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Sliding-Seeking Control: Model-Free Optimization with Safety Constraints

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Abstract

This paper considers the design of online model-free algorithms for the solution of convex optimization problems with a time-varying cost function. We propose an *online switched zeroth-order algorithm* where: i) different vector fields are implemented based on whether constraints are satisfied; and, ii) zeroth-order dynamics are leveraged to obtain estimates of the (time-varying) gradients in the algorithmic updates. The zeroth-order strategy is suitable for cases where the optimizer has access to functional evaluations of the cost and constraints, but has no knowledge of their functional form. The proposed online algorithm guarantees finite-time feasibility (while avoiding projections) and it exhibits asymptotic stability to a neighborhood of the optimal trajectory of the time-varying problem. Results are established for cost functions that are strictly convex and twice continuously differentiable. Illustrative numerical results are presented to showcase the main properties of the algorithm.

Keywords: Constrained Optimization, Zeroth-Order Methods, Switched Dynamical Systems

1. Introduction

This paper considers a convex optimization problem with a cost function that may change over time (Popkov, 2005). This optimization model is prevalent for networked systems operating under data streams and reflects dynamic performance objectives or dynamic physical and engineering constraints. Application domains include, for example, power grids (Dall'Anese and Simonetto, 2018; Hauswirth et al., 2018; Tang et al., 2017), transportation systems (Galarza-Jimenez et al., 2021b), robotics Zheng et al. (2019), and communication networks (Chen and Lau, 2012).

The works of Rahili and Ren (2017); Fazlyab et al. (2016) are examples where gradient flows were utilized to solve unconstrained time-varying optimization problems. The work of Li et al. (2021) proposes an algorithm for constrained time-varying optimization problems in discrete-time by implementing a finite prediction window. Time-varying constraints were considered in Hauswirth et al. (2018) using forward Lipschitz continuous sets. In Vaquero and Cortés (2018), the authors considered a (time-varying) saddle-flow method based on a Tikhonov-regularized Lagrangian function. Constraints were handled via the Moreau envelope in Colombino et al. (2020), where an online gradient descent algorithm was implemented in closed-loop with a linear time-invariant dynamical system. A common assumption in the literature is that the gradient information of the cost is *available* and the projection operator (or the proximal operator in Colombino et al. (2020)) is computationally cheap (and can afford a real-time implementation). In contrast to this setting, a model-free approach based on gradient descent was developed in Grushkovskaya et al. (2017); Scheinker and Krstić (2012); Poveda and Teel (2017) for time-varying unconstrained problems. Practical fixed-time results were recently presented in Poveda and Krstić (2021) using non-smooth model-free algorithms.

In this paper, we present an *online switched zeroth-order algorithm* for solving time-varying constrained optimization problems. The algorithm switches between two different zeroth-order algorithms depending on whether or not the current iteration is feasible. Both algorithms have access only to functional evaluations of the cost function and the constraint functions (via, e.g., measurements). The proposed approach is also applicable to the case where the constraint function is known, but the projection on the feasible set is computationally expensive, or it requires the collection of a large number of inputs (Mahdavi et al., 2012). Related non-smooth model-free optimization algorithms were studied in Fu and Özgüner (2011) and Oliveira et al. (2011) using tools from sliding mode control, and in Gerard et al. (2009) for model-based optimization. Unlike smooth standard saddle-flow dynamics (Cherukuri et al., 2016; Wang and Elia, 2011), the proposed switching algorithms can achieve finite-time convergence to the feasible set, which is also rendered forward invariant. These properties are particularly appealing in problems with safety constraints. Given that the zeroth-order dynamics that we introduce are modeled as discontinuous differential equations, stability and convergence properties of the algorithms are analyzed by leveraging tools from *non-smooth analysis*, *constrained differential inclusions*, and *averaging theory*. Our main result establishes semi-global practical asymptotic stability properties for the dynamics, provided the time-variation of the cost function is sufficiently slow. The theoretical results are illustrated in a numerical example where the cost function is strictly convex uniformly in time, and one convex constraint is enforced.

The rest of this paper is organized as follows: the preliminaries and problem statement are presented in Section 2 and Section 3, respectively. Section 4 presents the main results and Section 5 presents the analysis. Numerical examples are presented in Section 6, while Section 7 concludes the paper. Auxiliary results and the proofs of the results are reported in the Appendix of the extended version (Galarza-Jimenez et al., 2021a).

2. Preliminaries

For a given vector $z \in \mathbb{R}^n$, we denote the Euclidean norm (2-norm) as $|z|$. Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, we use $|z|_{\mathcal{A}} := \min_{s\in\mathcal{A}} |z - s|$ to denote the minimum distance of z to $\mathcal{A}$. For a matrix $A \in \mathbb{R}^{n\times m}$ we denote the spectral norm (induced 2-norm) as $|A|$. We use $\mathbb{S}^1 := \{z \in \mathbb{R}^2 : z_1^2 + z_2^2 = 1\}$ to denote the unit circle in $\mathbb{R}^2$, and $\mathbb{T}^n \subset \mathbb{R}^{2n}$ to denote the n^{th} Cartesian product of $\mathbb{S}^1$. The concatenation of two vectors x, y is denoted as $(x, y) = [x^\top, y^\top]^\top$. We use $z + r\mathbb{B}$ to denote a closed ball in the Euclidean space, centered at z and with radius $r > 0$. We define the interior of a set $\mathcal{S} \subset \mathbb{R}^n$ as $\text{int}(\mathcal{S}) := \{x \in \mathcal{S} : \exists r > 0 \text{ such that } x + r\mathbb{B} \subseteq \mathcal{S}\}$, and we denote the closure of $\mathcal{S}$ as $\overline{\mathcal{S}} := \bigcap \{\mathcal{S} + \varepsilon\mathbb{B} \,:\, \varepsilon > 0\}$ and its boundary if $\mathcal{S}$ is not empty as $\text{bd}(\mathcal{S}) := \{x \in \overline{\mathcal{S}} \backslash \text{int}(\mathcal{S})\}$. We define the convex hull of the set $\mathcal{S}$ as $\text{con}(\mathcal{S}) := \left\{\sum_{i=1}^k \lambda_i x_i \,:\, k > 0,\ x_i \in \mathcal{S},\ \lambda_i \geq 0,\ \sum_{i=1}^k \lambda_i = 1\right\}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is strictly increasing in its first argument, decreasing in its second argument, $\lim_{r\to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. For the analysis of our algorithms, we will leverage the framework of constrained differential inclusions (DIs) in Goebel et al. (2012). In particular, we consider DIs with state $x \in \mathbb{R}^n$, and dynamics of the form

$$x \in C, \quad \dot{x} \in F(x), \tag{1}$$

where $C \subset \mathbb{R}^n$ is a closed set, the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to C, and $F(x)$ is nonempty and convex for all $x \in C$. A solution to system (1) is an absolutely continuous function $x : \mathrm{dom}(x) \to \mathbb{R}^n$ that satisfies: a) $x(0) \in C$; b) $x(t) \in C$ for all $t \in \mathrm{dom}(x)$; and c) $\dot{x}(t) \in F(x(t))$ for almost all $t \in \mathrm{dom}(x)$. A solution is said to be complete if $\mathrm{dom}(x) = [0, \infty)$. A set $\mathcal{A} \subset C$ is said to be strongly forward-invariant if for every initial condition $x_0 \in \mathcal{A}$, every complete solution starting at x_0 remains in $\mathcal{A}$ for all $t \geq 0$. Given a compact set $\mathcal{A} \subset C$, system (1) is said to render $\mathcal{A}$ uniformly globally asymptotically stable (UGAS) if there exists a class $\mathcal{KL}$ function β such that every solution of (1) satisfies $|x(t)|_{\mathcal{A}} \leq \beta(|x(0)|_{\mathcal{A}}, t)$ for all $t \in \mathrm{dom}(x)$. In this paper, we will also consider ε-perturbed or parameterized differential inclusions (Goebel et al., 2012, Definition 6.27) of the form

$$x \in C, \quad \dot{x} \in F_{\varepsilon}(x), \quad F_{\varepsilon}(x) := \overline{\mathrm{con}}\left(F((x + \varepsilon\mathbb{B}) \cap C)\right) + \varepsilon\mathbb{B}, \;\; \varepsilon > 0. \tag{2}$$

Given a compact set $\mathcal{A} \subset C$, system (2) is said to render $\mathcal{A}$ Semi-Globally Practically Asymptotically Stable (SGPAS) as $\varepsilon \to 0^+$, if there exists a class $\mathcal{KL}$ function β such that, for each pair $\delta > \nu > 0$, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ every solution of (2) with $|x(0)|_{\mathcal{A}} \leq \delta$ satisfies

$$|x(t)|_{\mathcal{A}} \leq \beta(|x(0)|_{\mathcal{A}}, t) + \nu, \quad \forall\, t \in \mathrm{dom}(x). \tag{3}$$

The notion of SGPAS can be extended to systems that are parametrized by multiple parameters $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\ell)$. In this case, and with some abuse of notation, we say that system (2) renders the set $\mathcal{A}$ SGPAS as $(\varepsilon_\ell, \ldots, \varepsilon_2, \varepsilon_1) \to 0^+$, where the parameters are tuned in order starting from ε_1, i.e., ε_2^* may depend on ε_1, ε_3^* may depend on $\varepsilon_2 \in (0, \varepsilon_2^*)$, etc. Given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with domain C, we define the *Krasovskii regularization* of F ($\mathcal{K}[F]$) (Goebel et al., 2012, Def. 4.13) as

$$\dot{x} \in \mathcal{K}[F](x) := \bigcap_{\delta > 0} \overline{\mathrm{con}}\; \left(F(x + \delta\mathbb{B}) \cap C\right).$$

3. Problem Formulation

Consider the following constrained optimization problem:

$$\min_{z \in \mathbb{R}^n} \;\; \phi_t(z) \;\;\text{s.t.}\;\; c(z) \leq 0, \tag{4}$$

where $c : \mathbb{R}^n \to \mathbb{R}$ is a convex function and ϕ_t is a time-varying cost. The functional forms of ϕ_t and c are assumed to be *unknown*. However, for each $t \geq 0$, we assume to have access to *measurements* of $\phi_t(z)$ and $c(z)$. Additionally, we assume that the temporal variability of ϕ_t is driven by a parameter $\theta \in \mathbb{R}^p$ with exosystem dynamics

$$\dot{\theta} = \rho\Psi(\theta), \quad \theta \in \Theta, \tag{5}$$

where $\rho > 0$, $\Psi : \mathbb{R}^p \to \mathbb{R}^p$, and $\Theta \subset \mathbb{R}^p$. Under this parameterization, we rewrite problem (4) as

$$h(\theta) := \arg\min_{z \in \mathbb{R}^n} \;\; \phi(z, \theta) \;\;\text{s.t.}\;\; c(z) \leq 0, \tag{6}$$

where $h : \mathbb{R}^p \to \mathbb{R}^n$ maps θ to the optimal solutions of (6). The function ϕ may not necessarily depend on the whole vector θ, but only on some components. We define the feasible set of (6) as:

$$\mathcal{A} := \{z \in \mathbb{R}^n : c(z) \leq 0\} \tag{7}$$

We will make the following assumptions in our problem:

Assumption 1 *The functions ϕ and c are twice continuously differentiable.* □

Assumption 2 *The mapping $z \mapsto c(z)$ is convex, and the mapping $z \mapsto \phi(z, \theta)$ is strictly convex for any $\theta \in \Theta$.* □

Assumption 3 *The interior of $\mathcal{A}$ is nonempty, i.e., int$(\mathcal{A}) \neq \emptyset$.* □

Under Assumption 2 the set of solutions of (6) is a singleton. On the other hand, under Assumption 3, strong duality of the problem is guaranteed (Ruszczynski, 2006). The following assumption will provide enough regularity to the exosystem (5).

Assumption 4 *The function $\theta \mapsto \Psi(\theta)$ is Lipschitz continuous, and the set Θ is compact and strongly forward-invariant under the dynamics $\dot{\theta} = \rho\Psi(\theta)$.* □

The following lemma guarantees that the function h in (6) is well defined. We omit the proof due to space limitations.

Lemma 1 *Suppose Assumptions 1-4 hold and $\theta' \in \Theta$. Then, there exists a neighborhood $\mathcal{N}'$ around θ' and a continuously differentiable function $h : \mathcal{N}' \to \mathbb{R}^n$ that solves (6).* □

The previous result implies that $\nabla h : \mathbb{R}^p \to \mathbb{R}^{p\times n}$ is continuous in a neighborhood of $\theta' \in \Theta$. Therefore, there exists $\tilde{H} > 0$ such that $|\nabla h(\theta)| \leq \tilde{H}$, $\forall\, \theta \in \Theta$. For a given $\theta \in \Theta$, we will define $z^* := h(\theta)$ as the optimal solution to (6). Given the time variability of θ under Assumption 4, our goal is to design an online and model-free algorithm that guarantees convergence to a neighborhood of the optimal trajectory $z^*(t) := h(\theta(t))$.

Lemma 2 *Suppose Assumptions 1-3 hold, then, there exists $\underline{c} > 0$ such that for all $z \notin \mathcal{A}$, $|\nabla c(z)| \geq \underline{c}$.* □

We omit the proof of this lemma due to space limitations.

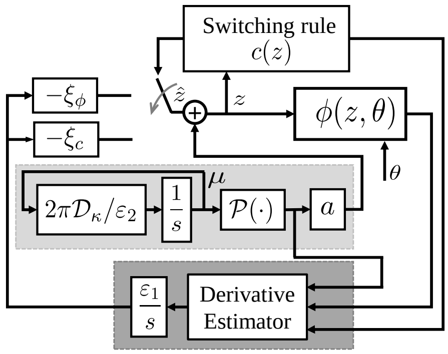
4. Algorithm and Main Result

To solve problem (6), we consider a class of model-free switched dynamics described by

$$\dot{\hat{z}} = F(\hat{z}) := \begin{cases} -\xi_\phi & \text{if } c(z) \leq 0 \\ -\xi_c & \text{if } c(z) > 0, \end{cases} \tag{8}$$

where the states ξ_ϕ and ξ_c are generated by the following dynamical systems

$$\varepsilon_1\dot{\xi}_\phi = -\xi_\phi + U_\phi(z, \mu, \theta), \quad \varepsilon_1\dot{\xi}_c = -\xi_c + U_c(z, \mu), \qquad \varepsilon_1 > 0, \tag{9}$$



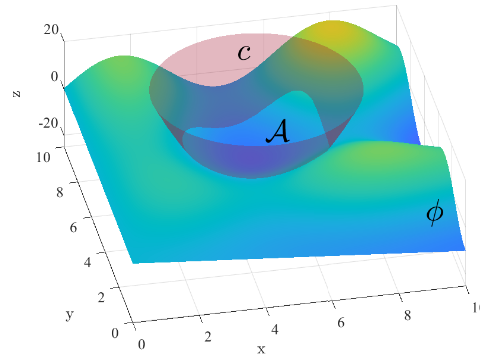


Figure 1: Scheme of the proposed algorithm in (8) (left side). An example of ϕ and c (right side), we note that the strict convexity of ϕ must be satisfied only in $\mathcal{A}$.

where $(\xi_\phi, \xi_c) \in M\mathbb{B}$ for an arbitrarily large $M > 0$,

$$U_\phi(z,\mu,\theta) := \frac{2}{a}\mathcal{P}(\mu)\phi(z,\theta), \quad U_c(z,\mu) := \frac{2}{a}\mathcal{P}(\mu)c(z), \qquad a > 0, \tag{10}$$

with $\mu \in \mathbb{T}^n$ is generated by a linear dynamic oscillator of the form

$$\varepsilon_2\dot{\mu} = 2\pi D_\kappa \mu, \quad \mu \in \mathbb{T}^n, \qquad \varepsilon_2 > 0, \tag{11}$$

and where the matrix $D_\kappa \in \mathbb{R}^{2n\times 2n}$ is block diagonal with i^{th} block given by $D_{\kappa_i} := [0\ \kappa_i; -\kappa_i\ 0]$, $\kappa_i > 0$, and $\mathcal{P}(\mu) : \mathbb{T}^n \to \mathbb{R}^n$ is defined as $\mathcal{P}(\mu) := [\mu_1, \mu_3, \mu_5, \ldots, \mu_{2n-1}]^\top$. Since the n oscillators are uncoupled, the odd entries μ_i of the solutions μ generated by (11) with $\mu(0) \in \mathbb{T}^n$ are given by $\mu_i(t) = \mu_i(0)\cos\left(\frac{2\pi}{\varepsilon_2}\kappa_i t\right) + \mu_{i+1}(0)\sin\left(\frac{2\pi}{\varepsilon_2}\kappa_i t\right)$, with $\mu_i(0)^2 + \mu_{i+1}(0)^2 = 1$, for all $i \in \{1,3,5,\ldots,n-1\}$. The states μ_i have the role of exploratory signals injected into the dynamics (9) via the mappings (10). We make the following assumption on the parameters κ_i.

Assumption 5 *For each $i \in \{1,2,\ldots,n\}$, κ_i is a positive rational number, and $\kappa_i \neq \kappa_j$ for all $i \neq j$.* □

Finally, using (8) and (11), the argument z of problem (6) is updated via the feedback law $z = \hat{z} + a\mathcal{P}(\mu)$.

The behavior of the switched zeroth-order dynamics is highly dependent on the parameters $(\varepsilon_1, \varepsilon_2, a)$; they need to be tuned to guarantee suitable convergence properties. Figure 1 shows a scheme of the proposed algorithm, as well as an example of a pair of functions ϕ and c. The following theorem characterizes the stability and convergence properties of the switched zeroth-order algorithm. In particular, we guarantee convergence to a ν-neighborhood of $\{z^*\}$. The proof is presented in Section 5.

Theorem 3 *Suppose Assumptions 1-5 hold, and consider the switched zeroth-order dynamics (8)-(11). Then, for each $\delta > \nu > 0$ such that $|\hat{z}(0) - z^*(0)| \leq \delta$, there is $\rho^* > 0$ such that for all $\rho \in [0, \rho^*]$, there is $\varepsilon_1^* > 0$ such that for all $\varepsilon_1 \in (0, \varepsilon_1^*]$, there is $a^* > 0$, such that for all $a \in (0, a^*]$, there is $\varepsilon_2^* > 0$ such that for all $\varepsilon_2 \in (0, \varepsilon_2^*]$, we have $|\hat{z}(t) - z^*(t)| \leq \beta(|\hat{z}(0) - z^*(0)|, t) + \nu$ for every $t \geq 0$, where β is a $\mathcal{KL}$ function.* □

Theorem 3 guarantees a practical stability result provided the variation of θ is sufficiently small, and provided the parameters $(\varepsilon_1, \varepsilon_2, a)$ are appropriately tuned. The result guarantees convergence of the dynamics (8)-(11) to a $\nu-$neighborhood of the singleton $\{z^*\}$ that solves (6). Numerical examples illustrating the role of these parameters are presented in Section 6. The following corollary is a direct consequence of the proof of Theorem 3.

Corollary 4 *Suppose Assumptions 1-5 hold, and that $\rho = 0$ and Θ is a singleton. Consider the switched zeroth-order averaging dynamics (8)-(11). Then, for each $\delta > \nu > 0$ such that $|\hat{z}(0) - z^*(0)| \leq \delta$, there is $\varepsilon_1^* > 0$ such that for all $\varepsilon_1 \in (0, \varepsilon_1^*]$, there is $a^* > 0$, such that for all $a \in (0, a^*]$, there is $\varepsilon_2^* > 0$ such that for all $\varepsilon_2 \in (0, \varepsilon_2^*]$, we have $|\hat{z}(t) - z^*(t)| \leq \beta(|\hat{z}(0) - z^*(0)|, t) + \nu$ for every $t \geq 0$, where β is a $\mathcal{KL}$ function.* □

The switched dynamics (8) have suitable transient properties that guarantee finite-time convergence to a small neighborhood of the feasible set $\mathcal{A}$, where the trajectories are further restricted to evolve.

Remark 5 *Since our main result establishes convergence to a small neighborhood of the optimal trajectory $h(\theta(t))$, in cases where the optimal point is on the boundary of $\mathcal{A}$, one needs to consider a restricted set $\bar{\mathcal{A}} := \mathcal{A} - \varepsilon\mathbb{B}$ characterized by a modified constraint function $\tilde{c} := c + \varepsilon$ to guarantee that the original constraints are satisfied.* □

We also note that the proposed zeroth-order algorithm can enforce the hard constraint $c(z) \leq 0$, or a tightened constraint as explained in the previous remark. This may not be the case for standard smooth algorithms such as saddle-flow dynamics or gradient dynamics based on approximate barrier functions.

Remark 6 *An unavoidable feature of switched algorithms of the form (8) is the emergence of chattering whenever the trajectories evolve in the boundary of the feasible set (Liberzon, 2003). However, several strategies can be used to avoid chattering, including spatial regularizations based on hysteresis mechanisms, or temporal regularizations based on clocks that restrict how fast the dynamics can switch; see (Goebel et al., 2012; Ferrara et al., 2019). Due to space limitations, we do not study these strategies in this paper.* □

5. Analysis

In this section, we present the stability and convergence analysis of the zeroth-order averaging based algorithms. First, we consider a system with known functions ϕ, c, and fixed $\theta \in \Theta$, and we provide suitable stability and convergence properties. Subsequently, we prove that these properties hold, in a semi-global practical way, for the original zeroth-order dynamics under sufficiently slow variations of θ.

We start by considering the following switched system:

$$\dot{z} = F(z, \theta) := \begin{cases} -\nabla\phi(z, \theta) & \text{if } c(z) \leq 0 \\ -\nabla c(z) & \text{if } c(z) > 0. \end{cases} \qquad \dot{\theta} = 0. \tag{12}$$

In particular, for each fixed $\theta \in \Theta$, we are interested in establishing that (12) renders the set $\{z^*\} \times \Theta$ UGAS. However, since the vector field (12) is discontinuous, even the problem of existence of solutions is not trivial; we will then work with generalized solutions of (12) obtained via *Krasovskii* *regularization*. Under Assumption 1, the Krasovskii regularization of the discontinuous differential equation (12) is given by the following differential inclusion:

$$\dot{z} \in \mathcal{K}[F](z,\theta) := \begin{cases} -\nabla\phi(z,\theta) & \text{if } c(z) < 0 \\ -\mathcal{C}(z,\theta) & \text{if } c(z) = 0 \\ -\nabla c(z) & \text{if } c(z) > 0, \end{cases} \qquad \dot{\theta} = 0, \tag{13}$$

where $\mathcal{C} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined as $\mathcal{C}(z,\theta) := \{f \in \mathbb{R}^n : f = (1-\lambda)\nabla\phi(z,\theta)+\lambda\nabla c(z),\ \lambda \in [0,1]\}$.

The following lemma follows by the boundedness of the mapping F in (12), and by (Goebel et al., 2012, Ex. 6.6).

Lemma 7 *Under Assumption 1, the set-valued mapping* (13) *is convex-valued, outer semicontinuous, and locally bounded.* □

5.1. Set of Equilibria

First, we analyze the set of equilibrium points of the system (13), which we denote as $\mathcal{E} := \{\bar{z} \in \mathbb{R}^n : 0 \in \mathcal{K}[F](z,\theta)\}$. Under Assumptions 1-3, the result of the following lemma follows from the KKT characterization of solutions of (6).

Lemma 8 *Suppose that Assumptions 1-3 hold. For any fixed $\theta \in \Theta$, we have that $\mathcal{E} = \{z^*\}$.* □

5.2. Finite Time Convergence to the Feasible Set and Uniform Asymptotic Stability

The following lemma establishes that the dynamics (13) guarantee finite-time convergence to the set $\mathcal{A}$.

Lemma 9 *Suppose that Assumptions 1-3 hold. Then, for each $\theta \in \Theta$ and each compact set $K \subset \mathbb{R}^n$ there exits a $T > 0$ such that every solution of* (13) *with $z(0) \in K\mathbb{B}$ satisfies $|z(t)|_{\mathcal{A}} = 0$ for all $t \geq T$. Moreover, the feasible set $\mathcal{A}$ is strongly forward-invariant.* □

Next, we establish UGAS when $\dot{\theta} = 0$.

Theorem 10 *Let $\theta \in \Theta$ be fixed, and consider the differential inclusion* (13)*. Suppose that Assumptions 1-4 hold. Then, every solution of* (13) *with $z(0) \in \mathbb{R}^n$, satisfies $|z(t) - z^*| \leq \beta(|z(0) - z^*|, t)$, for all $t \geq 0$, where β is a $\mathcal{KL}$ function.* □

The previous result holds for a constant parameter $\theta \in \Theta$ (and, consequently, for the corresponding constant optimal solution z^*). In the following, we leverage robustness properties with respect to slow variations on the parameter θ to provide a result on the convergence of z to a neighborhood of the optimal trajectory.

5.3. Robustness to Slow-Varying Parameters

By Theorem 10, for each $\theta \in \Theta$, the system (13) renders the compact set $\{z^*\} \times \Theta$ UGAS. Since the differential inclusion (13) is outer semicontinuous, locally bounded, and convex valued (see Lemma 7), the stability properties of $\{z^*\} \times \Theta$ are preserved, in a semi-global practical way, under small additive inflations of (13) (Goebel et al., 2012, Corollary 7.27).

Lemma 11 *Consider the differential inclusion*

$$\dot{z} \in \mathcal{K}[F](z,\theta) := \begin{cases} -\nabla\phi(z,\theta) & \textit{if } c(z) < 0 \\ -\mathcal{C}(z,\theta) & \textit{if } c(z) = 0 \\ -\nabla c(z) & \textit{if } c(z) > 0, \end{cases}, \qquad \dot{\theta} = \rho\Psi(\theta), \quad \theta \in \Theta. \tag{14}$$

and suppose that Assumptions 1-4 hold. Then, there exists $\rho^* > 0$ *such that for all* $\rho \in (0, \rho^*)$*, all solutions* z *satisfy* $|z(t) - z^*(t)| \leq \beta(|z(0) - z^*(0)|, t) + \nu$. □

This result establishes the existence of a $\rho^* > 0$ such that $\forall\ \rho \in (0, \rho^*)$, the trajectory $z(t)$ converges to a neighbourhood of $z^*(t)$ for some $\beta \in \mathcal{KL}$.

5.4. Averaging Analysis

Given the result of Lemma 11, we next utilize averaging theory for non-smooth dynamical systems to investigate the stability properties of the zeroth-order dynamics. First, we consider the Krasovskii regularization of the dynamics (8)-(11) given by

$$\dot{\hat{z}} \in F(\hat{z}) := \begin{cases} -\xi_\phi & \text{if } c(z) < 0 \\ \mathcal{G}(\xi_\phi, \xi_c) & \text{if } c(z) = 0 \\ -\xi_c & \text{if } c(z) > 0, \end{cases} \tag{15a}$$

$$\dot{\theta} = \rho\Psi(\theta), \quad \theta \in \Theta, \tag{15b}$$

$$\left.\begin{aligned} \varepsilon_1\dot{\xi}_\phi &= -\xi_\phi + U_\phi(z,\mu,\theta) \\ \varepsilon_1\dot{\xi}_c &= -\xi_c + U_c(z,\mu) \end{aligned}\right\}, \ (\xi_\phi, \xi_c) \in M\mathbb{B}, \tag{15c}$$

$$\varepsilon_2\dot{\mu} = 2\pi D_\kappa \mu, \qquad \mu \in \mathbb{T}^n, \tag{15d}$$

with $\mathcal{G}(\xi_\phi, \xi_c) := \{\xi \in \mathbb{R}^n : \xi = \lambda\xi_\phi + (1-\lambda)\xi_c,\ \lambda \in [0,1],\}$, and where the states ξ_ϕ and ξ_c are restricted to the compact set $M\mathbb{B}$ ($M > 0$ can be taking arbitrarily large to contain every solution from any initial condition of practical interest). We average the dynamics (15) along the trajectories of μ. Since the dynamics (15a) are independent of μ, computing the average of system (15) only affects the dynamics (15c). To compute the average, we consider a Taylor expansion of the functions $\phi(\hat{z} + aP(\mu))$ and $c(\hat{z} + aP(\mu))$ around $\hat{z}$, that is $\phi(\hat{u} + a\mathcal{P}(\mu), \theta) = \phi(\hat{u}, \theta) + \mathcal{P}(\mu)^\top \nabla_{\hat{z}}\phi(\hat{z}, \theta) + \mathcal{O}(a^2)$, and $c(\hat{u} + a\mathcal{P}(\mu)) = c(\hat{u}) + \mathcal{P}(\mu)^\top \nabla_{\hat{z}} c(\hat{z}) + \mathcal{O}(a^2)$, where $\mathcal{O}(a^2)$ represents higher order terms that are bounded on compact sets, and which can be made arbitrarily small by decreasing a^2. Using the average properties of μ and equations (10), the average of the dynamics (15) are given by

$$\dot{\hat{z}} \in F(\hat{z}) := \begin{cases} -\xi_\phi & \text{if } c(z) < 0 \\ \mathcal{G}(\xi_\phi, \xi_c) & \text{if } c(z) = 0 \\ -\xi_c & \text{if } c(z) > 0, \end{cases} \tag{16a}$$

$$\dot{\theta} = \rho\Psi(\theta), \quad \theta \in \Theta, \tag{16b}$$

$$\left.\begin{aligned} \varepsilon_1\dot{\xi}_\phi &= -\xi_\phi + \nabla_{\hat{z}}\phi(\hat{z},\theta) + \mathcal{O}(a) \\ \varepsilon_1\dot{\xi}_c &= -\xi_c + \nabla_{\hat{z}} c(\hat{z}) + \mathcal{O}(a) \end{aligned}\right\}, \ (\xi_\phi, \xi_c) \in M\mathbb{B}. \tag{16c}$$

By averaging theory for differential inclusions (Poveda and Teel, 2017, Lemma 6), we know that if the average system (16) has suitable stability properties, then the original dynamics (15) will preserve the stability properties in a semi-global practical way with respect to ε_2. Hence, we study the stability properties of the differential inclusion (16).

5.5. Singular Perturbation Analysis

System (16) is in standard form for the application of singular perturbation theory. Fist, we set $\mathcal{O}(a) = 0$ in the dynamics (16c). By linearity, these dynamics render the equilibrium points $\xi_\phi^* = \nabla_{\hat{z}}\phi(\hat{z}, \theta)$ and $\xi_c^* = \nabla_{\hat{z}} c(\hat{z})$ exponentially stable when $(\hat{z}, \theta)$ is frozen. By substituting these equilibrium points into the dynamics (16a), we obtain the reduced system

$$\dot{\hat{z}} \in F(\hat{z}) := \begin{cases} -\nabla_{\hat{z}}\phi(\hat{z}, \theta) & \text{if } c(z) < 0 \\ \mathcal{G}(\nabla_{\hat{z}}\phi(\hat{z}, \theta), \nabla_{\hat{z}} c(\hat{z})) & \text{if } c(z) = 0 \\ -\nabla_{\hat{z}} c(\hat{z}) & \text{if } c(z) > 0, \end{cases} \tag{17a}$$

$$\dot{\theta} = \rho\Psi(\theta), \quad \theta \in \Theta, \tag{17b}$$

which is precisely system (14). By singular perturbation theory for differential inclusions (Sanfelice and Teel, 2011, Thm. 1), we obtain that the dynamics (16) (with $\mathcal{O}(a) = 0$) render the set $\{z^*\} \times \Theta \times M\mathbb{B}$ SGPAS as $(\rho, \varepsilon_1) \to 0^+$. Since the right-hand side of the dynamics (16) is outer-semicontinuous, locally bounded, and convex-valued, the stability properties are preserved, in a semi-global practical way, for values of a sufficiently small, i.e., system (16) renders the set $\{z^*\} \times \Theta \times M\mathbb{B}$ SGPAS as $(\rho, \varepsilon_1, a) \to 0^+$. Finally, we use again averaging theory for differential inclusions (Poveda and Teel, 2017, Lemma 6), and obtain that the original dynamics (15) also retain the stability properties of its average system, in a semi-global practical way with respect to ε_2. Thus, the dynamics (15) renders the set $\{z^*\} \times \Theta \times M\mathbb{B} \times \mathbb{T}^n$ SGPAS as $(\rho, \varepsilon_1, a, \varepsilon_2) \to 0^+$. This establishes the result of Theorem 3.

The results of Corollary 4 follow by setting $\rho = 0$.

6. Illustrative Numerical Examples

In this section, we present two numerical examples that illustrate our main theoretical results.

Example 1. We consider the cost function ϕ_t of the form:

$$\phi(z, \theta) = \frac{\alpha_2}{2}(z_1 - \theta_1)^2 + \frac{\alpha_3}{2}(z_2 - \theta_2)^2, \;\; \dot{\theta}_1 = K\rho\cos(\rho t + \pi/2), \;\; \dot{\theta}_2 = 2K\rho\cos(2\rho t). \tag{18}$$

The constraint function $c : \mathbb{R}^2 \to \mathbb{R}$ is a convex function given by $c(z) = (z_1 - 10)^2 + (z_2 - 10)^2 - r^2$. The function $\theta(t)$ is initialized at the point $(15, 10)$. The parameters $\alpha_2 = \alpha_3 = 0.5$, $K = 5$, and $\rho = 0.1$, and the constraint function has $r = 5$. The parameters of our algorithm are $\frac{2\pi}{\varepsilon_2}\kappa_1 = 500$, $\frac{2\pi}{\varepsilon_2}\kappa_2 = 200$, $a = 1/10$, and $\varepsilon_1 = 1/3$. The feasible set $\mathcal{A}$, as well as the trajectories $\theta(t)$, are shown in the left plot of Figure 2. We also show in color red the trajectory generated by the switched zeroth-order dynamics. According to our cost function ϕ, the optimal trajectories inside $\mathcal{A}$ are given by $\theta(t)$, and we observe that our algorithm is able to track them.

Example 2. We now consider the cost function given by $\phi(z, \theta) = \frac{c_0}{2}|z - \theta|^2$, with constraint function given by $c(z) = a^\top z + b$ where $a = [-1 \;\; -2]$, $b = 2$. The resulting system that we aim to emulate via the zeroth-order dynamics is given by:

$$\dot{z} = \begin{cases} -\nabla_z\phi(z, \theta) := -c_0(z - \theta) & \text{if } a^\top z + b \leq 0, \\ -\nabla c(z) := -a & \text{if } a^\top z + b > 0, \end{cases} \qquad \dot{\theta} = \rho \begin{bmatrix} c_1\theta_1 - c_2\theta_1\theta_2 \\ c_3\theta_1\theta_2 - c_4\theta_2 \end{bmatrix}, \tag{19}$$

where $\rho = 0.2$, $c_0 = 10$, $c_1 = 2/3$, $c_2 = 4/3$, and $c_3 = c_4 = 1$. The parameters of our algorithm are $\frac{2\pi}{\varepsilon_2}\kappa_1 = 800$, $\frac{2\pi}{\varepsilon_2}\kappa_2 = 1000$, $a = 1/100$, and $\varepsilon_1 = 2/3$. The right plot of Figure 2 shows the

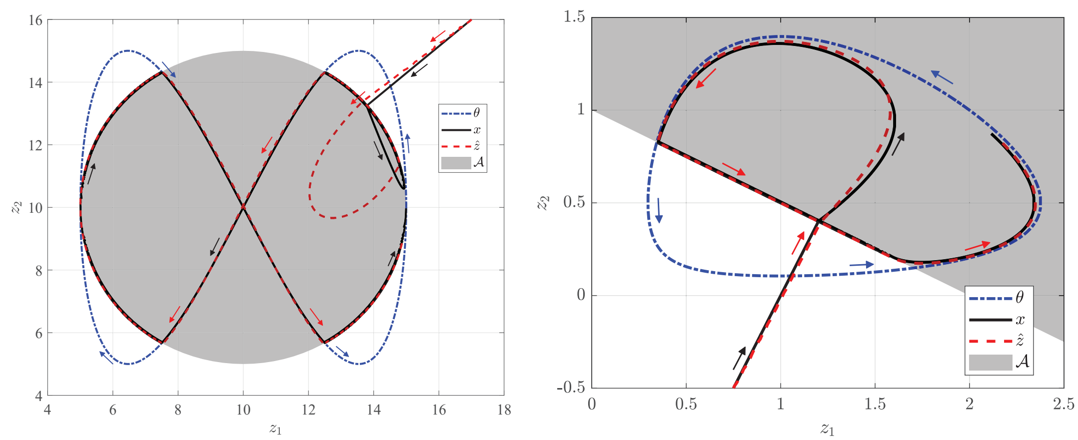


Figure 2: Evolution of the algorithm for the dynamical system (18) (left plot), and (19) (right plot). The grey area is the feasible set $\mathcal{A}$. The black solid lines are the solution of the model-based dynamics (13), with state x.

trajectories of the zeroth-order dynamics, as well as the feasible set of the optimization problem. It can be observed that the trajectories converge to the feasible set, and proceed to track the optimal trajectory z^*.

7. Conclusions

In this paper, we presented a switched zeroth-order method for the solution of constrained time-varying convex optimization problems with one convex constraint. Given that the optimization dynamics implement a different vector field depending on the current operational point of the system, the resulting dynamical system is discontinuous and therefore it is analyzed using tools for nonsmooth dynamical systems. For these dynamics, we establish semi-global practical asymptotic stability properties with respect to the time-varying minimizer of the problem for the case when its time-variation is sufficiently slow.

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