Bayesian Spillover Graphs for Dynamic Networks: Supplementary Material

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A MOVING AVERAGE REPRESENTATION OF VAR(1)

We can rewrite a VAR(1) model with a moving average representation [Tsay, 2013] using the mean-adjusted model, which is useful for computing variances of forecast errors.

We define the **mean-adjusted model** $\tilde{\mathbf{z}}_t = \mathbf{z}_t - \mu$, where $\mu = (I_d - \phi_1)^{-1}\phi_0$.

Then,

$$\begin{aligned}
\tilde{\mathbf{z}}_t &= \mathbf{a}_t + \phi_1 \tilde{\mathbf{z}}_{t-1} \\
&= \mathbf{a}_t + \phi_1(\mathbf{a}_{t-1} + \phi_1 \tilde{\mathbf{z}}_{t-2}) \\
&= \mathbf{a}_t + \phi_1 \mathbf{a}_{t-1} + \phi_1^2(\mathbf{a}_{t-2} + \phi_1 \tilde{\mathbf{z}}_{t-3}) \\
&= \mathbf{a}_t + \phi_1 \mathbf{a}_{t-1} + \phi_1^2 \mathbf{a}_{t-2} + \phi_1^3 \mathbf{a}_{t-3} + ...
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{z}_t &= \mu + \tilde{\mathbf{z}}_t \\
&= \mu + \mathbf{a}_t + \phi_1 \mathbf{a}_{t-1} + \phi_1^2 \mathbf{a}_{t-2} + \phi_1^3 \mathbf{a}_{t-3} + ... \\
&= \mu + \mathbf{a}_t + \psi_1 \mathbf{a}_{t-1} + \psi_2 \mathbf{a}_{t-2} + ... \\
&= \mu + \sum_{i=0}^{\infty} \psi_i \mathbf{a}_{t-i}
\end{aligned}$$

where $\psi_i = \phi_1^i$ for $i \geq 0$.

B PROOF OF THEOREM 1

Theorem 1. *If ϕ_1 is a DAG, then (1) no autocorrelation exists, (2) ϕ_1 can be specified by a strictly triangular matrix, (3) all eigenvalues of ϕ_1 are 0 and hence z_t is stationary.*

Proof: By definition of DAG, no cycles can exist in the adjacency matrix, in this case, ϕ_1. Hence, the diagonal entries which indicate dependency of z_{it} on $z_{i,t+1}$ is necessarily 0, and thereby proving point (1).

Note that by definition, there exists a topological ordering on the vertices if and only if a graph has no directed cycles. Because ϕ_1 is a DAG, we can relabel the d vertices (time series components) as $v_1, v_2, ..., v_d$. If $v_i v_{i'}$ is a directed edge into i from i' (indicating Granger-causality), then $i > i'$. Hence, all entries above the main diagonal are also 0 because these are entries for which $i < i'$. Combined with point (1) where main diagonal entries are also 0, this satisfies the definition of a strictly lower-triangular matrix (2).

We've shown that the adjacency matrix of a DAG is strictly lower-triangular via permutation, and note that the order of individual time series components does not matter, although in this case the d vertices are ordered from source to sink nodes.

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The eigenvalues of any lower-triangular matrix is just its diagonal components [Axler, 1997], meaning that all eigenvalues for ϕ_1 is just 0. Since these are strictly less than 1 in magnitude, we can conclude that z_t is stationary (3).

C EVALUATING ACCURACY FOR SOURCE & SINK NODE IDENTIFICATION

First, define Discounted Cumulative Gain (DCG) at position d, for d nodes arranged in a particular order:

$$\mathrm{DCG_d} = \sum_{i=1}^{d} \frac{rel_i}{\log_2(i+1)}$$

where rel_i is the graded precision score of node at position i, e.g. $\{1, 0.5, 0\}$ for $\{$source, intermediary, sink$\}$ nodes respectively. Greater penalty is given for source or sink nodes ranked in lower positions. NDCG [Valizadegan et al., 2009] then equals DCG divided by Ideal Discounted Cumulative Gain (IDCG):

$$\mathrm{NDCG_d} = \frac{DCG_d}{IDCG_d}, \ \ \mathrm{IDCG_d} = \sum_{i=1}^{|rel_d|} \frac{rel_i}{\log_2(i+1)}$$

and $|rel_d|$ represents the optimal order of nodes, which is given by the ground truth labels of each node.

D BSG FOR IDENTIFYING SINK AND SOURCE NODES

D.1 ABLATION EXPERIMENT - ERROR COVARIANCE Σ_a

Table 1: Average NDCG (Accuracy) for Identifying Sink & Source Nodes with Dependent Errors, 5 Rep.

Directed Acyclic	A. Weak Dependency $\sigma_{jk} = 0.1$		B. Moderate Dependency $\sigma_{jk} = 0.5$		C. Strong Dependency, $\sigma_{jk} = 0.9$	
Method	NDCG@24 Source Nodes	NDCG@24 Sink Nodes	NDCG@24 Source Nodes	NDCG@24 Sink Nodes	NDCG@24 Source Nodes	NDCG@24 Sink Nodes
BSG, $h = 1$	0.938 ± 0.04	**1** ± 0	0.951 ± 0.004	**1** ± 0	0.925 ± 0.016	1 ± 0
BSG, $h = 5$	0.995 ± 0.006	0.999 ± 0.001	**0.993** ± 0.004	0.997 ± 0.002	0.961 ± 0.011	**0.993** ± 0.001
BSG, $h = 10$	**0.99** ± 0.004	0.994 ± 0.002	0.989 ± 0.006	0.991 ± 0.003	**0.975** ± 0.01	0.988 ± 0.004
VAR-Between	0.778 ± 0.068	0.796 ± 0.068	—	—	—	—
VAR-Closeness	0.648 ± 0.024	0.926 ± 0.024	—	—	—	—
VAR-Degree	0.8 ± 0.045	0.868 ± 0.053	—	—	—	—
VAR-Eigen	0.71 ± 0.063	0.864 ± 0.063	—	—	—	—
DBN-Between	0.75 ± 0.036	0.825 ± 0.036	0.747 ± 0.085	0.827 ± 0.085	0.721 ± 0.075	0.853 ± 0.075
DBN-Closeness	0.842 ± 0.07	0.733 ± 0.07	0.827 ± 0.071	0.747 ± 0.071	0.801 ± 0.114	0.773 ± 0.114
DBN-Degree	0.85 ± 0.06	0.82 ± 0.05	0.834 ± 0.08	0.849 ± 0.031	0.827 ± 0.092	0.879 ± 0.05
DBN-Eigen	0.752 ± 0.031	0.822 ± 0.031	0.73 ± 0.081	0.845 ± 0.081	0.713 ± 0.071	0.862 ± 0.071
GVAR-Between	0.729 ± 0.066	0.845 ± 0.066	0.684 ± 0.078	0.891 ± 0.078	0.729 ± 0.04	0.845 ± 0.04
GVAR-Closeness	0.685 ± 0.037	0.89 ± 0.037	0.632 ± 0.04	0.943 ± 0.04	0.689 ± 0.062	0.885 ± 0.062
GVAR-Degree	†	†	†	†	†	†
GVAR-Eigen	0.935 ± 0.016	0.639 ± 0.016	0.953 ± 0.039	0.621 ± 0.039	0.89 ± 0.04	0.685 ± 0.04

— indicates retrieved NGC graph is degenerate, e.g., only edges are self-directed.
† indicates network measure cannot distinguish between nodes, e.g., all in/out degrees are equal.

D.2 MULTISPECIES LOTKA-VOLTERRA - NONLINEAR DYNAMIC SYSTEMS

E EVALUATING KINCADE FIRE SPILLOVERS

Table 2: Average NDCG (Accuracy) for Identifying Sink & Source Nodes with Nonlinear Systems, 5 Rep.

Multi-species LV	$d = 20, T = 50$		$d = 20, T = 200$		$d = 20, T = 1000$	
Method	NDCG@20 Source (Predator)	NDCG@20 Sink (Prey)	Source (Predator)	Sink (Prey)	Source (Predator)	Sink (Prey)
BSG, $h = 1$	0.995 ± 0.004	0.865 ± 0.045	**0.973** ± 0.013	**0.939** ± 0.039	0.982 ± 0.015	**0.811** ± 0.069
BSG, $h = 5$	**0.995** ± 0.002	0.905 ± 0.046	0.945 ± 0.021	0.931 ± 0.047	0.967 ± 0.024	0.755 ± 0.035
BSG, $h = 10$	0.989 ± 0.01	**0.946** ± 0.015	0.892 ± 0.058	0.907 ± 0.056	0.932 ± 0.031	0.711 ± 0.074
VAR-Between	0.71 ± 0.058	0.84 ± 0.058	0.721 ± 0.145	0.828 ± 0.145	—	—
VAR-Closeness	0.781 ± 0.093	0.768 ± 0.093	0.78 ± 0.09	0.769 ± 0.09	—	—
VAR-Degree	0.768 ± 0.091	0.748 ± 0.071	0.679 ± 0.084	0.737 ± 0.077	—	—
VAR-Eigen	0.812 ± 0.087	0.738 ± 0.087	0.881 ± 0.037	0.669 ± 0.037	—	—
DBN-Between	0.796 ± 0.125	0.753 ± 0.125	0.808 ± 0.091	0.742 ± 0.091	0.892 ± 0.107	0.657 ± 0.107
DBN-Closeness	0.796 ± 0.075	0.754 ± 0.075	0.806 ± 0.074	0.743 ± 0.074	0.854 ± 0.086	0.696 ± 0.086
DBN-Degree	0.801 ± 0.072	0.756 ± 0.101	0.825 ± 0.093	0.724 ± 0.112	0.891 ± 0.061	0.704 ± 0.072
DBN-Eigen	0.753 ± 0.086	0.797 ± 0.086	0.8 ± 0.111	0.75 ± 0.111	0.797 ± 0.067	0.748 ± 0.073
GVAR-Between	0.736 ± 0.077	0.814 ± 0.077	0.816 ± 0.111	0.733 ± 0.111	0.741 ± 0.063	0.809 ± 0.063
GVAR-Closeness	0.744 ± 0.093	0.806 ± 0.093	0.83 ± 0.114	0.72 ± 0.114	**0.996** ± 0.01	0.554 ± 0.01
GVAR-Degree	†	†	†	†	†	†
GVAR-Eigen	0.791 ± 0.129	0.758 ± 0.129	0.746 ± 0.098	0.803 ± 0.098	0.816 ± 0.077	0.734 ± 0.077

— indicates retrieved NGC graph is degenerate, e.g., only edges are self-directed.
† indicates network measure cannot distinguish between nodes, e.g., all in/out degrees are equal.

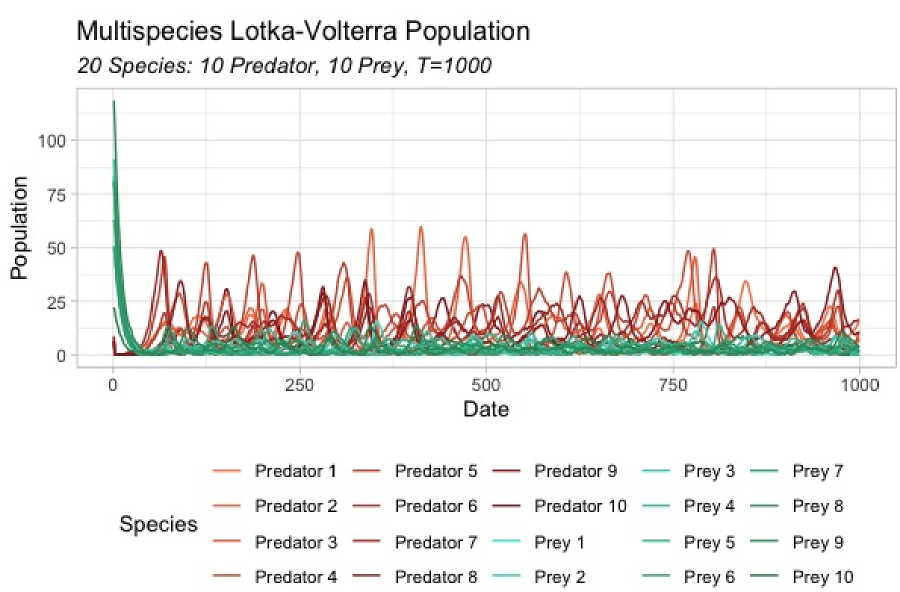

Figure 1: Example Multi-species Lotka-Volterra Population with $d = 20$ and $T = 1000$. Warm colors refer to the 10 predator species and cool colors refer to the 10 prey species. Each predator hunts 2 prey and each prey is hunted by 2 predators.

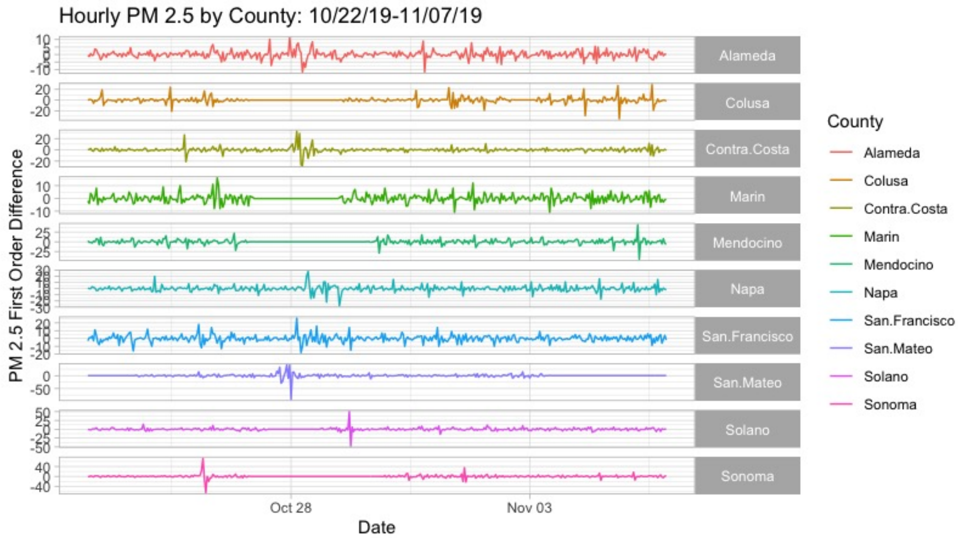

Figure 2: Hourly PM 2.5 Concentration (FOD) by County During Kincade Fire - Oct. 22 to Nov. 7, 2019.

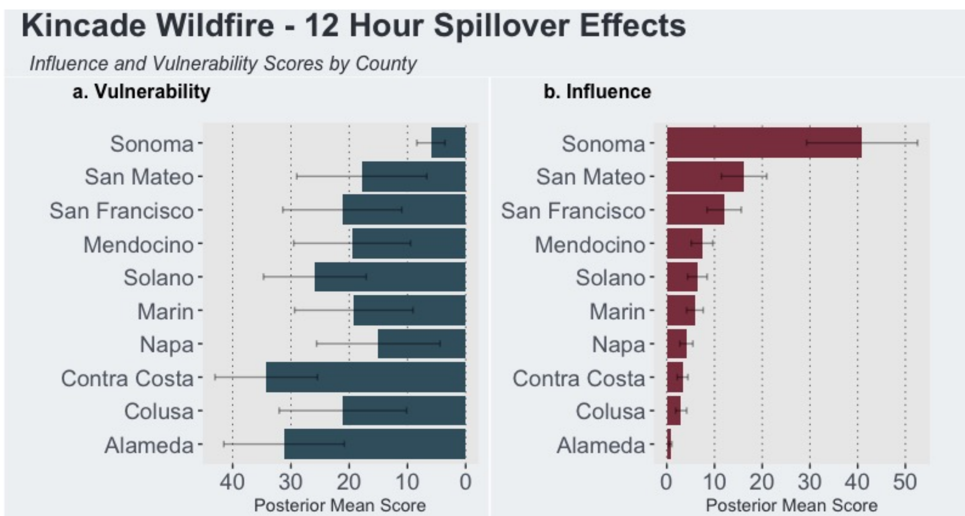

Figure 3: County Ranking by BSG Importance and Vulnerability Scores, $h = 12$.

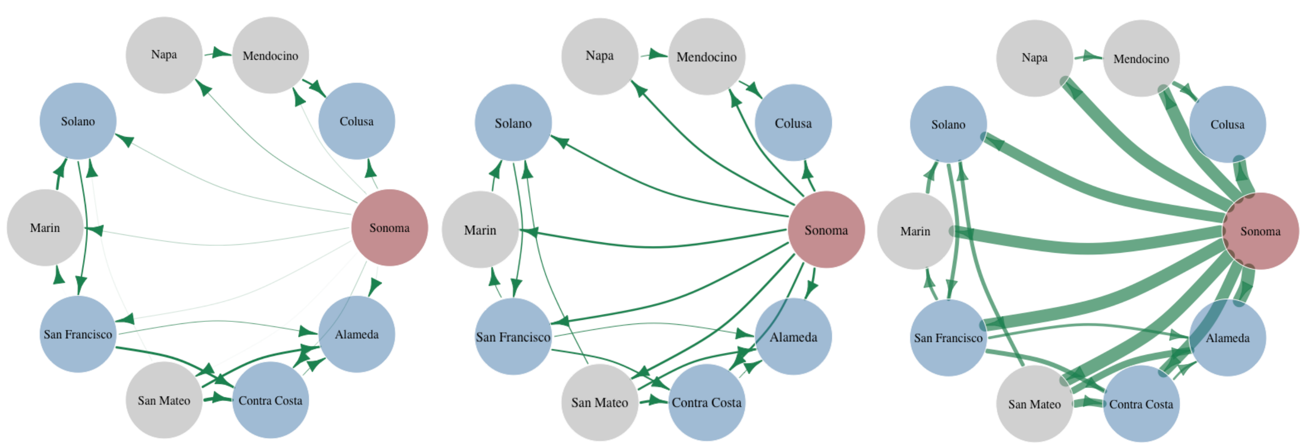

Figure 4: From left to right: Lower 95% HPDI Bound, Posterior Mean, and Upper 95% HPDI Bound. BSG for Kincade Fire, h=12 hours ahead. Note the strong variability in spillovers (edge weights) originating from Sonoma County and tighter intervals for indirect spillovers between San Francisco, Contra Costa, and Alameda counties.

References

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