Data-Dependent Randomized Smoothing

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Abstract

Randomized smoothing is a recent technique that achieves state-of-art performance in training certifiably robust deep neural networks. While the smoothing family of distributions is often connected to the choice of the norm used for certification, the parameters of these distributions are always set as global hyper parameters independent from the input data on which a network is certified. In this work, we revisit Gaussian randomized smoothing and show that the variance of the Gaussian distribution can be optimized at *each* input so as to maximize the certification radius for the construction of the smooth classifier. Since the data dependent classifier does not directly enjoy sound certification with existing approaches, we propose a memory-enhanced data dependent smooth classifier that is certifiable by construction. This new approach is generic, parameter-free, and easy to implement. In fact, we show that our data dependent framework can be seamlessly incorporated into 3 randomized smoothing approaches, leading to consistent improved certified accuracy. When this framework is used in the training routine of these approaches followed by a data dependent certification, we achieve 9% and 6% improvement over the certified accuracy of the strongest baseline for a radius of 0.5 on CIFAR10 and ImageNet.

1 INTRODUCTION

[1] Despite the success of Deep Neural Networks (DNNs) in various learning tasks [Krizhevsky et al., 2012, Long et al., 2015], they were shown to be vulnerable to small carefully crafted adversarial perturbations [Goodfellow et al., 2015, Szegedy et al., 2013]. For a DNN f that correctly classifies an image x, f can be fooled to produce an incorrect prediction for $x + \eta$ even when the adversary η is so small that x and $x + \eta$ are indistinguishable to the human eye. To circumvent this nuisance, there have been several works proposing heuristic training procedures to build networks that are *robust* against such perturbations [Cisse et al., 2017, Madry et al., 2018]. However, many of these works provided a false sense of security as they were subsequently broken, *i.e.* shown to be ineffective against stronger adversaries [Athalye et al., 2018, Tramer et al., 2020, Uesato et al., 2018]. This has inspired researchers to develop networks that are *certifiably robust*, *i.e.* networks that provably output constant predictions over a characterized region around every input. Among many certification methods, a probabilistic approach to certification called *randomized smoothing* has demonstrated impressive state-of-the-art certifiable robustness results [Cohen et al., 2019, Lecuyer et al., 2019, Li et al., 2019]. In a nutshell, given an input x and a base classifier f, *e.g.* a DNN, randomized smoothing constructs

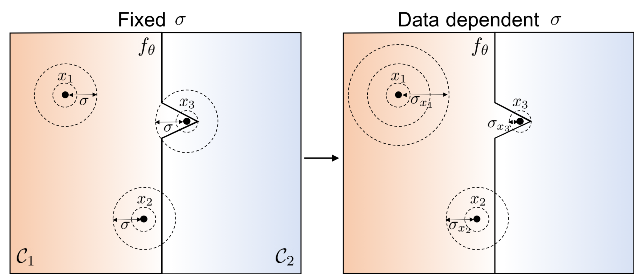

Figure 1: **From fixed to data dependent smoothing.** Using a fixed σ for all inputs to smooth f_θ may under certify (results in smaller certification radius) inputs far from decision boundary *e.g.* x_1, decrease in prediction confidence as for x_2 or produce incorrect predictions as for x_3. Thus, smoothing should vary per input (right figure) to alleviate the aforementioned issues.

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[1]Code: https://github.com/MotasemAlfarra/DDRS.

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a "smooth classifier" $g(x) = \mathbb{E}_{\epsilon\sim\mathcal{D}}\left[f(x+\epsilon)\right]$ such that, and under some choices of $\mathcal{D}$, $g(x) = g(x+\delta)\ \forall\delta \in \mathcal{R}$. As such, g is certifiable within the certification region $\mathcal{R}$ characterized by x and the smoothing distribution $\mathcal{D}$. While there has been considerable progress in devising a notion of "optimal" smoothing distribution $\mathcal{D}$ for when $\mathcal{R}$ is characterized by an ℓ_p certificate [Yang et al., 2020], a common trait among all works in the literature is that the choice of $\mathcal{D}$ is independent from the input x. For example, one of the earliest works on randomized smoothing grants ℓ_2 certificates under $\mathcal{D} = \mathcal{N}(0, \sigma^2 I)$, where σ is a free parameter that is constant for all x [Cohen et al., 2019]. That is to say, the classifier f is smoothed to a classifier g uniformly (same variance σ^2) over the entire input space of x. The choice of σ used for certification is often set either arbitrarily or via cross validation to obtain best certification results [Salman et al., 2019a]. We believe this is suboptimal and that σ should vary with the input x (data dependent), since using a fixed σ may under-certify inputs (*i.e.* the constructed smooth classifier g produces smaller certification radii), which are far from the decision boundaries as exemplified by x_1 in Figure 1. Moreover, this fixed σ could be large for inputs x close to the decision boundaries resulting in a smooth classifier g that incorrectly classifies x (refer to x_3 in Figure 1).

In this paper, we aim to introduce more structure to the smoothing distribution $\mathcal{D}$ by rendering its parameters data dependent. That is to say, the base classifier f is smoothed with a family of smoothing distributions to produce: $g(x) = \mathbb{E}_{\epsilon\sim\mathcal{N}(0,\sigma_x^2 I)}\left[f(x+\epsilon)\right]$ [2]. Note here that the variance of the Gaussian is now dependent on the data input x. Moreover, given that σ_x varies with x, classical randomized smoothing based certification does not apply directly. We propose a simple memory-based approach to certify the resultant data dependent smooth classifier g. We show that our memory-enhanced data dependent smooth classifier can boost certification performance of several randomized smoothing techniques. Our contributions can thus be summarized in three folds. (**i**) We propose a parameter free and generic framework that can easily turn several randomized smoothing techniques into their data dependent variants. In particular, given a network f and an input x, we propose to optimize the smoothing distribution parameters for every x, *e.g.* σ_x^*, so they maximize the certification radius. This choice of σ_x^* is then used to smooth f at x and construct a smoothed classifier g. Moreover, as the data dependent smooth classifier is not directly certifiable using Cohen et al. [2019] MCMC approaches, we propose a memory-enhanced data dependent smooth classifier for certification. (**ii**) We demonstrate the effectiveness of our memory-enhanced data dependent smoothing by showing that we can improve the certified accuracy of several models, specifically models trained with Gaussian augmentation (COHEN) [Cohen et al., 2019], adversaries on the smoothed classifier (SMOOTHADV) [Salman et al., 2019a], and radius regularization (MACER) [Zhai et al., 2020] *without any model retraining*. We boost the certified accuracy of the best baseline by 5.4% on CIFAR10 and by 2.8% on ImageNet for ℓ_2 perturbations with less than 0.5 (=127/255) ball radius. (**iii**) We show that incorporating the proposed data dependent smoothing in the training pipeline of COHEN, SMOOTHADV and MACER can further boost results to get certified accuracies of 68.3% on CIFAR10 and 64.2% on ImageNet at ℓ_2 perturbations less than 0.25.

[2]The paper mainly focuses on Gaussian smoothing, but the idea holds for other parameterized distributions.

2 RELATED WORK

Certified Defenses. Certified defenses aim to guarantee that an adversary does not exist in a certain region around a given input. Certified defenses can be divided into exact [Cheng et al., 2017, Lomuscio and Maganti, 2017, Huang et al., 2017, Ehlers, 2017] and relaxed certification [Salman et al., 2019b, Wong and Kolter, 2018]. Generally, exact certification suffers from poor scalability with networks that are at most 3 hidden layers deep [Tjeng et al., 2019]. On the other hand, relaxed methods resolve this issue by aiming at finding an upper bound to the worst adversarial loss over all possible bounded perturbations around a given input [Weng et al., 2018]. However, the latter is too expensive for any mixed certification-training routine.

Randomized Smoothing. The earliest work on randomized smoothing [Lecuyer et al., 2019] was from a differential privacy perspective, where it was demonstrated that adding Laplacian noise enjoys an ℓ_1 certification radius in which the average classifier prediction under this noise is constant. This work was later followed by the tight ℓ_2 certificate radius for Gaussian smoothing [Cohen et al., 2019]. Since then, there has been a body of work on randomized smoothing with empirical defenses [Salman et al., 2019a] to certify black box classifiers [Salman et al., 2020]. Other works derived certification guarantees for ℓ_1 bounded [Teng et al., 2019], ℓ_∞ bounded [Zhang et al., 2019], and ℓ_0 bounded [Levine and Feizi, 2020] perturbations. Even more recently, a novel framework that finds the optimal smoothing distribution for a given ℓ_p norm [Yang et al., 2020] was proposed showing state-of-art certification results on ℓ_1 perturbations. We deviate from the common literature by introducing the notion of smoothing, particularly Gaussian smoothing for ℓ_2 perturbations, which varies depending on the input. In particular, since an input x that is far from the decision boundaries should tolerate larger smoothing (and equivalently have a larger certification radius) as compared to inputs closer to these boundaries, we optimize for the amount of smoothing per input (specifically σ_x) that maximizes the certification radius. This proposed process is denoted as *data dependent smoothing* where we provide a procedure for certifying the resultant smooth classifier.

3 DATA DEPENDENT SMOOTHING

3.1 PRELIMINARIES AND NOTATIONS

Let $x \in \mathbb{R}^d$ and the labels $y \in \mathcal{Y} = \{1, \ldots, k\}$ be the input-label pairs (x, y) sampled from an unknown data distribution. Unless explicitly mentioned, we consider a classifier $f_\theta : \mathbb{R}^d \to \mathcal{P}(\mathcal{Y})$ parameterized by θ where $\mathcal{P}(\mathcal{Y})$ is a probability simplex over k labels. We say that f_θ is ℓ_p^r certifiably accurate for an input x, if and only if, $\arg\max_c f_\theta^c(x) = \arg\max_c f_\theta^c(x + \delta) = y \;\; \forall \, \|\delta\|_p \le r$, where f_θ^c is the c^{th} element of f_θ. That is to say, the classifier correctly predicts the label of x and enjoys a constant prediction for all perturbations δ that are in the ℓ_p ball of radius r from x. As such, the overall ℓ_p^r certification accuracy is defined as the average certified accuracy over the data distribution. Following prior art [Cohen et al., 2019, Salman et al., 2019a, Zhai et al., 2020], we focus on ℓ_2^r certification.

3.2 OVERVIEW OF RANDOMIZED SMOOTHING

Randomized smoothing constructs a certifiable classifier g_θ by smoothing a base classifier f_θ. For any $\sigma > 0$, the smooth classifier is defined as: $g_\theta(x) = \mathbb{E}_{\epsilon \sim \mathcal{N}(0, \sigma^2 I)} \left[f_\theta(x + \epsilon) \right]$. Let g_θ predict label c_A for input x with some confidence, *i.e.* $\mathbb{E}_\epsilon[f_\theta^{c_A}(x + \epsilon)] = p_A \ge p_B = \max_{c \ne c_A} \mathbb{E}_\epsilon[f_\theta^c(x + \epsilon)]$, then, g_θ is certifiably robust at x with certification radius:

$$R = \frac{\sigma}{2} \left(\Phi^{-1}(p_A) - \Phi^{-1}(p_B) \right). \tag{1}$$

Here, $g(x + \delta) = g(x) \; \forall \|\delta\|_2 \le R$, where Φ is the CDF of the standard Gaussian.

3.3 ROBUSTNESS-ACCURACY TRADE-OFF

Note that Equation 1 holds regardless of the prediction c_A made by the smooth classifier g_θ. This suggests that one can perhaps improve the robustness of g_θ, *i.e.* increase certification radius R where g_θ is constant, by increasing the hyper parameter σ in Equation 1. However, to reason about ℓ_2^r certification accuracy, it is not enough to increase the certification radius R, as this requires c_A to be the correct prediction for x by g_θ. This reveals the robustness-accuracy trade-off as one cannot improve ℓ_2^r certified accuracy by only increasing the certification radius R (robustness) through the increase in σ. This is because it comes at the expense of requiring a classifier g_θ that correctly classifies x with correct label y under large Gaussian perturbations (accuracy). As such, the following inequality should hold $\mathbb{E}_\epsilon[f_\theta^{y}(x + \epsilon)] \ge p_A \ge p_B \ge \max_{c \ne y} \mathbb{E}_\epsilon[f^c(x + \epsilon)]$.

Algorithm 1: Data Dependent Certification

Function `OptimizeSigma` $(f_\theta, x, \alpha, \sigma_0, n)$**:**
- **Initialize:** $\sigma_x^0 \leftarrow \sigma_0$, K
- **for** $k = 0 \ldots K - 1$ **do**
  - sample $\hat{\epsilon}_1, \ldots \hat{\epsilon}_n \sim \mathcal{N}(0, I)$
  - $\psi(\sigma_x^k) = \frac{1}{n} \sum_{i=1}^n f_\theta(x + \sigma_x^k \hat{\epsilon}_i)$
  - $E_A(\sigma_x^k) = \max_c \psi^c$; $y_A = \arg\max_c \psi^c$;
  - $E_B(\sigma_x^k) = \max_{c \ne y_A} \psi^c$
  - $R(\sigma_x^k) = \frac{\sigma_x^k}{2} \left(\Phi^{-1}(E_A) - \Phi^{-1}(E_B) \right)$
  - $\sigma_x^{k+1} \leftarrow \sigma_x^k + \alpha \nabla_{\sigma_x^k} R(\sigma_x^k)$
- $\sigma_x^* \leftarrow \sigma_x^K$
- **return** σ_x^*

3.4 DATA DEPENDENT SMOOTHING FOR CERTIFICATION

The certification region $\mathcal{R} = \{\delta : \|\delta\|_2 \le R\}$ at an input x is fully characterized by the classifier f_θ and the standard deviation of the Gaussian distribution σ. Moreover, for a given f_θ, the certification region $\mathcal{R}$ varies at different x, when σ is fixed, due to the nonlinear dependence of the prediction gap $\Phi^{-1}(p_A(x; \sigma)) - \Phi^{-1}(p_B(x; \sigma))$ on x. This hints that, for a given f_θ, different inputs x may enjoy a different optimal σ_x^* that maximizes the certification region. To see this, consider the three inputs x_1, x_2 and x_3 all correctly classified by the binary classifier f_θ as $\mathcal{C}_1$ in Figure 1. Using a fixed σ to smooth the predictions of f_θ, *i.e.* predict with g_θ, reveals that inputs, depending on how close they are from the decision boundaries, can enjoy different levels of smoothing without affecting the prediction of g_θ. For instance, as shown in Figure 1 for constant σ, the input far from the decision boundary x_1 could have still been classified correctly with similarly large prediction gap even if f_θ were to be smoothed with a larger σ. This indicates that perhaps the certification radius at x_1 could have been enlarged with a larger smoothing σ. As for x_2, we can observe that while the prediction under this choice of σ by g_θ is still correct, the prediction gap $\Phi^{-1}(p_A(x; \sigma)) - \Phi^{-1}(p_B(x; \sigma))$ drops, due to having more Gaussian samples fall in the $\mathcal{C}_2$ region. Thus, a different choice of σ could have been used to trade-off the drop in prediction gap and certification radius. Last, for the input x_3 that is very close to the decision boundary, the sub optimal choice of σ (too large for x_3) could result in an incorrect prediction by g_θ. Despite the observations that σ plays a significant role in ℓ_2^r certification accuracy, certification methods generally (**i**) choose σ arbitrarily and (**ii**) set it to be constant for all x. Based on this observation, for a given smooth classifier with a specific σ_0, where σ_0 can be zero reducing the smooth classifier to f_θ, we seek to construct another smooth classifier with parameter σ_x^* for every input x such that: (**i**) the prediction of both smooth classifiers (smoothing with σ_0 and σ_x^*) is identical for all x. (**ii**) The certification radius of the new smooth classifier at every x is maximized. To construct a classifier smoothed with σ_x^* enjoying the two previous properties, let c_A be the prediction under σ_0 smoothing, *i.e.* $c_A = \arg\max_c \mathbb{E}_{\epsilon\sim\mathcal{N}(0,\sigma_0 I)}[f^c(x+\epsilon)]$. We maximize R in Equation 1 over σ for every x by solving:

$$\sigma_x^* = \arg\max_\sigma \frac{\sigma}{2}\Bigg(\Phi^{-1}\left(\mathbb{E}_{\epsilon\sim\mathcal{N}(0,\sigma^2 I)}[f_\theta^{c_A}(x+\epsilon)]\right) - \Phi^{-1}\left(\max_{c\neq c_A}\mathbb{E}_{\epsilon\sim\mathcal{N}(0,\sigma^2 I)}[f_\theta^{c}(x+\epsilon)]\right)\Bigg). \tag{2}$$

Since Φ^{-1} is a strictly increasing function, it is important to note that solving Equation 2 for a fixed c_A can at worst yield a smooth classifier of an identical radius to when the classifier is smoothed with σ_0 both predicting c_A for x.

Solver. While our proposed Objective 2 has a similar form to the MACER regularizer [Zhai et al., 2020] used during training, ours differs in that we optimize σ for every x and not the network parameters θ, which are fixed here. A natural solver for 2 is stochastic gradient ascent with the expectation approximated with n Monte Carlo samples. As such, the gradient of the objective at the k^{th} iteration will be approximated as follows: $\nabla_{\sigma^k}\frac{\sigma^k}{2}\left[\Phi^{-1}\left(\gamma^{c_A}(\sigma^k)\right) - \Phi^{-1}\left(\max_{c\neq c_A}\gamma^c(\sigma^k)\right)\right]$, where $\gamma^c(\sigma^k) = \frac{1}{n}\sum_{i=1}^n f^c(x+\epsilon_i)$ for $\epsilon_1,\dots,\epsilon_n \sim \mathcal{N}(0,(\sigma^k)^2 I)$. However, this estimation of the gradient suffers from high variance due to the dependence of the expectation on the optimization variable σ that parameterizes the smoothing distribution $\mathcal{N}(0,\sigma^2 I)$ [Williams, 1992]. To alleviate this, we use the *reparameterization trick* suggested by Kingma and Welling [2014], Rezende et al. [2014] to compute a lower variance gradient estimate for our Objective 2. In particular, with the change of variable $\epsilon = \sigma\hat{\epsilon}$ where $\hat{\epsilon}\sim\mathcal{N}(0,I)$, Objective 2 is *equivalent to*:

$$\sigma_x^* = \arg\max_\sigma \frac{\sigma}{2}\Bigg(\Phi^{-1}\left(\mathbb{E}_{\hat{\epsilon}\sim\mathcal{N}(0,I)}[f_\theta^{c_A}(x+\sigma\hat{\epsilon})]\right) - \Phi^{-1}\left(\max_{c\neq c_A}\mathbb{E}_{\hat{\epsilon}\sim\mathcal{N}(0,I)}[f_\theta^{c}(x+\sigma\hat{\epsilon})]\right)\Bigg) \tag{3}$$

Note that, unlike before, the expectation over the distribution $\hat{\epsilon}\sim\mathcal{N}(0,I)$ no longer depends on the optimization variable σ. This allows the gradient of 3 to enjoy a lower variance compared to the gradient of 2 [Kingma and Welling, 2014, Rezende et al., 2014]. Algorithm 1 summarizes the updates for optimizing σ for each x by solving 3 with K steps of stochastic gradient ascent. It is worthwhile to mention that the function `OptimizeSigma` in Algorithm 1 is agnostic of the choice of architecture f_θ and of the training procedure that constructed f_θ.

Algorithm 2: Training with Data Dependent σ_{x_i}

Function `TrainBatch(`f_θ, $\{x_i,y_i\}_{i=1}^B$, $\{\sigma_{x_i}\}_{i=1}^B$, α, n`)`**:**
  for $i = 1,\dots,B$ **do**
    $\sigma_{x_i}^* =$ `OptimizeSigma`$(f_\theta, x_i, \alpha, \sigma_{x_i}, n)$
  `TrainFunction` $\left(\{x_i,y_i\}_{i=1}^B, \{\sigma_{x_i}^*\}_{i=1}^B\right)$
    `// any training routine e.g. MACER`

3.5 MEMORY-BASED CERTIFICATION FOR DATA DEPENDENT CLASSIFIERS

Unlike previous approaches where σ is constant for all inputs, the data dependent classifier g_θ with varying σ per input can not be directly certified by the classical Monte Carlo algorithms proposed by Cohen et al. [2019]. This is since the data dependent classifier g_θ does not enjoy a constant σ within the given certification region, *i.e.* g_θ tailors a new σ_x for every input x including within the certified region of x. Informally, let $R(\sigma_{x_1}^*)$ be the radius of certification at x_1 granted by the data dependent classifier g_θ. The data dependent classifier *does not guarantee* that there *can not exist* x_2 within the region of certification of x_1, *i.e.* $\|x_1 - x_2\|_2 \leq R(\sigma_{x_1}^*)$, where g_θ with $\sigma_{x_2}^*$ predicts x_2 differently from x_1 breaking the soundness of certification. To circumvent this problem, we propose a memory-based procedure to certifying our proposed data dependent classifier. Let $\{x_i\}_{i=1}^N$ be a set of previously predicted inputs and $\{\mathcal{C}_i\}_{i=1}^N$ be their corresponding predictions with mutually exclusive ℓ_2 certified regions $\mathcal{R}_i$ for differently predicted inputs, *i.e.* $\mathcal{R}_i\cap\mathcal{R}_j = \emptyset\ \forall i\neq j, \mathcal{C}_i\neq\mathcal{C}_j$. Let x_{N+1} be a new input with a certified region $\mathcal{R}_{N+1}$ computed by the Monte Carlo algorithms of Cohen et al. [2019] for the data dependent classifier g_θ with prediction $\mathcal{C}_{N+1}$. If there exists an i such that $\mathcal{R}_{N+1}\cap\mathcal{R}_i\neq\emptyset$, $x_{N+1}\in\mathcal{R}_i$, and $\mathcal{C}_{N+1}\neq\mathcal{C}_i$, we adjust the prediction of the data dependent classifier g_θ to be $\mathcal{C}_i$ and update $\mathcal{R}_{N+1}$ to be the largest subset of $\mathcal{R}_{N+1}$ that is a subset of $\mathcal{R}_i$ (see middle example in Figure 2). On the other hand, if $\mathcal{R}_{N+1}\cap\mathcal{R}_i\neq\emptyset$, $x_{N+1}\notin\mathcal{R}_i$, and that $\mathcal{C}_{N+1}\neq\mathcal{C}_i$, we update $\mathcal{R}_{N+1}$ to be the largest subset of $\mathcal{R}_{N+1}$ not intersecting with $\mathcal{R}_i$ (see right example in Figure 2). We perform the previous operations for all elements in the memory and add $x_{N+1}, \mathcal{C}_{N+1}, \mathcal{R}_{N+1}$ to memory. The aforementioned procedure grants a sound certification for the data dependent classifier preventing by construction overlapping certified regions with different predictions.While the memory-based certification is essential for a sound certification, empirically, we never found in any of the later experiments a case where two inputs predicted differently suffer from intersecting certified regions. That is to say while our sound certificate works on the memory-enhanced data dependent smooth classifier, we found that the certified radius of the memory classifier for every input is the radius granted by the Monte Carlo certificates of Cohen

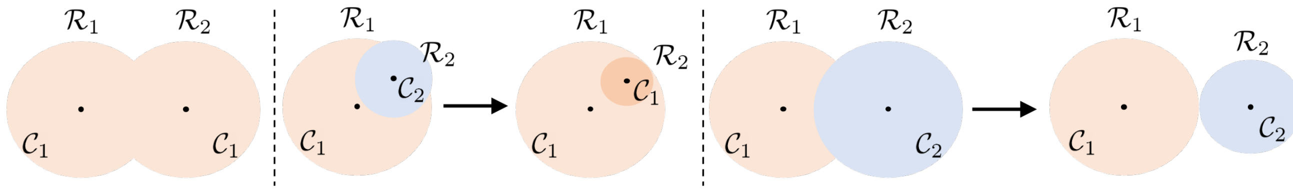

Figure 2: **Memory-based certification of the data dependent classifier.** Given a memory of an input x_1 with a certified region $\mathcal{R}_1$ and another input x_2 with a certified region $\mathcal{R}_2$. Three scenarios could arise where $\mathcal{R}_1$ and $\mathcal{R}_2$ intersect. **Left**: The certified regions intersect while both x_1 and x_2 share the same prediction. In this case, x_2 along with its certified region are directly added to memory. **Middle**: x_2 lies inside $\mathcal{R}_1$ with a different prediction from x_1. In this case, x_2 is predicted with the same prediction as x_1 and added to memory along with the largest subset of $\mathcal{R}_2$ that is within $\mathcal{R}_1$. **Right**: x_2 lies outside the $\mathcal{R}_1$ with a different prediction from x_1 In this case, x_2 with its prediction are added to memory along with the largest certified region in $\mathcal{R}_2$ not intersecting with $\mathcal{R}_1$.

et al. [2019] for the data dependent classifier. Therefore and throughout, we refer to the memory-enhanced data dependent smooth classifier and data dependent smooth classifier interchangeably. We elaborate more on this and provide an algorithm in the **Appendix**.

3.6 TRAINING WITH DATA DEPENDENT SMOOTHING

Models that enjoy a large ℓ_2^r certification accuracy under the randomized smoothing framework need to enjoy a large certification radius R in Equation 1 for all x and be able to correctly classify inputs corrupted with Gaussian noise, *i.e.* $g_\theta(x) = y$. While there are several approaches to train f_θ (or directly g_θ) so as to output correct predictions for inputs corrupted with noise sampled from $\mathcal{N}(0, \sigma^2 I)$, all existing works fix σ for all inputs during training. We are interested in complementing these approaches with smoothing distributions that are data dependent. As such, we can employ the training procedure of these approaches but with σ_x^* computed by `OptimizeSigma`. Algorithm 2 summarizes this proposed training pipeline. The function `TrainFunction` proceeds by performing backpropagation using any training scheme, given the estimated $\sigma_{x_i}^*$ for each x_i. We note that whenever Algorithm 2 is used, we initialize σ_{x_i} at each epoch with $\sigma_{x_i}^*$ computed at the previous epoch. Since COHEN, SMOOTHADV and MACER are among the most popular approaches that embed randomized smoothing certificates as part of the training routine, `TrainFunction` refers here to any of these three training methods. Empirically, we show that we can boost all three methods even further when models are trained with Algorithm 2.

4 EXPERIMENTS

We conduct two sets of experiments to validate our key contributions. (**i**) We show that we can boost certified accuracy for several pre-trained models by using Algorithm 1 for data dependent smoothing only during certification, *i.e.* without employing any additional training. (**ii**) Once data dependent smoothing is employed during training, we can improve the certified accuracy even further. Since our framework is agnostic to the training routine, we incorporate it into (**i**) COHEN [Cohen et al., 2019], (**ii**) SMOOTHADV [Salman et al., 2019a] and (**iii**) MACER [Zhai et al., 2020]. Throughout, we use DS to refer to when data dependent smoothing is used only in certification and DS^2 when it is used during both training and certification.

Setup. We conduct experiments with ResNet-18 and ReNet-50 [He et al., 2016] on CIFAR10 [Krizhevsky and Hinton, 2009] and ImageNet [Russakovsky et al., 2015], respectively. For CIFAR10 experiments, we train from scratch for 200 epochs. For ImageNet, we initialize using the network parameters provided by the authors. When σ is fixed and following prior art, *e.g.* COHEN, SMOOTHADV, and MACER, we set $\sigma \in \{0.12, 0.25, 0.50\}$ and $\sigma \in \{0.25, 0.50, 1.0\}$ for CIFAR10 and ImageNet, respectively, for training and certification. We set $\alpha = 10^{-4}$ in Algorithm 1 and the initial σ_0 to the σ used in training the respective model. Unless stated otherwise, we set $n = 1$ in Algorithm 1. Following COHEN and SMOOTHADV, we compare models using the approximate certified accuracy curve (simply referred to as certified accuracy) followed by the envelope curve over all σ. We also report the Average Certified Radius (ACR) proposed by MACER $1/|\mathcal{S}_{test}| \sum_{(x,y)\in\mathcal{S}_{test}} R(f_\theta, x).\mathbb{1}\{\arg\max_c g_\theta^c(x) = y\}$, where $\mathbb{1}\{.\}$ is an indicator function. Following COHEN and all randomized smoothing methods, we certify all results using $N_0 = 100$ Monte Carlo samples for prediction and $N = 100,000$ estimation samples to estimate the radius with a failure probability of 0.001 given a smoothing σ.

4.1 COHEN + DS

We combine data dependent smoothing with COHEN. Following Gaussian augmentation, this method trains f_θ on $(x + \epsilon)$, where $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$, with the cross entropy loss.

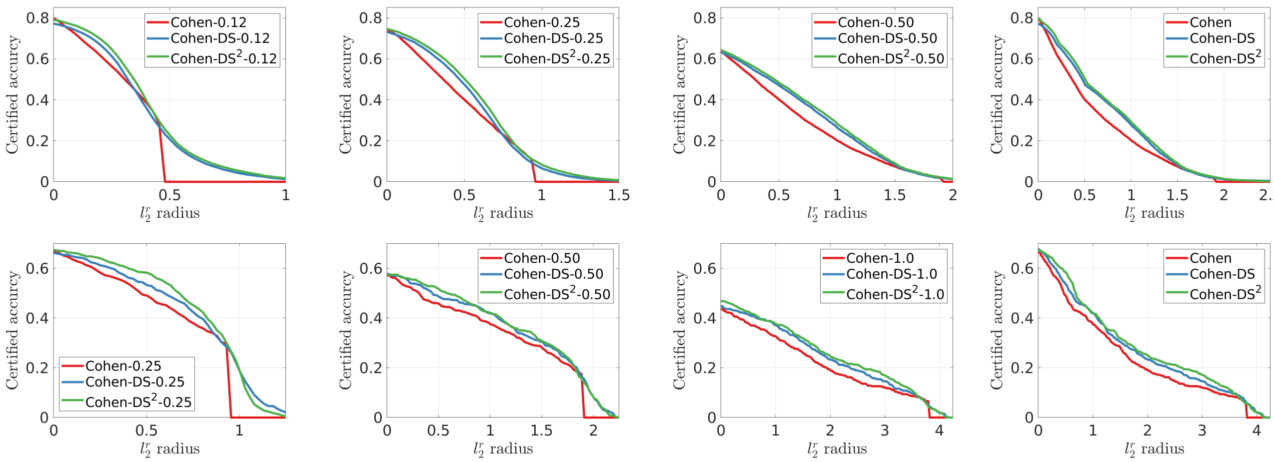

Figure 3: **Certified accuracy comparison against Cohen per radius per σ.** We compare Cohen against our data dependent certification Cohen-DS and when data dependency is incorporated in both training and certification Cohen-DS^2 for several σ. The value of σ shown for our models in the legend refers to the optimization initialization σ_0 in Algorithm 1. We show CIFAR10 and ImageNet results in first and second rows, respectively, where the last column is the envelope.

Table 1: **Best certified accuracy per radius and ACR** of Cohen, Cohen-DS and Cohen-DS^2.

CIFAR10	Radius Train	Certify	0.0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	ACR
Cohen	FS	FS	**79.9**	58.3	40.1	29.2	20.2	13.1	7.3	3.3	0.0	0.0	0.0	0.591
Cohen-DS	FS	DS	77.2	64.5	47.8	38.3	27.6	16.5	8.0	3.2	1.2	**0.7**	**0.5**	**0.784**
Cohen-DS^2	DS	DS	79.8	**66.5**	**50.4**	**39.2**	**29.1**	**18.3**	**8.8**	**3.8**	**1.4**	0.6	0.2	0.764
ImageNet	**Radius Train**	**Certify**	**0.0**	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**	**2.0**	**2.5**	**3.0**	**3.50**	**4.0**	**ACR**
Cohen	FS	FS	66.6	58.2	49.0	42.4	37.4	27.8	19.4	14.4	12.0	8.6	0.0	1.098
Cohen-DS	FS	DS	**67.8**	61.4	53.6	45.6	**42.0**	30.4	23.4	18.8	14.6	10.2	**2.0**	1.257
Cohen-DS^2	DS	DS	67.4	**64.2**	**58.4**	**47.4**	41.8	**31.8**	**25.0**	**21.2**	**17.2**	**11.0**	**2.0**	**1.319**

DS for certification only. We first certify the trained models with the same fixed σ used in training for all inputs, dubbed COHEN. Then, we certify using the memory based certification the same trained models with the proposed data dependent σ_x^* produced by Algorithm 1, which we refer to as COHEN-DS. Figure 3 plots the certified accuracy for CIFAR10 and ImageNet in the first and second rows, respectively. Even though the base classifier f_θ is identical for COHEN and COHEN-DS, Figure 3 shows that COHEN-DS is superior to COHEN in certified accuracy across almost all radii and for all training σ on both datasets. This is also evident from the envelope plots in the last column of Figure 3. In Table 1, we report the best certified accuracy per radius over all training σ for COHEN (envelope figure) against our best COHEN-DS, cross-validated over all training σ and the number of iterations in Algorithm 1 K, accompanied with the corresponding ACR score. For instance, we observe that data dependent certification COHEN-DS can significantly boost certified accuracy at radii 0.5 and 0.75 by 7.7% (from 40.1 to 47.8) and 9.1% (from 29.2% to 38.3%), respectively, and by 0.193 ACR points on CIFAR10. Moreover, we boost the certified accuracy on ImageNet by 4.6% and 3.2% at 0.5 and 0.75 radii, respectively, and by 0.159 ACR points.

DS for training and certification. We employ data dependent smoothing in both training and certification for COHEN models (denoted as COHEN-DS^2) by running Algorithm 2. For CIFAR10, we train COHEN first with fixed σ for 50 epochs, *i.e.* $K = 0$ in Algorithm 1, and then we perform data dependent smoothing with $K = 1$ for the remaining 150 epochs. For ImageNet experiments, we only finetune the provided models for 30 epochs using Algorithm 2 with $K = 1$. Once training is complete, we certify all trained models with Algorithm 1 using the memory based certification. In Figure 3, we observe that COHEN-DS^2 can further improve certified accuracy across all trained models on both CIFAR10 and ImageNet. This is also evident in the last column of Figure 3 that shows the best certified accuracy per radius

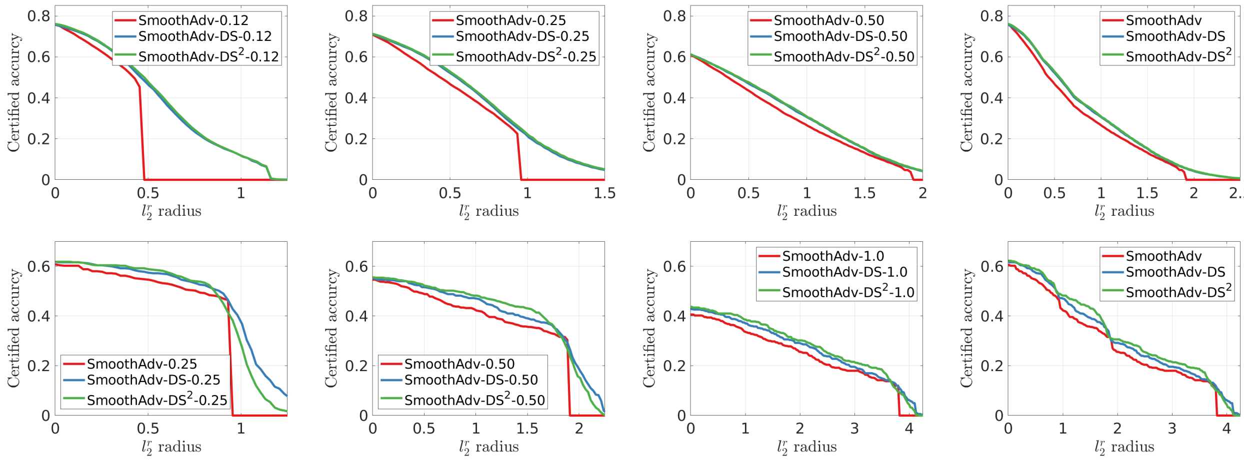

Figure 4: **Certified accuracy comparison against SmoothAdv per radius per σ.** We compare SmoothAdv against SmoothAdv-DS and SmoothAdv-DS^2. We show CIFAR10 and ImageNet results in first and second rows, respectively.

Table 2: **Best certified accuracy per radius and ACR** of SmoothAdv, SmoothAdv-DS and SmoothAdv-DS^2.

CIFAR10	Radius Train	Radius Certify	0.0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	ACR
SmoothAdv	FS	FS	76.0	62.4	46.7	34.6	26.5	19.5	12.9	7.5	0.0	0.0	0.0	0.681
SmoothAdv-DS	FS	DS	75.7	66.4	52.1	38.8	30.6	22.2	15.0	8.5	4.2	1.8	0.6	0.799
SmoothAdv-DS^2	DS	DS	**76.2**	**66.8**	**52.8**	**39.3**	**30.8**	**22.6**	**15.1**	**8.8**	**4.3**	**2.0**	**0.7**	**0.812**
ImageNet	**Radius Train**	**Radius Certify**	**0.0**	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**	**2.0**	**2.5**	**3.0**	**3.50**	**4.0**	**ACR**
SmoothAdv	FS	FS	60.8	57.8	54.6	50.4	42.2	35.6	25.6	20.4	18.0	14.2	0.0	1.287
SmoothAdv-DS	FS	DS	62.0	60.4	57.4	53.2	47.0	39.2	29.2	23.8	19.6	15.2	**6.2**	1.445
SmoothAdv-DS^2	DS	DS	**62.2**	**60.6**	**58.8**	**54.2**	**48.2**	**43.0**	**30.6**	**25.4**	**21.6**	**18.6**	4.2	**1.514**

(envelope) over all training σ. We note that COHEN-DS^2 improves the certification accuracy of COHEN-DS by 2.6% and by 0.9% at radii 0.5 and 0.75 respectively on CIFAR10, and by 4.8% and 1.8% at radii 0.5 and 0.75 respectively on ImageNet. The improvements are consistently present over a wide range of radii on both datasets. We do observe that the ACR score for COHEN-DS^2 on CIFAR10 marginally drops compared to COHEN-DS. We believe that this is due to the fact that some inputs that are classified correctly at the small radii have an overall larger certification radius for COHEN-DS compared to COHEN-DS^2 on CIFAR10. Regardless, COHEN-DS^2 substantially outperforms COHEN by 0.173 ACR points. As compared to COHEN-DS, COHEN-DS^2 improves the ACR on ImageNet from 1.257 to 1.319.

4.2 SMOOTHADV + DS

We combine our data dependent smoothing strategy with the more effective SMOOTHADV, which trains the smoothed classifier for every x on the adversarial example $\hat{x}$ that maximizes $-\log \mathbb{E}_{\epsilon\sim\mathcal{N}(0,\sigma^2 I)}\left[f_\theta^y(x'+\epsilon)\right]$, where $\|x'-x\| \leq \zeta$.For CIFAR10 experiments, we follow the training procedure of SMOOTHADV, where the adversary $\hat{x}$ is computed with 2 PGD (proximal gradient descent) steps with $\zeta = 0.25$ and one augmented sample to estimate the expectation. For ImageNet experiments, we use the best reported models, in terms of certified accuracy, provided by the authors, which correspond to $\zeta = 0.5$ for $\sigma = 0.25$ and $\zeta = 1.0$ for $\sigma \in \{0.5, 1.0\}$.

DS for certification only. Similar to COHEN, we first certify SMOOTHADV models trained with the same fixed σ. Then, we certify the proposed data dependent σ_x^* models using the memory-based certification, which we refer to as SMOOTHADV-DS. In Figure 4, we show the certified accuracy for both CIFAR10 and ImageNet in the first and second rows, respectively. The last column shows the envelopes per radius. Even though they both share the same classifier f_θ, SMOOTHADV-DS significantly improves upon SMOOTHADV over all radii and all values of σ in training for both CIFAR10 and ImageNet. In particular, for models trained with $\sigma = 0.25$, SMOOTHADV achieves a zero cer-

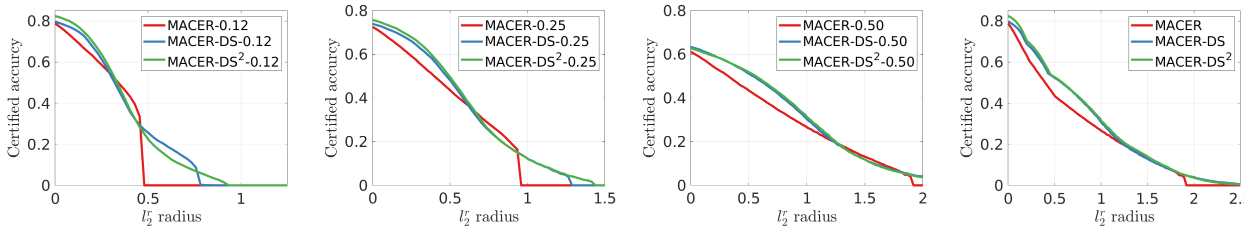

Figure 5: **Certified accuracy comparison against MACER per radius per σ.** We compare MACER against MACER-DS and MACER-DS^2 for several σ on CIFAR10 with the last column showing the envelope.

Table 3: **Best certified accuracy per radius and ACR** of MACER, MACER-DS and MACER-DS^2 on CIFAR10.

CIFAR10	Radius Train	Certify	0.0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	ACR
MACER	FS	FS	78.8	59.3	43.6	34.7	26.6	19.4	13.0	7.50	0.0	0.0	0.0	0.702
MACER-DS	FS	DS	79.5	66.7	52.3	43.0	30.8	19.5	12.8	7.55	**3.97**	**1.67**	**0.5**	**0.841**
MACER-DS^2	DS	DS	**82.4**	**68.3**	**52.7**	**43.5**	**31.7**	**20.6**	**13.8**	**7.92**	3.65	1.39	0.4	0.807

tified accuracy for large certification radii (≥ 1.0), while SMOOTHADV-DS achieves non-trivial certified accuracy in these cases. Similar to the earlier setup, we report the best certified accuracy along with the ACR scores in Table 2. We improve over SMOOTHADV by large margins. For example, the certified accuracy at 0.5 radius increases by 5.4% and 2.8% on CIFAR10 and Imagenet, respectively. The improvement is consistent over all radii. The ACR also improves by 0.118 and 0.158 on CIFAR10 and ImageNet, respectively.

DS for training and certification. We fine tune the SMOOTHADV trained models (either the retrained CIFAR10 models or the ImageNet models provided by SMOOTHADV) using Algorithm 2, where σ_x^* is computed using Algorithm 1. We report the per σ certification accuracy comparing SMOOTHADV-DS^2 (certified also using memory based certification) to both SMOOTHADV-DS and SMOOTHADV. SMOOTHADV-DS^2 further improves the certified accuracy as compared to SMOOTHADV-DS with performance gains more prominent on ImageNet. While the improvement of SMOOTHADV-DS^2 over SMOOTHADV-DS is indeed small, *e.g.* 0.7% at radius 0.5 on CIFAR10, we observe that the performance gaps are much larger on ImageNet reaching 1.4% at 0.5 radius as shown in Table 2. We see a similar trend in ACR with improvements of 0.013 and 0.069 on CIFAR10 and ImageNet, respectively. SMOOTHADV-DS^2 boosts the certified accuracy of SMOOTHADV at radius 0.5 by 6.1% and 4.2% on CIFAR10 and ImageNet, respectively.

4.3 MACER + DS

We integrate data dependent smoothing within MACER which trains g_θ by minimizing over the parameters θ the following objective $-\log g_\theta(x) + \frac{\lambda\sigma}{2}\max\left(\gamma - \frac{2R}{\sigma}, 0\right).\mathbb{1}\{\arg\max_c\ g_\theta^c(x) = y\}$. where R

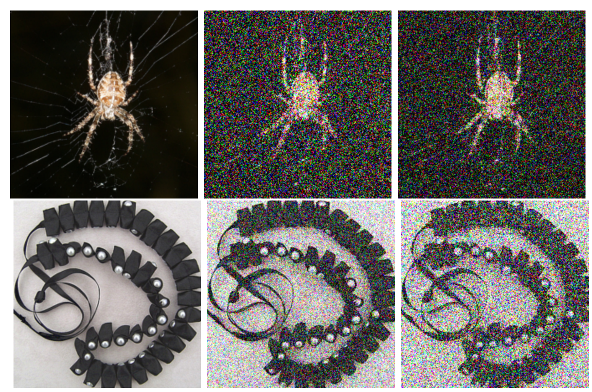

Figure 6: **Qualitative examples of estimated σ_x^* on different inputs.** From left to right of first row: clean image, fixed $\sigma = 0.5$ and estimated $\sigma_x^* = 0.368$ maximizing certification radius. Similarly for second row but with $\sigma = 0.25$ and $\sigma_x^* = 0.423$. This demonstrates that σ^* that maximizes the radius should vary per input x.

also depends on θ. While this seems to be similar in spirit to our approach, we in fact maximize the certification radius over σ with fixed parameters θ for every x. We conduct experiments on CIFAR10 following the training procedure of MACER estimating the expectation with 64 samples, $\lambda = 12$, and $\gamma = 8$. We set $n = 8$ in Algorithm 1 with ablations on $n = 1$ in the **appendix**.

DS for certification only. Similar to the earlier setup in COHEN and SMOOTHADV, we certify models with fixed σ and then with data dependent σ_x^* using the memory based certification, referred to as MACER-DS. In Figure 5, we observe that MACER-DS significantly outperforms MACER particularly in the large radius region. This can also be seen in the envelope figure reporting the best certified accuracy

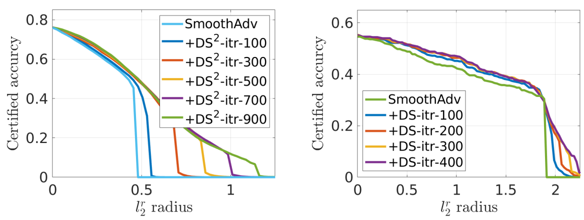

Figure 7: **Varying K in Algorithm 2.** Left figure shows certification with $\sigma_0 = 0.12$ on CIFAR10 and $\sigma_0 = 0.5$ on ImageNet is shown at the right.

per radius over σ. Similarly, Table 3 demonstrates the benefits of data dependent smoothing, where it boosts certified accuracy by 7.4% (from 59.3% to 66.7%) and 8.7% (43.6 to 52.3) at 0.25 and 0.5 radii, respectively. Moreover, we improve ACR by 0.139 points.

DS for training and certification. We incorporate data dependent smoothing as part of MACER training and certification in a similar fashion to the earlier setup, dubbed MACER-DS^2. Figure 5 shows the improvement of MACER-DS^2 over the certification only MACER-DS over all trained models. Table 3 summarizes the best certified accuracy per radius. Overall, we find that the performance is comparable or slightly better than MACER-DS, which is still significantly better than MACER by 8.67% at radius 0.5. We also observe that MACER-DS enjoys better ACR than MACER-DS^2 with both being far better than the MACER baseline.

4.4 DS FOR ℓ_1 CERTIFICATES

At last, we extend our methodology to ℓ_1 certification. We leveraged the results of Yang et al. [2020] that derived the tightest ℓ_1 certificate using randomized smoothing with uniform distribution $\mathcal{U}[-\lambda, \lambda]^d$. The certified radius in that case has the form $\mathcal{R}_1 = \lambda(p_A - p_B)$. We replace our objective in Equation (3) with:

$$\lambda_x^* = \arg\max_{\lambda} \lambda \Bigg(\mathbb{E}_{\epsilon \sim \mathcal{U}[-\lambda,\lambda]^d} (f_\theta^{c_A}(x + \epsilon)) - \max_{c \neq c_A} \mathbb{E}_{\epsilon \sim \mathcal{U}[-\lambda,\lambda]^d} (f_\theta^{c}(x + \epsilon)) \Bigg). \quad (4)$$

We solved our objective in Eq (3) in an identical fashion to our Algorithm 1 with the same hyperparameters for $\lambda \in \{0.25, 0.5, 1.0\}$ in certification on both CIFAR10 and ImageNet. Further, we combine our data-dependent smooth classifier with the memory based algorithm proposed in Section 3.5. It is worthwhile mentioning that similar to the ℓ_2 case, the memory based algorithm did not find any overlap between the certified regions of any pair of instances. We report the results in Table 4. We observe that, similar to our extensive experiments on the ℓ_2 certificate, our proposed memory-enhanced data-dependent smoothing yields consistent improvement in the ℓ_1 certified accuracy. We report an improvement of 7% and 3% over the state of the art certified accuracy at ℓ_1 radius of 0.5 on CIFAR10 and ImageNet, respectively. At last, we note similar improvement to the ℓ_1 ACR as reported in Table 4.

Table 4: **Best certified accuracy per ℓ_1 radii and ACR** of YANG and YANG-DS.

ℓ_1^r(CIFAR10)	0.0	0.25	0.5	0.75	1.0	1.5	2.0	ACR
YANG	92	83	75	71	46	0	0	0.775
YANG-DS	92	**89**	**82**	**76**	**58**	**6**	**2**	**0.946**
ℓ_1^r(ImageNet)	0.0	0.25	0.5	0.75	1.0	1.5	2.0	ACR
YANG	78	73	67	63	0	0	0	0.683
YANG-DS	**79**	**76**	**70**	**65**	**46**	0	0	**0.729**

4.5 DISCUSSION AND ABLATION

Varying K. We pose the question: does attaining better solutions to our proposed Objective 3 improve certified accuracy? To answer this, we control the solution quality of σ_x^* by certifying trained models with a varying number of stochastic gradient ascent iterations K in Algorithm 1. In particular, we certify the trained models SMOOTHADV-DS^2 and SMOOTHADV-DS on CIFAR10 and ImageNet, respectively, with a varying K. We leave the rest of the experiments for other models to the **appendix**. We observe in Figure 7 that the certified accuracy per radius consistently improves as K increases, particularly in the large radius regime. This is expected, since Algorithm 1 produces better optimal smoothing σ_x^* per input x with larger K, which in turn improves the certification radius leaving room for improvements with more powerful optimizers.

Visualizing σ_x^*. We show the variation of σ_x^* that maximizes the certification radius over different inputs x. Figure 6 shows two examples, where the first and fourth columns contain the clean images. In the second column, a choice of fixed $\sigma = 0.5$ is too large compared to our estimated $\sigma_x^* = 0.368$ that maximizes the certification radius as per Algorithm 1. As for the fifth column, we observe that a constant $\sigma = 0.25$ is far less than $\sigma_x^* = 0.423$. This indicates that indeed the σ_x^* maximizing the certification radius varies significantly over inputs.

5 CONCLUSION

In this work, we presented a simple and generic framework to equip randomized smoothing techniques with data dependency. We demonstrated that combining data dependent smoothing with 3 randomized smoothing techniques provided substantial improvement in their certified accuracy.

Acknowledgements

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A IMPLEMENTATION DETAILS

For reproducibility, we will release the full code upon acceptance. Nevertheless, we give the detailed implementation of Algorithm 1 in PyTorch Paszke et al. [2019] below.

```
import torch
from torch.autograd import Variable
from torch.distributions.normal import Normal
def OptimzeSigma(model, batch, alpha, sig_0, K, n):
    device='cuda:0'
    batch_size = batch.shape[0]

    sig = Variable(sig_0, requires_grad=True).view(batch_size, 1, 1, 1)
    m = Normal(torch.zeros(batch_size).to(device), torch.ones(batch_size).to(device))

    #Reshaping for n > 1
    new_shape = [batch_size * n]
    new_shape.extend(batch_size)
    new_batch = batch.repeat((1,n, 1, 1)).view(new_shape)
    sigma_repeated = sig.repeat((1, n, 1, 1)).view(-1,1,1,1)

    for _ in range(K):
        eps = torch.randn_like(new_batch)*sigma_repeated #Reparamitrization trick
        out = model(new_batch + eps).reshape(batch_size, n, 10).mean(1) #10 for CIFAR10

        vals, _ = torch.topk(out, 2)
        vals.transpose_(0, 1)
        gap = m.icdf(vals[0].clamp_(0.02, 0.98)) - m.icdf(vals[1].clamp_(0.02, 0.98))
        radius = sig.reshape(-1)/2 * gap  # The radius formula
        grad = torch.autograd.grad(radius.sum(), sig)

        sig.data += alpha*grad[0]  # Gradient Ascent step

    return sig.reshape(-1)
```

For comparisons against Cohen et al. [2019], we followed their official code in https://github.com/locuslab/smoothing. We also followed the common practice in using their provided code for certifying all models in all of our experiments. For comparisons against *SmoothAdv* [Salman et al., 2019a], we also followed their official implementation in https://github.com/Hadisalman/smoothing-adversarial and similarly for *MACER* https://github.com/RuntianZ/macer.

B DATA DEPENDENT GREEDY SEARCH OVER σ

We observe that the optimization problem (3) that we solve for every input x is one dimensional in σ_x^*. In this section, we show that heuristic grid search procedures are far inferior to solving Equation (3) with our solver in Algorithm 1. In particular, we show that under the same sample complexity as our approach for data dependent certification a trivial heuristic grid search as a baseline does not work. We conduct experiments where we only certify with data dependent smoothing a pre-trained model SmoothAdv with training $\sigma \in \{0.12, 0.25.0.50\}$ on CIFAR10. We examine a single model SmoothAdv-DS which is certified with $K = 100$ iterations and with $n = 1$ to approximate the expectation in Algorithm 1. Observe that since $n = 1$, and including the forward and backward passes computation, our data dependent certification of SmoothAdv-DS has a total of 200, since $K = 100$, evaluations for every given x before performing the certification with the optimized σ_x^*. To that end, we compare against a crude grid search baseline over $\hat{\sigma}_x^*$, and for a fair comparison, with a total of 200 evaluations. We restrict the grid search to $\hat{\sigma}_x^* \in [0, 1]$ with a resolution of $n/200$ so that the total number of evaluations is always exactly 200 similar to our SmoothAdv-DS. That is to say, the grid heuristic search solves the following problem:

$$\hat{\sigma}_x^* = \underset{\sigma_i \in \{0, \frac{n}{200}, \frac{2n}{200}, \dots, 1-\frac{n}{200}, 1\}}{\arg\max} \frac{\sigma_i}{2} \left(\Phi^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} \hat{f}_\theta^{c_A}(x + \sigma_i \hat{\epsilon})] \right) - \Phi^{-1} \left(\max_{c \neq c_A} \frac{1}{n} \sum_{i=1}^{n} \hat{f}_\theta^{c}(x + \sigma_i \hat{\epsilon})] \right) \right). \tag{5}$$

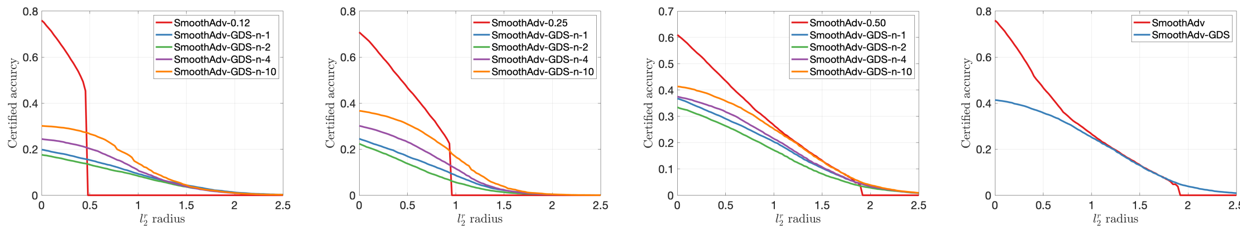

Figure 8: **Certified accuracy per radius per σ comparison of SmoothAdv against greedy search heuristic data dependent smoothing on CIFAR10.** We observe that when the number of samples n in Equation 5 the better the performance. Moreover, and despite that the total number of evaluations of the data dependent certification matches our approach SmoothAdv-DS reported in Figure 4, the performance is still far inferior to even data dependent certification as also evident from the last envelope figure.

We also explore with the number of samples to $n \in \{1, 2, 4, 10\}$ for the grid search pipeline. Note that this trades-off the accuracy of the expectation approximation to the resolution of the solution $\hat{\sigma}_x^*$.

We summarize our results in Figure 8. Note that in the first three figures, we report certified accuracies for when the model is certified with the same $\sigma = \{0, 12, 0.25, 0.50\}$ used in training without data dependent smoothing, i.e. fixed σ for all inputs. We refer to these plots as SmoothAdv-0.12, SmoothAdv-0.25 and SmoothAdv-0.50. In addition, we refer to the data dependent baseline grid search heuristic as SmoothAdv-GDS-n-1, SmoothAdv-GDS-n-2, SmoothAdv-GDS-n-4, and SmoothAdv-GDS-n-10 where n refers to the number of samples approximating the expectation in Equation (5). We report the envelops in the last figure.

At first we observe that the larger n used to approximate the expectation, the better the overall certification accuracy. This is regardless of the σ used to train the model. However, the performance is still far inferior to the baseline that is data *independent* which is inferior to our approach. This is also evident from the envelope last figure. This indicates that while data dependent smoothing is essential towards improving performance, a careful optimization is necessary for it to work. We reiterate here that both the grid search heuristic and our approach use the same number of evaluations, *i.e.* 200, when certifying the model; however, our approach reported in Figure 4 are far more superior.

C MEMORY-BASED CERTIFICATION FOR DATA DEPENDENT CLASSIFIERS

Algorithm 3: Memory-Based Certification

Input: input point x_{N+1}, certified region $\mathcal{R}_{N+1}$, prediction $\mathcal{C}_{N+1}$, and memory $\mathcal{M}$
Result: Prediction for x_{N+1} and certified region at x_{N+1} that does not intersect with any certified region in $\mathcal{M}$.
for $(x_i, \mathcal{C}_i, \mathcal{R}_i) \in \mathcal{M}$ **do**
  if $\mathcal{C}_{N+1} \neq \mathcal{C}_i$ **then**
    if $x_{N+1} \in \mathcal{R}_i$ **then**
      $\tilde{\mathcal{R}}_{N+1}$ = $\texttt{LargestInSubset}(\mathcal{R}_i, \mathcal{R}_{N+1})$,
      $\mathcal{R}_{N+1} \leftarrow \tilde{\mathcal{R}}_{N+1}$;
      $\mathcal{C}_{N+1} \leftarrow \mathcal{C}_i$
    else if $\textit{Intersect}(\mathcal{R}_{N+1}, \mathcal{R}_i)$ **then**
      $\mathcal{R}'_{N+1}$ = $\texttt{LargestOutSubset}(\mathcal{R}_i, \mathcal{R}_{N+1})$;
      $\mathcal{R}_{N+1} \leftarrow \mathcal{R}'_{N+1}$;
end
add $(x_{N+1}, \mathcal{C}_{N+1}, \mathcal{R}_{N+1})$ to $\mathcal{M}$;
return $\mathcal{C}_{N+1}, \mathcal{R}_{N+1}$;

Let $\mathcal{M} = \{(x_i, \mathcal{C}_i, \mathcal{R}_i)\}_{i=1}^N$ be set of the triplets: the input x_i, the prediction of x_i denoted by $\mathcal{C}_i$ and the certification region at x_i denoted as $\mathcal{R}_i$ which is characterized by the certification radius R_i and the center x_i. Moreover, we assume that $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset, \forall i \neq j, \mathcal{C}_i \neq \mathcal{C}_j$. That is to say, none of the certification regions of the inputs stored in the memory $\mathcal{M}$ intersect for inputs with different predictions. This is the key property for a sound certification procedure. Otherwise, if such a property does not hold, then this implies that the data dependent classifier produces different predictions within the same certified region. In what follows, and to circumvent this nuisance in the data dependent classifier g_θ, we rely on updating the memory while enforcing this property to hold. In particular, we certify the data dependent classifier using the classical Monte Carlo approach by Cohen et al. [2019] while guaranteeing that the certified regions does not intersect with the certification region of any previously predicted inputs.

In what follows, we present Algorithm 3 that enforces the non-intersection property of certified regions $\mathcal{M}$. Let $\mathcal{R}_{N+1}$ be the certified region at x_{N+1} of g_θ. **(i)** If $x_{N+1} \in \mathcal{R}_i$ and $\arg\max_c g_\theta^c(x_{N+1}) \neq \mathcal{C}_i$, we find the largest $\tilde{\mathcal{R}}_{N+1}$ such that the following two properties hold $\tilde{\mathcal{R}}_{N+1} \subset \mathcal{R}_{N+1}$ and $\tilde{\mathcal{R}}_{N+1} \subset \mathcal{R}_i$. For when the certified regions $\mathcal{R}_i$ are simple ℓ_2-balls, finding the largest $\tilde{\mathcal{R}}_{N+1}$ satisfying previous two properties is straightforward. We denote this with the function `LargestInSubset` (second example in Figure 2). We then update $\mathcal{R}_{N+1}$ with the refined $\tilde{\mathcal{R}}_{N+1}$ and change $\mathcal{C}_{N+1}$ to $\mathcal{C}_i$. **(ii)** Otherwise, if $\mathcal{R}_{N+1} \cap \mathcal{R}_i \neq \emptyset$ where $x_{N+1} \notin \mathcal{R}_i$ and $\arg\max_c g_\theta^c(x_{N+1}) \neq \mathcal{C}_i$, we find $\mathcal{R}'_{N+1}$ such that $\mathcal{R}'_{N+1} \subseteq \mathcal{R}_{N+1}$ is the largest subset of $\mathcal{R}_{N+1}$ non-intersecting with $\mathcal{R}_i$. We denote this function `LargestOutSubset` (third example in Figure 2). We then update $\mathcal{R}_{N+1}$ with the refined $\mathcal{R}'_{N+1}$. Moreover, computing `LargestOutSubset` for when $\mathcal{R}_i$ are ℓ_2-balls is straightforward. At last, note that `Intersect` is a function that returns whether two ℓ_2-balls intersect. At last, we then add $(x_{N+1}, \mathcal{C}_{N+1}, \mathcal{R}_{N+1})$ to memory. We provide below a pytorch implementation of the memory-based certification of the pseudo-algorithm 3.

While the memory-based certification is essential for a sound certification, empirically on CIFAR10 and ImageNet, we never found in any of the experiments a case where two inputs predicted differently suffer from intersecting certified regions. That is to say, the certified regions in the memory for every input is the certified regions granted by the Monte Carlo certificates of Cohen et al. [2019] for the data dependent classifier. We hypothesize that this is due to the following reasons: **(i)** Image datasets have very high dimensionality, resulting in samples very far apart, compared with the certified radius that randomized smoothing could provide. Thus, it is very unlikely to find two samples that have intersecting certified regions. **(ii)** Even if the rare case where two image inputs are close to one another that their certified regions intersect, we found that the data dependent classifier g_θ predicts these inputs similarly (the left example of Figure 2). This is since the data dependent classifier is trained to output smooth prediction, *i.e.* prediction changes are small for small input changes, resulting in a shared prediction. **(iii)** To maintain reasonable test accuracy on clean samples, the values of σ, and correspondingly optimized σ_x^* used in smoothing are moderately low ($\sigma_x^* \leq 1.0$). This results in limited smaller certified regions, ℓ_2 balls of radii $\approx 4\sigma_x^*$ which is much smaller than the distance between inputs in higher dimensional data (*e.g.* ImageNet).

It is worthwhile mentioning that while the memory-based certificate could work, in principle, independently without being combined with the data dependent smooth classifier under any arbitrary choice of a certification radius for every input; this results in a sub-optimal certification. This is since this may result in one of the following situations: **(i)** Assigning large radii for every input will yield a classifier that is very robust, but inaccurate. This is since several new points to be certified later will more likely fall in the certification region requiring either changing their prediction (inaccurate predictions) or reducing their certification radius. Therefore, measuring the certified accuracy for such a classifier will be very poor since it counts for both accuracy and robustness. **(ii)** Assigning, on the other hand, small certified radii for every input will result in a highly accurate classifier but very low robustness. Hence, this also results in a very small certified accuracy at large radii. Therefore, we combine the memory based certificate with our data-dependent smooth classifier that has a better robustness/accuracy tradeoff.

For completeness, we provide the full implementation of the memory-based algorithm using PyTorch.

```
import torch
class Memory_Based_Certification(object):
    def __init__(self):
        self.saved_radii = []
        self.saved_images = []
        self.saved_predictions = []


    def _internal_adjustment(self, img, rad, pre):
        diff = torch.norm(img.reshape(1, -1) - torch.stack(self.saved_images).reshape(len(
    self.saved_radii), -1), dim=1)

        where_overlap =  diff < (torch.tensor(self.saved_radii) + rad)
        #Check whether this image is with overlap with any other instances
        if where_overlap.any():
            preds_overlap = self.saved_predictions[where_overlap]
```

```
            where_overlap_diff_class = preds_overlap != pre

            #Check whether this image is with overlap with instances with different prediction
            if where_overlap_diff_class.any():
                #Get the radii, differences where the overlap
                saved_radii_with_overlap = self.saved_radii[where_overlap]
                dif_with_overlap = diff[where_overlap]

                preds_overlap_with_diff_class = preds_overlap[where_overlap_diff_class]
                rad_with_overlap_diff_class = saved_radii_with_overlap[where_overlap_diff_class]
                dif_with_overlap_diff_class = dif_with_overlap[where_overlap_diff_class]

                rad, rad_idx = torch.min(dif_with_overlap_diff_class - rad_with_overlap_diff_class)

                if rad.item() < 0:
                    pre = preds_overlap_with_diff_class[rad_idx]

                rad = torch.abs(rad).item()
        return rad, pre

    def adjust_radius(self, img, rad, pre):
        #The img already exists in the saved dictionary
        if img in self.saved_images:
            idx = self.saved_images == img
            return self.saved_radii[idx], self.saved_predictions[idx]

        if self.saved_radii != []: #Saved dictionaries are not empty
            rad, pre = self._internal_adjustment(img, rad, pre)

        self.saved_radii.append(rad)
        self.saved_images.append(img)
        self.saved_predictions.append(pre)
        return rad, pre
```

D ADDITIONAL VISUALIZATIONS

Here, we show similar results to the one in Figure 6. Similar to the earlier observations, while model parameters are fixed, optimal smoothing parameters vary per sample.

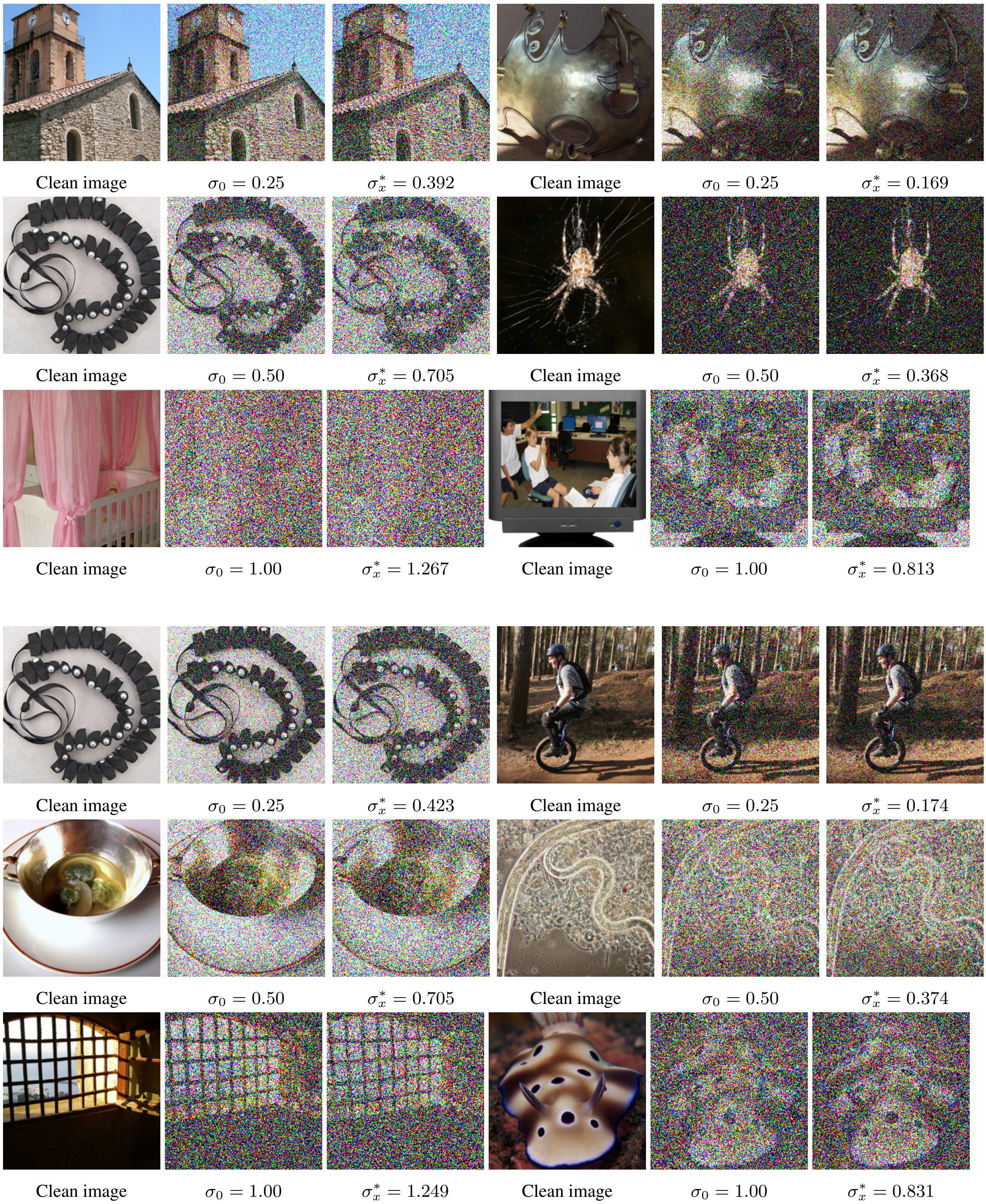

Figure 10: **Visualizing the extreme σ.** We report the visual comparison between the constant σ_0 and the optimal attained σ_x^* for Cohen Models (first three rows) and SmoothAdv (last three rows) both on ImageNet.

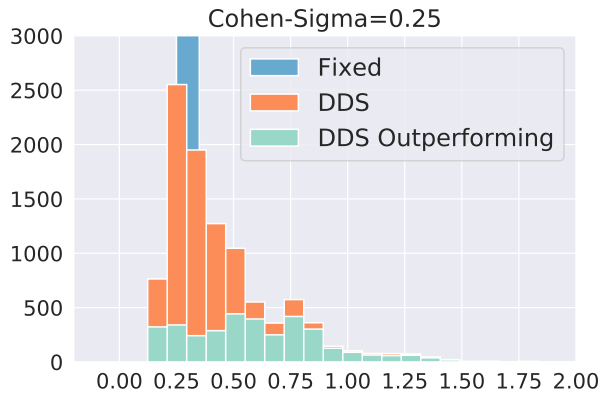

Figure 11: **Where are the good σ_x^*?** We plot a histogram of the σ_x^* in orange highlighting σ_x^* at which the certified radius is improved in green.

E WHERE ARE THE GOOD σ_x^*?

At last, one natural question that arises is that which σ_x^* is yielding better certified radii? To that regard, we conduct the following experiment for Cohen baseline at $\sigma = 0.25$. We plot the histogram of the obtained σ_x^* for CIFAR10 in orange. We also plot a histogram of the σ_x^* at which the certified radius is improved in green. We report the results in Figure 11. We found that the certified robustness improvements happen at the full spectrum of σ_x^* showing the efficacy of our proposed data-dependent smoothing.

F RUNTIME

We measure the certification runtime on an NVIDIA Quadro RTX-6000 GPU for our proposed data dependent smoothed classifier (time includes Algorithm 1 in addition to the memory based certification) compared to the certification of a fixed σ classifier. Certifying one CIFAR10 test input with ResNet18 takes 1.6 and an average of 1.8 seconds for a fixed σ classifier and for the data dependent classifier ($K = 900$), respectively. Certifying an ImageNet test input on ResNet50 takes 109.5 and an average of 136 seconds for a fixed σ classifier and our data dependent classifier ($K = 400$), respectively. The runtime overhead added by using Algorithm 1 and memory based certification is negligible compared to the gains in certified accuracy.

G LIMITATIONS, BROADER IMPACT AND COMPUTE POWERS USED.

Limitations. Similar to any certification framework, the main limitation of this kind of work is its running time to compute the certified radius. The proposed memory-based certification is at the cost of both memory and computational complexity. While the memory cost is of order $\mathcal{O}(N)$ the computational complexity is more involved. Let p be the probability that a new point x_{N+1} be in one of the certification regions $\mathcal{R}_i$, n is the complexity of computing a certification region at x_{N+1}, *i.e.* $\mathcal{R}_{N+1}$, using the classical Monte Carlo Algorithms of Cohen et al. [2019], then the expected computational complexity of prediction or certification will be $\mathcal{O}(Np + (1-p)(2N+n))$. The factor $2N + n$ is due to performing N comparisons to check that x_{N+1} is not in any $\mathcal{R}_i$, then computing $\mathcal{R}_{N+1}$ of complexity n, and at last a complexity of N for computing $\mathcal{R}'_{N+1}$. Informally, for larger N and in small dimensional input, $p \approx 1$ leading to a complexity of order N. When N is smaller compared to the input dimension, we have $p \approx 0$ with an expected complexity of order $2N + n$. Moreover, the memory-based data dependent smooth classifier is order dependent. That is, the certified accuracy depends on the order at which the data at test time is presented. However, this is not the case if there is no overlap between the certified regions of differently predicted inputs (middle and right scenarios of Figure 2 do not occur). We found that is the case in all of our experiments making our memory-enhanced data dependent smooth classifier order invariant. We elaborated on why we believe that is the case in Appendix C. We plan in future extension to delve into more practical solution to this problem. We postpone the design for more efficient algorithms that validate the soundness of data dependent certification to future work.

Broader Impact. While the performance of Deep Neural networks is dominating over several fields, the existence of adversarial examples hinders their deployment in lots of applications. This raise the attention to build networks that are not only accurate, but also robust to such perturbations. This work takes a step towards a remedy for this nuisance by improving the certified robustness of deep neural networks.

Compute Powers. In our experiment on CIFAR10, we used either NVIDIA Quadro RTX-600 GPU or NVIDIA 1080TI GPU. For ImageNet experiments, we used NVIDIA-V100 GPU. Note that one GPU was enough to run any of our experiments.

H DETAILED ABLATIONS

H.1 COHEN VS COHEN-DS VS COHEN-DS^2

In this section, we detail the certified accuracy per radius for all trained models per σ for Cohen and per σ and number of iterations K for Cohen et al. [2019], Cohen-DS and Cohen-DS^2 in Algorithm 1 on both CIFAR10 and ImageNet.

Table 5: **Certified accuracy per radius on CIFAR10.** We compare Cohen against Cohen-DS under varying σ and number of iterations K in Algorithm 1.

	ℓ_2^r (CIFAR10)		0.0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
Cohen	$\sigma = 0.12$		79.89	56.26	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$		74.45	58.34	40.13	22.85	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.50$		63.72	52.15	40.13	29.17	20.18	13.08	7.33	3.33	0.0	0.0	0.0
Cohen-DS	σ =0.12	K=100	77.19	61.27	20.8	5.47	1.23	0.02	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=200	74.98	60.67	19.75	4.05	0.94	0.22	0.01	0.0	0.0	0.0	0.0
	σ =0.12	K=300	73.56	60.08	19.63	4.37	1.12	0.36	0.08	0.0	0.0	0.0	0.0
	σ =0.12	K=400	72.11	59.38	19.58	4.27	1.39	0.58	0.13	0.0	0.0	0.0	0.0
	σ =0.12	K=500	70.78	58.77	19.5	4.77	1.51	0.68	0.14	0.0	0.0	0.0	0.0
	σ =0.12	K=600	70.17	58.46	19.51	4.75	1.63	0.85	0.18	0.03	0.0	0.0	0.0
	σ =0.12	K=700	69.83	58.25	19.91	5.05	1.83	0.88	0.21	0.04	0.0	0.0	0.0
	σ =0.12	K=800	69.25	57.97	19.75	5.04	1.99	0.95	0.17	0.03	0.0	0.0	0.0
	σ =0.12	K=900	68.27	57.51	19.91	5.07	1.94	0.93	0.21	0.04	0.0	0.0	0.0
	σ =0.25	K=100	73.17	64.54	47.48	22.58	6.53	1.82	0.47	0.0	0.0	0.0	0.0
	σ =0.25	K=200	71.62	64.2	47.3	21.66	5.45	1.15	0.34	0.13	0.06	0.01	0.0
	σ =0.25	K=300	70.23	63.91	47.44	21.75	5.38	1.37	0.43	0.21	0.13	0.04	0.02
	σ =0.25	K=400	69.41	63.42	47.43	22.23	5.83	1.38	0.49	0.26	0.1	0.04	0.02
	σ =0.25	K=500	68.88	63.53	47.56	22.19	5.8	1.54	0.53	0.26	0.1	0.06	0.03
	σ =0.25	K=600	68.09	63.21	47.78	22.05	6.16	1.59	0.51	0.25	0.12	0.07	0.02
	σ =0.25	K=700	67.57	63.02	47.6	22.25	5.97	1.63	0.57	0.29	0.11	0.04	0.02
	σ =0.25	K=800	67.36	62.93	47.64	22.04	6.29	1.62	0.6	0.27	0.11	0.04	0.02
	σ =0.25	K=900	67.22	62.93	47.45	22.55	6.19	1.62	0.54	0.26	0.11	0.05	0.03
	σ =0.50	K=100	63.18	55.88	47.07	37.2	26.56	16.43	8.0	3.21	1.23	0.55	0.19
	σ =0.50	K=200	61.26	55.08	47.25	37.86	27.25	16.49	7.49	2.56	1.07	0.53	0.23
	σ =0.50	K=300	59.52	54.25	47.35	38.28	27.29	16.23	7.16	2.39	0.96	0.48	0.24
	σ =0.50	K=400	58.29	53.67	47.19	38.05	27.45	16.39	7.41	2.44	0.93	0.45	0.24
	σ =0.50	K=500	57.46	53.53	47.38	38.28	27.47	16.45	7.38	2.38	0.87	0.48	0.24
	σ =0.50	K=600	56.68	53.11	47.04	38.21	27.47	16.34	7.21	2.37	1.03	0.55	0.32
	σ =0.50	K=700	55.83	52.37	46.88	38.12	27.43	16.37	7.21	2.3	1.01	0.57	0.37
	σ =0.50	K=800	55.26	52.11	46.8	38.3	27.26	16.19	7.18	2.45	1.04	0.62	0.4
	σ =0.50	K=900	54.83	51.83	46.62	38.15	27.55	16.5	7.37	2.52	1.21	0.69	0.5

H.2 SMOOTHADV VS SMOOTHADV-DS VS SMOOTHADV-DS^2

In a similar spirit to the previous section, we report the certified accuracy for the SmoothAdv variants, namely, SmoothAdv Salman et al. [2019a], SmoothAdv-DS and SmoothAdv-DS^2 on CIFAR10 and ImageNet.

Table 6: **Certified accuracy per radius on CIFAR10.** We report Cohen-DS^2 under varying σ and number of iterations K in Algorithm 1.

	ℓ_2^r (CIFAR10)		0.0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
Cohen-DS^2	σ =0.12	K=100	79.8	60.56	26.87	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=100	79.8	60.56	26.87	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=200	79.83	62.05	28.13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=300	79.74	62.81	26.96	0.03	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=400	79.56	63.07	25.47	6.66	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=500	79.4	63.24	24.23	7.74	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=600	79.14	63.23	23.58	7.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=700	78.95	63.34	22.96	7.12	0.86	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=800	78.77	63.34	22.57	6.48	1.26	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=900	79.06	64.6	22.69	6.61	1.68	0.01	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=1000	79.02	64.54	22.27	6.27	1.69	0.02	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=1100	78.81	64.41	21.9	5.89	1.58	0.22	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=1200	78.7	64.37	21.88	5.45	1.42	0.27	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=1300	78.53	64.39	21.67	5.15	1.31	0.26	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=1400	78.39	64.46	21.55	4.96	1.13	0.29	0.01	0.0	0.0	0.0	0.0
	σ =0.12	K=1500	78.31	64.41	21.56	4.73	1.05	0.3	0.03	0.0	0.0	0.0	0.0
	σ =0.25	K=100	74.99	61.47	43.92	24.54	7.93	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=200	75.13	63.21	45.94	25.75	9.35	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=300	75.0	64.03	46.96	25.96	10.05	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=400	75.04	64.58	47.59	25.63	9.93	1.92	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=500	74.79	64.9	47.85	25.41	9.42	2.6	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=600	74.7	65.15	48.38	25.05	8.88	2.69	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=700	74.51	65.35	48.47	24.71	8.34	2.62	0.01	0.0	0.0	0.0	0.0
	σ =0.25	K=800	74.46	65.42	48.5	24.72	7.98	2.43	0.52	0.0	0.0	0.0	0.0
	σ =0.25	K=900	74.58	66.42	50.23	25.57	8.25	2.83	0.74	0.0	0.0	0.0	0.0
	σ =0.25	K=1000	74.39	66.47	50.17	25.41	7.9	2.63	0.75	0.01	0.0	0.0	0.0
	σ =0.25	K=1100	74.2	66.42	50.31	25.13	7.65	2.41	0.72	0.14	0.0	0.0	0.0
	σ =0.25	K=1200	74.12	66.37	50.37	24.92	7.36	2.31	0.65	0.18	0.0	0.0	0.0
	σ =0.25	K=1300	73.98	66.41	50.38	24.75	7.14	2.25	0.58	0.19	0.0	0.0	0.0
	σ =0.25	K=1400	73.81	66.39	50.41	24.79	6.85	2.2	0.51	0.15	0.03	0.0	0.0
	σ =0.25	K=1500	73.67	66.33	50.31	24.63	6.68	2.03	0.54	0.19	0.03	0.0	0.0
	σ =0.50	K=100	63.92	53.49	42.6	31.83	22.15	14.12	7.48	3.51	0.0	0.0	0.0
	σ =0.50	K=200	64.14	54.36	44.3	33.52	23.79	15.14	7.93	3.6	1.09	0.0	0.0
	σ =0.50	K=300	64.21	54.95	45.32	35.04	24.71	15.81	8.16	3.65	1.36	0.0	0.0
	σ =0.50	K=400	64.22	55.56	45.92	35.86	25.45	16.28	8.35	3.79	1.41	0.0	0.0
	σ =0.50	K=500	64.14	55.84	46.35	36.29	25.88	16.56	8.39	3.69	1.42	0.3	0.0
	σ =0.50	K=600	64.14	56.07	46.7	36.71	26.18	16.76	8.48	3.61	1.41	0.45	0.0
	σ =0.50	K=700	64.04	56.2	46.94	37.09	26.54	16.76	8.4	3.63	1.37	0.45	0.0
	σ =0.50	K=800	63.93	56.32	47.23	37.33	26.69	16.91	8.35	3.46	1.28	0.5	0.09
	σ =0.50	K=900	64.26	57.26	48.27	38.85	28.41	17.97	8.82	3.66	1.37	0.58	0.14
	σ =0.50	K=1000	64.06	57.26	48.41	38.96	28.49	18.1	8.65	3.64	1.33	0.6	0.21
	σ =0.50	K=1100	63.72	57.21	48.47	39.0	28.69	18.26	8.57	3.55	1.36	0.59	0.2
	σ =0.50	K=1200	63.56	57.15	48.67	38.96	28.81	18.18	8.53	3.52	1.32	0.58	0.23
	σ =0.50	K=1300	63.29	57.01	48.81	39.07	28.98	18.31	8.44	3.44	1.3	0.6	0.21
	σ =0.50	K=1400	63.09	56.9	48.88	39.11	29.07	18.22	8.62	3.3	1.34	0.56	0.22
	σ =0.50	K=1500	62.94	56.87	48.9	39.21	29.04	18.1	8.55	3.27	1.28	0.53	0.23

Table 7: **Certified accuracy per radius on ImageNet.** We compare Cohen against Cohen-DS and Cohen-DS^2 under varying σ and number of iterations K in Algorithm 1.

	ℓ_2^r (ImageNet)		0.0	0.25	0.50	0.75	1.00	1.50	2.0	2.5	3.0	3.50	4.0
Cohen	σ = 0.25		66.6	58.2	49.0	38.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ = 0.50		57.2	51.4	45.8	42.4	37.4	27.8	0.0	0.0	0.0	0.0	0.0
	σ = 1.0		43.6	40.6	37.8	35.4	32.6	25.8	19.4	14.4	12.0	8.6	0.0
Cohen-DS	σ =0.25	K=100	67.8	61.0	53.6	42.8	18.8	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=200	67.0	61.4	53.6	43.0	18.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=300	66.8	61.2	53.4	42.2	18.6	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=400	66.2	61.4	53.2	42.2	18.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.50	K=100	58.4	54.0	48.2	45.2	40.6	30.4	1.8	0.0	0.0	0.0	0.0
	σ =0.50	K=200	58.0	53.4	48.2	45.2	41.4	29.8	9.0	0.0	0.0	0.0	0.0
	σ =0.50	K=300	58.0	54.0	48.8	45.4	41.4	30.2	9.0	0.0	0.0	0.0	0.0
	σ =0.50	K=400	57.8	53.8	48.8	45.6	42.0	30.4	8.2	0.0	0.0	0.0	0.0
	σ =1.0	K=100	45.0	42.6	40.4	39.0	36.4	29.6	22.4	17.8	13.8	10.0	0.2
	σ =1.0	K=200	45.2	43.0	41.8	39.4	36.8	29.6	23.0	18.6	14.2	10.2	0.6
	σ =1.0	K=300	45.0	43.4	41.2	39.6	37.2	30.0	23.4	18.8	14.4	9.4	2.0
	σ =1.0	K=400	44.8	43.2	41.4	39.6	37.2	30.4	23.2	18.8	14.6	9.8	1.8
Cohen-DS^2	σ =0.25	K=100	67.2	64.2	58.4	45.4	17.8	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=200	66.8	64.2	58.2	45.6	18.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=300	66.6	64.2	58.0	45.2	18.4	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=400	67.4	64.2	58.2	45.0	18.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.50	K=100	58.0	55.2	51.6	46.2	41.2	30.2	2.2	0.0	0.0	0.0	0.0
	σ =0.50	K=200	57.6	55.2	51.8	47.0	41.8	30.4	8.0	0.0	0.0	0.0	0.0
	σ =0.50	K=300	57.6	55.0	51.8	46.8	41.8	30.4	8.0	0.0	0.0	0.0	0.0
	σ =0.50	K=400	57.4	55.4	51.6	47.4	41.8	30.6	8.2	0.0	0.0	0.0	0.0
	σ =1.0	K=100	46.4	44.6	41.4	38.6	37.2	31.4	24.8	20.6	16.6	11.0	0.4
	σ =1.0	K=200	46.6	44.4	42.0	39.2	37.6	31.2	25.0	20.8	17.0	10.8	0.4
	σ =1.0	K=300	46.0	44.6	41.8	39.2	37.4	31.4	24.6	20.8	17.2	11.0	1.8
	σ =1.0	K=400	46.8	45.0	42.6	39.4	37.6	31.8	24.8	21.2	16.8	11.0	2.0

Table 8: **Certified accuracy per radius on CIFAR10.** We compare SmoothAdv against SmoothAdv-DS and SmoothAdv-DS^2 under varying σ and number of iterations K in Algorithm 1.

	ℓ_2^r (CIFAR10)		0.0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
SmoothAdv	$\sigma = 0.12$		75.97	62.44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$		70.82	59.55	46.71	33.66	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.50$		60.96	52.6	43.5	34.62	26.53	19.49	12.9	7.47	0.0	0.0	0.0
SmoothAdv-DS	σ =0.12	K=100	75.74	63.58	40.88	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=200	75.7	64.39	45.05	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=300	75.69	64.97	46.13	0.55	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=400	75.73	65.43	46.39	22.49	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=500	75.74	65.75	46.57	25.06	0.03	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=600	75.72	66.04	46.64	25.16	0.24	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=700	75.66	66.23	46.74	24.64	7.26	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=800	75.65	66.3	46.61	23.97	11.54	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=900	75.64	66.44	46.43	23.48	11.75	0.03	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=100	71.34	60.81	48.38	35.14	17.76	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=200	71.32	61.38	49.44	36.24	20.71	0.01	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=300	71.3	62.01	50.16	36.9	21.69	0.28	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=400	71.32	62.45	50.76	37.24	22.33	8.37	0.01	0.0	0.0	0.0	0.0
	σ =0.25	K=500	71.23	62.82	51.27	37.46	22.42	10.67	0.01	0.0	0.0	0.0	0.0
	σ =0.25	K=600	71.26	63.02	51.66	37.66	22.04	11.06	0.06	0.0	0.0	0.0	0.0
	σ =0.25	K=700	71.12	63.26	51.72	37.61	21.82	11.0	0.52	0.0	0.0	0.0	0.0
	σ =0.25	K=800	71.07	63.4	51.94	37.5	21.43	10.6	4.26	0.0	0.0	0.0	0.0
	σ =0.25	K=900	71.04	63.54	52.1	37.38	21.18	10.1	4.63	0.0	0.0	0.0	0.0
	σ =0.50	K=100	61.16	53.06	44.28	35.42	27.29	20.18	13.6	7.82	0.02	0.0	0.0
	σ =0.50	K=200	61.22	53.44	44.96	36.11	28.0	20.81	13.98	8.25	2.85	0.0	0.0
	σ =0.50	K=300	61.24	53.74	45.39	36.81	28.72	21.21	14.23	8.29	3.29	0.0	0.0
	σ =0.50	K=400	61.22	53.95	45.65	37.29	29.22	21.58	14.53	8.42	3.78	0.05	0.0
	σ =0.50	K=500	61.21	54.15	46.03	37.8	29.54	21.72	14.73	8.43	3.99	1.02	0.0
	σ =0.50	K=600	61.2	54.3	46.42	38.11	29.83	21.94	14.95	8.42	4.07	1.57	0.0
	σ =0.50	K=700	61.23	54.47	46.58	38.39	30.24	22.04	14.95	8.5	4.12	1.77	0.03
	σ =0.50	K=800	61.19	54.58	46.73	38.65	30.39	22.15	14.86	8.49	4.09	1.83	0.43
	σ =0.50	K=900	61.25	54.65	46.88	38.82	30.6	22.19	14.89	8.49	4.17	1.84	0.6
SmoothAdv-DS^2	σ =0.12	K=100	76.04	63.62	41.88	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=200	76.03	64.54	46.4	0.01	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=300	76.0	65.36	47.36	0.82	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=400	75.99	65.85	47.98	23.18	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=500	76.11	66.15	48.16	26.09	0.07	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=600	76.14	66.39	48.13	26.08	0.47	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=700	76.15	66.52	48.1	25.73	7.99	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=800	76.15	66.69	47.84	25.16	11.89	0.02	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=900	76.05	66.77	47.9	24.34	11.82	0.06	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=100	71.2	60.56	48.36	35.19	17.76	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=200	71.27	61.55	49.57	36.45	21.31	0.01	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=300	71.3	62.16	50.75	37.41	22.66	0.48	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=400	71.41	62.76	51.44	37.85	23.18	9.12	0.01	0.0	0.0	0.0	0.0
	σ =0.25	K=500	71.37	63.0	51.89	38.06	23.16	11.37	0.04	0.0	0.0	0.0	0.0
	σ =0.25	K=600	71.37	63.36	52.25	38.31	22.8	11.84	0.16	0.0	0.0	0.0	0.0
	σ =0.25	K=700	71.35	63.45	52.43	38.33	22.58	11.61	0.73	0.0	0.0	0.0	0.0
	σ =0.25	K=800	71.25	63.65	52.67	38.26	22.35	11.17	4.69	0.0	0.0	0.0	0.0
	σ =0.25	K=900	71.21	63.85	52.81	38.21	22.21	10.75	4.92	0.03	0.0	0.0	0.0
	σ =0.50	K=100	61.08	53.0	44.33	35.59	27.49	20.09	13.74	7.98	0.04	0.0	0.0
	σ =0.50	K=200	61.07	53.41	44.95	36.33	28.39	20.86	14.05	8.22	2.9	0.0	0.0
	σ =0.50	K=300	61.1	53.8	45.61	37.13	28.9	21.5	14.32	8.56	3.57	0.0	0.0
	σ =0.50	K=400	61.1	54.14	46.02	37.77	29.3	21.94	14.66	8.63	3.89	0.06	0.0
	σ =0.50	K=500	61.15	54.21	46.52	38.15	29.79	22.21	14.91	8.66	4.23	1.05	0.0
	σ =0.50	K=600	61.2	54.33	46.89	38.59	30.08	22.35	15.01	8.74	4.28	1.56	0.01
	σ =0.50	K=700	61.18	54.56	47.11	38.93	30.4	22.51	15.12	8.85	4.34	1.77	0.03
	σ =0.50	K=800	61.15	54.72	47.41	39.17	30.59	22.56	15.14	8.75	4.31	1.85	0.53
	σ =0.50	K=900	61.12	54.78	47.62	39.32	30.78	22.64	15.14	8.73	4.26	1.94	0.71

Table 9: **Certified accuracy per radius on ImageNet.** We compare SmoothAdv against SmoothAdv-DS and SmoothAdv-DS^2 under varying σ and number of iterations K in Algorithm 1.

	ℓ_2^r (ImageNet)		0.0	0.25	0.50	0.75	1.00	1.50	2.0	2.5	3.0	3.50	4.0
SmoothAdv	σ = 0.25		60.8	57.8	54.6	50.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ = 0.50		54.6	52.6	48.8	44.6	42.2	35.6	0.0	0.0	0.0	0.0	0.0
	σ = 1.0		40.6	39.6	38.6	36.4	33.6	29.8	25.6	20.4	18.0	14.2	0.0
SmoothAdv-DS	σ =0.25	K=100	61.6	59.6	56.8	52.6	31.4	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=200	61.6	59.8	57.2	52.8	35.8	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=300	62.0	60.2	57.2	52.8	36.6	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=400	61.8	60.4	57.4	53.2	36.8	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.50	K=100	55.0	53.6	51.2	47.2	45.2	38.0	4.8	0.0	0.0	0.0	0.0
	σ =0.50	K=200	55.0	53.8	51.6	48.4	46.4	39.2	16.6	0.0	0.0	0.0	0.0
	σ =0.50	K=300	55.4	54.0	51.6	48.6	47.0	39.2	18.0	0.0	0.0	0.0	0.0
	σ =0.50	K=400	55.2	54.0	51.6	48.8	47.0	39.0	18.6	0.0	0.0	0.0	0.0
	σ =1.0	K=100	41.8	41.0	39.4	37.6	35.2	31.6	28.0	22.6	19.2	15.2	0.8
	σ =1.0	K=200	42.4	41.8	40.2	38.4	36.6	32.4	28.8	23.4	19.0	14.6	1.2
	σ =1.0	K=300	42.6	41.8	40.4	38.8	36.8	32.4	29.2	23.8	19.6	15.2	6.2
	σ =1.0	K=400	42.8	42.2	40.8	38.8	37.0	33.2	29.0	23.8	19.6	14.8	6.2
SmoothAdv-DS^2	σ =0.25	K=100	62.2	60.4	58.8	54.0	27.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=200	62.0	60.6	58.6	54.2	27.4	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=300	62.0	60.4	58.8	54.0	27.4	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=400	61.8	60.4	58.8	54.0	27.4	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.50	K=100	55.8	54.2	52.6	50.4	48.2	43.0	7.8	0.0	0.0	0.0	0.0
	σ =0.50	K=200	55.2	54.0	51.8	49.8	47.8	42.6	14.2	0.0	0.0	0.0	0.0
	σ =0.50	K=300	55.6	54.0	52.0	49.8	47.8	42.6	15.0	0.0	0.0	0.0	0.0
	σ =0.50	K=400	55.6	54.4	52.2	50.2	48.2	43.0	15.0	0.0	0.0	0.0	0.0
	σ =1.0	K=100	44.0	43.0	41.2	40.6	38.4	34.6	30.6	25.4	21.6	18.6	1.2
	σ =1.0	K=200	44.4	43.2	41.6	40.6	38.6	34.8	30.6	25.0	21.6	18.4	1.6
	σ =1.0	K=300	44.2	43.0	41.8	41.2	38.6	34.6	30.6	25.2	21.4	17.8	4.2
	σ =1.0	K=400	43.8	43.0	41.0	40.8	38.6	34.6	30.2	25.2	21.4	18.2	4.0

H.3 MACER VS MACER-DS VS MACER-DS^2 (N=1) VS MACER-DS^2 (N=8)

We report ℓ_2^r certified accuracy per radius r for MACER [Zhai et al., 2020] variants on CIFAR10. Note that as highlighted in the main manuscript, for certification only, *i.e.* $MACER - DS$, we set $n = 8$ for all experiments in Algorithm 1. Moreover, in the main paper and for ease of computation we set $n = 1$ for when training is employed, *i.e.* $-DS^2$. In here we also explore the variant where when data dependent smoothing is introduced during training we set $n = 8$ for ablations. We refer to when $n = 1$ and $n = 8$ for when data dependent smoothing is used in training and certification as MACER $- DS(n = 1)$ and MACER $- DS(n = 8)$, respectively.

Table 10: **Certified accuracy per radius on CIFAR10.** We compare MACER against MACER-DS and MACER-DS^2($n = 1$) under varying σ and number of iterations K in Algorithm 1.

	ℓ_2^r (CIFAR10)		0.0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
MACER	$\sigma = 0.12$		78.75	58.51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$		72.51	59.25	43.64	28.25	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.50$		61.23	52.52	43.44	34.65	26.57	19.39	13.0	7.5	0.0	0.0	0.0
MACER-DS	$\sigma = 0.12$	K=100	79.21	60.57	30.95	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=200	79.3	60.98	30.18	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=300	79.39	61.33	27.9	0.07	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=400	79.45	61.27	25.62	10.07	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=500	79.48	61.4	23.43	11.02	0.01	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=600	79.44	61.55	22.22	10.66	0.11	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=700	79.5	61.39	21.79	9.94	3.82	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=800	79.47	61.25	21.83	9.33	5.38	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=900	79.48	61.34	21.59	8.89	6.02	0.1	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=100	73.41	63.59	46.37	27.96	12.76	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=200	73.72	65.1	47.51	27.19	13.85	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=300	73.9	65.63	47.81	26.42	13.19	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=400	73.96	66.03	48.12	25.14	12.2	4.17	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=500	74.0	66.18	47.97	23.98	11.01	4.59	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=600	74.04	66.41	48.23	23.4	9.74	4.23	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=700	74.02	66.47	48.18	22.86	8.65	3.78	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=800	74.07	66.68	48.12	22.58	7.62	3.25	1.06	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=900	74.01	66.74	48.24	22.37	6.88	2.74	1.08	0.0	0.0	0.0	0.0
	$\sigma = 0.50$	K=100	62.62	55.99	47.65	38.37	28.3	19.54	12.75	7.55	0.0	0.0	0.0
	$\sigma = 0.50$	K=200	63.07	57.27	49.54	40.25	29.36	19.44	12.35	7.43	3.23	0.0	0.0
	$\sigma = 0.50$	K=300	63.28	57.91	50.46	41.4	30.0	19.41	11.99	7.08	3.54	0.0	0.0
	$\sigma = 0.50$	K=400	63.39	58.25	51.18	41.98	30.22	19.11	11.69	6.9	3.97	0.0	0.0
	$\sigma = 0.50$	K=500	63.5	58.51	51.51	42.4	30.66	18.7	11.13	6.73	3.83	1.06	0.0
	$\sigma = 0.50$	K=600	63.57	58.72	51.83	42.62	30.51	18.66	10.85	6.44	3.7	1.61	0.0
	$\sigma = 0.50$	K=700	63.65	58.9	52.06	42.79	30.63	18.25	10.57	6.25	3.53	1.67	0.0
	$\sigma = 0.50$	K=800	63.74	59.02	52.19	42.96	30.62	18.2	10.18	5.84	3.35	1.66	0.45
	$\sigma = 0.50$	K=900	63.79	59.09	52.28	43.03	30.75	18.21	9.89	5.53	3.13	1.62	0.52
MACER-DS^2 (n=1)	$\sigma = 0.12$	K=100	79.57	61.25	34.66	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=200	79.58	61.57	36.29	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=300	79.42	61.35	36.21	0.06	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=400	79.44	61.1	35.32	12.32	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=500	79.2	60.64	34.22	13.65	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=600	79.09	60.23	33.75	13.25	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=700	78.98	60.01	32.89	12.66	1.46	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=800	78.85	59.65	32.65	12.07	2.24	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=900	78.78	59.52	32.3	11.4	2.25	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.12$	K=1000	78.73	59.15	31.58	10.63	2.05	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=100	71.45	59.44	45.71	30.76	14.57	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=200	71.81	60.13	46.5	31.3	16.2	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=300	71.81	60.13	46.5	31.3	16.2	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=400	71.91	60.48	46.51	30.83	16.73	5.66	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=500	71.84	60.56	46.3	30.26	16.43	7.07	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=600	71.77	60.38	45.92	29.84	16.11	7.14	0.0	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=700	71.69	60.12	45.66	29.41	15.6	7.0	0.04	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=800	71.73	60.19	45.41	28.91	15.07	6.68	2.15	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=900	71.68	60.11	45.14	28.51	14.54	6.23	2.34	0.0	0.0	0.0	0.0
	$\sigma = 0.25$	K=1000	71.63	59.98	44.97	28.21	14.08	5.9	2.12	0.0	0.0	0.0	0.0
	$\sigma = 0.50$	K=100	60.96	53.69	44.96	36.64	28.13	20.46	14.44	8.73	0.01	0.0	0.0
	$\sigma = 0.50$	K=200	61.37	54.35	46.07	37.43	28.55	20.58	14.26	8.65	3.6	0.0	0.0
	$\sigma = 0.50$	K=300	61.52	54.74	46.53	37.9	28.91	20.62	14.12	8.42	3.9	0.0	0.0
	$\sigma = 0.50$	K=400	61.42	54.81	46.83	38.02	28.98	20.51	13.69	8.3	4.27	0.0	0.0
	$\sigma = 0.50$	K=500	61.39	54.74	47.03	38.2	28.85	20.25	13.45	8.14	4.16	1.0	0.0
	$\sigma = 0.50$	K=600	61.44	54.8	46.96	38.2	28.83	19.97	13.23	7.94	4.1	1.53	0.0
	$\sigma = 0.50$	K=700	61.35	54.75	46.89	38.04	28.7	19.53	12.95	7.64	3.98	1.7	0.0
	$\sigma = 0.50$	K=800	61.24	54.75	46.94	38.1	28.49	19.3	12.59	7.46	3.9	1.69	0.4
	$\sigma = 0.50$	K=900	61.25	54.73	46.85	37.94	28.19	18.87	12.29	7.13	3.69	1.7	0.51
	$\sigma = 0.50$	K=1000	61.21	54.72	46.84	37.87	27.97	18.74	12.08	6.82	3.43	1.66	0.71

Table 11: **Certified accuracy per radius on CIFAR10.** We report MACER-DS$^2(n = 8)$ under varying σ and number of iterations K in Algorithm 1.

	ℓ_2^r (CIFAR10)		0.0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
MACER-DS^2 (n=8)	σ =0.12	K=100	81.9	62.52	29.38	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=200	82.16	63.09	29.72	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=300	82.2	63.4	28.47	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=400	82.21	63.51	26.29	6.84	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=500	82.34	63.76	24.13	7.62	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=600	82.34	63.66	22.6	7.05	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=700	82.32	63.84	21.61	6.48	0.6	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=800	82.35	63.9	21.06	5.4	0.83	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=900	82.37	63.9	20.79	4.67	0.82	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.12	K=1000	82.39	63.89	20.57	3.88	0.77	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=100	75.18	64.79	47.14	29.31	12.56	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=200	75.36	66.23	48.55	28.91	13.99	0.0	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=300	75.53	66.87	49.24	28.31	14.26	0.01	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=400	75.57	67.36	49.53	27.35	13.94	3.86	0.0	0.0	0.0	0.0	0.0
	σ =0.25	K=500	75.65	67.71	49.59	26.24	13.23	4.68	0.01	0.0	0.0	0.0	0.0
	σ =0.25	K=600	75.72	67.81	49.64	25.46	12.12	4.73	0.01	0.0	0.0	0.0	0.0
	σ =0.25	K=700	75.84	67.93	49.85	24.92	11.1	4.58	0.01	0.01	0.0	0.0	0.0
	σ =0.25	K=800	75.81	68.08	49.84	24.79	10.18	4.33	1.05	0.01	0.0	0.0	0.0
	σ =0.25	K=900	75.87	68.16	49.84	24.53	9.38	3.81	1.02	0.01	0.0	0.0	0.0
	σ =0.25	K=1000	75.87	68.26	49.94	24.27	8.66	3.32	0.93	0.01	0.01	0.0	0.0
	σ =0.50	K=100	61.79	55.41	47.8	39.03	29.04	20.62	13.83	7.92	0.0	0.0	0.0
	σ =0.50	K=200	62.11	56.53	49.36	40.68	30.08	20.55	13.38	7.84	3.07	0.0	0.0
	σ =0.50	K=300	62.31	57.21	50.54	41.6	30.79	20.46	13.03	7.56	3.44	0.0	0.0
	σ =0.50	K=400	62.49	57.68	51.12	42.08	31.12	20.21	12.45	7.34	3.65	0.0	0.0
	σ =0.50	K=500	62.62	57.98	51.61	42.41	31.3	20.13	12.09	6.94	3.65	0.74	0.0
	σ =0.50	K=600	62.71	58.28	51.82	42.85	31.52	19.78	11.58	6.54	3.56	1.28	0.0
	σ =0.50	K=700	62.84	58.35	52.15	43.04	31.42	19.6	11.12	6.33	3.29	1.37	0.0
	σ =0.50	K=800	62.91	58.45	52.33	43.29	31.48	19.47	10.62	5.95	3.21	1.39	0.27
	σ =0.50	K=900	62.93	58.54	52.56	43.44	31.49	19.14	10.17	5.73	3.09	1.34	0.38
	σ =0.50	K=1000	63.0	58.66	52.67	43.47	31.66	19.04	9.85	5.49	2.85	1.21	0.4