

## Random Probabilistic Circuits (Supplementary material)

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### A AN EXAMPLE OF DETERMINISTIC AND STRUCTURED DECOMPOSABLE XPC

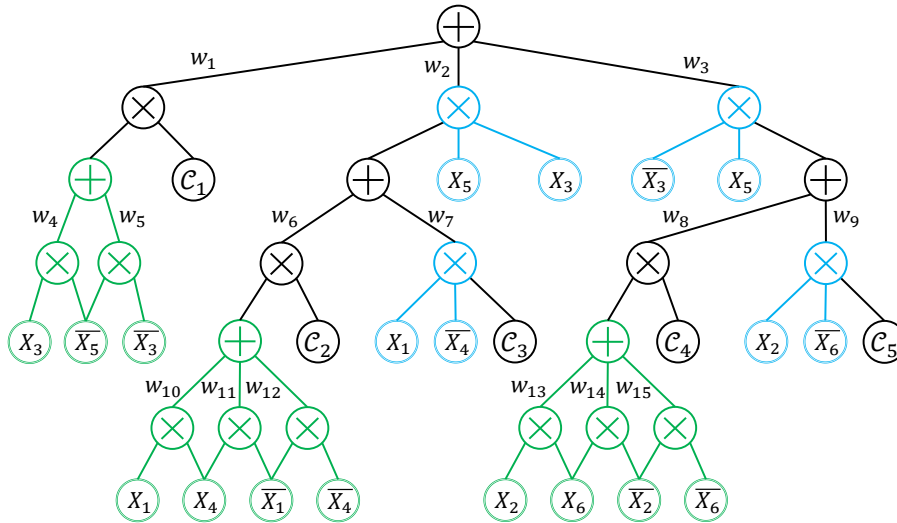


Figure 2: A deterministic and non-SD XPC learned setting  $l = 2$  and  $k = 3$  over RVs  $\mathbf{X} = \{X_i\}_{i=1}^7$ . The sub-circuits in blue (resp., green) encode  $\mathcal{Q}$ -regions (resp.,  $\mathcal{R}$ -regions). Every CLT (represented as a multivariate node  $\mathcal{C}_i$ ) is learned by following the traditional Chow-Liu tree algorithm [Chow and Liu, 1968].

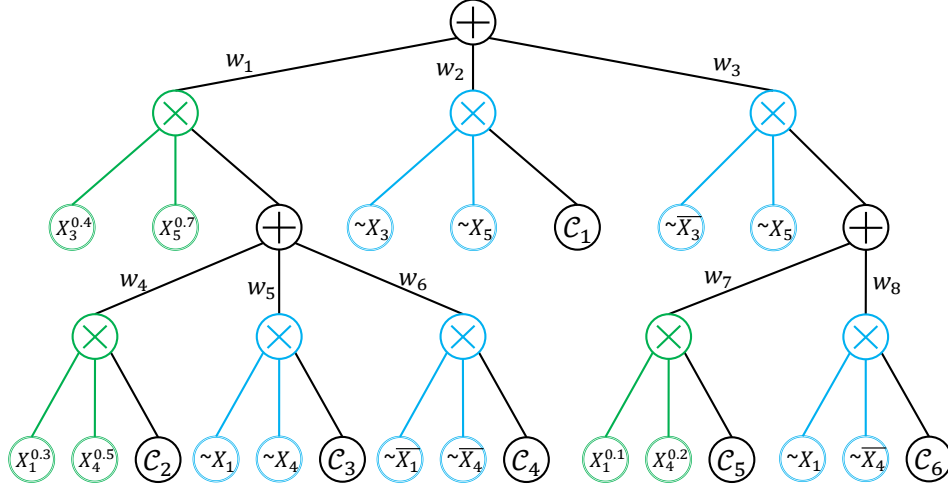


Figure 3: A non-deterministic and SD XPC learned setting  $l = 2$  and  $k = 3$  over RVs  $\mathbf{X} = \{X_i\}_{i=1}^7$ . The sub-circuits in blue (resp., green) encode  $\mathcal{Q}$ -regions (resp.,  $\mathcal{R}$ -regions).  $\sim X_i$  denotes the smoothed indicator of  $X_i$  and  $X_i^p$  denotes a Bernoulli node of  $X_i$  with parameter  $p$ . The CLTs (represented as multivariate nodes  $\mathcal{C}_i$ ) have the same dependency tree—are normalized for the same vtree.

## B PROOF OF THEOREM 1.

We recall from [Chow and Liu, 1968] that the best first-order dependency tree  $\widehat{\mathcal{T}}$  over a set of RVs  $\mathbf{X} = \{X_i\}_{i=1}^d$  is:

$$\widehat{\mathcal{T}} = \arg \max_{\mathcal{T}} \sum_{i=1}^d \text{MI}(X_i, X_{\tau(i)}), \quad (1)$$

where  $X_{\tau(i)}$  is the parent of  $X_i$  in  $\mathcal{T}$ . Let  $\{p_k\}_{k=1}^n$  be probability distributions on  $\mathbf{X}$  and  $\{q_k\}_{k=1}^n$  their corresponding approximations based on a same first-order dependency tree  $\mathcal{T}$ . We want to find  $\widehat{\mathcal{T}}$  such that:

$$\widehat{\mathcal{T}} = \arg \min_{\mathcal{T}} \sum_{k=1}^n \mathbb{KL}(p_k || q_k) = \arg \max_{\mathcal{T}} \sum_{k=1}^n \mathbb{E}_{\mathbf{x} \sim p_k} [\log q_k(\mathbf{x})], \quad (2)$$

where  $\mathbb{KL}$  denotes the Kullback-Leibler divergence. Thus considering a dependency tree  $\mathcal{T}$ :

$$\sum_{k=1}^n \mathbb{E}_{\mathbf{x} \sim p_k} [\log q_k(\mathbf{x})] = \sum_{k=1}^n \mathbb{E}_{\mathbf{x} \sim p_k} \left[ \sum_{i=1}^d \log p_k(X_i | X_{\tau(i)}) \right] \quad (3)$$

$$= \sum_{k=1}^n \left( \sum_{i=1}^d \text{MI}_k(X_i; X_{\tau(i)}) + \mathbb{E}_{\mathbf{x} \sim p_k} \left[ \sum_{i=1}^d \log p_k(X_i) \right] \right), \quad (4)$$

where  $\text{MI}_k(X, Y)$  is the mutual information of  $X$  and  $Y$  on  $p_k$ . Therefore:

$$\widehat{\mathcal{T}} = \arg \min_{\mathcal{T}} \sum_{k=1}^n \mathbb{KL}(p_k || q_k) = \arg \max_{\mathcal{T}} \sum_{k=1}^n \sum_{i=1}^d \text{MI}_k(X_i; X_{\tau(i)}). \quad (5)$$

## C TWENTY DENSITY ESTIMATION DATASETS

We report in Table 3 the adopted 20 real-world density estimation datasets [Haaren and Davis, 2012] and their statistics, already used to evaluate different tractable density estimators.

Table 3: Standard benchmark datasets and their statistics.

DATASET	RVS	TRAIN	VALID	TEST
NLTCS	16	16181	2157	3236
MSNBC	17	291326	38843	58265
KDD	64	180092	19907	34955
PLANTS	69	17412	2321	3482
AUDIO	100	15000	2000	3000
JESTER	100	9000	1000	4116
NETFLIX	100	15000	2000	3000
ACCIDENTS	111	12758	1700	2551
RETAIL	135	22041	2938	4408
PUMSB-STAR	163	12262	1635	2452
DNA	180	1600	400	1186
KOSAREK	190	33375	4450	6675
MSWEB	294	29441	3270	5000
BOOK	500	8700	1159	1739
EACHMOVIE	500	4525	1002	591
WEBKB	839	2803	558	838
ROUTERS-52	889	6532	1028	1540
20NEWS-GRP	910	11293	3764	3764
BBC	1058	1670	225	330
AD	1556	2461	327	491

## D SINGLE MODELS

### D.1 DENSITY ESTIMATION PERFORMANCES

We compare all the variants of XPCs. The average test-set log-likelihood is reported in Table 4

Table 4: Means and standard deviations of the best average test-set log-likelihoods over 40 runs for each variant of XPCs type in the space formed by  $l = \{1, 2, 3\}$ ,  $k = \{2, 3, 4, 8\}$  and  $\delta = \{16, 32, 64, 128, 256, 512\}$ . For every dataset, each row corresponds (from top to down) to XPC, XPC<sub>Det</sub>, XPC<sup>SD</sup> and XPC<sub>Det</sub><sup>SD</sup>. For each model (row) we report the best value in bold.

	$l = 1$ $k = 2$	$l = 2$ $k = 2$	$l = 2$ $k = 3$	$l = 2$ $k = 4$	$l = 3$ $k = 2$	$l = 3$ $k = 3$	$l = 3$ $k = 4$	$l = 3$ $k = 8$
NLTCS	<b>-6.10±0.03</b>	-7.03±0.36	-6.35±0.23	-6.14±0.02	-7.75±0.57	-6.92±0.41	-6.55±0.33	-6.15±0.02
	<b>-6.10±0.03</b>	-6.76±0.23	-6.26±0.13	-6.11±0.02	-7.16±0.31	-6.59±0.21	-6.37±0.17	-6.12±0.02
	<b>-6.09±0.00</b>	-7.03±0.58	-6.42±0.26	-6.12±0.01	-7.57±0.84	-6.69±0.35	-6.36±0.06	-6.14±0.02
	<b>-6.09±0.00</b>	-6.70±0.31	-6.27±0.13	-6.10±0.00	-6.77±0.31	-6.40±0.13	-6.27±0.03	-6.10±0.01
MSNBC	<b>-6.18±0.04</b>	-6.68±0.06	-6.38±0.07	-6.21±0.03	-6.78±0.06	-6.70±0.07	-6.54±0.10	-6.22±0.03
	<b>-6.18±0.04</b>	-6.51±0.06	-6.29±0.06	-6.19±0.03	-6.62±0.05	-6.49±0.08	-6.36±0.08	-6.19±0.03
	<b>-6.21±0.02</b>	-6.72±0.08	-6.45±0.10	-6.25±0.02	-6.75±0.07	-6.73±0.09	-6.62±0.08	-6.26±0.04
	<b>-6.21±0.02</b>	-6.50±0.06	-6.33±0.06	-6.23±0.02	-6.57±0.04	-6.49±0.06	-6.40±0.06	-6.24±0.04

KDD	<b>-2.21±0.01</b>	-2.23±0.02	<b>-2.21±0.01</b>	<b>-2.21±0.00</b>	-2.24±0.02	-2.22±0.01	<b>-2.21±0.01</b>	<b>-2.21±0.01</b>
	-2.22±0.01	-2.23±0.02	-2.22±0.01	-2.22±0.01	-2.23±0.02	<b>-2.21±0.01</b>	<b>-2.21±0.01</b>	-2.22±0.01
	-2.20±0.00	-2.24±0.02	-2.21±0.01	<b>-2.19±0.00</b>	-2.28±0.04	-2.24±0.03	-2.21±0.01	-2.20±0.00
	<b>-2.20±0.00</b>	-2.24±0.02	-2.21±0.01	<b>-2.20±0.00</b>	-2.26±0.03	-2.22±0.02	<b>-2.20±0.01</b>	<b>-2.20±0.00</b>
PLANTS	-13.83±0.15	-15.15±0.63	-14.02±0.26	-13.67±0.17	-16.50±0.90	-14.98±0.60	-14.33±0.49	<b>-13.56±0.09</b>
	-14.11±0.14	-15.26±0.34	-14.26±0.21	-13.97±0.09	-16.22±0.71	-14.94±0.45	-14.41±0.26	<b>-13.96±0.13</b>
	-14.93±0.00	-15.55±0.59	-14.75±0.25	<b>-14.55±0.04</b>	-15.55±0.81	-14.91±0.31	-14.67±0.07	-14.52±0.05
	-14.93±0.01	-15.37±0.47	-14.82±0.16	-14.67±0.03	-15.25±0.34	-14.86±0.15	-14.73±0.05	<b>-14.59±0.05</b>
AUDIO	-42.54±0.10	-42.36±0.24	-42.30±0.21	-42.38±0.12	-42.98±0.44	-42.33±0.29	<b>-42.20±0.20</b>	-42.32±0.13
	<b>-42.63±0.11</b>	-43.05±0.24	-42.78±0.13	-42.67±0.15	-43.44±0.35	-42.97±0.22	-42.85±0.25	-42.65±0.14
	<b>-41.96±0.00</b>	-42.31±0.47	-42.13±0.16	-42.08±0.03	-43.07±0.71	-42.58±0.38	-42.22±0.26	-42.11±0.03
	<b>-41.97±0.00</b>	-42.67±0.44	-42.29±0.16	-42.16±0.03	-43.00±0.51	-42.73±0.31	-42.49±0.24	-42.25±0.03
JESTER	-55.94±0.21	-55.52±0.34	-55.62±0.23	-55.73±0.14	-56.26±0.49	-55.65±0.34	<b>-55.47±0.29</b>	-55.61±0.21
	-56.09±0.22	-56.59±0.38	-56.25±0.25	-56.06±0.15	-56.85±0.45	-56.37±0.37	-56.19±0.33	<b>-56.00±0.23</b>
	-54.89±0.00	-55.15±0.52	-54.86±0.22	-54.84±0.02	-56.11±0.69	-55.20±0.32	-54.95±0.33	<b>-54.67±0.00</b>
	<b>-54.94±0.00</b>	-55.72±0.45	-55.16±0.23	-55.01±0.02	-56.12±0.52	-55.59±0.29	-55.42±0.31	-54.93±0.01
NETFLIX	-59.28±0.10	-59.23±0.20	-59.22±0.13	<b>-59.21±0.13</b>	-60.08±0.31	-59.44±0.19	-59.22±0.17	-59.22±0.13
	-59.29±0.10	-59.79±0.17	-59.43±0.15	<b>-59.28±0.13</b>	-60.26±0.24	-59.82±0.14	-59.57±0.16	-59.33±0.12
	<b>-58.72±0.00</b>	-59.17±0.51	-58.88±0.15	-58.79±0.02	-60.29±0.72	-59.27±0.27	-58.99±0.31	-58.87±0.01
	<b>-58.73±0.00</b>	-59.37±0.36	-58.99±0.17	-58.84±0.01	-59.85±0.41	-59.34±0.20	-59.17±0.21	-58.97±0.01
ACCIDENTS	-31.90±0.27	-32.76±0.38	-32.03±0.34	<b>-31.84±0.24</b>	-33.83±0.63	-32.77±0.40	-32.31±0.34	-31.92±0.27
	<b>-31.88±0.37</b>	-32.76±0.36	-32.09±0.34	-31.91±0.25	-33.77±0.56	-32.72±0.35	-32.32±0.33	-32.03±0.28
	<b>-31.01±0.00</b>	-32.02±0.37	-31.35±0.01	-31.35±0.01	-34.40±1.02	-32.06±0.28	-31.38±0.22	-31.17±0.21
	<b>-31.03±0.00</b>	-31.88±0.24	-31.35±0.01	-31.35±0.01	-32.78±0.44	-31.73±0.13	-31.30±0.15	-31.15±0.08
RETAIL	<b>-10.95±0.00</b>	-10.96±0.02	-10.96±0.02	-10.96±0.02	-11.01±0.04	-11.02±0.04	-11.02±0.03	-11.02±0.03
	<b>-10.95±0.00</b>	-10.96±0.01	-10.96±0.01	-10.96±0.01	-10.99±0.04	-11.00±0.03	-11.00±0.03	-11.00±0.03
	<b>-10.97±0.00</b>	-11.17±0.09	-11.04±0.07	-10.98±0.00	-11.24±0.09	-11.11±0.07	-11.06±0.03	-11.01±0.01
	<b>-10.98±0.00</b>	-11.06±0.09	-10.99±0.01	-10.99±0.00	-11.07±0.07	-11.02±0.05	-11.00±0.01	-11.00±0.01
PUMSB	-25.95±0.48	-27.32±0.76	-26.00±0.47	-25.69±0.37	-28.37±0.74	-27.03±0.65	-26.18±0.57	<b>-25.62±0.27</b>
	-26.14±0.52	-27.42±0.88	-26.18±0.46	<b>-25.90±0.41</b>	-28.29±0.68	-27.04±0.62	-26.27±0.58	-25.92±0.37
	-27.06±0.00	-27.19±0.06	-27.19±0.06	-27.19±0.06	-26.65±0.47	<b>-26.55±0.50</b>	<b>-26.55±0.50</b>	<b>-26.55±0.50</b>
	-27.06±0.00	-27.19±0.06	-27.19±0.06	-27.19±0.06	-26.66±0.48	<b>-26.56±0.51</b>	<b>-26.56±0.51</b>	<b>-26.56±0.51</b>
DNA	<b>-87.74±0.12</b>	-87.79±0.11	-87.79±0.11	-87.79±0.11	-87.92±0.11	-87.92±0.11	-87.92±0.11	-87.92±0.11
	<b>-87.75±0.12</b>	-87.79±0.08	-87.79±0.08	-87.79±0.08	-87.95±0.10	-87.95±0.10	-87.95±0.10	-87.95±0.10
	<b>-87.46±0.00</b>	-87.53±0.00	-87.53±0.00	-87.53±0.00	-87.54±0.00	-87.54±0.00	-87.54±0.00	-87.54±0.00
	<b>-87.46±0.00</b>	-87.53±0.00	-87.53±0.00	-87.53±0.00	-87.47±0.00	-87.47±0.00	-87.47±0.00	-87.47±0.00
KOSAREK	-11.25±0.07	-11.11±0.06	-11.09±0.09	-11.12±0.08	-11.08±0.08	<b>-11.03±0.05</b>	-11.04±0.06	-11.04±0.06
	-11.38±0.08	-11.35±0.10	-11.31±0.08	-11.34±0.08	-11.34±0.10	<b>-11.29±0.08</b>	<b>-11.29±0.09</b>	<b>-11.29±0.09</b>
	-10.97±0.00	-11.23±0.22	-11.05±0.16	<b>-10.92±0.02</b>	-11.40±0.15	-11.16±0.14	-11.04±0.06	-10.96±0.01
	-11.01±0.00	-11.25±0.14	-11.06±0.09	<b>-10.99±0.01</b>	-11.42±0.14	-11.20±0.14	-11.08±0.05	-11.03±0.00
MSWEB	<b>-10.19±0.02</b>	<b>-10.19±0.02</b>	<b>-10.19±0.02</b>	<b>-10.19±0.02</b>	<b>-10.19±0.02</b>	<b>-10.19±0.02</b>	<b>-10.19±0.02</b>	<b>-10.19±0.02</b>
	<b>-10.19±0.01</b>	<b>-10.19±0.02</b>	<b>-10.19±0.02</b>	<b>-10.19±0.02</b>	-10.20±0.02	-10.20±0.02	-10.20±0.03	-10.20±0.03
	<b>-10.12±0.00</b>	-10.43±0.13	-10.13±0.03	<b>-10.12±0.01</b>	-10.65±0.14	-10.44±0.13	-10.27±0.11	-10.13±0.02
	<b>-10.12±0.00</b>	-10.31±0.07	-10.15±0.02	-10.15±0.00	-10.38±0.06	-10.29±0.06	-10.20±0.05	-10.13±0.01
BOOK	-37.66±0.32	-37.14±0.32	-37.34±0.40	-37.34±0.40	-36.73±0.37	<b>-36.68±0.34</b>	<b>-36.68±0.34</b>	-36.68±0.34
	-37.79±0.19	-37.69±0.21	-37.69±0.21	-37.69±0.21	-37.51±0.29	<b>-37.50±0.21</b>	<b>-37.50±0.21</b>	<b>-37.50±0.21</b>
	-36.35±0.00	-36.00±0.17	-36.00±0.17	-36.00±0.17	<b>-35.90±0.00</b>	<b>-35.90±0.00</b>	<b>-35.90±0.00</b>	<b>-35.90±0.00</b>
	-36.97±0.00	-36.83±0.10	-36.83±0.10	-36.83±0.10	-36.60±0.00	<b>-36.60±0.00</b>	<b>-36.60±0.00</b>	<b>-36.60±0.00</b>

EMOVIE	-62.19±2.13	-60.61±1.69	-60.98±1.87	-61.38±1.99	-59.78±1.95	-59.64±1.92	-59.41±2.01	<b>-59.20±1.21</b>
	-63.91±1.24	-63.37±1.41	-63.46±1.31	-63.46±1.31	-62.86±1.54	<b>-62.62±1.58</b>	-62.72±1.30	-62.74±1.31
	-59.07±0.01	-58.44±0.54	-57.94±0.39	-57.80±0.15	-57.90±0.89	-58.06±0.20	<b>-57.71±0.57</b>	-57.98±0.00
	-60.56±0.00	-60.86±0.24	-60.49±0.06	-60.49±0.06	<b>-59.99±0.65</b>	-60.47±0.22	-60.35±0.00	-60.35±0.00
WEBKB	-163.31±0.35	-162.82±0.88	-162.88±0.88	-162.88±0.88	<b>-162.42±1.10</b>	-162.53±0.92	-162.44±1.00	-162.44±1.00
	-163.43±0.23	<b>-163.33±0.29</b>	<b>-163.33±0.29</b>	<b>-163.33±0.29</b>	-163.33±0.34	-163.33±0.34	-163.33±0.34	-163.33±0.34
	-159.56±0.00	-158.44±0.35	-159.11±0.41	-159.11±0.41	<b>-158.30±0.09</b>	-158.52±0.63	-158.48±0.00	-158.48±0.00
	-161.30±0.00	-161.19±0.22	-161.91±0.09	-161.91±0.09	<b>-160.93±0.29</b>	-161.26±0.00	-161.26±0.00	-161.26±0.00
ROUTERS	-94.05±0.69	-93.45±1.15	-93.69±1.13	-93.71±1.11	<b>-93.00±1.34</b>	-93.13±0.80	-93.12±0.84	-93.12±0.84
	-94.32±0.33	-94.32±0.51	<b>-94.29±0.37</b>	<b>-94.29±0.37</b>	-94.29±0.48	-94.29±0.48	-94.29±0.48	-94.29±0.48
	-88.13±0.00	-87.94±0.52	-87.95±0.45	-87.95±0.45	-88.89±0.80	-88.18±0.61	-87.91±0.52	<b>-87.91±0.29</b>
	<b>-88.92±0.00</b>	-89.71±0.58	-89.33±0.42	-89.33±0.42	-90.66±0.55	-90.58±0.65	-89.47±0.35	-89.43±0.18
20NEWS	-162.96±1.12	-161.58±0.77	-162.04±0.96	-162.04±0.96	<b>-160.73±0.56</b>	-161.06±0.67	-161.06±0.63	-161.06±0.63
	-164.26±0.61	<b>-163.81±0.53</b>	-163.95±0.44	-163.95±0.44	-163.86±0.84	-163.95±0.85	-164.03±0.87	-164.03±0.87
	-158.38±0.00	<b>-157.31±0.58</b>	-157.55±0.43	-157.44±0.16	-157.79±0.80	-157.86±0.41	-157.71±0.52	-158.16±0.01
	-159.58±0.00	-160.61±1.07	-159.73±0.46	<b>-159.37±0.17</b>	-161.12±0.65	-160.68±0.44	-160.41±0.42	-160.20±0.01
BBC	-261.88±0.34	<b>-261.81±0.28</b>	<b>-261.81±0.28</b>	<b>-261.81±0.28</b>	-261.96±0.48	-261.96±0.48	-261.96±0.48	-261.96±0.48
	-262.04±0.77	<b>-261.96±0.58</b>	<b>-261.96±0.58</b>	<b>-261.96±0.58</b>	-262.24±0.98	-262.24±0.98	-262.24±0.98	-262.24±0.98
	-262.33±0.00	-260.52±0.00	-260.52±0.00	-260.52±0.00	-259.45±1.26	<b>-258.79±0.00</b>	<b>-258.79±0.00</b>	<b>-258.79±0.00</b>
	-262.91±0.00	-261.41±0.00	-261.41±0.00	-261.41±0.00	<b>-260.62±0.00</b>	<b>-260.62±0.00</b>	<b>-260.62±0.00</b>	<b>-260.62±0.00</b>
AD	-16.40±0.04	<b>-16.38±0.11</b>	<b>-16.38±0.11</b>	<b>-16.38±0.11</b>	-16.40±0.12	-16.40±0.12	-16.40±0.12	-16.40±0.12
	-16.62±0.13	<b>-16.61±0.13</b>	<b>-16.61±0.13</b>	<b>-16.61±0.13</b>	-16.66±0.26	-16.66±0.26	-16.66±0.26	-16.66±0.26
	-16.02±0.00	-16.02±0.00	-16.02±0.00	-16.02±0.00	-16.26±0.10	-16.26±0.10	-16.05±0.20	<b>-15.97±0.08</b>
	<b>-16.28±0.00</b>	<b>-16.28±0.00</b>	<b>-16.28±0.00</b>	<b>-16.28±0.00</b>	-16.39±0.07	-16.39±0.07	-16.39±0.07	-16.39±0.07

## D.2 TRAINING TIME AND CIRCUIT SIZES

The training time and the circuit size for our models are reported in Table 5.

Table 5: Each row reports  $\delta$ -values/training times in seconds (top) and circuit sizes (bottom) obtained for learning for the best XPC models reported in Table 4.

DATASET	XPC	XPC <sub>Det</sub>	XPC <sup>SD</sup>	XPC <sup>SD</sup> <sub>Det</sub>
NLTC5	64 / 0.06	64 / 0.08	64 / 0.05	64 / 0.05
	2765±458	2765±458	4335±67	4401±67
MSNBC	512 / 0.36	512 / 0.71	512 / 0.31	512 / 0.31
	3575±546	3575±546	4887±125	4887±125
KDD	64 / 1.29	128 / 2.09	128 / 0.64	128 / 1.02
	21331±1591	16089±1312	12951±109	13040±109
PLANTS	64 / 0.25	128 / 0.18	64 / 0.12	128 / 0.1
	31403±1301	19138±2135	22242±532	13960±644
AUDIO	256 / 0.18	512 / 0.12	128 / 0.17	128 / 0.17
	20897±1232	11396±740	29275±42	29317±42
JESTER	512 / 0.06	256 / 0.11	128 / 0.13	128 / 0.11
	6530±722	12584±870	25351±69	20273±29
NETFLIX	256 / 0.17	512 / 0.12	128 / 0.21	128 / 0.21
	20239±1655	12510±976	39811±57	39868±57
ACCIDENTS	256 / 0.15	256 / 0.09	128 / 0.09	128 / 0.09
	16252±2742	8994±2417	11906±15	11921±15
RETAIL	512 / 0.04	512 / 0.04	512 / 0.09	512 / 0.09
	1098±416	1098±416	6645±7	6651±7
PUMSB-STAR	128 / 0.45	128 / 0.35	128 / 0.1	128 / 0.12
	43737±6494	32356±7266	8703±1442	8866±1480
DNA	256 / 0.04	256 / 0.04	512 / 0.02	512 / 0.02
	3927±888	3927±888	2615±1	2616±1
KOSAREK	256 / 0.28	256 / 0.29	256 / 0.27	256 / 0.3
	13730±3381	13617±2872	20898±34	20938±34
MSWEB	512 / 0.13	512 / 0.13	512 / 0.27	512 / 0.27
	2298±714	2298±714	12130±5	12135±5
BOOK	256 / 0.37	256 / 0.35	256 / 0.39	256 / 0.38
	11610±4205	11227±4306	13505±4	13678±28
EACHMOVIE	128 / 0.53	256 / 0.31	64 / 1.58	256 / 0.5
	19913±8846	11375±5785	80294±9009	21369±3124
WEBKB	256 / 0.89	512 / 0.42	128 / 2.59	512 / 0.87
	17026±6503	7719±3334	61915±13804	17122±2
ROUTERS-52	128 / 2.88	512 / 0.65	256 / 2.91	512 / 1.52
	55112±18036	10479±5049	67594±2067	36440±5
20NEWS-GRP	512 / 2.11	512 / 2.2	512 / 3.28	512 / 3.34
	37138±10983	38557±10433	65838±2880	65881±2884
BBC	512 / 0.49	512 / 0.51	128 / 1.86	256 / 0.77
	7755±1930	7755±1932	36578±6	14578±1
AD	64 / 0.94	64 / 1.01	16 / 21.59	128 / 1.69
	11661±2170	12368±3157	298553±10013	22093±20

## E MIXTURE MODELS

### E.1 EXPCS

The best average test-set log-likelihoods for every EXPC type are reported in Table 6.

Table 6: Best average test-set log-likelihoods for every EXPC type in the space formed by  $l = \{1, 2, 3\}$ ,  $k = \{2, 3, 4, 8\}$ ,  $\delta = \{16, 32, 64, 128, 256, 512\}$  and  $M = \{2, 5, 10, 15, 20, 25, 30, 40\}$ . For every dataset, each row corresponds (from top to down) to: EXPC, EXPC<sub>Det</sub>, EXPC<sup>SD</sup> and EXPC<sub>Det</sub><sup>SD</sup>.

	$l = 1$		$l = 2$		$l = 3$			
	$k = 2$	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$	$k = 8$
NLTC5	-6.00	-6.30	-6.04	-6.00	-6.54	-6.20	-6.10	<b>-6.01</b>
	<b>-6.00</b>	-6.26	-6.04	-6.00	-6.48	-6.18	-6.09	-6.01
	-	-6.43	-6.09	<b>-6.05</b>	-6.50	-6.29	-6.15	-6.10
MSNBC	-	-6.38	-6.08	<b>-6.05</b>	-6.42	-6.23	-6.11	-6.08
	<b>-6.12</b>	-6.40	-6.19	-6.12	-6.56	-6.38	-6.25	-6.12
	<b>-6.12</b>	-6.35	-6.17	-6.12	-6.46	-6.33	-6.21	-6.12
KDD	-	-6.43	-6.22	<b>-6.18</b>	-6.54	-6.44	-6.33	-6.19
	-	-6.38	-6.20	<b>-6.17</b>	-6.48	-6.38	-6.29	-6.19
	-2.13	-2.14	-2.13	<b>-2.13</b>	-2.13	-2.13	-2.13	-2.13
PLANTS	-2.13	-2.13	-2.13	<b>-2.14</b>	-2.13	-2.13	-2.13	-2.13
	-	-2.15	-2.17	-2.19	-2.17	<b>-2.15</b>	-2.16	-2.18
	-	-2.15	-2.17	-2.20	-2.17	<b>-2.16</b>	-2.16	-2.19
AUDIO	-12.41	-13.14	-12.53	-12.35	-13.91	-13.03	-12.59	<b>-12.24</b>
	-12.41	-13.11	-12.51	-12.35	-13.78	-12.96	-12.57	<b>-12.24</b>
	-	-14.73	<b>-14.19</b>	-14.55	-14.82	-14.45	-14.39	-14.46
JESTER	-	-14.61	<b>-14.21</b>	-14.62	-14.76	-14.46	-14.43	-14.50
	-39.84	-40.03	-39.80	-39.78	-40.46	-40.06	-39.89	<b>-39.75</b>
	-39.84	-40.12	-39.86	-39.80	-40.53	-40.14	-39.97	<b>-39.77</b>
NETFLIX	-	-40.96	-40.96	-41.74	-41.66	<b>-40.91</b>	-40.96	-41.91
	-	-41.00	-41.05	-41.76	-41.65	<b>-40.97</b>	-41.07	-42.01
	-52.72	-52.89	-52.65	-52.67	-53.27	-52.89	-52.73	<b>-52.60</b>
RETAILACCIDENTS	-52.72	-52.96	-52.75	-52.68	-53.35	-52.97	-52.84	<b>-52.63</b>
	-	<b>-53.43</b>	-53.52	-54.59	-54.01	-53.49	-53.41	-54.68
	-	<b>-53.51</b>	-53.68	-54.63	-54.03	-53.61	-53.59	-54.76
PUMSB	-56.43	-56.79	-56.50	<b>-56.38</b>	-57.39	-56.77	-56.64	-56.41
	-56.43	-56.93	-56.58	<b>-56.40</b>	-57.41	-56.88	-56.75	-56.43
	-	-57.66	-57.68	-58.46	-58.18	-57.67	<b>-57.58</b>	-58.60
KOSAREK	-	-57.68	-57.80	-58.46	-58.14	-57.73	<b>-57.69</b>	-58.62
	-29.63	-30.63	-29.76	<b>-29.54</b>	-31.33	-30.35	-29.97	-29.55
	-29.63	-30.58	-29.74	-29.53	-31.28	-30.31	-29.93	<b>-29.54</b>
DNA	-	-31.71	-31.36	-31.36	-33.12	-31.45	<b>-31.02</b>	-31.08
	-	-31.60	-31.36	-31.36	-31.99	-31.32	<b>-30.99</b>	-31.08
	-10.83	-10.82	-10.82	-10.82	-10.81	-10.81	<b>-10.81</b>	-10.81
NETFLIX	-10.83	-10.81	-10.81	-10.81	-10.81	<b>-10.80</b>	-10.79	-10.79
	-	-10.94	-10.96	-10.97	-11.00	-10.94	<b>-10.94</b>	-10.96
	-	-10.91	-10.95	-10.98	<b>-10.90</b>	-10.91	-10.95	-10.97
RETAILACCIDENTS	-23.78	-24.87	-23.86	-23.58	-25.61	-24.56	-23.90	<b>-23.55</b>
	-23.78	-24.78	-23.83	-23.57	-25.49	-24.44	-23.85	<b>-23.53</b>
	-	-27.21	-27.21	-27.21	-26.16	<b>-26.06</b>	-26.06	-26.06
KOSAREK	-	-27.21	-27.21	-27.21	-26.16	<b>-26.05</b>	-26.05	-26.05
	<b>-84.83</b>	-85.75	-85.47	-85.47	-86.25	-85.69	-85.62	-85.53
	<b>-84.83</b>	-85.78	-85.48	-85.48	-86.24	-85.73	-85.62	-85.54
NETFLIX	-	-87.25	-87.53	-87.53	-86.72	<b>-86.61</b>	-87.54	-87.54
	-	-86.85	-87.25	-87.25	<b>-85.09</b>	-86.30	-87.47	-87.47
	-10.68	-10.64	-10.66	-10.66	-10.65	-10.63	<b>-10.61</b>	-10.62
NETFLIX	-10.68	-10.66	-10.66	-10.66	-10.67	-10.64	<b>-10.63</b>	-10.63
	-	-10.87	<b>-10.77</b>	-10.89	-11.04	-10.85	-10.81	-10.96
	-	-10.86	<b>-10.81</b>	-10.95	-11.01	-10.87	-10.85	-11.03

MSWEB	-10.09	-9.94	-9.90	-9.90	-9.84	<b>-9.78</b>	-9.78	-9.77
	-10.09	-9.94	-9.90	-9.90	-9.84	<b>-9.78</b>	-9.78	-9.78
	-	-10.09	<b>-9.93</b>	-10.07	-10.24	-10.05	-9.97	-10.05
	-	-10.02	<b>-9.94</b>	-10.12	-10.06	-9.97	-9.93	-10.09
BOOK	-35.72	-35.16	-35.23	-35.23	<b>-34.85</b>	-35.02	-35.02	-35.02
	-35.63	-35.40	-35.40	-35.40	<b>-34.98</b>	-35.16	-35.16	-35.16
	-	-34.86	-35.37	-35.68	<b>-34.75</b>	-35.08	-35.41	-35.90
	-	-35.25	-36.28	-36.28	<b>-35.14</b>	-35.71	-36.60	-36.60
EMOVIE	-54.50	-53.78	-53.78	-53.87	-53.08	-53.23	-53.15	<b>-52.93</b>
	-54.65	-54.06	-54.13	-54.30	-53.35	-53.65	<b>-53.44</b>	-53.45
	-	-55.35	-56.05	-57.27	-55.23	<b>-54.82</b>	-55.42	-57.98
	-	-55.88	-57.39	-60.15	<b>-55.26</b>	-55.70	-56.47	-60.35
WEBKB	-153.97	-153.92	-155.16	-155.23	<b>-154.03</b>	-153.93	-154.17	-154.17
	-154.29	-154.36	-155.33	-155.49	<b>-154.42</b>	-154.49	-154.66	-154.66
	-	-153.62	-157.22	-157.22	<b>-153.67</b>	-154.17	-156.28	-157.89
	-	-154.45	-159.88	-159.88	<b>-154.23</b>	-155.58	-158.22	-161.26
ROUTERS	-85.61	-85.32	-85.71	-85.28	-85.07	<b>-84.82</b>	-85.31	-85.04
	-85.89	-85.96	-86.12	-85.72	-85.53	<b>-85.27</b>	-85.74	-85.65
	-	-85.19	-85.17	-86.48	-85.38	<b>-84.70</b>	-85.02	-86.94
	-	-85.51	-85.93	-87.25	-85.49	<b>-85.09</b>	-85.96	-88.61
20NEWS	-154.48	-153.98	-154.47	-154.35	-153.85	<b>-154.13</b>	-154.45	-154.48
	-154.60	-154.54	-154.80	<b>-154.71</b>	-154.64	-154.91	-155.09	-155.07
	-	-153.78	-155.10	-156.86	<b>-153.75</b>	-154.02	-154.52	-157.26
	-	-154.35	-156.26	-158.63	<b>-154.21</b>	-154.56	-155.49	-160.10
BBC	<b>-237.96</b>	-242.58	-243.68	-243.72	-250.91	-251.07	-251.33	-251.33
	<b>-239.01</b>	-247.14	-246.98	-247.43	-251.35	-251.64	-251.85	-251.85
	-	-248.43	-253.19	-260.52	<b>-248.34</b>	-250.16	-251.50	-258.79
	-	-249.46	-255.95	-261.41	<b>-248.79</b>	-252.21	-255.59	-260.62
AD	-15.90	-14.39	-14.23	-14.23	-13.95	<b>-13.61</b>	-13.60	-13.60
	-15.94	-14.28	-14.14	-14.14	-13.78	<b>-13.45</b>	-13.45	-13.45
	-	-16.03	-16.03	-16.03	-15.66	-15.59	<b>-15.50</b>	-15.67
	-	-16.31	-16.31	-16.31	<b>-15.59</b>	-15.73	-15.68	-15.87



## E.2 COMPETITORS

The average test-set log-likelihoods for best EXPCs are reported in Table 7.

Table 7: Average test-set log-likelihoods for best EXPCs, SOTA ensembling techniques and single model competitors. In the first half of the table the best results for ensembles of bagged (CNet<sup>40</sup>) and boosted (CNet<sup>40</sup><sub>boost</sub>) entropy based CNets taken from [Rahman and Gogate, 2016], ensembles of bagged CNets learned with dCSN and ensemble of extremely randomized CNets as in [Di Mauro et al., 2017]. The remaining columns report the comparison to other models employing more sophisticated models such as ID-SPN [Rooshenas and Lowd, 2014], RAT-SPN [Peharz et al., 2019], and Learn-SPN [Gens and Domingos, 2013].

DATASET	ensembles					competitors		
	EXPC	XCNet <sup>40</sup>	CNet <sup>40</sup>	CNet <sup>40</sup> <sub>boost</sub>	dCSN <sup>40</sup>	RAT	ID-SPN	Learn-SPN
NLTCS	<b>-6.00</b>	<b>-6.00</b>	<b>-6.00</b>	-6.01	<b>-6.00</b>	-6.01	-6.02	-6.11
MSNBC	-6.12	-6.06	-6.08	-6.15	-6.05	<b>-6.04</b>	<b>-6.04</b>	-6.11
KDD	<b>-2.12</b>	-2.13	-2.14	-2.15	-2.15	-2.13	-2.13	-2.18
PLANTS	-12.24	<b>-11.99</b>	-12.32	-12.67	-12.59	-13.44	-12.54	-12.99
AUDIO	<b>-39.75</b>	-39.77	-40.09	-39.84	-40.19	-39.96	-39.79	-40.50
JESTER	<b>-52.51</b>	-52.65	-52.88	-52.82	-52.99	-52.97	-52.86	-75.98
NETFLIX	-56.38	-56.38	-56.55	-56.44	-56.69	-56.85	<b>-56.36</b>	-57.33
ACCIDENTS	-29.54	-29.31	-29.88	-29.45	-29.27	-35.49	-26.98	-30.04
RETAIL	<b>-10.79</b>	-10.93	-10.84	-10.81	-11.17	-10.91	-10.85	-11.04
PUMSB-STAR	-23.53	-23.44	-23.98	-23.46	-23.78	-32.53	-22.41	-24.78
DNA	-84.83	-84.96	-81.07	-85.67	-85.95	-97.23	-81.21	-82.52
KOSAREK	-10.61	-10.72	-10.74	<b>-10.60</b>	-10.97	-10.89	<b>-10.60</b>	-10.99
MSWEB	-9.78	<b>-9.66</b>	-9.77	-9.74	-9.93	-10.12	-9.73	-10.25
BOOK	<b>-34.14</b>	-36.35	-35.55	-34.46	-37.38	-34.68	<b>-34.14</b>	-35.89
EACHMOVIE	-52.38	-51.72	-53.00	-51.53	-54.14	-53.63	<b>-51.51</b>	-52.49
WEBKB	-152.44	-153.01	-153.12	-152.53	-155.47	-157.53	<b>-151.84</b>	-158.21
ROUTERS-52	-84.70	-84.05	-83.71	-83.69	-86.19	-87.37	<b>-83.35</b>	-85.07
20NEWS-GRP	-152.26	-153.89	-156.09	-153.12	-156.46	-152.06	<b>-151.47</b>	-155.93
BBC	-237.96	-238.47	<b>-237.42</b>	-247.01	-248.84	-252.14	-248.93	-250.69
AD	<b>-13.45</b>	-14.20	-15.28	-14.36	-15.55	-48.47	-19.05	-19.73
AVG. RANK	<b>2.00</b>	2.25	3.25	2.85	4.25			
	2.75	3.15	4.3	3.8	5.85	6	<b>2.5</b>	6.9

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