
Lightweight Projective Derivative Codes for Compressed Asynchronous Gradient Descent

Pedro Soto¹ Ilia Ilmer¹ Haibin Guan² Jun Li³

Abstract

Coded distributed computation has become common practice for performing gradient descent on large datasets to mitigate stragglers and other faults. This paper proposes a novel algorithm that *encodes the partial derivatives* themselves and furthermore optimizes the codes by performing *lossy compression* on the derivative codewords by maximizing the information contained in the codewords while minimizing the information between the codewords. The utility of this application of coding theory is a geometrical consequence of the observed fact in optimization research that noise is tolerable, sometimes even helpful, in gradient descent based learning algorithms since it helps avoid overfitting and local minima. This stands in contrast with much current conventional work on distributed coded computation which focuses on recovering all of the data from the workers. A second further contribution is that the *low-weight* nature of the coding scheme allows for *asynchronous gradient updates* since the code can be iteratively decoded; *i.e.*, a worker’s task can immediately be updated into the larger gradient. The directional derivative is always a linear function of the direction vectors; thus, our framework is robust since it can apply linear coding techniques to general machine learning frameworks such as deep neural networks.

1. Motivation

The majority of machine learning problems take the form: find a function h_{w_0, \dots, w_k} in some family of hypothesis functions \mathcal{H} that are parameterized over the w_0, \dots, w_k which

¹Department of Computer Science, The Graduate Center, CUNY, New York, USA ²Icahn School of Medicine at Mount Sinai, New York, USA ³Department of Computer Science, CUNY Queens College & Graduate Center, New York, USA. Correspondence to: Pedro Soto <psoto@gradcenter.cuny.edu>.

best explains the data,

$$D = \begin{array}{c|cccccc} & x_0 & \dots & x_u & y_0 & \dots & y_v \\ \hline D_0 & x_1^0 & \dots & x_u^0 & y_1^0 & \dots & y_v^0 \\ D_1 & x_1^1 & \dots & x_u^1 & y_1^1 & \dots & y_v^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D_N & x_1^{(N)} & \dots & x_u^{(N)} & y_1^{(N)} & \dots & y_v^{(N)} \end{array}, \quad (1)$$

where the D_i are the datapoints, the x_i are the input features, and the y_i are the output features.

If the h_{w_0, \dots, w_k} are smoothly parameterized by the w_0, \dots, w_k , then this is usually accomplished by performing gradient descent on the summation of loss functions of the form $l_i(w) = l(h_w(x^{(i)}), y^{(i)})$ to compute

$$\begin{aligned} \min_{w \in \mathcal{W}} \mathcal{L}(D, w) &\stackrel{\text{def}}{=} \min_{w \in \mathcal{W}} \sum_{D_i} l(h_w(x^{(i)}), y^{(i)}) = \\ &= \min_{w \in \mathcal{W}} \sum_{D_i} l_i(w); \end{aligned} \quad (2)$$

i.e., find the w that best fits D . For example, one of the most common loss functions, $l(f, y) = \frac{1}{2} \|f - y\|^2$, gives us the mean squared error and the ubiquitous method of least squares.

If the dataset D has many datapoints D_i then the overall computation, or *job*, is distributed as *tasks* amongst *workers*, which model a distributed network of computing devices. This solution creates a new problem; stragglers and other faults can severely impact the performance and overall training time. An emerging technique is to use distributed coded computation to mitigate stragglers and other failures in the network. Many of the current algorithms only encode the data; this paper proposes further encoding the directional derivatives as well in such a way that allows for asynchronous gradient updates using low weight codes. Furthermore the number of weights usually grow quite large as well¹, which necessitates a “2D” coding scheme which codes both the data and the derivatives.

¹As a matter of fact it grows proportionately with the number of features or *dimension of the dataset*

1.1. Related Work

The two algorithms which we use to benchmark our algorithm are Gradient Coding (Tandon et al., 2017) and K -Asynchronous Gradient Descent (Dutta et al., 2018b; 2021); however, many of the design of our coding scheme is also influenced by the works in (Lee et al., 2018), (Yu et al., 2017), and (Dutta et al., 2020).

1.1.1. GRADIENT CODING

In the gradient coding Gradient Coding (Tandon et al., 2017) scheme the main idea is to encode the derivatives with respects to the data partitions $\frac{\partial}{\partial D_i}$ from Eq. (1); since the loss function in Eq. (2) splits up into a *sum* of smaller loss functions, l_i , in terms of the partitions, D_i , linear codes can be efficiently applied to the gradients $\frac{\partial}{\partial D_i}$. This work has gone on to spawn many works (Atallah & Rahnvard, 2018; Charles & Papailiopoulos, 2018; Ye & Abbe, 2018; Halbawi et al., 2018; Ozfatura et al., 2019b; Karakus et al., 2019; Ozfatura et al., 2019a;c; Horii et al., 2019; Raviv et al., 2020; Reisizadeh et al., 2019a; Maity et al., 2019; Bitar et al., 2019; Amiri & Gündüz, 2019; Reisizadeh et al., 2019b; Sasi et al., 2020; Wang et al., 2021; Ozfatura et al., 2020; Bitar et al., 2020; Zhang & Simeone, 2021); gradient coding is currently a vibrant topic of research. The main improvements of our coding scheme over the state of the art in Gradient Coding is that our code: can perform asynchronous coded updates, allows the backpropagation itself to be coded (which greatly reduces the communication complexity for high dimensional data), our code has 0 encoding and decoding overhead in terms of multiplications, and has an overall reduction in the redundancy of data/memory overhead.

1.1.2. ASYNCHRONOUS GRADIENT DESCENT

The main idea in Asynchronous Gradient Descent (Ferdinand et al., 2017; Dutta et al., 2018b; Ferdinand & Draper, 2018; Ferdinand et al., 2020; Dutta et al., 2021) is to simply perform a gradient update when whenever a specified number, k , workers have returned. The name ‘‘asynchronous’’ comes from the eponymous concept in distributed computing where communication rounds are not synchronized.

1.1.3. GENERAL CODED DISTRIBUTED FUNCTION COMPUTATION SCHEMES

The main idea in (Lee et al., 2018), (Yu et al., 2017), and (Dutta et al., 2020) is that one can distribute large matrix multiplications amongst workers and encode the smaller block matrix operations. The works initiated much research in distributed coded matrix multiplication (Dutta et al., 2020; Lee et al., 2017; Baharav et al., 2018; Dutta et al., 2018a; Wang et al., 2018; Soto et al., 2019; Dutta et al., 2019; Das & Ramamoorthy, 2019; Hong et al., 2021). Further work has been extended to include batch matrix multiplication as

well (Yu et al., 2019; Jia & Jafar, 2020). The main drawback of these (multi-)linear methods is the non-linear activation functions; in particular, these methods can only encode the linear computations between the layers of a network. Another interesting approach is to attempt to encode the neural network itself (Kosaian et al., 2018; 2019a;b); however, this approach suffers from long training times dues to combinatorial explosion of different fault patterns is there are enough stragglers.

1.2. Contribution

The main contributions of this paper are to introduce a novel coding scheme for gradient descent that: allows for asynchronous gradient updates, maximizes the amount of information contained by random subsets of vectors, minimizes the weight of the code, compresses the gradient in a manner that scales well with the number of nodes, and achieves a lower a communication complexity and memory (storage) overhead with respect to the state of the art. Another improvement of our algorithm over the state of the art is to consider the correct information metric; all of the other coding schemes assume that the Hamming distance is the correct metric, which does not consider the natural (differential) geometry of the gradient. We will show that the correct distance is the one given by the *real projective space*² \mathbb{RP}^n . Furthermore, we will show that our coding scheme, *i.e.*, our choice of coefficients, maximizes the amount of information returned by the workers and furthermore has zero decoding overhead (in terms of multiplications) since the master can just directly add and subtract the results returned by the workers without needing to decode the information.

1.3. Background

We quickly give some important definitions and background from coding theory, information theory, and geometry. In coded distributed computing an *erasure code* is a pair of functions $\mathcal{C} = (\mathcal{E}, \mathcal{D})$ where the workers tasks are given by the encoding procedure

$$\{\tilde{\theta}_0, \dots, \tilde{\theta}_{n-1}\} := \mathcal{E} \{\theta_0, \dots, \theta_k\}$$

and a decoding procedure for some family of fault-tolerant subsets, $\mathcal{F}_{\mathcal{C}}$, such that

$$\{\tilde{\theta}_{i_1}, \dots, \tilde{\theta}_{i_m}\} \in \mathcal{F}_{\mathcal{C}} \implies \mathcal{D} \{\tilde{\theta}_{i_1}, \dots, \tilde{\theta}_{i_m}\} = \{\theta_0, \dots, \theta_k\}.$$

If $\mathcal{F}_{\mathcal{C}}$ consists of all the m -subsets (for some integer r) of $\{\tilde{\theta}_0, \dots, \tilde{\theta}_{n-1}\}$, then \mathcal{C} can correct any $r := n - m$ erasures or stragglers; furthermore, if $r = n - k$ then the code is a *maximum distance separable* (MDS) code. If the encoder \mathcal{E}

² \mathbb{RP}^n is defined as the set of all vectors in \mathbb{R}^{n+1} quotiented by the equivalence relation $v \sim w \iff (\exists \lambda) v = \lambda w$.

is given by a generator matrix \mathcal{G}_C , *i.e.*, if

$$\mathcal{E} [\theta_0 \quad \dots \quad \theta_k]^T = \mathcal{G}_C [\theta_0 \quad \dots \quad \theta_k]^T$$

then \mathcal{C} is called a *linear code*. The *weight* of a linear code is the maximum number of 0's in the rows of the matrix \mathcal{G}_C ; the importance of the weight metric stems from the fact that it measures the amount of work that the workers do since the rows of \mathcal{G}_C are the worker tasks $\tilde{\theta}_i$. Thus, in order to avoid confusion we will use t to denote the weight of the code as well as the number of *tasks* that each worker does; equivalently t is the number of data partitions on the workers. To further simplify notation we abuse notation and use \mathcal{C} in place of \mathcal{G}_C and \mathcal{E}_C when the context is clear.

A potential point of confusion is that the θ *need not be* the weights w of the h_w . This is because *the derivative of loss function \mathcal{L} also implicitly takes the data D_i as an input*; this is an important insight used in all gradient coding algorithms. One of the key insights of this paper is to allow the coded gradient to be linear combinations of both $\frac{\partial}{\partial D_i}$ and the $\frac{\partial}{\partial w_i}$. An important notational convention is that we let the D_i be partitions (or batches) of the data set instead of just datapoints as is common in the gradient coding literature; in particular, D_0, \dots, D_{t-1} denotes a partitioning of the data set into t pieces.

The reason for the name “maximum distance separable code” is that an MDS maximizes the distances between the codewords $\mathcal{E} \{\theta_0, \dots, \theta_k\}$ using the Hamming distance; in particular, maximum distance separable means that the code words $\tilde{\theta} \in \mathcal{C}$ have achieve the maximum $\max_{\mathcal{C}': \text{code on } \Theta} \min_{\tilde{\theta}, \tilde{\theta}' \in \mathcal{C}'} d(\tilde{\theta}, \tilde{\theta}')$ where d is the Hamming distance. There are two problems with this approach: the first is that MDS codes in this context require arbitrarily large amount of work, *i.e.*, they have a large weight, and the second is that the classical *discrete* MDS codes are using the wrong metric. This paper proposes to use the metric given by the projective geometry³ on the space of derivatives. Here we mean maximum distance separable with respect to the distance function $d(\theta, \theta') = \min\{\arccos\langle \tilde{\theta}, \tilde{\theta}' \rangle, \arccos\langle -\tilde{\theta}, \tilde{\theta}' \rangle\}$.

2. General Overview of the Design Principles

Consider the case where there are two derivatives and we wish to create two parity tasks using only summation and subtraction in the encoding procedure. Such a code is given

³See (Kühnel et al., 2006) for the case \mathbb{RP}^2 and Appx. 3 of (Vogtmann et al., 2013) or Thm. 10.2 in ch. 3 of (Suetin et al., 1989) for the more general case \mathbb{CP}^n .

by the following generator matrix

$$\mathcal{C} = \begin{matrix} \Theta_0 \\ \tilde{\Theta}_0 \\ \tilde{\Theta}_1 \\ \tilde{\Theta}_2 \\ \tilde{\Theta}_3 \end{matrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{matrix} \tilde{\theta}_0 \\ \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{matrix} \begin{bmatrix} \theta_0 & \theta_1 \\ 1 & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

which adds fault tolerance to the job $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with

the *parity* tasks $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$. This code has the

serendipitous property of having negligible decoding complexity and negligible communication complexity! For example, if the master receives ∇ in the direction $\tilde{\theta}_3 = \frac{1}{\sqrt{2}}(\theta_0 + \theta_1)$, then the master can decrease both θ_0 and θ_1 by the value returned by \mathcal{W}_2 , *i.e.*, $\tilde{\theta}_2$, if the master receives ∇ in the direction $\tilde{\theta}_2 = \frac{1}{\sqrt{2}}(\theta_0 - \theta_1)$, then the master can decrease θ_0 and increase θ_1 by the value returned by \mathcal{W}_3 , *i.e.*, $\tilde{\theta}_3$. The master need only perform 2 additions/subtractions, and more generally (see Eq. 4) if there are t “sub-tasks” the master only needs to perform t additions/subtractions. The multiplication by $\frac{1}{\sqrt{2}}$ can be subsumed by the learning rate; thus, our code has zero multiplication overhead. Furthermore, this information can be communicated using only one float, since the master knows which direction/worker the derivative was computed from.

2.1. What is the Optimal Choice of Directions?

Looking at Fig. 1, we see that the code \mathcal{G} is MDS in the sense that it maximizes the independence between the vectors. Equivalently ⁴ \mathcal{G} minimizes the confusion between codewords or minimizes the mutual information between codewords; thereby maximizing the entropy or the information content. As we will soon see this has the effect of allowing *lossy low-distortion compression* for larger codes. A second contribution of this paper is to show how to preserve an *approximate* MDS property for larger codes which allows for this form of compression.

In what sense does \mathcal{C} being MDS imply fault tolerance? The following example illustrates one kind of error which the code is immune to:

Consider the case where two workers return the derivatives in the directions $\tilde{\theta} = \left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^T$, $\tilde{\theta}' = \left(\cos\left(\frac{5\pi}{4} - \epsilon\right) \quad \sin\left(\frac{5\pi}{4} - \epsilon\right)\right)^T$. By inspecting the second diagram in Fig. 1, it is easy to see that $\lim_{\epsilon \rightarrow 0} \tilde{\theta}' \rightarrow -\tilde{\theta}$, so that $\lim_{\epsilon \rightarrow 0} \arccos\langle \tilde{\theta}, \tilde{\theta}' \rangle \rightarrow \pi$. We will also show that the

⁴Under the assumption that the data D_i are i.i.d. See (Cover & Thomas, 2006) for why maximal entropy maximizes information sent through a message.

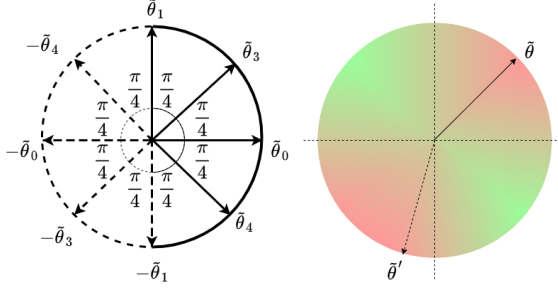


Figure 1. The code \mathcal{C} is MDS with respect to the the distance between the codewords. It is easy to see that $-\tilde{\theta}_i$ carries the same information as $\tilde{\theta}_i$ and it is therefore inefficient to include it since it is a form of replication coding. Likewise we should not include a vector that makes a small angle with $\tilde{\theta}_i$ for the same reason. The geometry of this problem is that of \mathbb{RP}^1 , i.e., the real projective line. This is because the directions $\tilde{\theta}$ and $-\tilde{\theta}$ are information theoretically equivalent. The second figure displays the relationship between $\tilde{\theta}$ and $\tilde{\theta}'$. As $\tilde{\theta}'$ becomes more linearly independent from $\tilde{\theta}$ it begins to carry more novel information that cannot be inferred from $\tilde{\theta}$. At the “greenest” extremes $\tilde{\theta}$ and $\tilde{\theta}'$ become statistically independent which is the maximum entropy configuration. Maximum entropy is equivalent (see Fn. 4) to maximum information about the θ_i .

error in the derivative can get bigger and bigger as $\epsilon \rightarrow 0$.

If worker one computes $\frac{\partial \mathcal{L}}{\partial \theta}$, worker two computes $\frac{\partial \mathcal{L}}{\partial \theta'}$, and $\frac{\partial \mathcal{L}}{\partial \theta_i} = (-1)^i L$, then we have that $\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{\sqrt{2}}(L + (-L)) = 0$ and that $\frac{\partial \mathcal{L}}{\partial \theta'} = \cos\left(\frac{5\pi}{4} - \epsilon\right)L - \sin\left(\frac{5\pi}{4} - \epsilon\right)L = (\cos\left(\frac{5\pi}{4} - \epsilon\right) - \sin\left(\frac{5\pi}{4} - \epsilon\right))L$. Therefore if both ($\epsilon \approx 0$) and ($L \gg 1$), then $\frac{\partial \mathcal{L}}{\partial \theta} \approx \frac{\partial \mathcal{L}}{\partial \theta'} \approx 0$, which is an error; when the master receives the messages from workers one and two she will think she has arrived at a optimal fit since both $\frac{\partial \mathcal{L}}{\partial \theta}$ and $\frac{\partial \mathcal{L}}{\partial \theta'}$ are very small; i.e., the master may halt the algorithm on a terrible fit. Furthermore it is easy to see that $\max_{\epsilon} \frac{\partial \mathcal{L}}{\partial \theta'}$ occurs when $\epsilon = \frac{\pm\pi}{2}$, i.e., when $\arccos\langle \tilde{\theta}, \tilde{\theta}' \rangle = \frac{\pi}{2}$ so that our previous choice is optimal.

2.2. Motivating Example: Base Case of LWP Codes

The last example did not allow us to show the more general *lossy compression* phenomenon that can occur for more general codes. Also we will soon prove that it is impossible to have MDS codes for large dimensions where the workers perform a small amount of work⁵. In a sense, $k = 2$ is a very special case. Therefore before showing the general compression phenomenon, let us show how to *compress the derivative* for $k = 4$.

⁵This follows from a general rule of thumb in coding theory which states that a code cannot simultaneously have a sparse matrix and be MDS; however we will prove it rigorously for our case.

Suppose that we have 8 workers $\tilde{\theta}_i$, loss function $l(h, y) = \frac{1}{2}\|h - y\|^2$, u input features x_i , and space of hypothesis functions

$$\mathcal{H} = \left\{ h_w = (y_1, \dots, y_v) \mid y_i = \frac{e^{w_i^T x}}{1 + \sum_j w_j^T x}, w_i \in \mathbb{R}^2 \right\},$$

where $w_j^T x = w_{j,1}x_1 + \dots + w_{j,u}x_u$; i.e., \mathcal{H} is the space of multinomial logistic regression functions (however, this procedure will work for any feed-forward deep neural network, see Fig. 2). Similarly to the previous design we can give the following directions to the workers

$$\mathcal{C}^{(8,4,2)} = \begin{matrix} & \theta_0 & \theta_1 & \theta_2 & \theta_3 \\ \begin{matrix} \tilde{\theta}_0 \\ \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \\ \tilde{\theta}_4 \\ \tilde{\theta}_5 \\ \tilde{\theta}_6 \\ \tilde{\theta}_7 \end{matrix} & \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \end{matrix}; \quad (3)$$

however, this time we let $\frac{\partial}{\partial \theta_0}$ = “the derivative of the first half of the output nodes with respect to the first half of the dataset”, $\frac{\partial}{\partial \theta_1}$ = “the derivative of the second half of the output nodes with respect to the first half of the dataset”, $\frac{\partial}{\partial \theta_2}$ = “the derivative of the first half of the output nodes with respect to the second half of the dataset”, and $\frac{\partial}{\partial \theta_3}$ = “the derivative of the second half of the output nodes with respect to the second half of the dataset”. We can see that “lossy” part of the lossy compression is that the 4 workers don’t necessarily return the gradient perfectly, but we will later prove that they return a pretty good approximation of it. However, we had a second further lossy compression step to our code; we give workers 5 and 6 the data partitions D_2, D_3 and give workers 7 and 8 the data partitions D_1, D_4 instead of giving these workers all of the partitions as the code in Eq. 3 suggests.

3. General Code Construction

We first show how to construct what we will denote as a $[n, k, t]$ -projective derivative code, or $[n, k, t]$ -code, for $n = 2^m$, $k = 2^l$ and $t = 2^p$, i.e., n, k, t are all powers of 2, and then show how to use “cyclic” and “toroidal” permutations to construct the code for more general k, n ; however, the t is always chosen to be a power of two for reasons that will soon become clear. The parameter n is the number of workers, k the number of derivatives/features, and t is the number of sub-tasks that each worker will perform; i.e., the number of derivatives per worker.

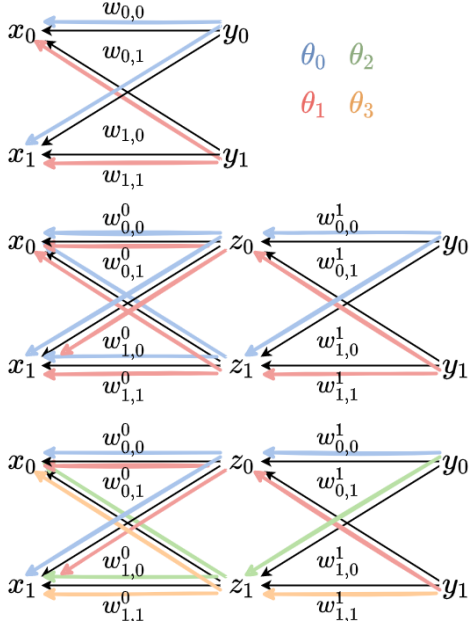


Figure 2. The different ways to partition the backpropagation gradient. The first partition shows how to partition the gradient for a simple neural network with no hidden nodes. The two other partition corresponds to a more general deep-neural network with hidden nodes. The last partition shows how to apply the recursive step in Alg. 2.

3.1. The Characteristic Vectors

To construct the code we first construct *the characteristic vectors* from the following family of functions $\chi_\alpha : \mathbb{F}_2^p \rightarrow \mathbb{C}$, defined by the lambda expression

$$\chi_\alpha : \beta \mapsto \frac{1}{\sqrt{2^p}} e^{\langle \alpha, \beta \rangle \pi i} = \frac{(-1)^{\langle \alpha, \beta \rangle}}{\sqrt{2^p}},$$

where $\alpha, \beta \in \mathbb{F}_2^p$ are defined as binary strings of length t and $\langle \alpha, \beta \rangle$ is the *dot product* on $\alpha, \beta \in \mathbb{F}_2^p$, which is equivalent to taking the *bit-wise AND*⁶ of α and β and then taking the *XOR*⁷ of the result. In particular, $\langle \alpha, \beta \rangle$ is defined as $\langle \alpha, \beta \rangle = (\alpha_0 \wedge \beta_0) \oplus (\alpha_1 \wedge \beta_1) \oplus \dots \oplus (\alpha_{p-1} \wedge \beta_{p-1})$, where \oplus is as defined in fn. 7.

It is an elementary fact of representation theory that the vectors, $\sqrt{2^p} \chi_\alpha$ correspond to the *irreducible representations* of \mathbb{F}_2^p in \mathbb{C} and are therefore an orthogonal basis see Thm. 6 of (Serre, 2012) or Thm. 2.12 of (Fulton & Harris, 1991). One can also prove this fact by direct computation using discrete Fourier analysis see ch. 4 of (Tao & Vu, 2006).

⁶AND is the *logical conjunction*, denoted by \wedge , i.e., $a \wedge b = 1$ if $a = b = 1$ and $a \wedge b = 0$ otherwise. The bit-wise AND of two sequences $\alpha = \alpha_0 \dots \alpha_{p-1}, \beta = \beta_0 \dots \beta_{p-1}$ is $\alpha \wedge \beta = (\alpha_0 \wedge \beta_0)(\alpha_1 \wedge \beta_1) \dots (\alpha_{p-1} \wedge \beta_{p-1})$

⁷XOR is the *exclusive or*, denoted by \oplus , i.e., $a \oplus b = 1$ if $a \neq b$ and $a \oplus b = 0$ if $a = b$.

These functions are well-studied in discrete mathematics and usually referred to as the (additive) characters of \mathbb{F}_2^p .

Let us construct these vectors for $p = 2$ and verify the veracity of these statements for that case. The binary strings of length 2 are $\alpha \in \{00, 01, 10, 11\}$ and this corresponds to the functions

$$X^4 = \begin{matrix} & \theta_0 & \theta_1 & \theta_2 & \theta_3 \\ \chi_{00} & (-2)^{\frac{-p}{2} \langle 00,00 \rangle} & (-2)^{\frac{-p}{2} \langle 00,01 \rangle} & (-2)^{\frac{-p}{2} \langle 00,10 \rangle} & (-2)^{\frac{-p}{2} \langle 00,11 \rangle} \\ \chi_{01} & (-2)^{\frac{-p}{2} \langle 01,00 \rangle} & (-2)^{\frac{-p}{2} \langle 01,01 \rangle} & (-2)^{\frac{-p}{2} \langle 01,10 \rangle} & (-2)^{\frac{-p}{2} \langle 01,11 \rangle} \\ \chi_{10} & (-2)^{\frac{-p}{2} \langle 10,00 \rangle} & (-2)^{\frac{-p}{2} \langle 10,01 \rangle} & (-2)^{\frac{-p}{2} \langle 10,10 \rangle} & (-2)^{\frac{-p}{2} \langle 10,11 \rangle} \\ \chi_{11} & (-2)^{\frac{-p}{2} \langle 11,00 \rangle} & (-2)^{\frac{-p}{2} \langle 11,01 \rangle} & (-2)^{\frac{-p}{2} \langle 11,10 \rangle} & (-2)^{\frac{-p}{2} \langle 11,11 \rangle} \end{matrix}$$

$$= \begin{matrix} & \theta_0 & \theta_1 & \theta_2 & \theta_3 \\ \chi_{00} & 1/2 & 1/2 & 1/2 & 1/2 \\ \chi_{01} & 1/2 & -1/2 & 1/2 & -1/2 \\ \chi_{10} & 1/2 & 1/2 & -1/2 & -1/2 \\ \chi_{11} & 1/2 & -1/2 & -1/2 & 1/2 \end{matrix},$$

and it is straightforward to see that all of the vectors $\chi_{00}, \chi_{01}, \chi_{10},$ and χ_{11} are orthonormal.

3.2. Construction for Powers of Two

If we identify the binary strings α, β with the integers that they represent and let $X^{(t)}$ be the matrix defined coordinate-wise by the equation

$$X_{\alpha, \beta}^{(t)} = \chi_\alpha(\beta) = \frac{1}{\sqrt{t}} e^{\langle \alpha, \beta \rangle \pi i},$$

where $p = \log(t)$ and $\chi_\alpha : \mathbb{F}_2^p \rightarrow \mathbb{C}$ and let $L^{(t)}$ and $R^{(t)}$ be the matrices defined by the equation

$$L^{(2t)} = \begin{bmatrix} 0^{(t)} & \frac{1}{\sqrt{2}} X^{(t)} \\ 0^{(t)} & \frac{1}{\sqrt{2}} X^{(t)} \end{bmatrix}, \quad R^{(2t)} = \begin{bmatrix} \frac{1}{\sqrt{2}} X^{(t)} & 0^{(t)} \\ -\frac{1}{\sqrt{2}} X^{(t)} & 0^{(t)} \end{bmatrix},$$

then we can define $\mathcal{C}^{(2k, k, t)}$, the generator for the $[2k, k, t]$ -code, as

$$\begin{matrix} \theta_0^{(t)} & \theta_1^{(t)} & \theta_2^{(t)} & \dots & \theta_{s-2}^{(t)} & \theta_{s-1}^{(t)} \\ \tilde{\theta}_0^{(t)} & X^{(t)} & 0 & \dots & 0 & 0 \\ \tilde{\theta}_1^{(t)} & 0 & X^{(t)} & \dots & 0 & 0 \\ \tilde{\theta}_2^{(t)} & 0 & 0 & X^{(t)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{\theta}_{s-2}^{(t)} & 0 & 0 & 0 & X^{(t)} & 0 \\ \tilde{\theta}_{s-1}^{(t)} & 0 & 0 & 0 & \dots & X^{(t)} \\ \tilde{\theta}_s^{(t)} & L^{(t)} & R^{(t)} & 0 & \dots & 0 \\ \tilde{\theta}_{s+1}^{(t)} & 0 & L^{(t)} & R^{(t)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{\theta}_{2s-2}^{(t)} & 0 & 0 & 0 & L^{(t)} & R^{(t)} \\ \tilde{\theta}_{2s-1}^{(t)} & R^{(t)} & 0 & 0 & \dots & L^{(t)} \end{matrix}, \quad (4)$$

where⁸ s is the ratio of tasks to sub-tasks, $\theta_i^{(t)}$ is the sequence of sub-tasks θ_{it} through $\theta_{(i+1)t-1}$, and $\tilde{\theta}_i^{(t)}$ is similarly defined as a sequence of t consecutive workers. Equivalently if we define the ‘‘rectangles’’ $\mathcal{R}_{u,v}^{(t)}$ as

$$\mathcal{R}_{u,v}^{(t)} = \{(i, j) \in \mathbb{N}^2 \mid ut \leq i < (u+1)t, vt \leq j < (v+1)t\},$$

then we can define $\mathcal{C}^{(2k,k,t)}$ coordinate-wise as

$$\mathcal{C}_{\theta_i, \tilde{\theta}_j}^{(2k,k,t)} = \begin{cases} X_{i\%t, j\%t}^{(t)} & \text{if } i < k \text{ and } (i, j) \in \mathcal{R}_{\lfloor \frac{i}{t} \rfloor, \lfloor \frac{j}{t} \rfloor}^{(t)} \\ L_{i\%t, j\%t}^{(t)} & \text{if } k \leq i \text{ and } (i, j) \in \mathcal{R}_{\lfloor \frac{i}{t} \rfloor + \frac{k}{t}, \lfloor \frac{j}{t} \rfloor}^{(t)} \\ R_{i\%t, j\%t}^{(t)} & \text{if } (i, j) \in \mathcal{R}_{\frac{2k}{t}-1, 0}^{(t)} \text{ or } k \leq i \\ & \text{and } t \leq j \text{ and } (i, j) \in \mathcal{R}_{\lfloor \frac{i}{t} \rfloor + \frac{k}{t}, \lfloor \frac{j}{t} \rfloor}^{(t)} \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to prove the following beautiful property

Lemma 3.1. The matrices $X^{(t)}$ satisfy the following recursion relation $X^{(2t)} = X^{(2)} \otimes X^{(t)}$.

Proof. This is a direct consequence of Thm. 10 in (Serre, 2012). \square

An alternative is the weaker statement⁹ ‘‘ $X^{(2)}$ is a Hadamard matrix and the tensor product of two Hadamard matrices is a Hadamard matrix’’ whose proof can be found in (Huffman & Pless, 2003).

3.2.1. DATA-&-GRADIENT-PARTITION FOR POWERS OF TWO

Similar to the example given in Sec. 2.2 we give the workers $\tilde{\theta}_i$ the data partition given by Alg. 1. The

Algorithm 1 Data_Partition_Assignment

Input: data D , code_parameters (n, k, t)
 Partition the data D into D_0, \dots, D_{k-1}
 Set $\mathcal{C} := \mathcal{C}^{(n,k,t)}$
for $i \leq n$ **do**
 Data $[\tilde{\theta}_i] := \emptyset$
 for $j \leq k$ **do**
 if $\mathcal{C}_{i,j} \neq 0$ **then**
 Set Data $[\tilde{\theta}_i] := \text{Data}[\tilde{\theta}_i] \cup D_j$
 end if
 end for
end for

idea behind Alg. 1 is simple; we give the first worker

⁸Equivalently, we can write these definitions as $s = \frac{k}{t}$ and $\theta_i^{(t)} = \theta_{it}\theta_{it+1}\dots\theta_{(i+1)t-1}$

⁹Although a weaker statement it suffices to prove the claim of optimally, *i.e.*, Thm. 4.5.

Data $[\tilde{\theta}_0] = D_0, \dots, D_{t-1}$, and the second worker Data $[\tilde{\theta}_1] = D_t, \dots, D_{2t-1}$, and so on up to worker k , at which point we give the workers k, \dots, n a cyclic shift of the previous assignment, *e.g.* worker k gets Data $[\tilde{\theta}_k] = D_{\frac{k}{2}}, \dots, D_{t+\frac{k}{2}}$.

The procedure for partitioning and encoding the gradients, Alg. 2, is slightly more involved; however, the main idea is illustrated in Fig. 2. The main intuition behind Alg. 2 is to

Algorithm 2 Gradient_Partition_Assignment

Input: network x, z^0, \dots, z^m, y , code_parameters (n, k, t)
 Set $\mathcal{C} := \mathcal{C}^{(n,k,t)}$
 Partition y into t groups $y^{(i)}$
 where $y^{(0)} = (y_0, \dots, y_{t-1}); \dots; y^{(t)} = (y_{v-t}, \dots, y_v)$
for $i \leq n$ **do**
 Encode grad $[\tilde{\theta}_i]$ according to row i in \mathcal{C} as in Fig. 2
end for
if network == x, y **then**
 End Procedure
else
 for $i \leq m$ **do**
 Recursively call ‘‘Gradient_Partition_Assignment’’ on the network x, z^0, \dots, z^m parameters (n, k, t) as in Fig. 2 to encode grad $[\tilde{\theta}_i]$ according to row i in \mathcal{C} by repeatedly splitting the (non-zero-)row by t
 end for
end if

encode the gradient in the *manner in which backpropagation occurs*; this allows for the iterative decoding/gradient update at the master node, which in turn allows for asynchronous gradient updating.

3.3. Construction for General Parameters

Given some general (n, k, t) we construct the matrix $\mathcal{C}^{(n',k',t)}$, where n' and k' are the next nearest powers of 2 (repeating rows if necessary) and use a ‘‘2-D’’ permutation algorithm similar to (Fan et al., 2020) to distribute the sub-tasks in each round; however our algorithm uses more general (prime number) step-sizes chosen in each round and the permutations now occur in ‘‘higher dimensions¹⁰.’’ In particular; we now use a similar procedure to permute tasks amongst workers if n and k are not powers of 2. For example if we have $n = 6$ workers and $k = 3$ tasks we can add extra virtual tasks $\theta_3 = \theta_0, \theta_4 = \theta_1, \dots, \theta_x = \theta_{x\%3}$ and perform the following toroidal permutations on $\mathcal{C}^{(5,3,2)}$ so that at round r worker i performs task $\theta_{i+5r\%n'}$ and similarly at round r we have $t_i = \theta_{i+r\%k}$.

¹⁰*I.e.*, our algorithm permutes more than one index; in particular, it permutes the subtask, worker, and output indices in order to create ‘‘3-D’’ permutations. The step-sizes are more general because they must be co-prime to one another to ensure every blue

$$\begin{aligned}
 & \begin{array}{c} \tilde{\vartheta}_0 \quad \tilde{\vartheta}_1 \quad \tilde{\vartheta}_2 \quad \tilde{\vartheta}_3 \quad \tilde{\vartheta}_4 \\ \tilde{\theta}_0 \quad \tilde{\theta}_1 \quad \tilde{\theta}_2 \quad \tilde{\theta}_3 \quad \tilde{\theta}_4 \quad \tilde{\theta}_5 \quad \tilde{\theta}_6 \quad \tilde{\theta}_7 \end{array} \\
 t_0 \theta_0 & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
 t_1 \theta_1 & \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
 t_2 \theta_2 & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \end{bmatrix} \\
 t_3 \theta_3 & \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix} \\
 & \Rightarrow \begin{array}{c} \tilde{\vartheta}_3 \quad \tilde{\vartheta}_4 \\ \tilde{\theta}_0 \quad \tilde{\theta}_1 \quad \tilde{\theta}_2 \quad \tilde{\theta}_3 \quad \tilde{\theta}_4 \quad \tilde{\theta}_5 \quad \tilde{\theta}_6 \quad \tilde{\theta}_7 \end{array} \\
 t_0 \theta_1 & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
 t_1 \theta_2 & \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
 t_2 \theta_3 & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \end{bmatrix} \\
 & \Rightarrow \begin{array}{c} \tilde{\vartheta}_1 \quad \tilde{\vartheta}_2 \quad \tilde{\vartheta}_3 \quad \tilde{\vartheta}_4 \\ \tilde{\theta}_0 \quad \tilde{\theta}_1 \quad \tilde{\theta}_2 \quad \tilde{\theta}_3 \quad \tilde{\theta}_4 \quad \tilde{\theta}_5 \quad \tilde{\theta}_6 \quad \tilde{\theta}_7 \end{array} \\
 t_2 \theta_4 & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
 \theta_1 & \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
 t_0 \theta_2 & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \end{bmatrix} \\
 t_1 \theta_3 & \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix} \\
 & \Rightarrow \begin{array}{c} \tilde{\vartheta}_1 \quad \tilde{\vartheta}_2 \quad \tilde{\vartheta}_3 \quad \tilde{\vartheta}_4 \\ \tilde{\theta}_0 \quad \tilde{\theta}_1 \quad \tilde{\theta}_2 \quad \tilde{\theta}_3 \quad \tilde{\theta}_4 \quad \tilde{\theta}_5 \quad \tilde{\theta}_6 \quad \tilde{\theta}_7 \end{array} \\
 t_1 \theta_4 & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
 t_2 \theta_5 & \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
 \theta_2 & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \end{bmatrix} \\
 t_0 \theta_3 & \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}
 \end{aligned} \tag{5}$$

More generally we find a displacement d equal to an (odd) prime number that is co-prime¹¹ to k and we let worker i performs task $\theta_{i+dr \% n'}$ at round r and let $t_i = \theta_{i+r \% k}$ at round r . This allows gives the following statistical uniformity lemma:

Lemma 3.2. If the displacement, d , is equal to an (odd) prime number that is co-prime to k then the blue rectangle in (the general form of) Eq. 5 will visit every entry in the matrix with every possible pattern of $X^{(t)}$ and every cyclic permutation of the t_i contained inside of the blue rectangle.

Proof. The leftmost point of the blue rectangle is equal to $(i, i) + r(1, d) \equiv (i + r, i + dr) \pmod{\mathbb{Z}/n'\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}}$. By the Chinese remainder theorem (see (Ireland & Rosen, 1982) or (Dummit & Foote, 2003)) $(1, d)$ is a generator of $\pmod{\mathbb{Z}/n'\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}}$ since d is coprime to $1, k,$ and n' . \square

4. Analysis and Evaluation

4.1. Theoretical Analysis

In this section we give a theoretical comparison of the algorithms, see Table 1, and we prove theorems regarding the existence and non-existence of codes with certain properties. The following theorem, *i.e.*, Thm. 4.1, shows that Hamming-distance MDS coding schemes must have the workers do an arbitrarily large amount of work. We then later show that our codes are approximately MDS with respects to the projective geometry metric which maximize the amount of information sent back by the workers¹² while keeping the

rectangle (see Eq. 5) is visited.

¹¹Although it is notoriously hard to find a prime divisor of number, it is surprisingly easy to find a prime *non*-divisor. This easy to see since one can just test divisibility by 2,3,5,... and since the product of the first primes less than 100 is approximately equal to 2^{120} this will halt *very* quickly, *i.e.* it will halt in less than 25 steps for $k < 2^{120}$ since there are only 25 primes less than 100.

¹²This is because large angles gives us large conditional entropy.

amount of work done by the workers as low as possible; *i.e.*, there are approximately projective-MDS that have weights $t = 2, \dots, n$.

Theorem 4.1. If the parameters (n, k, t) satisfy $t \leq n - k$ then there is no Hamming-distance MDS (n, k) -code for the derivatives.

Proof. If $A(C)_i =$ “number of rows of weight i ”, then Theorem 7.4.1 in (Huffman & Pless, 2003) gives us that an MDS will have $A(C)_i = 0$ for $i \leq n - k$. \square

In particular; the proof of Thm. 4.1 can be strengthened to say that:

Corollary 4.2. In an MDS (n, k) -coding scheme $A(C)_i = 0$, for $i \leq n - k$, where $A(C)_i = 0$ is the weight distribution of a code.

The importance of Cor. 4.2 is made clear through the following interpretation:

Corollary 4.3. In an MDS (n, k) -coding scheme all of the workers must do at least $n - k$ amount of work.

However a simple observation of the construction given in Sec. 3.2 gives us that:

Theorem 4.4. There exists (n, k, t) -LWPD codes for any $t \geq 2$.

The next theorem proves that under the projective distance we have that our code achieves approximately maximal distance.

Theorem 4.5. The family (n, k, t) -code are approximately MDS (n, k) -code for the derivatives in the projective-distance for $n \leq 2k$.

Proof. By Lem. 3.2 it suffices to prove this for powers of two. The distance between any vectors is $\arccos \frac{1}{2} = \frac{\pi}{3}$ and this only happens for t out of n choices for any

Table 1. Comparison of main algorithms.

CODE SCHEME	ENCODING COMPLEXITY	COMMUNICATION COMPLEXITY	DECODING COMPLEXITY	WEIGHT RANGE	ASYNCHRONOUS ?	PARAMETER COMPRESSION?
LWPD	0	$\mathcal{O}(\frac{k}{t})$	0	$t \in [2, \frac{n}{4}]$	✓	✓
GC	$\mathcal{O}(nk)$	$\mathcal{O}(k)$	$\mathcal{O}(k^\omega) \leq \mathcal{O}(k^{2.38})$	$t = n - k + 1$	✗	✗
K-AC	0	$\mathcal{O}(k)$	0	$t \in [1, n]$	✓	✗

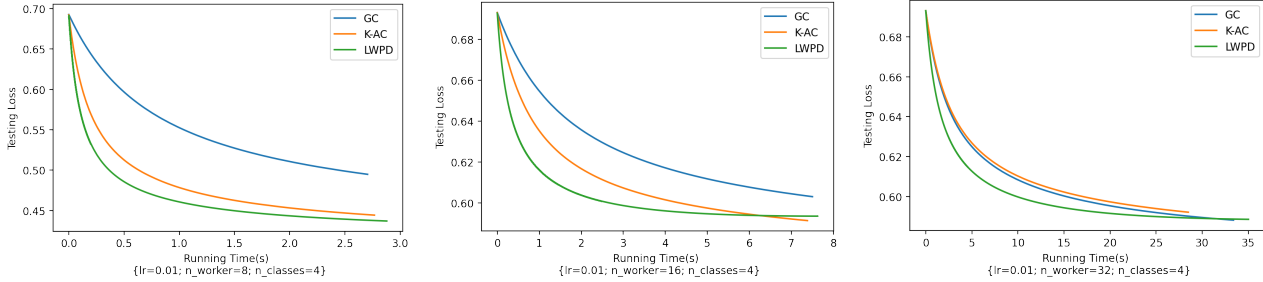


Figure 3. Experiments with 8, 16, and 32 workers.

vector; the distance is equal to the maximum $\frac{\pi}{2}$ for all other vectors. \square

The proof of Thm. 4.5 gives us that the distance between any two codewords $\tilde{\theta}, \theta$ is bounded above by $d(\mathcal{C}) = \min_{\tilde{\theta}, \theta \in \mathcal{C}} d(\tilde{\theta}, \theta) = \frac{\pi}{3}$ and thus in term of percentages of the optimal $\frac{\pi}{2}$ we have

$$\frac{\frac{\pi}{2} - d(\mathcal{C})}{\frac{\pi}{2}} = \frac{1}{6} \approx 16\% \quad (6)$$

of the ‘‘theoretical’’ optimal distance; however there can be no code that achieves the ‘‘theoretical’’ optimal distance:

Theorem 4.6. *The percentage in Eq. 6 cannot be made 100%; i.e., there are no projective MDS codes for $n > k$ which achieve distance $\frac{\pi}{2}$.*

Proof. If this statement was false there would be $n + 1$ linearly independent vectors in n -dimensional space, \mathbb{F}^n . \square

An interesting fact about the bound given by Thm. 4.5 is that is a constant independent of the dimension of the code and thus it scales well for larger and larger number of workers.

4.2. Experimental Results

The experiments were run on AWS Spot Instances; the workers are AWS EC2 c5a.large instances (compute optimized) and master is an AWS EC2 r3.large instance (Memory Optimized). The experimental procedure was written using Mpi4py (Dalcin et al., 2005; 2008) in Python. We used a

modification of the code in (Tandon, 2017) written by the first author of (Tandon et al., 2017) to implement Gradient Coding (GC) as well as the random data generation; the implementation only supported logistic regression and we generalized it to support multinomial logistic regression (i.e., more than one class). The software in (Tandon, 2017) used a Gaussian mixture model of two distributions to create input features for the logistic model; we generalized it to allow for an arbitrary number of Gaussian distributions in the mixture to create a robust data set. To be as fair as possible in our comparison with K -Asynchronous Gradient descent (K -AC) (Dutta et al., 2018b) we made setup for K -AC nearly identical with the exception of the coding scheme; i.e., K -AC and LWPD used the exact same data partitions and same number of k workers in the k -asynchronous batches.

We ran experiments with 8 workers, 16 workers, and 32 workers (see Fig. 3). The testing error (i.e., the workers do not train on the test data) is plotted against the time. In all of the experiments we ran LWPD codes converged far faster; however, it often overfitted and sometimes the other algorithms would eventually get a lower test error. There are two possible explanations for this: either LWPD converges so much faster than the other algorithms that they never get a chance to overfit or the noise that initially helps LWPD find a very quick solution eventually causes it to stay some distance from the optimal solution. There is evidence for both of these possibilities because there are experiments where the other algorithms do not catch up to LWPD; see Sec. A for more results.

We have modified and re-ran our experiments to allow for deep neural networks (see Fig. 4). The new architecture had

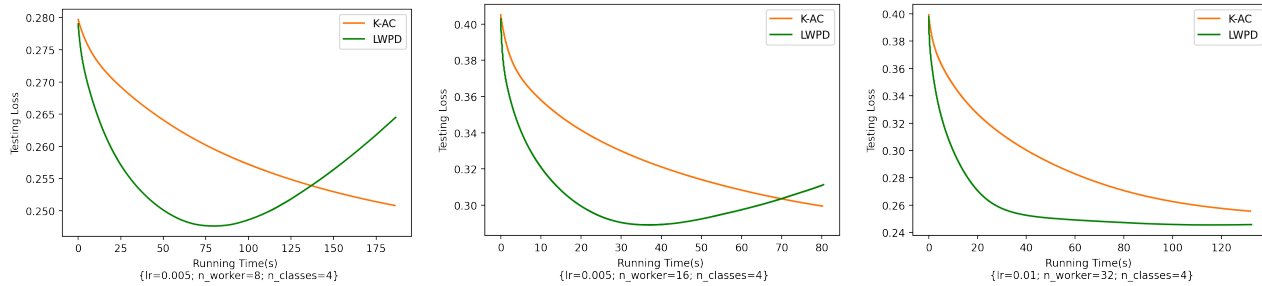


Figure 4. Further experiments with 8, 16, and 32 workers.

ReLU activations at the hidden nodes, Softmax at the output nodes, and cross entropy as the loss function. The experiments were run using `Mpi4py` on AWS Spot Instances with AWS EC2 `c5a.large` for the workers and AWS EC2 `r3.large` for the master. The original authors’ implementation of Gradient Coding did not support deep neural networks and thus we simply compared with K -Asynchronous Gradient descent for these experiments. Both the updated and original experiments contained stragglers and highlighted the asynchronous nature of the algorithm, since we used Amazon Spot instances, which are the most unreliable but cost-efficient machines that Amazon AWS can offer.

In our experimental setup, all workers compute training loss individually. These plots should not be considered a centralized measure of model performance because the data set is *decentralized*. Thus, plotting training metrics does not make as much sense as, for example, visualizing the testing/validation set error or loss. Furthermore, we were guided by the intuition that the validation error is the more important measure, since it measures how well the algorithm or model can perform on data it has not seen.

The experimental data plots show *test loss* (y -axis) versus the *time elapsed* (x -axis). The test loss refers to the error on the validation set (which the model is not trained on) and thus overfitting occurs when the training loss continues to decrease, but the generalization error stops decreasing and eventually begins to grow. In our experiments, we observed that other methods reach similar convergence point much later in terms of number of iterations. One hypothesis is that our gradient coding approach offers faster training. A second possible explanation comes from the lossy compression of our algorithm. We trade off error for runtime. Our hypothesis is that the noise coming from the trade-off is beneficial in that it prevents us from being stuck in any potential saddle point.

5. Conclusion and Open Problems

We propose LWPD codes which allow for asynchronous gradient updates by maximizing the amount of information

contained by random subsets of vectors and minimizing the weight of the code. Our code compresses the gradient in a manner that scales well with the number of nodes (and the dimension of the data) and achieves a lower a communication complexity and memory overhead with respect to the state of the art. Another improvement of our algorithm over the state of the art is our discovery of the correct information metric; all of the other coding schemes assume that the Hamming distance is the correct metric, which does not consider the natural (differential) geometry of the gradient. Furthermore, we showed that our code was very efficient since the master can just directly add and subtract the results returned by the workers without needing to decode the information. We proved many of the complexity guarantees theoretically and also provided much empirical evidence for the performance. For future work we would like to strengthen the theoretical results by proving stronger complexity bounds as well as further investigating the effect of noise (or lossy compression) on the performance; it seems that at first the lossy compression is a great help but eventually it causes over fitting.

Acknowledgments

This paper is based upon work supported by the National Science Foundation under Grant No. CCF-2101388.

References

- Amiri, M. M. and Gündüz, D. Computation scheduling for distributed machine learning with straggling workers. In *ICASSP 2019 - 2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 8177–8181, 2019. doi: 10.1109/ICASSP.2019.8682911.
- Atallah, E. and Rahnvard, N. A code-based distributed gradient descent method. In *2018 56th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pp. 951–958, 2018. doi: 10.1109/ALLERTON.2018.8635869.
- Baharav, T., Lee, K., Ocal, O., and Ramchandran, K.

- Straggler-proofing massive-scale distributed matrix multiplication with d-dimensional product codes. In *2018 IEEE International Symposium on Information Theory (ISIT)*, pp. 1993–1997, 2018. doi: 10.1109/ISIT.2018.8437549.
- Bitar, R., Wootters, M., and Rouayheb, S. E. Stochastic gradient coding for flexible straggler mitigation in distributed learning. In *2019 IEEE Information Theory Workshop (ITW)*, pp. 1–5, 2019. doi: 10.1109/ITW44776.2019.8989328.
- Bitar, R., Wootters, M., and el Rouayheb, S. Stochastic gradient coding for straggler mitigation in distributed learning. *IEEE Journal on Selected Areas in Information Theory*, 1:277–291, 2020.
- Charles, Z. and Papailiopoulos, D. Gradient coding using the stochastic block model. In *IEEE Symposium on Information Theory*, 2018.
- Cover, T. M. and Thomas, J. A. *Elements of information theory* (2. ed.). 2006.
- Dalcín, L., Paz, R., and Storti, M. Mpi for python. *Journal of Parallel and Distributed Computing*, 65(9):1108–1115, 2005. ISSN 0743-7315. doi: <https://doi.org/10.1016/j.jpdc.2005.03.010>. URL <https://www.sciencedirect.com/science/article/pii/S0743731505000560>.
- Dalcín, L., Paz, R., Storti, M., and D’Elía, J. Mpi for python: Performance improvements and mpi-2 extensions. *Journal of Parallel and Distributed Computing*, 68(5):655–662, 2008. ISSN 0743-7315. doi: <https://doi.org/10.1016/j.jpdc.2007.09.005>. URL <https://www.sciencedirect.com/science/article/pii/S0743731507001712>.
- Das, A. B. and Ramamoorthy, A. Distributed matrix-vector multiplication: A convolutional coding approach. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pp. 3022–3026, 2019. doi: 10.1109/ISIT.2019.8849395.
- Dummit, D. and Foote, R. *Abstract Algebra*. Wiley, 2003. ISBN 9780471433347. URL <https://books.google.com/books?id=KIGbCgAAQBAJ>.
- Dutta, S., Bai, Z., Jeong, H., Low, T. M., and Grover, P. A unified coded deep neural network training strategy based on generalized polydot codes. In *2018 IEEE International Symposium on Information Theory (ISIT)*, pp. 1585–1589, 2018a. doi: 10.1109/ISIT.2018.8437852.
- Dutta, S., Joshi, G., Ghosh, S., Dube, P., and Nagpurkar, P. Slow and stale gradients can win the race: Error-runtime trade-offs in distributed sgd. In Storkey, A. and Perez-Cruz, F. (eds.), *Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics*, volume 84 of *Proceedings of Machine Learning Research*, pp. 803–812. PMLR, 09–11 Apr 2018b. URL <https://proceedings.mlr.press/v84/dutta18a.html>.
- Dutta, S., Cadambe, V., and Grover, P. “short-dot”: Computing large linear transforms distributedly using coded short dot products. *IEEE Transactions on Information Theory*, 65(10):6171–6193, 2019. doi: 10.1109/TIT.2019.2927558.
- Dutta, S., Fahim, M., Haddadpour, F., Jeong, H., Cadambe, V., and Grover, P. On the optimal recovery threshold of coded matrix multiplication. *IEEE Transactions on Information Theory*, 66(1):278–301, 2020. doi: 10.1109/TIT.2019.2929328.
- Dutta, S., Joshi, G., Ghosh, S., Dube, P., and Nagpurkar, P. Slow and stale gradients can win the race. *IEEE Journal on Selected Areas in Information Theory*, 2:1012–1024, 2021.
- Fan, X., Soto, P., Zhong, X., Xi, D., Wang, Y., and Li, J. Leveraging stragglers in coded computing with heterogeneous servers. In *2020 IEEE/ACM 28th International Symposium on Quality of Service (IWQoS)*, pp. 1–10, 2020. doi: 10.1109/IWQoS49365.2020.9213028.
- Ferdinand, N. S. and Draper, S. C. Anytime stochastic gradient descent: A time to hear from all the workers. *2018 56th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pp. 552–559, 2018.
- Ferdinand, N. S., Gharachorloo, B., and Draper, S. C. Anytime exploitation of stragglers in synchronous stochastic gradient descent. *2017 16th IEEE International Conference on Machine Learning and Applications (ICMLA)*, pp. 141–146, 2017.
- Ferdinand, N. S., Al-Lawati, H., Draper, S. C., and Nokleby, M. S. Anytime minibatch: Exploiting stragglers in online distributed optimization. *CoRR*, abs/2006.05752, 2020. URL <https://arxiv.org/abs/2006.05752>.
- Fulton, W. and Harris, J. *Representation Theory: A First Course*. Graduate Texts in Mathematics. Springer New York, 1991. ISBN 9780387974958. URL <https://books.google.com/books?id=qGFzi20nMcYC>.
- Halbawi, W., Azizan, N., Salehi, F., and Hassibi, B. Improving distributed gradient descent using reed-solomon codes. In *2018 IEEE International Symposium on Information Theory (ISIT)*, pp. 2027–2031, 2018. doi: 10.1109/ISIT.2018.8437467.

- Hong, S., Yang, H., Yoon, Y., Cho, T., and Lee, J. Chebyshev polynomial codes: Task entanglement-based coding for distributed matrix multiplication. In Meila, M. and Zhang, T. (eds.), *Proceedings of the 38th International Conference on Machine Learning*, volume 139 of *Proceedings of Machine Learning Research*, pp. 4319–4327. PMLR, 18–24 Jul 2021.
- Horii, S., Yoshida, T., Kobayashi, M., and Matsushima, T. Distributed stochastic gradient descent using ldgm codes. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pp. 1417–1421, 2019. doi: 10.1109/ISIT.2019.8849580.
- Huffman, W. and Pless, V. *Fundamentals of Error-Correcting Codes*. Cambridge University Press, 2003. ISBN 9780521782807. URL https://books.google.com/books?id=B2FjPxtS_QUC.
- Ireland, K. and Rosen, M. *A classical introduction to modern number theory*. Graduate texts in mathematics. Springer-Verlag, 1982. ISBN 9783540906254. URL <https://books.google.com/books?id=WvjuAAAAMAAJ>.
- Jia, Z. and Jafar, S. A. Generalized cross subspace alignment codes for coded distributed batch matrix multiplication. In *ICC 2020 - 2020 IEEE International Conference on Communications (ICC)*, pp. 1–6, 2020. doi: 10.1109/ICC40277.2020.9149322.
- Karakus, C., Sun, Y., Diggavi, S., and Yin, W. Redundancy techniques for straggler mitigation in distributed optimization and learning. *Journal of Machine Learning Research*, 20(72):1–47, 2019. URL <http://jmlr.org/papers/v20/18-148.html>.
- Kosaian, J., Rashmi, K. V., and Venkataraman, S. Learning a code: Machine learning for approximate non-linear coded computation. *CoRR*, abs/1806.01259, 2018. URL <http://arxiv.org/abs/1806.01259>.
- Kosaian, J., Rashmi, K. V., and Venkataraman, S. Parity models: Erasure-coded resilience for prediction serving systems. In *Proceedings of the 27th ACM Symposium on Operating Systems Principles, SOSP '19*, pp. 30–46, New York, NY, USA, 2019a. Association for Computing Machinery. ISBN 9781450368735. doi: 10.1145/3341301.3359654. URL <https://doi.org/10.1145/3341301.3359654>.
- Kosaian, J., Rashmi, K. V., and Venkataraman, S. Parity models: A general framework for coding-based resilience in ml inference. *ArXiv*, abs/1905.00863, 2019b.
- Kühnel, W., Hunt, B., and Society, A. M. *Differential Geometry: Curves - Surfaces - Manifolds*. Student mathematical library. American Mathematical Society, 2006. ISBN 9780821839881. URL <https://books.google.com/books?id=TyqUnlyV4Y4C>.
- Lee, K., Suh, C., and Ramchandran, K. High-dimensional coded matrix multiplication. In *2017 IEEE International Symposium on Information Theory (ISIT)*, pp. 2418–2422, 2017. doi: 10.1109/ISIT.2017.8006963.
- Lee, K., Lam, M., Pedarsani, R., Papailiopoulos, D., and Ramchandran, K. Speeding Up Distributed Machine Learning Using Codes. *IEEE Transactions on Information Theory*, 64(3):1514–1529, 2018.
- Maity, R. K., Rawa, A. S., and Mazumdar, A. Robust gradient descent via moment encoding and ldpc codes. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pp. 2734–2738. IEEE, 2019.
- Ozfatura, E., Gündüz, D., and Ulukus, S. Gradient coding with clustering and multi-message communication. In *2019 IEEE Data Science Workshop (DSW)*, pp. 42–46. IEEE, 2019a.
- Ozfatura, E., Gündüz, D., and Ulukus, S. Speeding up distributed gradient descent by utilizing non-persistent stragglers. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pp. 2729–2733, 2019b. doi: 10.1109/ISIT.2019.8849684.
- Ozfatura, E., Ulukus, S., and Gündüz, D. Distributed gradient descent with coded partial gradient computations. *ICASSP 2019 - 2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 3492–3496, 2019c.
- Ozfatura, E., Ulukus, S., and Gündüz, D. Straggler-aware distributed learning: Communication–computation latency trade-off. *Entropy*, 22(5), 2020. ISSN 1099-4300. doi: 10.3390/e22050544. URL <https://www.mdpi.com/1099-4300/22/5/544>.
- Raviv, N., Tamo, I., Tandon, R., and Dimakis, A. G. Gradient coding from cyclic mds codes and expander graphs. *IEEE Transactions on Information Theory*, 66(12):7475–7489, 2020. doi: 10.1109/TIT.2020.3029396.
- Reisizadeh, A., Prakash, S., Pedarsani, R., and Avestimehr, A. S. Tree gradient coding. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pp. 2808–2812, 2019a. doi: 10.1109/ISIT.2019.8849431.
- Reisizadeh, A., Taheri, H., Mokhtari, A., Hassani, H., and Pedarsani, R. Robust and communication-efficient collaborative learning. In Wallach, H., Larochelle, H., Beygelzimer, A., d'Alché-Buc, F., Fox, E., and Garnett, R. (eds.), *Advances in Neural Information Processing Systems*, volume 32. Curran Associates, Inc., 2019b. URL <https://proceedings>.

- neurips.cc/paper/2019/file/3eb2f1a06667bfb9daba7f7effa0284b-Paper.pdf.
- Sasi, S., Lalitha, V., Aggarwal, V., and Rajan, B. S. Straggler mitigation with tiered gradient codes. *IEEE Transactions on Communications*, 68(8):4632–4647, 2020. doi: 10.1109/TCOMM.2020.2992721.
- Serre, J. *Linear Representations of Finite Groups*. Graduate Texts in Mathematics. Springer New York, 2012. ISBN 9781468494587. URL <https://books.google.com/books?id=9mT1BwAAQBAJ>.
- Soto, P., Li, J., and Fan, X. Dual entangled polynomial code: Three-dimensional coding for distributed matrix multiplication. In Chaudhuri, K. and Salakhutdinov, R. (eds.), *Proceedings of the 36th International Conference on Machine Learning*, volume 97 of *Proceedings of Machine Learning Research*, pp. 5937–5945. PMLR, 09–15 Jun 2019. URL <http://proceedings.mlr.press/v97/soto19a.html>.
- Suetin, P. K., Kostrikin, A. I., and Manin, Y. I. *Linear algebra and geometry*. 1989.
- Tandon, R. `gradient_coding`. https://github.com/rashisht1/gradient_coding, 2017.
- Tandon, R., Lei, Q., Dimakis, A. G., and Karampatziakis, N. Gradient coding: Avoiding stragglers in distributed learning. In Precup, D. and Teh, Y. W. (eds.), *Proceedings of the 34th International Conference on Machine Learning*, volume 70 of *Proceedings of Machine Learning Research*, pp. 3368–3376. PMLR, 06–11 Aug 2017. URL <https://proceedings.mlr.press/v70/tandon17a.html>.
- Tao, T. and Vu, V. *Additive Combinatorics*. Number v. 13 in *Additive combinatorics*. Cambridge University Press, 2006. ISBN 9780521853866. URL <https://books.google.com/books?id=WY8YnwEACAAJ>.
- Vogtmann, K., Weinstein, A., and Arnol’d, V. *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics. Springer New York, 2013. ISBN 9781475720648. URL <https://books.google.com/books?id=BXIAswEACAAJ>.
- Wang, H., Guo, S., Tang, B., Li, R., Yang, Y., Qu, Z., and Wang, Y. Heterogeneity-aware gradient coding for tolerating and leveraging stragglers. *IEEE Transactions on Computers*, pp. 1–1, 2021. doi: 10.1109/TC.2021.3063180.
- Wang, S., Liu, J., and Shroff, N. Coded sparse matrix multiplication. In Dy, J. and Krause, A. (eds.), *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pp. 5152–5160. PMLR, 10–15 Jul 2018. URL <https://proceedings.mlr.press/v80/wang18e.html>.
- Ye, M. and Abbe, E. Communication-computation efficient gradient coding. In *ICML*, 2018.
- Yu, Q., Maddah-Ali, M., and Avestimehr, S. Polynomial codes: an optimal design for high-dimensional coded matrix multiplication. In Guyon, I., Luxburg, U. V., Bengio, S., Wallach, H., Fergus, R., Vishwanathan, S., and Garnett, R. (eds.), *Advances in Neural Information Processing Systems*, volume 30. Curran Associates, Inc., 2017. URL <https://proceedings.neurips.cc/paper/2017/file/e6c2dc3dee4a51dcec3a876aa2339a78-Paper.pdf>.
- Yu, Q., Li, S., Raviv, N., Kalan, S. M. M., Soltanolkotabi, M., and Avestimehr, S. A. Lagrange coded computing: Optimal design for resiliency, security, and privacy. In Chaudhuri, K. and Sugiyama, M. (eds.), *Proceedings of the Twenty-Second International Conference on Artificial Intelligence and Statistics*, volume 89 of *Proceedings of Machine Learning Research*, pp. 1215–1225. PMLR, 16–18 Apr 2019. URL <http://proceedings.mlr.press/v89/yu19b.html>.
- Zhang, J. and Simeone, O. Lagc: Lazily aggregated gradient coding for straggler-tolerant and communication-efficient distributed learning. *IEEE Transactions on Neural Networks and Learning Systems*, 32:962–974, 2021.

A. Experimental Results

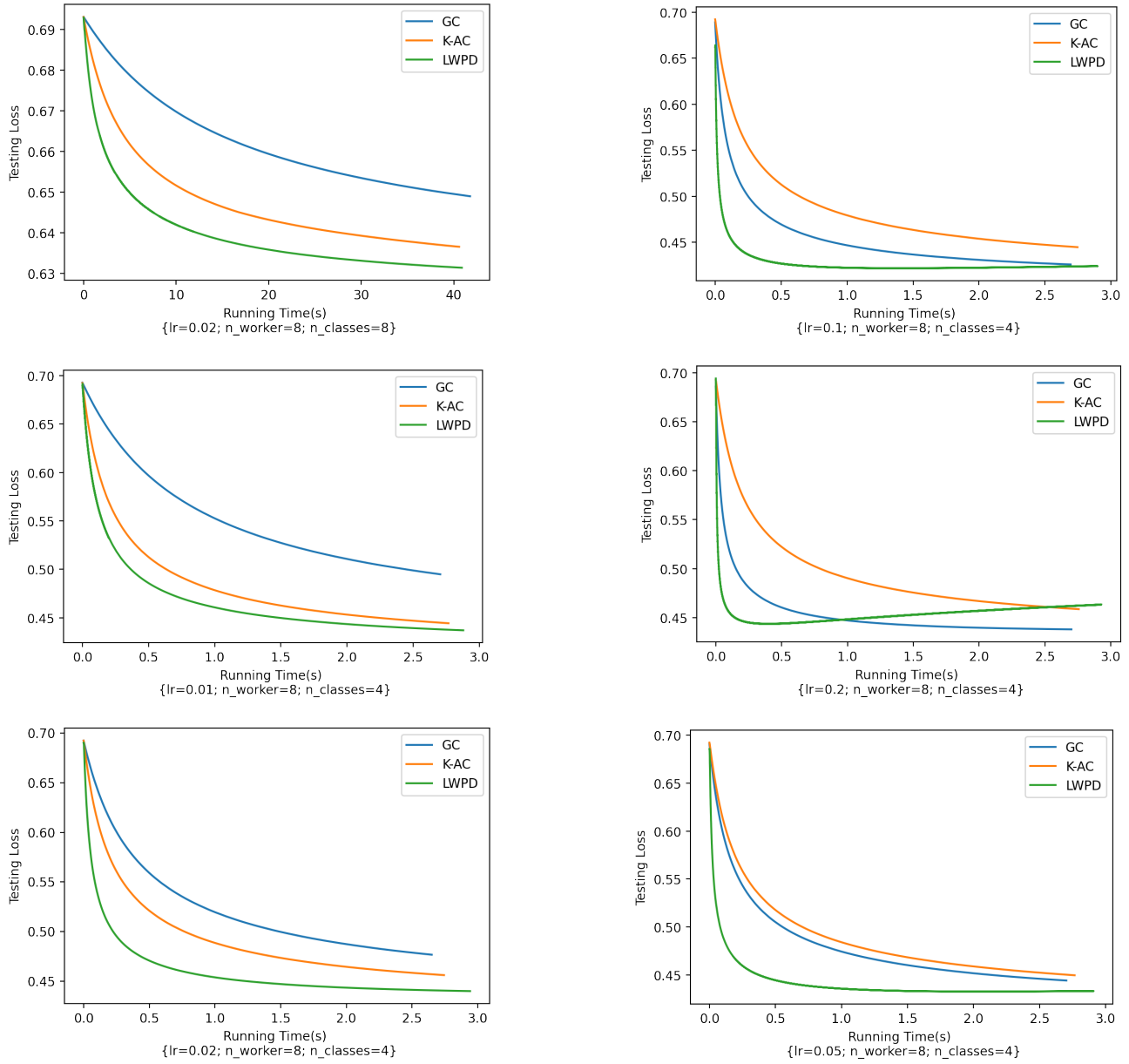


Figure 5. Experiments with 8 workers.

Lightweight Projective Derivative Codes

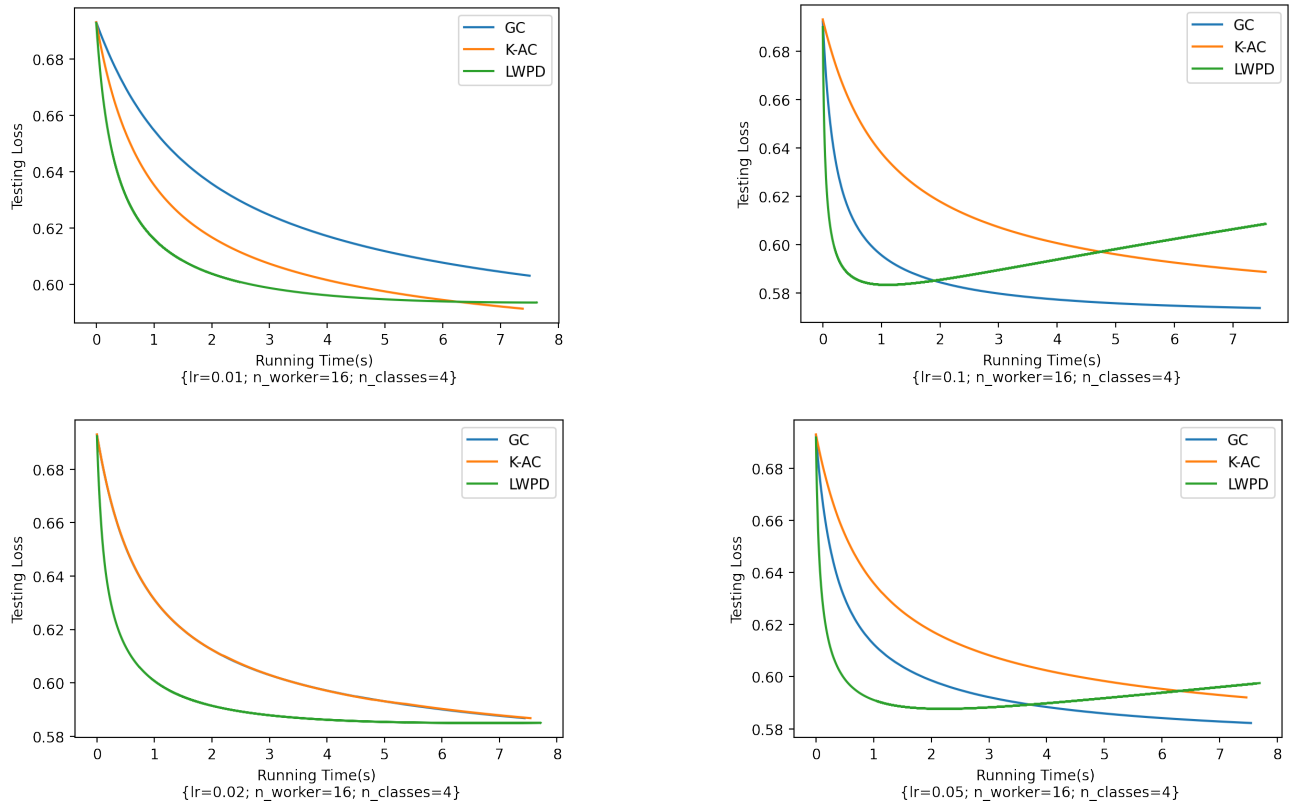


Figure 6. Experiments with 16 workers.

Lightweight Projective Derivative Codes

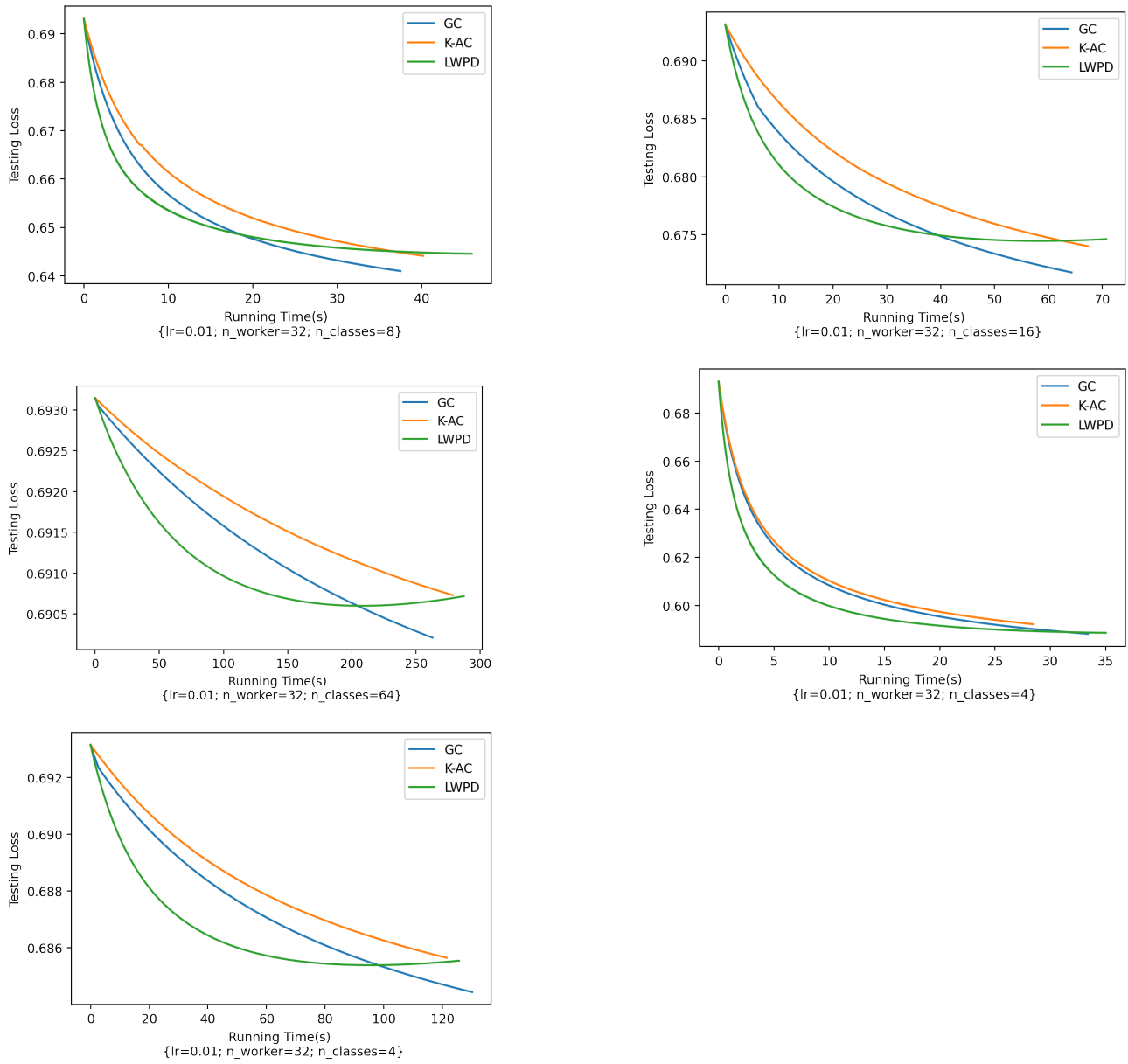


Figure 7. Experiments with 32 workers.