Min/Max Stability and Box Distributions (Supplementary Material)

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A PROOFS OF LEMMAS

Lemma 1. *If X is a real-valued random variable with finite mean then*

$$\lim_{x\to-\infty} xF(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} x(1-F(x)) = 0$$

Lemma 2. *Let X, Y be independent random variables a.c. with respect to the Lebesgue measure. Then*

$$\mathbb{E}[\max(X,Y)] = \int_{-\infty}^{\infty} z\Big(f_Y(z)F_X(z)+f_x(z)F_Y(z)\Big)\,dz.$$

Proof. We start by noting, for $x < 0$,

$$0 \geq xF(x) = x\int_{-\infty}^{x} f(z)\,dz \geq \int_{-\infty}^{x} zf(z)\,dz \tag{1}$$

Since $\mathbb{E}[X]$ is finite, we can calculate

$$\lim_{x\to-\infty}\int_{-\infty}^{x} zf(z)\,dz = \lim_{x\to-\infty}\left(\mathbb{E}[X] - \int_{x}^{\infty} zf(z)\,dz\right) = 0 \tag{2}$$

Applying the squeeze theorem to (1) yields

$$\lim_{x\to-\infty} xF(x) = 0. \tag{3}$$

The other limit can be obtained by applying this to $-X$. □

Proof. Let $Z = \max(X, Y)$, then

$$F_Z(z) = F_{\max(X,Y)} = F_X(z)F_Y(z), \qquad \text{for } z \in \mathbb{R}.$$

Thus,

$$\mathbb{E}[\max(X,Y)] = \int_{-\infty}^{\infty} z\frac{d}{dz}F_X(z)F_Y(z)\,dz \tag{4}$$

$$= \int_{-\infty}^{\infty} z(f_X(z)F_Y(z) + f_Y(z)F_X(z))\,dz. \tag{5}$$

□

B EXPECTED VOLUME OF GUMBEL BOX

Note that, with Lemma 3 in hand, we almost instantly can calculate the expected volume of a Gumbel box. If $X \sim \text{Gumbel}_{\max}(\mu_x, \beta)$ and $Y \sim \text{Gumbel}_{\min}(\mu_y, \beta)$, Lemma 3 implies

$$\mathbb{E}[\max(0, Y-X)] = \int_{-\infty}^{\infty} (1 - F_{\min}(z;\mu_y))F_{\max}(z;\mu_x)\,dz \tag{6}$$

$$= \int_{-\infty}^{\infty} \exp(-e^{\frac{z-\mu_y}{\beta}} - e^{-\frac{z-\mu_x}{\beta}})\,dz. \tag{7}$$

The remaining steps (which we include here for convenience) are to make the substitution $u = \frac{z-(\mu_y+\mu_x)/2}{\beta}$:

$$= \beta\int_{-\infty}^{\infty} \exp(-e^{\frac{\mu_x-\mu_y}{2\beta}}(e^u + e^{-u}))\,du \tag{8}$$

$$= 2\beta\int_{0}^{\infty} \exp(-2e^{\frac{\mu_x-\mu_y}{2\beta}}\cosh u)\,du. \tag{9}$$

By setting $z = 2e^{\frac{\mu_x-\mu_y}{2\beta}}$ this is a known integral representation of the modified Bessel function of the second kind of order zero, $K_0(z)$ (DLMF, eq 10.32.9).

C EXPLICIT CALCULATION OF INTERSECTION OF GUMBELBOX

We can compute this explicitly for Gumbel boxes, in which case we have

$$Z^- \sim \text{Gumbel}_{\max}(\mu_{Z^-}, \beta) \tag{10}$$

$$Z^+ \sim \text{Gumbel}_{\min}(\mu_{Z^+}, \beta), \tag{11}$$

where

$$\begin{aligned}\mu_{Z^-} &= \quad \beta \ln(e^{\frac{\mu_{X^-}}{\beta}} + e^{\frac{\mu_{Y^-}}{\beta}}), \qquad \text{and} \\ \mu_{Z^+} &= -\beta \ln(e^{-\frac{\mu_{X^+}}{\beta}} + e^{-\frac{\mu_{Y^+}}{\beta}}).\end{aligned}$$

Note that

$$\begin{aligned}\ln F_{Z^-}(z) &= -\exp\Big[-\frac{z-\mu_{Z^-}}{\beta}\Big] \\ &= -\exp\left[-\frac{z-\beta\ln(\exp(\frac{\mu_{X^-}}{\beta})+\exp(\frac{\mu_{Y^-}}{\beta}))}{\beta}\right] \\ &= -e^{-\frac{z-\mu_{X^-}}{\beta}} - e^{-\frac{z-\mu_{Y^-}}{\beta}},\end{aligned}$$

and

$$\begin{aligned}\ln F_{Z^+}(z) &= -\exp\Big[\frac{z+\mu_{Z^+}}{\beta}\Big] \\ &= -\exp\left[\frac{z+\beta\ln(\exp(-\frac{\mu_{X^+}}{\beta})+\exp(-\frac{\mu_{Y^+}}{\beta}))}{\beta}\right] \\ &= -e^{\frac{z-\mu_{X^+}}{\beta}} - e^{\frac{z-\mu_{Y^+}}{\beta}}.\end{aligned}$$

Thus, for $z \in \mathbb{R}$, we have

$$\begin{aligned}&\ln\big[(1-F_{Z^+}(z))F_{Z^-}(z)\big] = \\ &\quad = -e^{\frac{z-\mu_{X^+}}{\beta}} - e^{\frac{z-\mu_{Y^+}}{\beta}} - e^{-\frac{z-\mu_{X^-}}{\beta}} - e^{-\frac{z-\mu_{Y^-}}{\beta}} \\ &\quad = \ln\big[(1-F_{X^+}(z))F_{X^-}(z)(1-F_{Y^+}(z))F_{Y^-}(z)\big].\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}[\max(0, Z^+ - Z^-)] = \int_{\mathbb{R}} (1-F_{Z^+})(z)F_{Z^-}(z)\,dz \\ &\quad = \int_{\mathbb{R}} (1-F_{X^+}(z))F_{X^-}(z)(1-F_{Y^+}(z))F_{Y^-}(z)\,dz.\end{aligned}$$