Learning soft interventions in complex equilibrium systems (Supplementary material)

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A ADDITIONAL BACKGROUND

A.1 SMOOTH MANIFOLDS

While many non-equivalent definitions exist for smooth manifold, we follow Lee [2013] in defining smoothness as infinite continuously differentiability of functions. A diffeomorphism is then a smooth bijection whose inverse is also smooth.

For an n-dimensional topological manifold M, an atlas is a collection of coordinate charts (U_k, φ_k) such that U_k's are open sets of M covering it, and such that the mappings $\varphi_k : U_k \mapsto \varphi_k(U_k) \subset \mathbb{R}^n$ are homeomorphisms (continuous bijection with continuous inverse). Briefly, the atlas is smooth whenever $\varphi_k \circ \varphi_n^{-1}$ is are diffeomorphisms whenever well defined, and a smooth manifold is a topological manifold associated with a maximal smooth atlas.

A smooth map $F : M \to N$ between two smooth manifolds M and N is a function such that for any chart (U, φ) and (V, ψ), $\psi \circ F \circ \varphi^{-1}$ is smooth whenever well defined.

A.2 LIE GROUPS

We first provide a formal definition of groups.

Definition A.1 (Group)**.** A set G is a group if it is equipped with a binary operation "$\cdot$" $: G \times G \to G$ satisfying

1. Associativity: $\forall a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. Identity: There exists $e \in G$ such that $\forall a \in G$, $a \cdot e = e \cdot a = a$.
3. Inverse: $\forall a \in G$, there exists $b \in G$ such that $a \cdot b = b \cdot a = e$. This inverse is denoted a^{-1}.

Then a Lie group is essentially a group that is also a smooth manifold.

Definition A.2 (Lie Group)**.** A Lie Group G is a nonempty set satisfying the following conditions:

- G is a group.
- G is a smooth manifold.
- The group operation $\cdot : G \times G \to G$ and the inverse map $.^{-1} : G \to G$ are smooth.

We are often interested in sets of transformations, which respect a group structure, but are applied to objects that are not necessarily group elements. This can be studied through group actions, which describe how groups *act* on other mathematical entities.

Definition A.3 (Lie group Action)**.** Given a Lie group G and a set X, a Lie group action (or smooth group action) is a function $\cdot_X : G \times X \to X$ such that the following conditions are satisfied.

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1. Identity: If $e \in G$ is the identity element, then $e \cdot_X x = x, \forall x \in X$.
2. Compatibility: $\forall g, h \in G$ and $\forall x \in X$, $g \cdot_X (h \cdot_X x) = ((g \cdot h) \cdot_X x)$
3. Smoothness: the map $\cdot_X : G \times X \to X$ is smooth.

A.3 CYCLIC CAUSAL MODELS

A classical type of hard interventions are *perfect interventions*, which replace the structural assignments of a given variable X_k by an assignment $X_k := \xi_k$, with ξ_k constant [Blom et al., 2020]. It thus eliminates the arrows in the causal graph pointing to this variables, and makes this variable deterministic.

In particular, tracing the effects of perfect interventions requires special assumptions. In contrast, soft interventions may be read from the so-called causal ordering graph, which can be built from the original SCM graph. Broadly construed, a unique causal ordering graph can be constructed with several algorithms [Blom et al., 2020]. This is a directed cluster graph that contains groups of variables connected by oriented edges (starting from single variable in a given cluster, and pointing to another cluster). By construction, the resulting graph between clusters entailed by these edges is directed and contains no cycles. As a consequence, the effect of generic soft intervention on clustered variables can be easily read from this graph.

A.4 LINK BETWEEN EQUILIBRIUM AND DYNAMIC MODELS

The equilibrium of eq. (1) can be thought of as the asymptotic value of $\boldsymbol{x}$ in a dynamic model (see Appendix A)

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} + \boldsymbol{y} - \boldsymbol{x}\,,$$

where the increase or decrease of the sectors' activity is controlled by the imbalance between their demand $A\boldsymbol{x} + y$ and their current output $\boldsymbol{x}$. More generally, any fixed point-equation can be thought of a the equilibrium value of some dynamical system, for example by considering a numerical algorithm that converges to it. However, the relationship between dynamical systems and self-consistent equation is not one to one. Notably, we can rescale the time evolution of a stable dynamical system to create many other that converge to the same self-consistent equation. Moreover, by inverting the arrow of time, we can obtain systems for which the self-consistent equation is an unstable equilibrium. As mentioned in main text, in this work we leave aside the dynamical aspects to focus on the equilibrium properties.

A.5 MRIO MODELS

Multi-regional input-output models are built based on macro-economic information, notably the one provided by the National Accounts of the countries involved in the model. The technical coefficient matrix of eq. (1) is computed from so-called *Supply and Use Tables* that form the basis of National Accounts. The unit used to measure output is frequently monetary (e.g., EUR) due to the data collection process and to allow an homogeneous treatment of the economic flows. However, under homogeneity and linearity assumptions, the output of each sector may be converted in appropriate physical units using unit prices and material flow data. Moreover, there also exist hybrid MRIO models which include information regarding physical flows in the economy (energy, raw materials, ...) and the are combined with monetary information to ensure the best level of self-consistency.

B PROOF OF MAIN TEXT RESULTS

B.1 PROOF OF PROPOSITION 1

Proof. Assuming the SSCM is locally diffeomorphic entails that the Jacobian of $\boldsymbol{x} \to \boldsymbol{x} - \mathbf{f}(\boldsymbol{x}, \boldsymbol{\theta}^{\mathrm{ref}})$ is invertible at $\boldsymbol{x} = \boldsymbol{x}^{\mathrm{ref}}$. Then the Jacobian of $(\boldsymbol{x}, \boldsymbol{\theta}) \to (\boldsymbol{x} - \mathbf{f}(\boldsymbol{x}, \boldsymbol{\theta}),\, \boldsymbol{\theta})$ is also invertible at $(\boldsymbol{x}^{\mathrm{ref}}, \boldsymbol{\theta}^{\mathrm{ref}})$ (due to its block triangular structure). Using the inverse function theorem for smooth maps between smooth manifolds [Lee, 2013, Theorem 4.5], this implies that there exists connected open neighborhoods (U, V) of $(\boldsymbol{x}^{\mathrm{ref}}, \boldsymbol{\theta}^{\mathrm{ref}})$ and $(\mathbf{0}, \boldsymbol{\theta}^{\mathrm{ref}})$ such that

$$\begin{aligned} g\colon \quad & U \to V \\ (\boldsymbol{x}, \boldsymbol{\theta}) & \mapsto (\boldsymbol{x} - \mathbf{f}(\boldsymbol{x}, \boldsymbol{\theta}),\, \boldsymbol{\theta}) \end{aligned}$$

is a diffeomorphism. As a consequence, self-consistent solutions $(\boldsymbol{x}, \boldsymbol{\theta})$ in U are given by $S = g^{-1}((\{0\} \times \mathcal{T}) \cap V)$. It is a submanifold of same dimension as $\mathcal{T}$ for the following reasons:

- S is a manifold diffeomorphic to $(\{0\} \times \mathcal{T}) \cap V$ and thus has the same dimension [Lee, 2013, Theorem 2.17],
- $(\{0\} \times \mathcal{T}) \cap V$ is an open submanifold because V is open, and thus has the same dimension as $\{0\} \times \mathcal{T}$ [Lee, 2013, Proposition 5.1]
- $\{0\} \times \mathcal{T}$ has the same dimension as $\mathcal{T}$ because it is diffeomorphic to it [Lee, 2013, Propositions 5.3 and 2.17].

Let us now define the cartesian projection

$$\begin{aligned} \pi: \quad & U \to \mathcal{T} \\ & (\boldsymbol{x}, \boldsymbol{\theta}) \mapsto \boldsymbol{\theta}, \end{aligned}$$

we want to establish that there exist an open neighborhood $U_{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}^{\text{ref}}$ such that there is a unique self-consistent solution for each parameter choice in this set $\pi_{|S}$ is a smooth embedding because it is an injective smooth immersion, and is open[1], by Lee [2013, Proposition 4.22]). As a consequence $\pi(S)$ is an embedded submanifold of $\mathcal{T}$ diffeomorphic to S (by Lee [2013, Proposition 5.2]). Since we have shown that the dimension of S is the dimension of $\mathcal{T}$, then $\pi(S)$ is a submanifold of same codimension 0 (same dimension as its ambient manifold) and is thus an open submanifold of $\mathcal{T}$ (Proposition 5.1 in Lee [2013]). As a consequence, $\pi(S)$ is open, such that there is an open neighborhood of $\boldsymbol{\theta}^{\text{ref}}$ included in it. Then for any parameter chosen in this neighborhood, there is one solution to the self-consistency equation, by definition of the image. Assume there are two distinct solution for this parameter, then the mapping $(\boldsymbol{x}, \boldsymbol{\theta}) \to (\boldsymbol{x} - \mathbf{f}(\boldsymbol{x}, \boldsymbol{\theta}), \boldsymbol{\theta})$ would not be a diffeomorphism. □

B.2 PROOF OF PROPOSITION 2

Proof. We extend the smooth parameterization of function f by $\boldsymbol{\theta}$ to get a smooth parameterization of the intervened functional assignments by $\bar{\boldsymbol{\theta}} = (g, \boldsymbol{\theta})$. Indeed, the mapping

$$(x, \bar{\boldsymbol{\theta}}) \mapsto g \cdot f(x, \theta)$$

is smooth as a composition of the following smooth maps

$$(\boldsymbol{x}, \boldsymbol{\theta}, g) \underset{f \text{ smooth}}{\mapsto} (f(\boldsymbol{x}, \boldsymbol{\theta}), g) \underset{\varphi \text{ smooth}}{\mapsto} \varphi(g, f(\boldsymbol{x}, \boldsymbol{\theta})) = g \cdot f(\boldsymbol{x}, \boldsymbol{\theta})$$

where the smoothness of each transformation stem from the definition of SSCM and Lie interventions, respectively. Proposition 1 applied around the extend parameter $(e, \boldsymbol{\theta}^{\text{ref}})$ implies that there exists a neighborhood $U_{(e, \boldsymbol{\theta}^{\text{ref}})}$ of this point such that the intervened solution is uniquely solvable and the mapping from the extended parameter to the solution is smooth. There exists moreover a product neighborhood $U_L \times U_{\boldsymbol{\theta}} \subset U_{(e, \boldsymbol{\theta}^{\text{ref}})}$ (this is a basic property of neighborhoods on product spaces). By continuity of the partial derivative of the intervened functional assignment (due to smoothness of the Lie group action), dependency on the parents of the intervened variables is preserved in a neighborhood of the identity, such that the intervention is soft in the considered neighborhood. □

B.3 PROOF OF PROPOSTION 3

Proof. The Lie intervention parameterized by u guaranties solvability of the SSCM is preserved in a neighborhood of the identity (Proposition 2), and we denote $x^{(u)}(\boldsymbol{\theta})$ the unique solution in such neighborhood, with $x^{(e)}(\boldsymbol{\theta}) = x^*(\boldsymbol{\theta})$. The Jacobian $J^{\boldsymbol{\theta}}_{x^*_{\mathbf{Pa}_k}}(\boldsymbol{\theta}^{\text{ref}})$ is the Jacobian of the mapping from the parameters $\boldsymbol{\theta}$ to the vector consisting of the parent nodes of k at equilibrium. Because this Jacobian is full column rank, there exists a neighborhood of e such that for any fixed u in it, the mapping $\boldsymbol{\theta} \mapsto \boldsymbol{x}^{(u)}_{\mathbf{Pa}_k}(\boldsymbol{\theta})$ is injective in a neighborhood of the reference parameter. As a consequence the restriction to its image is a diffeomorphic map between manifolds. Let us denote $\psi^{(u)}$ its inverse.

Consider the SSCM obtained by performing a hard intervention $x_j := x^*_j(\boldsymbol{\theta})$. Because the original SSCM is locally diffeomorphic at $(\boldsymbol{x}^{\text{ref}}, \boldsymbol{\theta}^{\text{ref}})$, $\{x_j := x^*_j(\boldsymbol{\theta})\}$ is a smooth assignment, and because additionally the Jacobian of the mapping

[1] $\pi_{|S}$ is open because $\pi_{|S} \circ g^{-1}_{|(\{0\} \times \mathcal{T}) \cap V}$ is a smooth submersion and thus open by Proposition 4.28 in Lee [2013], and g_S is also open as the restriction of a diffeomorphism.

$\boldsymbol{x}_{-j} \to \boldsymbol{x}_{-j} - \mathbf{f}_{-j}(\boldsymbol{x}_{-j}, \boldsymbol{\theta}^{\text{ref}})$ is invertible, then this hard intervened system is also locally diffeomorphic at $(\boldsymbol{x}^{\text{ref}}, \boldsymbol{\theta}^{\text{ref}})$ (exploiting the block diagonality of the Jacobian of its assignment). As a consequence, Lie intervention with parameter u on node i of this (already hard-intervened) system leads to a smooth intervened equilibrium $x^{(u)}$.

Let us recall that the partial derivative $\frac{\partial x_j^*}{\partial x_k}_{|\boldsymbol{\theta}=\boldsymbol{\theta}^{\text{ref}}}$ corresponds to the derivative with respect to the hard interventions value. The assumption $\frac{\partial x_j^*}{\partial x_k}_{|\boldsymbol{\theta}=\boldsymbol{\theta}_{\text{ref}}} \neq 0$ thus entails, by the inverse function theorem, that there exists also a smooth mapping $\phi^{(u)}$ such that $x_k^{(u)} = \phi^{(u)}(x_j^*(\boldsymbol{\theta}))$ in a neighborhood of $(e, \boldsymbol{\theta}_{\text{ref}})$. As a consequence, the mapping defined as $f_k^{(u)} = \phi^{(u)} \circ x_j^*(.) \circ \psi^{(u)}$ is a soft intervention replacing f_k achieving the same equilibrium values as the above hard-intervened system under Lie interventions, and in particular the invariance constraint $x_j^{(u)}(\boldsymbol{\theta}) = x_j^*(\boldsymbol{\theta})$. □

B.4 PROOF OF PROPOSITION 4

Proof. We proceed iteratively by adding one intervention after the next. First intervention on compartment C_1 leaves invariant the equilibrium values of the remaining compartments C_{-1} as the only node from C_1 influencing them is invariant.

Given $C_1, ..., C_n$ satisfy invariance with respect to each others interventions, consider intervening on C_{n+1}. As C_{n+1} receives only inputs from intervened upon compartments $C_1, ..., C_n$ through invariant nodes, the invariant intervention on it can be designed identical as for the non-intervened system. Moreover, invariance of the nodes having outgoing arrows to other compartments ensures that the equilibrium values of other (potentially intervened upon) compartments $C_{-(n+1)}$ remains invariant. □

C ADDITIONAL THEORETICAL RESULTS

C.1 MOTIVATING EXAMPLE OF SEC. 3.3

Let us restate the unintervened assignments of this example.

$$\begin{aligned} x &= \tau\,, \\ y &= (\alpha x + \beta z)\,, \\ z &= \gamma y\,. \end{aligned}$$

The equilibrium solution then writes

$$\begin{aligned} x^* &= \tau\,, \\ y^* &= \frac{\alpha\tau}{1-\beta\gamma}\,, \\ z^* &= \frac{\gamma\alpha\tau}{1-\beta\gamma}\,. \end{aligned}$$

Applying multiplicative Lie interventions on both x an y leads to the assignments

$$\begin{aligned} x &= \tau\,, \\ y &= u_y(\alpha x + \beta z)\,, \\ z &= u_z\gamma y\,. \end{aligned}$$

which leads to the intervened equilibrium

$$\begin{aligned} x^{(\boldsymbol{u})} &= \tau\,, \\ y^{(\boldsymbol{u})} &= \frac{u_y\alpha\tau}{1-u_z u_y\beta\gamma}\,, \\ z^{(\boldsymbol{u})} &= \frac{u_y u_z\gamma\alpha\tau}{1-u_y u_z\beta\gamma}\,. \end{aligned}$$

We can thus notice that choosing $u_z = \frac{1}{u_y}$ makes the intervened equilibrium value invariant for any choice of parameters $(\tau, \alpha, \beta, \gamma)$.

D METHODS

Following [Bai et al., 2019], we implemented implicit layers using the pyTorch library [Paszke et al., 2019]. We use Anderson acceleration with $m = 5$ previous iterations and a relaxation parameter $\beta = 2.0$ (based on preliminary analysis) to compute iteratively the fixed points of the implicit layers, for both forward and backward passes, with a maximum number of iterations of 5000 and a tolerance of .0001. Our experiments were run with a fixed initialization of the equilibrium point (zero).

Optimization of interventions is done using backpropagation with adaptive moment estimation (Adam), with a learning rate of .001 and 10000-20000 iterations. Soft interventions to enforce invariance are learned with two hidden layer perceptrons, with 20 and 10 hidden units respectively for the first and second layers, and all layers have ReLU activation functions. At each iteration, free parameters $\boldsymbol{\theta}$ are sampled from an factorized Gaussian distribution whose means and variances are chosen to cover the neighborhood of the reference point. Optimization of invariant soft interventions is performed by sampling at each iteration from a range of values of the Lie intervention and unintervened model parameters.

Toy experiments use artificially generated parameters for few sectors economic models (this is the case of paragraphs "Control of rebound effects" and "Compartmentalized intervention design"), whose structure is described in Figs. 2a and 1b. Instead, the semi-synthetic experiments use 200 sector economic models based on realistic parameters from the Exiobase3 dataset, as described in the paragraphs "Evaluation of equilibrium estimation", and "Optimization of multiplicative Lie interventions".

Code for the toy optimization experiments is provided at https://github.com/mbesserve/lie-inter.

E SUPPLEMENTAL DISCUSSION

Socio-economic impacts of environmental policies In full generality, whether and which environmental policies have a negative socio-economic impact is a highly debated topic. In main text, we argue that straightforward measures that could be taken to significantly contribute to achieve environmental goals typically have a short-term socio-economic cost. The example repetitively used in our paper is activity reduction of greenhouse gas emitting sectors, wish has straightforward *short-term impacts* on their employment [Oei et al., 2020]. We are not aware of literature challenging the view that such classical socio-economic and environmental goals are at least to some extent at odds and require tradeoff from the standpoint of political decision makers. *On the longer term*, the feasibility of making these goals compatible based on economic concepts such as Green Growth is debated [Jakob and Edenhofer, 2014, Hickel and Kallis, 2020].

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