = \{j \in \mathcal{C}_N \mid j \text{ is a type-2 point}\}$. Therefore, $\mathcal{C}^1_N \cap \mathcal{C}^2_N = \emptyset$ and $\mathcal{C}^1_N \cup \mathcal{C}^2_N = \mathcal{C}_N$. Finally, the definition of a type-2 point implies:

Observation 2.9. For all $j \in \mathcal{C}^2_N$ and $c \in S_N$, we have $d(j, \pi(j)) \leq 2R$ and $d(j,c) > R$.

The distinction between type-1 and type-2 points is necessary for satisfying the fairness constraints. Notice that by construction of S_N type-1 points are more "privilleged", since they have an available center within distance at most R from them. On the other hand, type-2 points do not have such an advantage, and the assignment process must be aware of this discrepancy, so it can favor type-2 points in a controlled way that will satisfy everyone's fairness constraint.

Algorithm 3 demonstrates the full details of constructing $\phi_N : \mathcal{C}_N \mapsto S_N$. The high-level intuition behind it follows. At first, we try to provide each point j with an assignment distance in the range $[R, 5R]$, something that is possible due to Observation 2.2. However, since α might be less than 5, we are very careful in how we handle the assignment of similar points. The latter is achieved by considering type-1 and type-2 points independently, in a manner that is aware of where the potential similar points of each type may be.

Lemma 2.10. *Fix a $j \in \mathcal{C}_N$. If j gets assigned to $\phi_N(j)$ according to **Case (A)** then $d(j, \pi(j)) \leq R < d(j, \phi_N(j)) \leq 4R$. If j gets assigned to $\phi_N(j)$ according to **Case (B)** then $R < d(j, \pi(j)) \leq d(j, \phi_N(j)) \leq 2R$. If j gets assigned to $\phi_N(j)$ according to **Case (C)** then $d(j, \pi(j)) \leq 2R < d(j, \phi_N(j)) \leq 5R$.*

Lemma 2.10 immediately gives an upper bound of $5R$ for the maximum assignment distance. However, it is

Algorithm 3: Assignment for the points of $\mathcal{C}_N$

```
for every j ∈ C_N do
    if j ∈ C_N^1 then
        φ_N(j) ← argmin_{c∈(S_N \ {π(j)})} d(j, c) ;
        // Case (A)
    end
    if j ∈ C_N^2 then
        if ∃c ∈ (S_N \ {π(j)}) : d(j, c) ≤ 2R then
            φ_N(j) ← argmax_{c'∈S_N : d(j,c')≤2R} d(j, c')
            ;                                   // Case (B)
        else
            φ_N(j) ← argmin_{c'∈(S_N \ {π(j)})} d(j, c') ;
            // Case (C)
        end
    end
end
```

the rest of the inequalities shown there that allow us to prove satisfaction of the fairness constraints by ϕ_N. This is achieved in the following Lemma.

Lemma 2.11. *For all $j \in \mathcal{C}_N$, we have $d(j, \phi_N(j)) \leq \alpha \cdot d(j', \phi_N(j'))$ for all $j' \in \mathcal{S}_j$.*

Proof. Suppose we have some $j \in \mathcal{C}_N$ and some $j' \in \mathcal{S}_j$. The proof of the statement will be based on an exhaustive case analysis. Before we proceed, we mention two inequalities that we will repeatedly use. At first, $d(j, j') \leq d(j', \phi_N(j'))$, because $d(j, j') \leq R_m \leq R$ and by Lemma 2.10 we have $d(j', \phi_N(j')) > R$. Moreover, $d(j', \pi(j')) \leq d(j', \phi_N(j'))$, again by Lemma 2.10.

- Suppose that both j and j' are type-1 points

  At first let $\pi(j) \neq \pi(j')$. Then j can potentially be assigned to $\pi(j')$, and therefore we have $d(j, \phi_N(j)) \leq d(j, \pi(j')) \leq d(j, j') + d(j', \pi(j')) \leq 2d(j', \phi_N(j')) \leq \alpha \cdot d(j', \phi_N(j'))$.

  Now let $\pi(j) = \pi(j')$. Because j' is a type-1 point and gets assigned according to **Case (A)**, we know that $\phi_N(j') \neq \pi(j)$, Hence j can potentially be assigned to $\phi_N(j')$. Therefore, $d(j, \phi_N(j)) \leq d(j, \phi_N(j')) \leq d(j, j') + d(j', \phi_N(j')) \leq 2d(j', \phi_N(j')) \leq \alpha \cdot d(j', \phi_N(j'))$.

- Suppose that j is a type-1 and j' is a type-2 point.

  At first assume that j' received its assignment through **Case (C)**. Then, by Lemma 2.10 we know that $d(j', \phi_N(j')) > 2R$. In addition, again by Lemma 2.10, we have $d(j, \phi_N(j)) \leq 4R$. Thus, $d(j, \phi_N(j)) \leq 2d(j', \phi_N(j')) \leq \alpha \cdot d(j', \phi_N(j'))$.

  Now assume that j' got its assignment via **Case (B)**. Therefore, there exists $c \in S \setminus \{\pi(j')\}$ with $d(j', c) \leq 2R$. By the way **Case (B)** works and Observation 2.9, we also have $d(j', \phi_N(j')) \geq \max(d(j', \pi(j')), d(j', c))$. Let us now see what happens when $\pi(j') = \pi(j)$. Then $c \neq \pi(j)$, and thus j can potentially be assigned to c. Therefore, $d(j, \phi_N(j)) \leq d(j, c) \leq d(j, j') + d(j', c) \leq d(j, j') + d(j', \phi_N(j')) \leq 2d(j', \phi_N(j')) \leq \alpha \cdot d(j', \phi_N(j'))$. On the other hand, if $\pi(j') \neq \pi(j)$, then j can potentially get assigned to $\pi(j')$, and thus have $d(j, \phi_N(j)) \leq d(j, \pi(j')) \leq d(j, j') + d(j', \pi(j')) \leq \alpha \cdot d(j', \phi_N(j'))$.

- Suppose that j is a type-2 point, and also gets its assignment via **Case (B)**. By Lemma 2.10 we have $d(j, \phi_N(j)) \leq 2R$ and $d(j', \phi_N(j')) > R$. For $\alpha \geq 2$ the statement trivially follows.

- Suppose that j is a type-2 point, j' is a type-1 point, and j gets its assignment via **Case (C)**.

  First, let $\phi_N(j') \neq \pi(j)$. Then j can get assigned to $\phi_N(j')$, and $d(j, \phi_N(j)) \leq d(j, \phi_N(j')) \leq d(j, j') + d(j', \phi_N(j')) \leq \alpha \cdot d(j', \phi_N(j'))$.

  Now assume that $\phi_N(j') = \pi(j)$. Because j' is a type-1 points and so $\phi_N(j') \neq \pi(j')$, we can infer that $\pi(j) \neq \pi(j')$. Also, $d(j, \pi(j')) \leq d(j, j') + d(j', \pi(j')) \leq 2R$. However, the latter contradicts the assumption that j got its assignment according to **Case (C)**. Therefore, we know that $\phi_N(j') \neq \pi(j)$ necessarily.

- Suppose that both j, j' are type-2 points, and j gets its assignment via **Case (C)**.

  At first, assume $\pi(j') \neq \pi(j)$. Then j can potentially get assigned to $\pi(j')$, and therefore $d(j, \phi_N(j)) \leq d(j, \pi(j')) \leq d(j, j') + d(j', \pi(j')) \leq 2d(j', \phi_N(j')) \leq \alpha \cdot d(j', \phi_N(j'))$.

  Now let $\pi(j') = \pi(j)$. To begin with, assume that there exists a $c \in S \setminus \{\pi(j)\}$ such that $d(j', c) \leq 2R$. Moreover, because $c \neq \pi(j)$, j can potentially get assigned to c, and thus $d(j, \phi_N(j)) \leq d(j, c) \leq d(j, j') + d(j', c) \leq d(j, j') + d(j', \phi_N(j')) \leq \alpha \cdot d(j', \phi_N(j'))$. To get $d(j', c) \leq d(j', \phi_N(j'))$ we simply used the way **Case (B)** works. Finally, suppose that $\forall c \in S \setminus \{\pi(j)\}$ we have $d(j', c) > 2R$. Then $\phi_N(j') \neq \pi(j)$ and thus j can potentially get assigned to $\phi_N(j')$. Therefore, $d(j, \phi_N(j)) \leq d(j, \phi_N(j')) \leq d(j, j') + d(j', \phi_N(j')) \leq \alpha \cdot d(j, \phi_N(j'))$. □

Combining Lemmas 2.10 and 2.11 we immediately get

Lemma 2.12. *After the execution of Algorithm 3, for every $j \in \mathcal{C}_N$ we have that the constructed assignment ϕ_N will **1)** satisfy j's fairness constraint, and **2)** guarantee $d(j, \phi_N(j)) \leq 5R$.*

Finally, by combining Lemmas 2.7, 2.12 and 2.8 with the fact that the number of centers we use is $|S'_I| + |S_N|$, we see that we provide a procedure that for a guess $R \geq R_m$ works as follows. It either returns a feasible solution with maximum assignment distance $5R$, or returns an infeasibility message that indicates $R < R^*$.

2.1 Cases with Bounded PoF

As we have already shown in Theorem 1.4, the Price of Fairness for both variants of EqCenter can in general be unbounded. However, we are going to demonstrate that in certain scenarios we can provably achieve solutions with bounded PoF.

The first scenario we study is a small modification to our main algorithm, which consists of only changing Algorithm 2, and thus the construction of S'_I and ϕ_I. Specifically, if for some $c \in S_I$ we have $|G_c| = 1$, then we use c as a center and set $\phi_I(c) = c$. If for some $c \in S_I$ we have $|G_c| \geq 2$, then we immediately use the procedure of Lemma A.1, without checking if only one point of G_c can yield a feasible solution. This modification yields the results of Theorem 1.6.

Although the result of Theorem 1.6 is interesting in the sense of showing a scenario with bounded PoF, it is not a true approximation algorithm. We are now going to demonstrate another case, where we achieve a true feasible solution to EqCenter-AG, that additionally enjoys a bounded PoF. In this scenario, the radius R_j is the same for all points, i.e., for all $j \in \mathcal{C}$ we have $R_j = R_d$ for some R_d. Our algorithm here is actually identical to the one presented in the previous subsection, and the difficulty in proving Theorem 1.7 for it lies only on the analysis (see Appendix A).

3 EXPERIMENTAL EVALUATION

We implemented all algorithms in Python 3.8 and ran our experiments on Intel Xeon (Ivy Bridge) E3-12 @ 2.4 GHz with 20 cores and 96 GB 1200 MHz DDR4 memory. Our code can be found here.

Datasets: We used 5 datasets from the UCI Machine Learning Repository (Dua and Graff, 2017), namely: **(1)** Bank-4,521 points (Moro et al., 2014), **(2)** Adult-32,561 points (Kohavi, 1996), **(3)** Creditcard-30,000 points (Yeh and Lien, 2009), **(4)** Census1990-2,458,285 points (Meek et al., 2002) and **(5)** Diabetes-101,766 points (Strack et al., 2014). From Adult, Creditcard, Census and Diabetes we uniformly subsampled $25,000$ points, and performed our experiments with respect to those sampled sets. In order to construct the distances between points, we removed non-numeric features, standardized each of the remaining features, took the Euclidean distances between these modified points, and then normalized the distances to be in $[0, 1]$ for each dataset (by dividing the distances for a given dataset by the maximum pairwise distance).

Algorithms: We first implemented the two versions of the algorithm of Theorem 1.5, one solving EqCenter-AG and the other EqCenter-PP. We call Alg-AG the variant solving EqCenter-AG, and Alg-PP the variant solving EqCenter-PP. Furthermore, we implemented the algorithm of Theorem 1.6 and we refer to this as Pseudo-PoF-Alg. Finally, as baselines we used our own implementations of two "unfair" k-center algorithms, specifically the 2-approximation of Hochbaum and Shmoys (1985) and the 2-approximation of Gonzalez (1985).

Range of k and value of the fairness parameter α: We ran all of our experiments for every value of k in $\{2, 4, 8, 16, 32, 64, 128\}$, and in all our simulations we set $\alpha = 2$ for constraints (1) and (2). We did not test any other value for α, since in practice $\alpha > 2$ is unsuitable if strong fairness considerations are at play.

Constructing the similarity sets: For each combination of dataset and value of k that we are interested in, we need to construct the similarity sets $\mathcal{S}_j$, such that they satisfy Assumption 1.1. Our first step in doing so, was utilizing the filtering procedure from (Hochbaum and Shmoys, 1985), which for a given instance (combination of a dataset and a value k) returns a value R_f. If R^*_{unf} is the value of the optimal "unfair" k-center solution for the instance, the aforementioned filtering guarantees that $R_f \leq R^*_{unf}$. Then, for each point j we drew R_j uniformly at random from $[0, 2R_f]$, and then set $\mathcal{S}_j = \{j' \mid d(j, j') \leq R_j\}$. There were two reasons for constructing the sets $\mathcal{S}_j$ in this way. At first, this approach agrees with the canonical case for ψ. As described in Section 1.1.1, $\psi = 2$ is the most well-justified instantiation of Assumption 1.1. Second, this approach forces non-uniformity in the values of R_j, and thus we are able to test our algorithms in the most general setting.

Evaluated Metrics: Let S be the set of chosen centers and $\phi : \mathcal{C} \mapsto S$ the corresponding assignment function, that constituted the solution we got when we ran some particular algorithm on some problem instance. The quantities we evaluate are:

a) Max assignment distance: $\max_{j \in \mathcal{C}} d(j, \phi(j))$

b) Satisfaction of constraint (1): Here for each j we define $f^{\text{PP}}_j = \max_{j' \in \mathcal{S}_j} \frac{d(j,\phi(j))}{d(j',\phi(j'))}$.

c) Satisfaction of constraint (2): Here for each j we define $f^{\text{AG}}_j = \frac{|\mathcal{S}_j| d(j,\phi(j))}{\sum_{j' \in \mathcal{S}_j} d(j',\phi(j'))}$.

We now present our results that involve running all 5 mentioned algorithms on the Adult dataset. The corresponding plots for the other four datasets can be found in Appendix D, and they exhibit the ex-

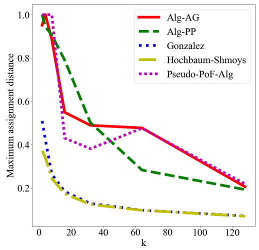

Figure 1: Max assignment distance for all algorithms

act behavior as the ones displayed here. In addition, the maximum runtime encountered in all our simulations was approximately 30 minutes (running Alg-PP on Census1990), and the bottleneck in all executions was computing the pairwise distances.

In Figure 1 we present the maximum assignment distance as a function of k for all algorithms. At first, we observe that even our algorithms with no PoF guarantees, i.e., Alg-PP and Alg-AG, perform very well in terms of an empirical PoF with respect to the baseline solutions. In addition, we want to compare the objective values of Alg-PP and Alg-AG. Recall that since a solution to EqCenter-PP also constitutes a solution to EqCenter-AG, we are theoretically expecting Alg-AG to perform better. However, we see that in practice there is no clear winner, and hence the use of Alg-PP is highly recommended, since the notion of fairness guaranteed by that algorithm is much stronger.

In Figures 2a-2c we demonstrate how all algorithms perform in terms of the fairness constraints.[1] Figure 2a shows $\max_j f_j^{PP}$ as a function of k for our two algorithms for EqCenter-PP, i.e., Alg-PP and Pseudo-PoF-Alg. Here we see that as the theory suggests, our algorithms always satisfy constraint (1) and have $\max_j f_j^{PP} \leq 2$. On the other hand, Figure 2b shows $\max_j f_j^{PP}$ as a function of k for the baselines. Here we see that the baselines are far from satisfying constraint (1), and specifically that there exist points that are treated very unfairly. Finally, Figure 2c shows $\max_j f_j^{AG}$ as a function of k for all algorithms that can be potentially used for EqCenter-AG. Here we see that our algorithms again satisfy the corresponding constraint (2), and furthermore have a better $\max_j f_j^{AG}$ value compared to the baselines. Finally, in the AG case the baselines seem to perform much better compared to the PP case, and this is reasonable because the notion of fairness described by (2) is much weaker. Nonetheless, we still see that in many cases the baselines are not able to satisfy (2).

[1] In these plots, for the two baseline algorithms we excluded points with $f_j^{PP} = +\infty$ or $f_j^{AG} = +\infty$ in the computation of $\max_j f_j^{PP}$ and $\max_j f_j^{AG}$. In other words, we were very lenient with the two baselines.

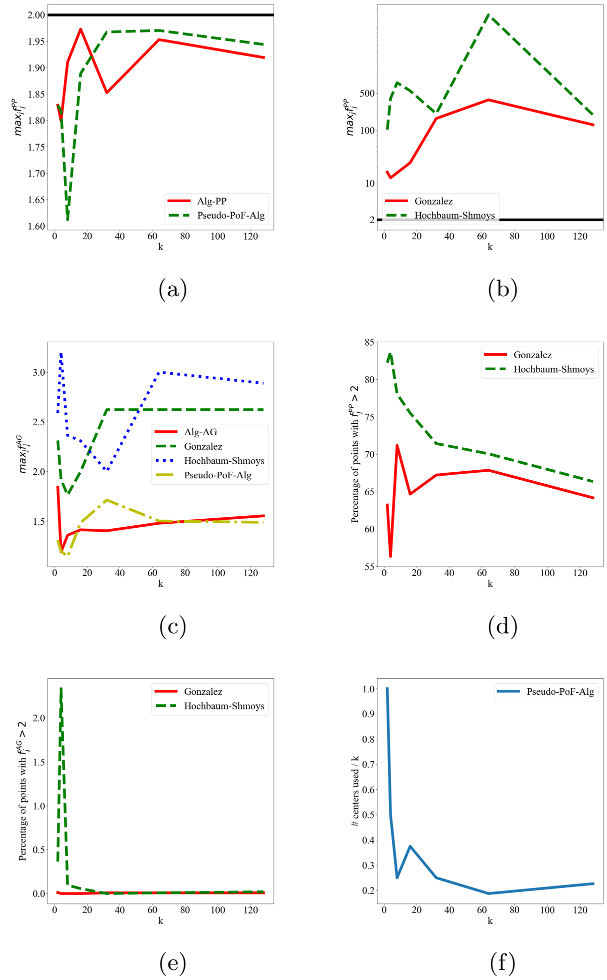

Figure 2: Satisfaction of fairness constraints

In Figures 2d and 2e we are interested in the percentage of points for which baselines do not satisfy the appropriate fairness constraint. Figure 2d demonstrates that for the stronger notion of PP-fairness, a substantial percentage of points gets unfair treatment ($f_j^{PP} > 2$). On the other hand, for the weaker notion of fairness captured by (2), the two baselines do much better. Nonetheless, even if one is interested only in the weaker AG concept of fairness, they should not use the baselines. *Even one unfairly treated point goes against the very nature of individual fairness.*

Finally, in Figure 2f we see by how much Pseudo-PoF-Alg violates the constraint $|S| \leq k$ on the set of chosen centers (recall that in theory Pseudo-PoF-Alg yields $|S| \leq 2k$). Here we plot the ratio of the number of centers used by the algorithm over the given value k, and see that in practice Pseudo-PoF-Alg does not actually incur any violation.

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Supplementary Material:
A New Notion of Individually Fair Clustering: α-Equitable k-Center

A MISSING PROOFS

Proof of Theorem 1.2. Let m be a very large even integer, with $\frac{m}{2}$ also being an even integer. We consider $2m$ points $\mathcal{C} = \{j_1, j_2, \dots, j_{2m-1}, j_{2m}\}$ in a cycle, where $d(j_i, j_{i+1}) = 1$ for all $i \in [2m-1]$, and also $d(j_{2m}, j_1) = 1$. The rest of the distances are set to be the shortest path ones, based on those already defined. This is a valid metric space, since it constitutes the shortest path metric resulting from a simple cycle graph of $2m$ vertices.

To construct the similarity sets, we map each point j to another point $\pi(j) \neq j$, such that the function $\pi : \mathcal{C} \mapsto \mathcal{C}$ is one-to-one and $\pi(\pi(j)) = j$. Given that, the similarity set of point j will be set to be $\mathcal{S}_j = \{\pi(j)\}$. Now let $\mathcal{C}_1 = \{j_i \mid i \text{ is odd}\}$ and $\mathcal{C}_2 = \{j_i \mid i \text{ is even}\}$. For every odd $i \in [m]$, set $\pi(j_i) = j_{i+m}$ and $\pi(j_{i+m}) = j_i$. In this way, because m is even, we map every point of $\mathcal{C}_1$ to some other point of $\mathcal{C}_1$. Also, note that for every $j \in \mathcal{C}_1$ we will have $d(j, \pi(j)) = m$. For the points $j_i \in \mathcal{C}_2$, consider them in increasing order of i. If j_i is not already mapped to some other point, set $\pi(j_i) = j_{i+\frac{m}{2}}$ and $\pi(j_{i+\frac{m}{2}}) = j_i$. This is a valid assignment because $\frac{m}{2}$ is assumed to be an even integer. At the end of the above process, we have created a one-to-one mapping between the points of $\mathcal{C}_2$, such that for every $j \in \mathcal{C}_2$ we have $d(j, \pi(j)) = \frac{m}{2}$. This concludes the description of the similarity sets. Finally, this pairing process for $\mathcal{C}_1$ and $\mathcal{C}_2$ is possible, because both sets include an even number of points. See Figure 3 for an example.

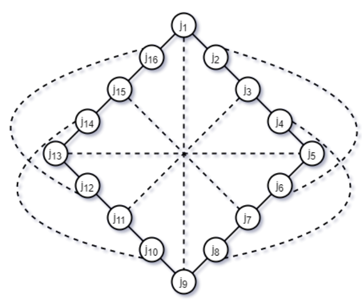


Figure 3: Here $m = 8$. The solid lines represent a distance of 1 between the corresponding points. The dashed lines correspond to similarity sets. E.g., the dashed line between j_1 and j_9 shows that $\pi(j_1) = j_9$ and $\pi(j_9) = j_1$.

To conclude the description of the input we assume $k = 2$. At this point observe that the constructed instance also satisfies Assumption 1.1 for $\psi \geq 2$, therefore covering the canonical case for ψ. This is because the optimal unfair value for the instance is easily seen to be $\frac{m}{2}$, while the maximum distance between similar points is m.

In addition, note that because for all j we have $|\mathcal{S}_j| = 1$, constraints (1) and (2) are equivalent and hence showing infeasibility for this instance covers both EqCenter-PP and EqCenter-AG. Finally, to prove the statement of the theorem, it suffices to show that for all possible choices of centers and all possible corresponding assignments ϕ, there will always be a point j_p for which $d(j_p, \phi(j_p)) \geq 2d(\pi(j_p), \phi(\pi(j_p)))$.

At first, notice that there exists no feasible solution that uses just one center. Supposing otherwise, let c be the only chosen center. Then there exists only one possible assignment for c, and that is $\phi(c) = c$. Hence $d(c, \phi(c)) = 0$, and the fairness constraint for $\pi(c)$ will never be satisfied.

Now we will show that even solutions that pick two centers c_1, c_2 cannot admit any feasible assignment. We proceed via a case analysis on $d(c_1, c_2)$.

- $d(c_1, c_2) \leq \frac{m}{3}$: Because the points of $\mathcal{C}_1$ and $\mathcal{C}_2$ alternate in the metric cycle, we know that there exists a $j \in \mathcal{C}_1$ such that $d(j, c_1) \leq 1$ (in the example of Figure 3 we might have $c_1 = j_2, c_2 = j_3$ and $j = j_1$,

$\pi(j) = j_9$). By the triangle inequality we also get $d(j, c_2) \leq \frac{m}{3} + 1$. As for the point $\pi(j)$, we have:

$$d(\pi(j), c_1) \geq d(\pi(j), j) - d(j, c_1) \geq m - 1$$
$$d(\pi(j), c_2) \geq d(\pi(j), j) - d(j, c_2) \geq m - \frac{m}{3} - 1 = \frac{2m}{3} - 1$$

From $\pi(j)$'s perspective, the best case situation regarding its fairness constraint is if $\pi(j)$ gets assigned to its closest center, and j gets assigned to its farthest one. Given all the previous inequalities, we see that the best possible service for $\pi(j)$ is $\frac{2m}{3} - 1$, and the worst possible service for j is $\frac{m}{3} + 1$. We next show that even in this ideal situation for $\pi(j)$, its fairness constraint with $\alpha < 2$ will never be satisfied if m is significantly large. To see this, note that $\frac{2m/3-1}{m/3+1}$ is an increasing function of m and also:

$$\lim_{m\to\infty} \Big(\frac{\frac{2m}{3} - 1}{\frac{m}{3} + 1}\Big) = \frac{2/3}{1/3} = 2$$

Therefore, for every given $\alpha < 2$, there exists an m_a such that $\frac{2m_a/3-1}{m_a/3+1} > \alpha$.

- $\frac{m}{3} < d(c_1, c_2) \leq \frac{2m}{3}$: In this case, because m is assumed to be significantly large and because the points of $\mathcal{C}_1, \mathcal{C}_2$ alternate in the metric cycle, we can find a point $j \in \mathcal{C}_1$ in the shortest path between c_1 and c_2, which will be approximately in the middle of the path. Letting $\gamma \in (\frac{1}{3}, \frac{2}{3}]$ such that $d(c_1, c_2) = \gamma m$, we have $\frac{\gamma m}{2} - 1 \leq d(j, c_1), d(j, c_2) \leq \frac{\gamma m}{2} + 1$ (in the example of Figure 3 we might have $c_1 = j_1, c_2 = j_5$ and $j = j_3$, $\pi(j) = j_{11}$). Regarding the possible assignments for $\pi(j)$ we have:

$$d(\pi(j), c_1) \geq d(j, \pi(j)) - d(j, c_1) \geq m - \frac{\gamma m}{2} - 1 = m\Big(\frac{2-\gamma}{2}\Big) - 1$$
$$d(\pi(j), c_2) \geq d(j, \pi(j)) - d(j, c_2) \geq m - \frac{\gamma m}{2} - 1 = m\Big(\frac{2-\gamma}{2}\Big) - 1$$

  Again we will focus on the best case situation for $\pi(j)$, which according to the previous analysis is $\pi(j)$ getting assigned to a center at distance $\frac{m(2-\gamma)}{2} - 1$ from it, and j getting assigned to a center at distance $\frac{\gamma m}{2} + 1$. Therefore, we consider the ratio $\frac{m(2-\gamma)/2-1}{\gamma m/2+1}$, and we are going to prove that even in this ideal case for $\pi(j)$, its fairness constraint for $\alpha < 2$ will not be satisfiable if m is suffieciently large. At first, because $\frac{2-\gamma}{2}, \frac{\gamma}{2} > 0$ the previous ratio will be an increasing function of m. In addition,

$$\lim_{m\to\infty} \Big(\frac{m(2-\gamma)/2 - 1}{\gamma m/2 + 1}\Big) = \frac{(2-\gamma)/2}{\gamma/2} = \frac{2-\gamma}{\gamma} \geq 2$$

  The last inequality follows since $\frac{2-\gamma}{\gamma}$ is a decreasing function, and for $\gamma \in (\frac{1}{3}, \frac{2}{3}]$ we have $\frac{2-\gamma}{\gamma} \in [2, 5)$. Hence, for every $\alpha < 2$ there exists an m_b such that $\frac{m_b(2-\gamma)/2-1}{\gamma m_b/2+1} > \alpha$.

- $\frac{2m}{3} < d(c_1, c_2) \leq m$: Because m is assumed to be significantly large and because the points of $\mathcal{C}_1, \mathcal{C}_2$ alternate in the metric cycle, we can find a point $j \in \mathcal{C}_2$ in the shortest path between c_1 and c_2, which will be approximately in the middle of the path. Letting $\gamma \in (\frac{2}{3}, 1]$ such that $d(c_1, c_2) = \gamma m$, we have $\frac{\gamma m}{2} - 1 \leq d(j, c_1), d(j, c_2) \leq \frac{\gamma m}{2} + 1$ (in Figure 3 we might have $c_1 = j_2, c_2 = j_{10}$ and $j = j_{14}$, $\pi(j) = j_{10}$). Consider now $\pi(j)$, and without loss of generality assume that $d(\pi(j), c_1) \geq d(\pi(j), c_2)$ (when $d(\pi(j), c_1) \leq d(\pi(j), c_2)$ the situation is symmetric, with the roles of c_1, c_2 switched.).

  At first, suppose that $\pi(j)$ is a point in the shortest path between c_1 and c_2 (in the example of Figure 3 $c_1 = j_2, c_2 = j_{10}$ and $j = j_{14}$ would result in that). Thus, because $d(j, \pi(j)) = m/2$, $d(\pi(j), c_1) \geq d(\pi(j), c_2)$ and $d(c_1, c_2) \leq m$, we can focus on the line segment $c_1, j, \pi(j), c_2$, where the triangle inequality holds with equality. Here we get,

$$d(\pi(j), c_2) = d(j, c_2) - d(j, \pi(j)) \leq \frac{\gamma m}{2} + 1 - \frac{m}{2} = \frac{(\gamma - 1)m}{2} + 1 \leq 1$$

  In addition,

$$d(\pi(j), c_1) = d(j, \pi(j)) + d(j, c_1) \geq \frac{m}{2} + \frac{\gamma m}{2} - 1 = \frac{(1+\gamma)m}{2} - 1$$

The second case we consider is when $\pi(j)$ is not on the shortest path between c_1 and c_2 (in Figure 3 take for instance $c_1 = j_1, c_2 = j_{11}$ and hence $j = j_{14}$ and $\pi(j) = j_{10}$). In that scenario, because $d(\pi(j), c_1) \geq d(\pi(j), c_2)$, we turn our attention to the line segment $c_1, j, c_2, \pi(j)$, where the triangle inequality holds with equality. Here we have

$$d(\pi(j), c_2) = d(j, \pi(j)) - d(j, c_2) \leq \frac{m}{2} - \frac{\gamma m}{2} + 1 = \frac{(1-\gamma)m}{2} + 1$$

In addition,

$$d(\pi(j), c_1) = d(j, \pi(j)) + d(j, c_1) \geq \frac{m}{2} + \frac{\gamma m}{2} - 1 = \frac{(1+\gamma)m}{2} - 1$$

Therefore, in every case we have the following:

$$d(\pi(j), c_1) \geq \frac{(1+\gamma)m}{2} - 1 \text{ and } d(\pi(j), c_2) \leq \frac{(1-\gamma)m}{2} + 1 \tag{3}$$

Now that we have the bounds (3) for the assignment distance of $\pi(j)$ to both centers, we proceed with the final case analysis.

Suppose that $\pi(j)$ gets assigned to c_1. Then from $\pi(j)$'s perspective, the best possible situation is if its own assignment distance is exactly $\frac{(1+\gamma)m}{2} - 1$, and j gets an assignment distance of $\frac{\gamma m}{2} + 1$. In this case, the ratio $\frac{(1+\gamma)m/2-1}{\gamma m/2+1}$ is an increasing function of m, because $(1+\gamma)/2, \gamma/2 > 0$. In addition we have:

$$\lim_{m\to\infty} \frac{(1+\gamma)m/2 - 1}{\gamma m/2 + 1} = \frac{1+\gamma}{\gamma} \geq 2$$

The last inequality is because $\frac{1+\gamma}{\gamma}$ is a decreasing function and $\gamma \leq 1$. Hence, for every $\alpha < 2$, there exists an m_c such that $\frac{(1+\gamma)m_c/2-1}{\gamma m_c/2+1} > \alpha$. Thus, even in the ideal situation for $\pi(j)$, if m is larger than m_c its fairness constraint for $\alpha < 2$ will be unsatisfiable.

On the other hand, suppose that $\pi(j)$ gets assigned to c_2. Then from j's perspective, the best possible situation is if it gets an assignment distance of $\frac{\gamma m}{2} - 1$, and $\pi(j)$ has assignment distance exactly $\frac{(1-\gamma)m}{2} + 1$. In this case, the ratio $\frac{\gamma m/2-1}{(1-\gamma)m/2+1}$ is an increasing function of m, because $(1-\gamma)/2, \gamma/2 > 0$. Also:

$$\lim_{m\to\infty} \frac{\gamma m/2 - 1}{(1-\gamma)m/2 + 1} = \frac{\gamma}{1-\gamma} > 2$$

The last inequality is because $\frac{\gamma}{1-\gamma}$ is an increasing function and $\gamma > 2/3$. Hence, for every $\alpha < 2$, there exists an m_d such that $\frac{\gamma m_d/2-1}{(1-\gamma)m_d/2+1} > \alpha$. Thus, even in the ideal situation for j, if m is larger than m_d, j's fairness constraint for $\alpha < 2$ will be unsatisfiable.

The analysis is exhaustive, because the maximum distance between two points in the metric is m. Further, we see that if we set $m = 4\max\{m_a, m_b, m_c, m_d\}$, then in every possible scenario there will exist a point whose fairness constraint for $\alpha < 2$ will not be satisfiable. □

The next Lemma is crucial for some of the remaining proofs.

Lemma A.1. *Consider a set of points $\mathcal{C}$ in a metric space with distance function d, where $|\mathcal{C}| \geq 2$. Then there exists an efficient way of finding two distinct points $c_1, c_2 \in \mathcal{C}$ and an assignment $\phi : \mathcal{C} \mapsto \{c_1, c_2\}$, such that for every $j \in \mathcal{C}$ we have $\frac{d(c_1,c_2)}{2} \leq d(j, \phi(j)) \leq d(c_1, c_2)$.*

Proof. At first, choose c_1, c_2 to be the two points of $\mathcal{C}$ that are the furthest apart, i.e. $(c_1, c_2) = \arg\max_{x,y\in\mathcal{C}} d(x, y)$. Then, for every $j \in \mathcal{C}$ set $\phi(j) = \arg\max_{c\in\{c_1,c_2\}} d(j, c)$. In other words, given the chosen centers, each point is assigned to the center that is furthest from it in the metric. Let also $\bar{\phi}(j)$ be the center to which j is not assigned to. For any $j \in \mathcal{C}$, combining the triangle inequality and the fact that $d(j, \bar{\phi}(j)) \leq d(j, \phi(j))$, will give us:

$$d(c_1, c_2) \leq d(j, \phi(j)) + d(j, \bar{\phi}(j)) \leq 2d(j, \phi(j)) \implies d(c_1, c_2)/2 \leq d(j, \phi(j))$$

Finally, by the way we chose c_1 and c_2 we also get $d(j, \phi(j)) \leq d(c_1, c_2)$. □

Proof of Theorem 1.3. Suppose that as an instance to either problem we are given a set of points $\mathcal{C}$ together with their associated similarity sets $\mathcal{S}_j$, $k \geq 2$ and $\alpha \geq 2$. W.l.o.g. we can assume that $|\mathcal{C}| \geq 2$, because otherwise the statement of the Lemma is trivially true. Since $k \geq 2$, we can use Lemma A.1 and get a set of two centers $\{c_1, c_2\}$ and an assignment function $\phi : \mathcal{C} \mapsto \{c_1, c_2\}$, such that for all $j \in \mathcal{C}$ we have $d(c_1, c_2)/2 \leq d(j, \phi(j)) \leq d(c_1, c_2)$. In the case of constraint (1), for every $j \in \mathcal{C}$ and any $j' \in \mathcal{S}_j$ we have $d(j, \phi(j)) \leq d(c_1, c_2) \leq 2d(j', \phi(j')) \leq \alpha d(j', \phi(j'))$. Furthermore, since any feasible solution for constraint (1) is also a feasible solution for constraint (2), the proof is concluded. □

Proof of Theorem 1.4. Consider an instance with four points j_1, j_2, j_3, j_4. For the distances we have $d(j_1, j_2) = d(j_3, j_4) = R$ and $d(j_1, j_3) = d(j_1, j_4) = d(j_2, j_3) = d(j_2, j_4) = D$, where $R \ll D$. Note that this is a valid metric space, where j_1, j_2 form a clique that is very far away from the clique of j_3, j_4. In addition, we assume $k = 2$ and $\alpha = 2$. For the similarity sets we have $\mathcal{S}_{j_1} = \{j_2\}$, $\mathcal{S}_{j_2} = \{j_1\}$, $\mathcal{S}_{j_3} = \{j_4\}$, $\mathcal{S}_{j_4} = \{j_3\}$.

Observe that the value of the optimal unfair solution is clearly R. This is achievable by choosing j_1, j_3 as centers. Given this, we see that the instance also satisfies Assumption 1.1 since $R \ll D$.

Moving forward, we are going to show that the optimal solution for the fair variants has value D (note that the existence of such a solution is guaranteed by Theorem 1.3). This implies that PoF is $\frac{D}{R}$, and since $R \ll D$ this ratio can be arbitrarily large. Furthermore, note that because all similarity sets have cardinality 1, constraints (1) and (2) are equivalent and hence we can solely focus on proving the result for (1).

At first, assume that the optimal fair solution uses only one center. Then, any assignment that uses only one center should necessarily yield a maximum assignment distance of D.

Let us now consider the case of the optimal fair solution using two centers. If both these centers are in the same clique, i.e., the centers are either $\{j_1, j_2\}$ or $\{j_3, j_4\}$, then trivially any assignment that uses those sets will lead to a maximum assignment distance of D. Therefore, we only need to see what happens when the optimal fair solution places one center in each clique, and without loss of generality let us assume that the chosen centers are $\{j_1, j_3\}$. Focus now on j_1. If the optimal solution assigns j_1 to itself, i.e., $\phi(j_1) = j_1$, then $d(j_1, \phi(j_1)) = 0$. The latter implies that the fairness constraint for j_2 cannot be satisfied. Thus, the optimal must set $\phi(j_1) = j_3$, hence leading to a maximum assignment distance of D. □

Proof of Lemma 2.4. Focus on such a $c \in S_I$, and for the sake of contradiction assume that there exists a $j \in G_c$ and a $j' \in \mathcal{C} \setminus G_c$ for which $d(j, j') \leq R$. Let $c' \neq c$ the center of S with $c' = \rho(j')$.

At first, suppose that during the execution of Algorithm 1 c entered S before c'. Having $|P_{t(c)}| = 1$ means that when $P_{t(c)} = \{c\}$, the algorithm tried to find a point in U within distance $3R$ from c but failed. However, at that time j' was still in U, because $j' \in G_{c'}$ and c' entered S after c. In addition $d(j', c) \leq d(j, j') + d(j, c) \leq 3R$, and thus we reached a contradiction.

Now assume that c' entered S before c. This implies that $t(c') < t(c)$, because $|P_{t(c)}| = 1$. When the algorithm stopped expanding $P_{t(c')}$, there was not any point of U within distance $3R$ from a center of $P_{t(c')}$. However, at that moment j was still in U, because $j \in G_c$ and $t(c') < t(c)$. In addition $d(j, c') \leq d(j, j') + d(j', c') \leq 3R$, and so we once again reach a contradiction. □

Proof of Lemma 2.7. At first, due to Observation 2.3, Algorithm 2 sets the value $\phi_I(j)$ for each $j \in \mathcal{C}_I$ exactly once. In addition, we know that for every $j \in \mathcal{C}_I$, all points of $\mathcal{S}_j$ will have their assignment set in the same iteration of Algorithm 2, since $\rho(j) \in S_I$ and by Corollary 2.5 we have $\mathcal{S}_j \subseteq G_{\rho(j)}$.

For a point $j \in \mathcal{C}_I$, when $\rho(j)$ is considered by Algorithm 2 there are two possible scenarios. In the first we have $|S'_I \cap G_{\rho(j)}| = 1$. If that happens, all points of $G_{\rho(j)}$ are assigned to the only point of $S'_I \cap G_{\rho(j)}$, and because of the first check of the algorithm we are also sure that the fairness constraint of all of them is satisfied. Otherwise, we have $|S'_I \cap G_{\rho(j)}| = 2$, as a result of running the algorithm of Lemma A.1 on $G_{\rho(j)}$. By using the assignment guarantees of that algorithm, it is easy to see that the fairness constraints for all $j' \in G_{\rho(j)}$ will again be satisfied. Hence, in both cases the corresponding fairness constraint is satisfied for j.

Finally, $d(j, \phi_I(j)) \leq d(j, \rho(j)) + d(\phi_I(j), \rho(j)) \leq 4R$, since $\phi_I(j) \in G_{\rho(j)}$ in each case. □

Proof of Lemma 2.8. Let S^* be the optimal set of centers, and ϕ^* the corresponding optimal assignment.

The following two statements rely on the fact that $R \geq R^*$. First, by Observation 2.1 note that for two distinct points $c, c' \in S_N$ we must have $\phi^*(c) \neq \phi^*(c')$. Second, due to Lemma 2.4 we also have $\phi^*(c) \notin \mathcal{C}_I$ for every $c \in S_N$. The two previous statements imply $|S_N| \leq |S^* \setminus \mathcal{C}_I|$.

Now focus on $S^* \cap \mathcal{C}_I$, and see that $|S^* \cap \mathcal{C}_I| = \sum_{c \in S_I} |S^* \cap G_c|$ due to Observation 2.3 and the definition of $\mathcal{C}_I$. Further, due to Lemma 2.4 and the fact that $R \geq R^*$, we have that $|S^* \cap G_c| \geq 1$ for every $c \in S_I$. If $|S^* \cap G_c| = 1$, then Lemma 2.4 implies that the optimal solution assigns all points of G_c to the unique point of $S^* \cap G_c$. This assignment is obviously feasible, and thus the first part of Algorithm 2 can identify it and give $|S'_I \cap G_c| = 1$. Otherwise, if $|S^* \cap G_c| \geq 2$, then Algorithm 2 ensures that $|S'_I \cap G_c| \leq 2$. Therefore, we get

$$|S'_I| = \sum_{c \in S_I} |S'_I \cap G_c| \leq \sum_{c \in S_I} |S^* \cap G_c| = |S^* \cap \mathcal{C}_I|$$

Putting everything together yields

$$|S'_I| + |S_N| \leq |S^* \cap \mathcal{C}_I| + |S^* \setminus \mathcal{C}_I| = |S^*| \leq k \qquad \square$$

Proof of Lemma 2.10. In **Case (A)** $d(j, \pi(j)) \leq R$ since $j \in H^1_{\pi(j)}$. Also, from the definition of type-1 points, there does not exist any center in $S_N \setminus \{\pi(j)\}$ that is within distance at most R from j, and hence $d(j, \phi_N(j)) > R \geq d(j, \pi(j))$. In addition, Observation 2.2 ensures that there exists a $c \in S_N \setminus \{\pi(j)\}$ such that $d(\pi(j), c) \leq 3R$. Therefore, $d(j, \phi_N(j)) \leq d(j, c) \leq d(j, \pi(j)) + d(\pi(j), c) \leq 4R$.

The assignment guarantee for **Case (B)** follows trivially from Observation 2.9, and the way the algorithm operates in that situation.

In **Case (C)** we have $d(j, c) > 2R \geq d(j, \pi(j))$ for all $c \in S_N \setminus \{\pi(j)\}$. In addition, Observation 2.2 ensures that there exists a $c' \in S_N \setminus \{\pi(j)\}$ such that $d(\pi(j), c') \leq 3R$. Hence, $d(j, \phi_N(j)) \leq d(j, c') \leq d(j, \pi(j)) + d(\pi(j), c') \leq 5R$, where $d(j, \pi(j)) \leq 2R$ follows from Observation 2.9. $\square$

Proof of Theorem 1.6. At first, note that due to Assumption 1.1 we have $R_m \leq \psi R^*_{unf}$, and hence the guess ψR^*_{unf} will be among the ones we test; recall that we test guesses $R \in [R_m, \max_{j,j'} d(j, j')]$. Assume for now that $\psi \geq 1$. For the iteration where the guess is ψR^*_{unf}, Lemmas 2.7 and 2.12 will clearly hold, thus ensuring that the returned solution has value $5\psi R^*_{unf}$, and the constructed assignment satisfies all fairness constraints. The only thing left to analyze is the number of centers we end up using when the guess is ψR^*_{unf}. Combining Observation 2.1, the fact that $\psi \geq 1$ and the fact that the optimal unfair solution uses at most k centers, we immediately get $|S_I| + |S_N| \leq k$. On the other hand, observe that the number of centers our modified algorithm uses is in the worst case is $2|S_I| + |S_N|$, and therefore at most $2k$.

When $\psi < 1$, then we know for sure that R^*_{unf} will be among the tested guesses. In that case, the previous analysis follows through, with the only difference being that now the maximum radius of our returned solution would be $5R^*_{unf}$.

Finally, to conclude the proof, we just need to make sure that for a radius guess that resulted in $|S_I| + |S_N| > k$, we return an infeasibility message. $\square$

Proof of Theorem 1.7. At first, note that due to Assumption 1.1 we have $R_d \leq \psi R^*_{unf}$, and hence the guess ψR^*_{unf} will be among the ones we test. As in the proof of Theorem 1.6 we can solely focus on the $\psi \geq 1$ case. For the iteration of ψR^*_{unf}, Lemma 2.12 clearly holds. We will show that Lemma 2.7 will hold as well, and furthermore that Algorithm 2 will always pick *just one* center in each G_c for $c \in S_I$. This will immediately imply that the returned solution has value at most $5\psi R^*_{unf}$, all constraints (2) are satisfied, and the centers we end up using are exactly $|S_I| + |S_N|$. Finally, note that by Observation 2.1, the fact that $\psi \geq 1$ and the fact that the optimal unfair solution uses at most k centers, we will also have $|S_I| + |S_N| \leq k$.

Therefore, all we need to show is that for every $c \in S_I$, Algorithm 2 is able to find exactly one center that satisfies constraint (2) for all $j \in G_c$ (recall that $\mathcal{S}_j \subseteq G_c$). To do that, we prove that there exists an $x \in G_c$, such that that for all $j \in G_c$ we have $\sum_{j' \in \mathcal{S}_j} d(j, j') \leq \sum_{j' \in \mathcal{S}_j} d(j', x)$. This suffices to prove the desired statement. To see why, assume that we make x the chosen center of G_c, and assign all points of G_c to it. Then for any point

Algorithm 4: Solving the assignment problem for EqCenter-PP

For every $j \in \mathcal{C}$ set $\phi(j) \leftarrow \arg\max_{i \in S^*: d(i,j) \leq R^*} d(i,j)$;
while *there exists a* $j \in \mathcal{C}$ *with a* $j' \in \mathcal{S}_j$ *such that* $d(j, \phi(j)) > \alpha d(j', \phi(j'))$ **do**
  Find such a pair $j \in \mathcal{C}$ and $j' \in \mathcal{S}_j$;
  Let $\Delta_{j,j'} = \{i \in S^* : d(i,j) < d(j, \phi(j)) \text{ and } d(i,j) \leq \alpha d(j', \phi(j'))\}$;
  Set $\phi(j) \leftarrow \arg\max_{i \in \Delta_{j,j'}} d(i,j)$;
end
Return ϕ;

$j \in G_c$ and any $j' \in \mathcal{S}_j$ we have $d(j,x) \leq d(j,j') + d(j',x)$ by the triangle inequality. Summing over all $j' \in \mathcal{S}_j$ and using the property of x gives:

$$d(j,x) \leq \frac{1}{|\mathcal{S}_j|} \sum_{j' \in \mathcal{S}_j} d(j,j') + \frac{1}{|\mathcal{S}_j|} \sum_{j' \in \mathcal{S}_j} d(j',x) \leq \frac{2}{|\mathcal{S}_j|} \sum_{j' \in \mathcal{S}_j} d(j',x) \leq \frac{\alpha}{|\mathcal{S}_j|} \sum_{j' \in \mathcal{S}_j} d(j',x)$$

For the sake of contradiction, assume now that for all $x \in G_c$ there exists a point $j \in G_c$ such that $\sum_{j' \in \mathcal{S}_j} d(j,j') > \sum_{j' \in \mathcal{S}_j} d(j',x)$. Based on this, we can create a dependency graph, where every point of G_c is a vertex, and there is a directed edge from x to j if $\sum_{j' \in \mathcal{S}_j} d(j,j') > \sum_{j' \in \mathcal{S}_j} d(j',x)$. The assumption for the contradiction implies that this dependency graph will contain a directed cycle $x_1, x_2, \dots, x_r$, for which we have $\sum_{j' \in \mathcal{S}_{x_t}} d(x_t, j') > \sum_{j' \in \mathcal{S}_{x_t}} d(j', x_{t-1})$ for all $t \in [2, r+1]$, assuming that $x_{r+1} = x_1$. If we add all the above inequalities we get

$$\sum_{t=2}^{r+1} \sum_{j' \in \mathcal{S}_{x_t}} d(j', x_t) > \sum_{t=2}^{r+1} \sum_{j' \in \mathcal{S}_{x_t}} d(j', x_{t-1})$$

Now focus on any j', and see that its contribution in the LHS of the above inequality is $A = \sum_{t: j' \in \mathcal{S}_{x_t}} d(j', x_t)$, and in the RHS is $B = \sum_{t: j' \in \mathcal{S}_{x_t}} d(j', x_{t-1})$. We argue that $A > B$ is impossible, and thus reach a contradiction. If $A > B$, we can first subtract from both A and B the common terms appearing in the sums. Then, in what is left of A we will only have terms $d(j', x_t)$ being added, for $j' \in \mathcal{S}_{x_t}$. In what is left of B we will only have terms $d(j', x_{t-1})$ being added, but for which $j' \notin \mathcal{S}_{x_{t-1}}$. Note also that the number of leftover terms is the same in both A and B. Moreover, since the similarity radius is the same for all points, for any two points $z, y \in G_c$ with $j' \in \mathcal{S}_z$ and $j' \notin \mathcal{S}_y$, we have $d(j', z) < d(j', y)$. Hence we reached the desired contradiction. □

B SOLVING THE ASSIGNMENT PROBLEM

Here we address the assignment problem for EqCenter. Specifically, for an instance with $\alpha, k \geq 2$, if we are given the set of centers S^* used in the optimal solution, can we efficiently find the optimal assignment $\phi^* : \mathcal{C} \mapsto S^*$? In other words, if R^* is the value of the optimal solution, we want to compute ϕ^* such that **1)** ϕ^* satisfies the appropriate fairness constraint for all points, and **2)** for every $j \in \mathcal{C}$ we have $d(j, \phi^*(j)) \leq R^*$. In what follows, we show in full detail a procedure that achieves this for EqCenter-PP. A similar process can handle EqCenter-AG, but for the sake of not repeating the same arguments, we are only going to sketch this.

Before we proceed with our assignment algorithm for EqCenter-PP, note that w.l.o.g. we can always assume that the optimal value R^* is known. This is because there are only polynomially many options for it, and thus we can efficiently guess the optimal one. Our process is presented in Algorithm 4, and it works iteratively. The high-level idea is that it always maintains an assignment of value at most R^*, and in each iteration it corrects one violated fairness constraint. As we show later, a polynomial number of iterations suffices in order to reach a feasible assignment.

Lemma B.1. *Every time the condition of the while loop in Algorithm 4 is checked, we have* $d(\phi(j), j) \geq d(\phi^*(j), j)$ *for every* $j \in \mathcal{C}$.

Proof. We are going to prove this via induction. For the first time we check the condition, the statement is obviously true by the way we initialized the mapping ϕ before the start of the loop, and the fact that $d(j, \phi^*(j)) \leq R^*$ for all $j \in \mathcal{C}$.

Consider now the t^{th} time we check the condition, for which by the inductive hypothesis the statement of the lemma holds. If at that time no violated fairness constraint is found, then we are done. Hence, we need to focus on the case where the main body of the while loop is executed, and show that after the changes that occur in ϕ, the statement will still be satisfied for the $(t+1)^{\text{th}}$ time we will check the condition.

Let j_t be the point chosen at that iteration, with $j'_t \in \mathcal{S}_{j_t}$ the point with $d(j_t, \phi(j_t)) > \alpha d(j'_t, \phi(j'_t))$. By the inductive hypothesis we have $d(j'_t, \phi(j'_t)) \geq d(j'_t, \phi^*(j'_t))$. Combining the two previous inequalities gives $d(j_t, \phi(j_t)) > \alpha d(j'_t, \phi^*(j'_t))$. Now because the optimal assignment satisfies $d(j_t, \phi^*(j_t)) \leq \alpha d(j'_t, \phi^*(j'_t))$, we finally get $d(j_t, \phi(j_t)) > d(j_t, \phi^*(j_t))$. In addition, we have $d(j_t, \phi^*(j_t)) \leq \alpha d(j'_t, \phi^*(j'_t)) \leq \alpha d(j'_t, \phi(j'_t))$. Therefore, we see that $\phi^*(j_t) \in \Delta_{j_t, j'_t}$. Let now $\phi'(j_t)$ be the updated assignment for j_t after the end of the iteration. From the way we update the assignment for j_t and the fact that $\phi^*(j_t) \in \Delta_{j_t, j'_t}$, we infer that $d(\phi'(j_t), j_t) \geq d(\phi^*(j_t), j_t)$. ☐

Theorem B.2. *Algorithm 4 terminates within $|\mathcal{C}||S^*|$ iterations, and the final assignment ϕ satisfies:* ***1)*** *$d(j, \phi(j)) \leq R^*$ for all $j \in \mathcal{C}$, and* ***2)*** *$d(j, \phi(j)) \leq \alpha d(j', \phi(j')$ for all $j \in \mathcal{C}$ and $j' \in \mathcal{S}_j$.*

Proof. From the condition of the while loop we know that when the algorithm terminates, the fairness constraints will be satisfied by the mapping ϕ. Also, because we never assign a point to a center that is further than R^* from it, we know that ϕ achieves the optimal value.

Now we are going to count the total possible number of iterations. We do that by considering how many times we changed the assignment of every single point j, i.e., how many times an iteration tried to fix one of j's violated constraints. By Lemma B.1, we see that for any j the minimum possible assignment distance we can provide to it is $d(j, \phi^*(j))$. Observe that if at any moment $d(j, \phi(j)) = d(j, \phi^*(j))$, then Lemma B.1 guarantees that j's assignment will never change again. This is because for every $j' \in \mathcal{S}_j$ we always have $d(j', \phi(j')) \geq d(j', \phi^*(j'))$, and thus using the properties of the optimal assignment we get $d(j, \phi(j)) = d(j, \phi^*(j)) \leq \alpha d(j', \phi^*(j')) \leq \alpha d(j', \phi(j'))$.

On the other hand, if at some point $d(j, \phi(j)) > d(j, \phi^*(j))$, then one of j's fairness constraints might be violated, and hence we might end up using an iteration to fix it. In this case, let $j' \in \mathcal{S}_j$ the point causing the problematic situation. In addition, note that Lemma B.1 and the properties of the optimal solution ensure that $d(j_t, \phi^*(j_t)) \leq \alpha d(j'_t, \phi^*(j'_t)) \leq \alpha d(j'_t, \phi(j'_t))$. Thus, for this iteration $\phi^*(j) \in \Delta_{j,j'}$, and the new assignment distance of j will be strictly smaller than the one it had at the beginning of the iteration. Thus, j can be chosen in at most $|S^*|$ iterations. ☐

The assignment procedure for EqCenter-AG is almost identical to Algorithm 4, with the only difference being that we should instead be looking for violated constraints (2). In addition, the analysis of that algorithm remains identical to that of Algorithm 4.

C EXPLICITLY ENFORCING ASSUMPTION 1.1

It is reasonable to assume that there will be situations in which a central planner is not certain that Assumption 1.1 holds. Furthermore, there may also be cases where the sets $\mathcal{S}_j$ are not explicitly provided, e.g., because individuals have a fuzzy understanding of similarity and cannot accurately determine their most comparable points. Nonetheless, even under such conditions, the central planner can help the points construct the sets $\mathcal{S}_j$, in way that is explainable and will also satisfy the necessary assumption. This is clearly described in what follows.

The planner can first compute a nearly-tight lower bound R_f for R^*_{unf} (note that computing R^*_{unf} exactly is NP-hard). This can be done efficiently in multiple ways, for example by using the thresholding technique of Hochbaum and Shmoys (1985). Afterwards, the planner publishes R_f and informs the agents that even under optimal conditions, the points that are considered similar to each of them are only within distance ψR_f, for some small constant ψ. Then, the points are asked to independently construct their similarity sets, such that $\mathcal{S}_j \subseteq \{j' \in \mathcal{C} : d(j, j') \leq \psi R^*_{unf}\}$.

This strategy certainly enjoys explainability merits. Besides having the planner compute, publish and clarify the meaning of ψR_f to the points, it also gives the planner a valid justification to turn down requests for $\mathcal{S}_j$ that do not satisfy Assumption 1.1, by clearly explaining to such an agent j why this choice is unreasonable.

D ADDITIONAL EXPERIMENTAL RESULTS

Experimental results for Bank:

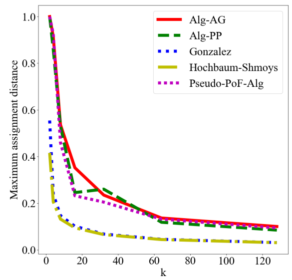
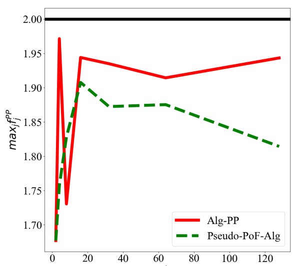
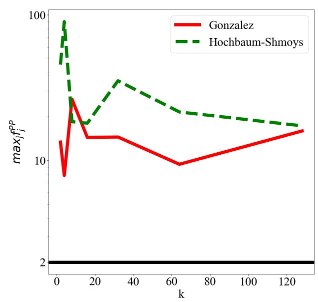


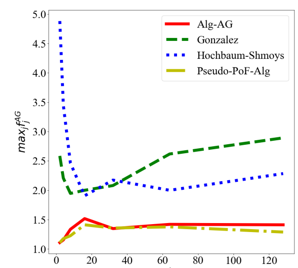
Figure 4: Maximum assignment distance for all algorithms



(a)

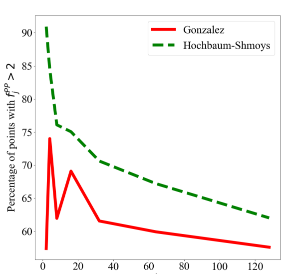


(b)

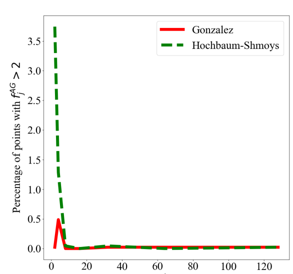


(c)

Figure 5: Satisfaction of fairness constraints



(a)



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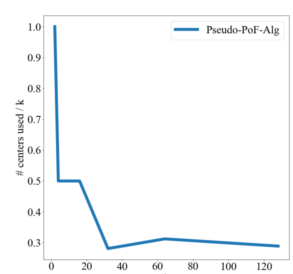


(c)

Figure 6: Amount of constraint violation

Experimental results for Creditcard:

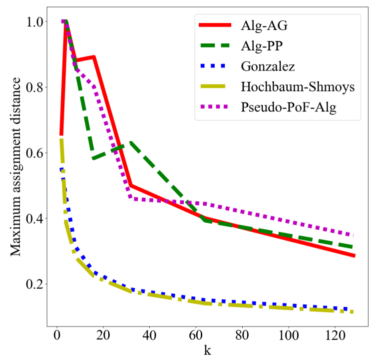
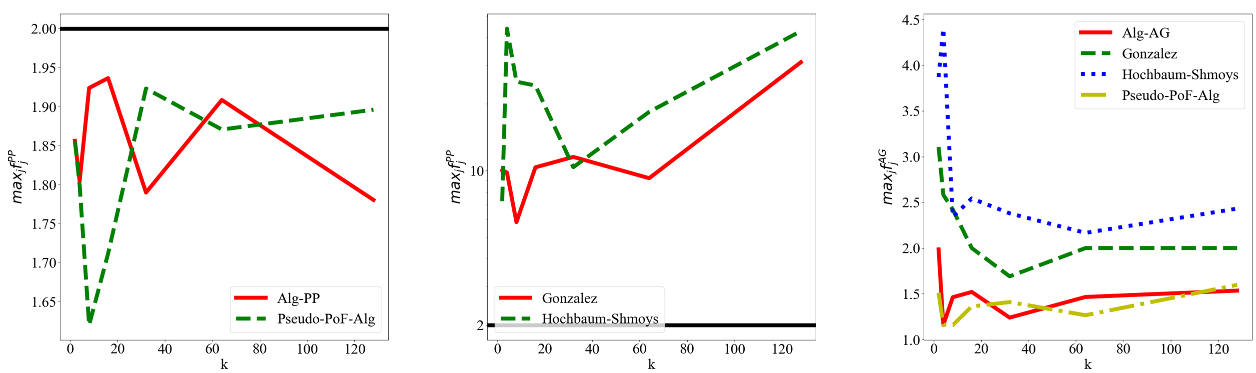
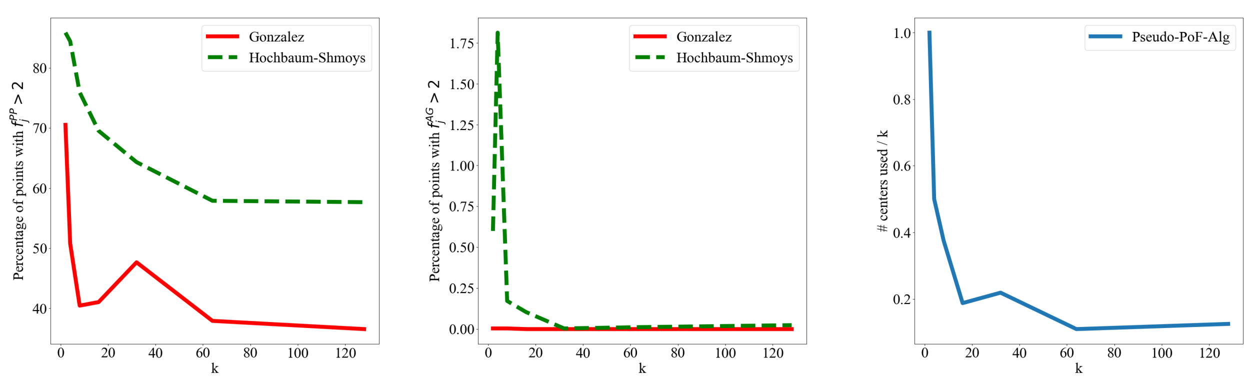


Figure 7: Maximum assignment distance for all algorithms



(a) (b) (c)

Figure 8: Satisfaction of fairness constraints



(a) (b) (c)

Figure 9: Amount of constraint violation

Experimental results for Census1990:

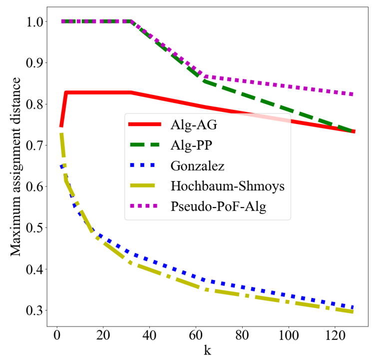
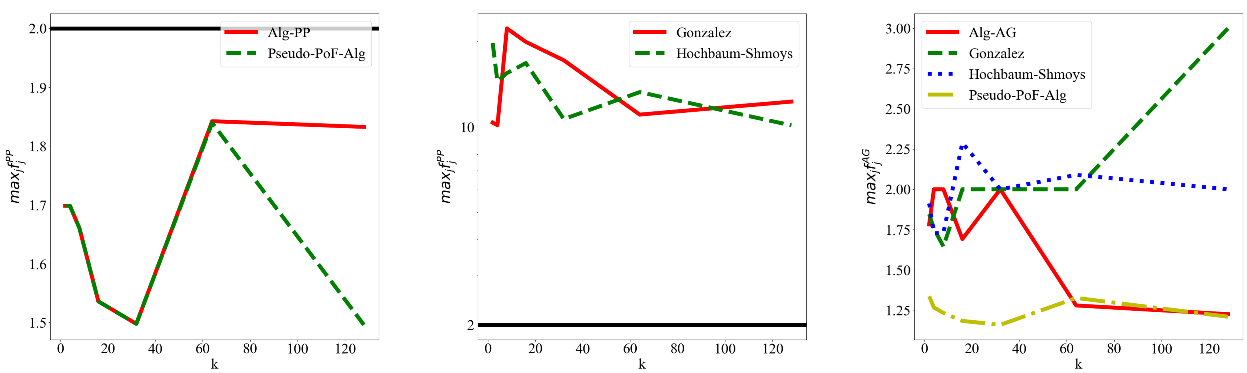
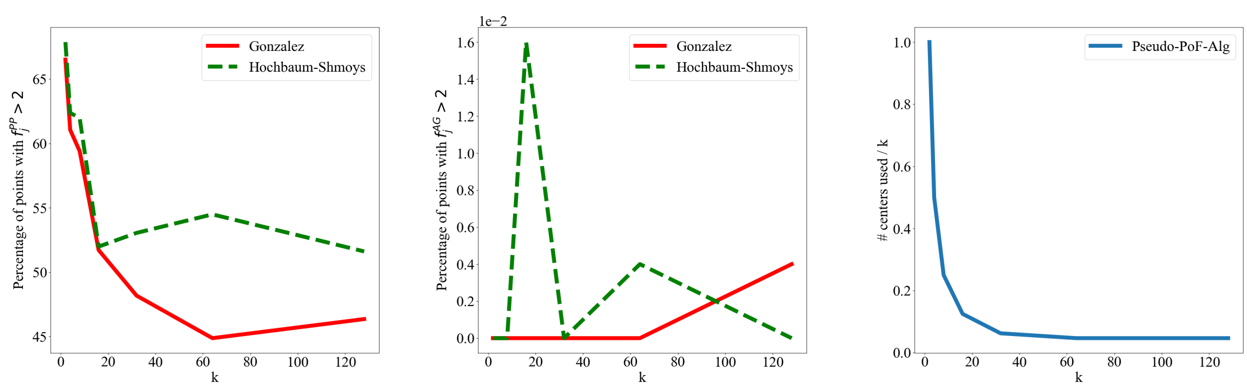


Figure 10: Maximum assignment distance for all algorithms



(a) (b) (c)

Figure 11: Satisfaction of fairness constraints



(a) (b) (c)

Figure 12: Amount of constraint violation

Experimental results for Diabetes:

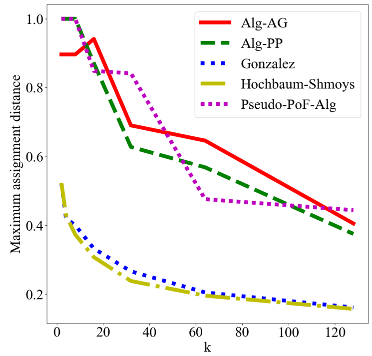
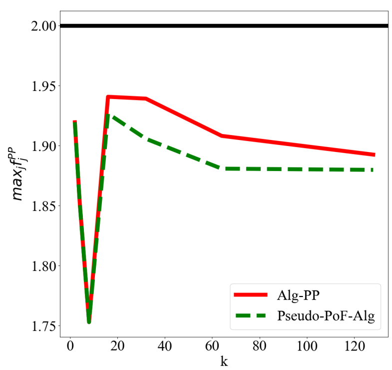
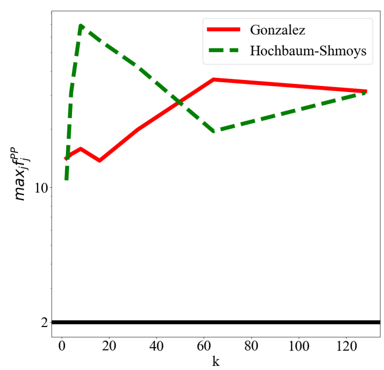


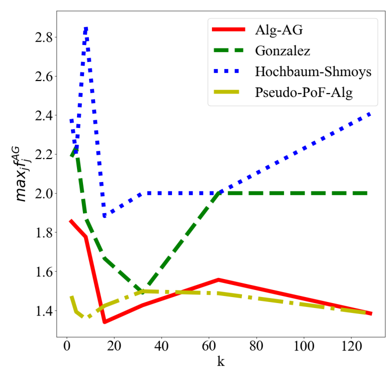
Figure 13: Maximum assignment distance for all algorithms



(a)

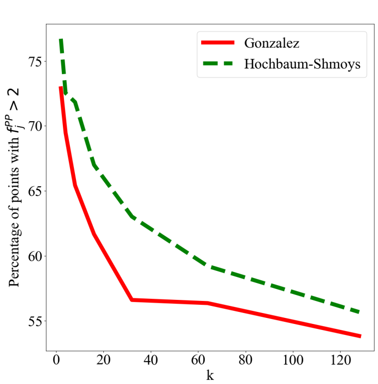


(b)

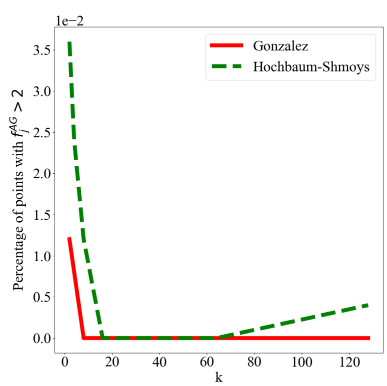


(c)

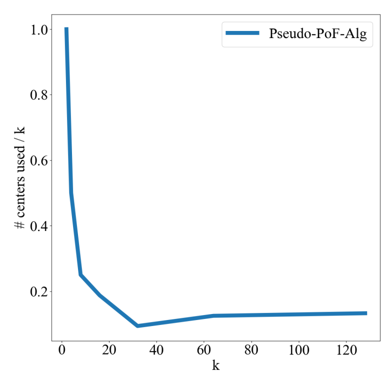
Figure 14: Satisfaction of fairness constraints



(a)



(b)



(c)

Figure 15: Amount of constraint violation