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Non-asymptotic convergence bounds for Sinkhorn iterates and their gradients: a coupling approach.

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Abstract

Computational optimal transport (OT) has recently emerged as a powerful framework with applications in various fields. In this paper we focus on a relaxation of the original OT problem, the entropic OT problem, which allows to implement efficient and practical algorithmic solutions, even in high dimensional settings. This formulation, also known as the Schrödinger Bridge problem, notably connects with Stochastic Optimal Control (SOC) and can be solved with the popular Sinkhorn algorithm. In the case of discrete-state spaces, this algorithm is known to have exponential convergence; however, achieving a similar rate of convergence in a more general setting is still an active area of research. In this work, we analyze the convergence of the Sinkhorn algorithm for probability measures defined on the d-dimensional torus $\mathbb{T}_L^d$, that admit densities with respect to the Haar measure of $\mathbb{T}_L^d$. In particular, we prove pointwise exponential convergence of Sinkhorn iterates and their gradient. Our proof relies on the connection between these iterates and the evolution along the Hamilton-Jacobi-Bellman equations of value functions obtained from SOC -problems. Our approach is novel in that it is purely probabilistic and relies on coupling by reflection techniques for controlled diffusions on the torus.

Keywords: optimal transport, Sinkhorn algorithm, stochastic optimal control, Schrödinger bridge

1. Introduction

Computational optimal transport (OT) has known great progress over these past few years (Peyré and Cuturi, 2019), and has thus become a popular tool in a wide range of fields such as machine learning (Adler et al., 2017; Arjovsky et al., 2017), computer vision (Dominitz and Tannenbaum, 2009; Solomon et al., 2015), or signal processing (Kolouri et al., 2017). Let μ and ν be two probability measures defined on a measurable state space $(\mathsf{X}, \mathcal{X})$. The primal OT problem (Villani, 2008) between μ and ν, corresponding to a measurable cost function $\mathsf{c} : \mathsf{X}^2 \to [0, +\infty)$, can be formulated as solving the optimization problem

$$\inf_{\pi \in \Pi(\mu,\nu)} \int \mathsf{c}(x, y)\mathrm{d}\pi(x, y)\,, \tag{1}$$

where $\Pi(\mu, \nu)$ is defined as the set of couplings between μ and ν, *i.e.*, $\pi \in \Pi(\mu, \nu)$ if $\pi(\mathsf{A} \times \mathsf{X}) = \mu(\mathsf{A})$ and $\pi(\mathsf{X} \times \mathsf{A}) = \nu(\mathsf{A})$ for any $\mathsf{A} \in \mathcal{X}$. This problem admits the following dual formulation

$$\sup_{(\varphi^\star,\psi^\star)\in\mathcal{R}(\mathsf{c})} \int \{\varphi^\star(x) + \psi^\star(y)\}\mathrm{d}(\mu \otimes \nu)(x, y)\,, \tag{2}$$

where

$$\mathcal{R}(\mathsf{c}) = \{(\varphi^\star, \psi^\star) \in \mathcal{C}(\mathsf{X})^2 : \text{ for any } (x, y) \in \mathsf{X}^2, \varphi^\star(x) + \psi^\star(y) \leq \mathsf{c}(x, y)\}$$

is the set of "Kantorovitch potentials" (Kellerer, 1984). In many applications of OT, $\mathsf{X} \subset \mathbb{R}^d$ and one chooses the Euclidean quadratic cost $\mathsf{c}(x, y) = \|x - y\|^2/2$. Under this setting, Monge-Kantorovich's theorem states that (1) admits a unique minimizer $\pi^\star$. In addition, in the case where μ admits a density with respect to the Lebesgue measure, Brenier's theorem (Brenier, 1991) established that this minimizer is also solution to the Monge problem, *i.e.*, there exists a convex function $\Psi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $\pi^\star$ is the pushforward of μ by the application $x \mapsto (x, \mathrm{T}(x))$ with $\mathrm{T}(x) = \nabla\Psi(x)$ if $\Psi(x) < \infty$ and $\mathrm{T}(x) = 0$ otherwise (referred to as the "Monge" map). Moreover, Ψ is related to a Kantorovitch potential $\varphi^\star$ solving (2) as $\Psi(x) = \|x\|^2/2 - \varphi^\star(x)$. Unfortunately, OT problems (1) and (2) suffer from the curse of dimensionality (Papadakis et al., 2014; Niles-Weed and Rigollet, 2022), which makes impossible to compute $\pi^\star$ or the map T in high-dimensional settings. Although recent works have been carried out this problem assuming regularity conditions on the domain X or on some densities of μ and ν, if they exist, solving efficiently (1) and (2) remains an open problem (Benamou et al., 2014; Niles-Weed and Rigollet, 2022; Forrow et al., 2019).

To circumvent these computational limits, an approach consists in computing a regularized version of the OT problem (1), which penalizes the entropy of the joint coupling π:

$$\inf_{\pi \in \Pi(\mu,\nu)} \left\{ \int \mathsf{c}(x, y)\mathrm{d}\pi(x, y) + \varepsilon \mathrm{KL}(\pi \mid \mu \otimes \nu) \right\} , \tag{3}$$

where KL denotes the Kullback-Leibler divergence and $\varepsilon > 0$ is a regularization parameter. The entropic regularization notably defines a convex minimization problem (in contrast to (1) in general settings), which admits a unique solution $\pi^\star_\varepsilon$. In addition, under appropriate conditions, $\{\pi^\star_\varepsilon\}_{\varepsilon>0}$ converges to a solution of (1); see e.g., Léonard (2012). The entropic OT problem (3) can be tracked back to Schrödinger (Schrödinger, 1931) and may be casted as a "static Schrödinger problem" (Léonard, 2014; Conforti, 2019; Carlier et al., 2017) given by

$$\inf_{\pi \in \Pi(\mu,\nu)} \mathrm{KL}(\pi \mid \rho_\varepsilon) , \tag{4}$$

where $\rho_\varepsilon \in \mathcal{P}(\mathsf{X} \times \mathsf{X})$ is the *reference* measure defined by $\mathrm{d}\rho_\varepsilon(x, y)/\mathrm{d}\{\mu \otimes \nu\} \propto \exp(-\mathsf{c}(x, y)/\varepsilon)$. Under some conditions on μ and ν, $\pi^\star_\varepsilon$ admits as density

$$\frac{\mathrm{d}\pi^\star_\varepsilon}{\mathrm{d}\rho_\varepsilon}(x, y) = \exp[-(\varphi_\varepsilon(x) + \psi_\varepsilon(y))] , \tag{5}$$

where $\varphi_\varepsilon \in \mathrm{L}^1(\mu)$ and $\psi_\varepsilon \in \mathrm{L}^1(\nu)$ are called the "Schrödinger potentials". These potentials are unique up to a trivial additive constant and can be considered as a regularized version of the Kantorovich potentials $\varphi^\star$ and $\psi^\star$. Indeed, under similar assumptions as Brenier's theorem, it holds that the (rescaled) Schrödinger potentials and their gradients respectively converge to the Kantorovich potentials and their gradients as ϵ goes to 0 (Chiarini et al., 2023), hence recovering the Monge map Ψ. In contrast to exact OT problems (1) and (2), (5) can be solved quickly using the Sinkhorn algorithm (Sinkhorn, 1964; Cuturi, 2013), and has thus become a popular alternative to the standard OT formulation. The Sinkhorn algorithm consists in defining sequences $(\varphi_{\varepsilon,n})_{n\in\mathbb{N}}$ and $(\psi_{\varepsilon,n})_{n\in\mathbb{N}}$ respectively approximating φ_ε and ψ_ε, relying that these two functions are fixed points of a particular functional. In this paper, we are interested in the convergence of these two sequences to the "Schrödinger potentials" φ_ε and ψ_ε. More precisely, our contributions are as follows.

Contributions. We provide a new approach to study the convergence of the Sinkhorn algorithm for the case where the state space X is chosen as the d-dimensional torus $\mathbb{T}_L^d = \mathbb{R}^d/(L\mathbb{Z}^d)$, for $L > 0$, endowed with its canonical Riemannian metric. In particular, our analysis exploits the relationship between the Schrödinger bridge problem and Stochastic Optimal Control (SOC). As shown by Léonard (2014) for the case $\mathsf{X} = \mathbb{R}^d$, $\pi_\varepsilon^\star$ is the distribution of the pair of random variables (X_0, X_1), where $(X_t)_{t\in[0,1]}$ evolves along the stochastic differential equation $\mathrm{d}X_t = u_\varepsilon^\star(t, X_t)\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t$ and $u_\varepsilon^\star$ is the *control* function solving

$$\inf_{u:[0,1]\times\mathbb{R}^d\to\mathbb{R}^d} \frac{1}{2}\int_0^1 \mathbb{E}\|u(t,X_t)\|^2\mathrm{d}t \quad \text{such that} \quad \begin{cases} \mathrm{d}X_t = u(t,X_t)\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t\,, \\ X_0 \sim \mu\,, X_1 \sim \nu\,, \end{cases} \tag{6}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion over $\mathbb{R}^d$. By establishing new convergence bounds for inhomogeneous controlled processes on $\mathbb{T}_L^d$ related to (6) using coupling techniques, we show the pointwise exponential convergence of the sequence of Sinkhorn iterates. While this result is not new, our approach is new in that it is essentially probabilistic in nature. More importantly, this approach allows us to prove our second contribution, namely the convergence of the gradient for the Sinkhorn iterates $(\nabla\varphi_{\varepsilon,n})_{n\in\mathbb{N}}$ and $(\nabla\psi_{\varepsilon,n})_{n\in\mathbb{N}}$ to the gradients of the Schrödinger potentials, $\nabla\varphi_\varepsilon$ resp. $\nabla\psi_\varepsilon$, which are used to estimate the Brenier map. To the best of our knowledge, our analysis is the first to derive convergence of gradients independently from iterates' convergence [1]. We highlight that this approach is of primary interest since it can be directly generalized to the unbounded setting.

Outline of the work. The paper is organized as follows. In Section 2, we introduce the theoretical setting of our analysis of Sinkhorn algorithm, detail our assumptions and present our main result. In Section 3, we discuss the dependence of the convergence rate in the parameters of the problem. We review related work and precisely detail our contributions in Section 4, and present the main steps of our proof in Section 5.

Notation. For any measurable space $(\mathsf{X}, \mathcal{X})$, we denote by $\mathcal{P}(\mathsf{X})$ the space of probability measures defined on $(\mathsf{X}, \mathcal{X})$. Denote by $\mathrm{L}^1(\mathsf{X})$ the set of function integrable with respect to μ. For any two distributions $\mu, \nu \in \mathcal{P}(\mathsf{X})$, we define the Kullback–Leibler divergence between μ and ν as $\mathrm{KL}(\mu \mid \nu) = \int_{\mathsf{X}} \mathrm{d}\mu \log(\mathrm{d}\mu/\mathrm{d}\nu)$ if $\mu \ll \nu$ and $\mathrm{KL}(\mu \mid \nu) = +\infty$ otherwise. In the case $\mathsf{X} = \mathbb{R}^d$, we denote by Leb the Lebesgue measure and define $\mathrm{Ent}(\mu) = \int \mathrm{dLeb}\log(\mathrm{d}\mu/\mathrm{dLeb})$ if $\mu \ll \mathrm{Leb}$ and $+\infty$ otherwise.

2. Theoretical framework and main results

Setting and Sinkhorn iterates. Throughout this paper, we consider two probability measures μ and ν defined on the torus $\mathbb{T}_L^d := \mathbb{R}^d/L\mathbb{Z}^d$ of length $L > 0$. Since $\mathbb{T}_L^d$, endowed with addition, is a compact Lie group (Bump, 2013, Chapter 15), we denote by $\mathtt{H}_L^d$ the left Haar measure which corresponds to its Riemannian volume form (Folland, 2013, Chapter 11.4). Furthermore, we consider

1. During the discussion period before acceptance of the present paper, one of the reviewers pointed out that it could be possible to adapt (del Barrio et al., 2022, Lemma 4.8) to obtain convergence of the sequence of the gradients from the convergence of Sinkhorn iterates. However, the constants that would appear in the resulting convergence bounds are not explicit.

in our paper the problem (4) for a particular class of reference measure ρ_ε: for a fixed time horizon $T > 0$, we aim at solving the static Schrödinger problem defined by

$$\inf_{\pi\in\Pi(\mu,\nu)} \mathrm{KL}(\pi \mid \mathrm{R}_{0,T})\,, \tag{7}$$

where $\mathrm{R}_{0,T}$ is a distribution on $\mathbb{T}_L^{2d}$ related to the Langevin stochastic differential equation (SDE)

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t + \mathrm{d}B_t\,, \tag{8}$$

for a twice continuously differentiable potential function $V : \mathbb{T}_L^d \to \mathbb{R}$. Note that for $V \equiv 0$ the above stochastic dynamics corresponds to Brownian motions (*i.e.*, the one associated to the Laplacian operator). If we further consider as state space $\mathsf{X} = \mathbb{R}^d$, then $\mathbf{m}$ equals the Lebesgue measure Leb and (7) is an equivalent formulation of the entropic transport problem

$$\inf_{\pi\in\Pi(\mu,\nu)} \left\{\frac{1}{2}\int \|x-y\|^2\,\mathrm{d}\pi(x,y) + T\,\mathrm{KL}(\pi \mid \mu\otimes\nu)\right\}\,.$$

For the general case $V \not\equiv 0$, we refer to García-Portugués et al. (2019) for an introduction to Langevin diffusion on $\mathbb{T}_L^d$. Since V is twice continuously differentiable and $\mathbb{T}_L^d$ is compact, by (Kent, 1978, Theorem 10.1), (8) admits a unique solution and define a Markov semigroup $(\mathrm{P}_t)_{t\geq 0}$ with bi-continuous transition density $(p_t)_{t>0}$ with respect to the stationary distribution, $\mathbf{m}(\mathrm{d}x) = \mathrm{e}^{-2V(x)}\mathtt{H}(\mathrm{d}x)$, of (8) that is symmetric, *i.e.*, for any x, y, $p_t(x,y) = p_t(y,x)$. As a result, for any $T > 0$, $(x,y) \mapsto \mathrm{e}^{-2V(x)}p_T(x,y)$ defines a joint density on $(\mathbb{T}_L^d)^2$ and $\mathrm{R}_{0,T}$ is the corresponding probability measure.

We now state our main assumption. In particular, we will suppose that the two distributions μ and ν are equivalent to $\mathbf{m}$.

Assumption 1 *The potential V is twice continuously differentiable and there exists two continuously differentiable functions from $\mathbb{T}_L^d$ to $\mathbb{R}$, U_μ and U_ν, such that*

$$\mu(\mathrm{d}x) = \exp(-U_\mu(x))\mathbf{m}(\mathrm{d}x)\;, \quad \nu(\mathrm{d}x) = \exp(-U_\nu(x))\mathbf{m}(\mathrm{d}x)\;. \tag{9}$$

Under Assumption 1, $\mathrm{KL}(\mu \mid \mathbf{m})$ and $\mathrm{KL}(\nu \mid \mathbf{m})$ are finite and (Léonard, 2014, Theorem 2.6) shows that Problem (7) admits a unique minimizer $\pi^\star \in \mathcal{P}(\mathbb{T}_L^d \times \mathbb{T}_L^d)$ dominated by $\mathrm{R}_{0,T}$, which can be expressed via Schrödinger potentials $\varphi^\star, \psi^\star : \mathbb{T}_L^d \to \mathbb{R}\cup\{\infty\}$ such that

$$\frac{\mathrm{d}\pi^\star}{\mathrm{d}\mathrm{R}_{0,T}}(x,y) = \exp(-\varphi^\star(x) - \psi^\star(y))\,. \tag{10}$$

Since p_T is continuously differentiable with respect to its both variables (Kent, 1978), (Nutz, 2021, Lemma 4.11) implies that $\varphi^\star$ and $\psi^\star$ are also continuous and even Lipschitz. In fact, we will recover this result as a corollary of our results.

Here, we assume the potentials $\varphi^\star, \psi^\star$ satisfying the symmetric normalization $\int \varphi^\star\mathrm{d}\mu + \mathrm{KL}(\mu \mid \mathbf{m}) = \int \psi^\star\mathrm{d}\nu + \mathrm{KL}(\nu \mid \mathbf{m})$. Then, the Sinkhorn algorithm (Sinkhorn, 1964; Di Marino and Gerolin, 2020) consists in defining the sequence of potentials $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$, starting from $\psi^0 = 0$[2], by the recursion: for $n \in \mathbb{N}$

$$\varphi^{n+1} := U_\mu + \log \mathrm{P}_T\mathrm{e}^{-\psi^n}\;, \qquad \psi^{n+1} := U_\nu + \log \mathrm{P}_T\mathrm{e}^{-\varphi^{n+1}}\;. \tag{11}$$

2. Let us point out the fact that our results hold true for any smooth choice of ψ^0. Here, we set $\psi^0 = 0$ for convenience.

where $(\mathrm{P}_t)_{t\geq 0}$ is the semigroup associated to the SDE (8). From (9) and (10), it is immediate to deduce that the couple $(\varphi^\star, \psi^\star)$ is a fixed point of the above iteration. Moreover, the algorithm can be interpreted as fixing one of the prescribed marginals at each step. More precisely, when ψ^n is given and we compute the next iterate φ^{n+1}, we are implicitly prescribing the couple (φ^{n+1}, ψ^n) to fit the first marginal constraint, *i.e.*, we are imposing that the first marginal of the probability measure $\mathrm{d}\pi^{n+1,n}/\mathrm{dR}_{0,T} \propto \exp(-\varphi^{n+1}(x) - \psi^n(y))$ is exactly μ. At the next iteration, when we compute ψ^{n+1} we forget about the first marginal and impose the constraint on the second one, which yields to imposing the second marginal of $\mathrm{d}\pi^{n+1,n+1}/\mathrm{dR}_{0,T} \propto \exp(-\varphi^{n+1}(x) - \psi^{n+1}(y))$ to be equal to ν. On the primal side this is also equivalent to minimizing at each step the KL-divergence from the previous plan subject to a one-sided marginal constraint, *i.e.*,

$$\pi^{n+1,n} := \arg\min_{\Pi(\mu,\star)} \mathrm{KL}(\cdot|\pi^{n,n}) \,, \qquad \pi^{n+1,n+1} := \arg\min_{\Pi(\star,\nu)} \mathrm{KL}(\cdot|\pi^{n+1,n}) \,, \tag{12}$$

where $\Pi(\mu,\star)$ (resp. $\Pi(\star,\nu)$) is the set of probability measures on $(\mathbb{T}^d_L)^2$ such that the first marginal is μ (resp. the second marginal is ν). Let us also point out that the choice of an optimal-enough regularization parameter T, which guarantees both fast convergence of Sinkhorn algorithm and accurate approximation of OT, is still a very active field of research. For instance, on a discrete setting (with n-atomic supports), Altschuler et al. (2017) suggests that choosing $T = \log(n)/\tau$ is enough in order to get a τ-accuracy with just $O(\log(n)/\tau^3)$ iterations. We refer to Peyré and Cuturi (2019) for a further discussion on this trade-off.

In order to be consistent with the normalization imposed on the Schrödinger potentials $\varphi^\star$ and $\psi^\star$, we might have to normalize at each step the obtained iterates by considering for any $n \in \mathbb{N}$

$$\varphi^{\diamond n} = \varphi^n - \left(\int \varphi^n \mathrm{d}\mu - \int \varphi^\star \mathrm{d}\mu\right) , \qquad \psi^{\diamond n} = \psi^n - \left(\int \psi^n \mathrm{d}\nu - \int \psi^\star \mathrm{d}\nu\right) ,$$

so that

$$\int \varphi^{\diamond n} \mathrm{d}\mu = \int \varphi^\star \mathrm{d}\mu \quad \text{and} \quad \int \psi^{\diamond n} \mathrm{d}\nu = \int \psi^\star \mathrm{d}\nu \,. \tag{13}$$

One may also consider other normalization options, such as the pointwise condition $\varphi^\star(0) = \psi^\star(0) = 0$, or the zero-mean normalization (Di Marino and Gerolin, 2020; Carlier and Laborde, 2020; Deligiannidis et al., 2021; Carlier, 2022)

We consider on the torus $\mathbb{T}^d_L$ a *sine-distance* which suits best our periodic situation. More precisely, for any pair $(x,\, y) \in \mathbb{T}^d_L \times \mathbb{T}^d_L$, we define

$$\delta(x,y) = L\sqrt{\sum_{i=1}^d \sin^2\left(\frac{\pi}{L}(x^i - y^i)\right)} \in [0, L\, d^{1/2}] \,, \tag{14}$$

where $x = (x^i)_{i\in[d]}$, $y = (y^i)_{i\in[d]}$ and the difference $(\pi/L)(x^i - y^i)$ has to be thought as an element of the one dimensional unit-torus $\mathbb{T}^1 = \mathbb{S}^1$ identified with the unit-circle. Note that the above sine-distance is indeed a distance (the triangular inequality follows from the properties of sin) and is equivalent to the flat-distance d induced by the Euclidean distance function:

$$(\pi\ L)^{1/2}\, \mathsf{d}(x,y) \leq \delta(x,y) \leq\ \pi\, \mathsf{d}(x,y) \,.$$

Let us remark here that our motivation behind adapting such equivalent metric comes from the coupling techniques considered in Appendix A, where we need to consider a smooth metric on $\mathbb{T}^d_L$.

Finally, we define the Lispchitz norm of a function $h : \mathbb{T}_L^d \to \mathbb{R}$ as

$$\|h\|_{\mathrm{Lip}} := \sup_{x \neq y \in \mathbb{T}_L^d} \frac{|h(x) - h(y)|}{\mathsf{d}(x,y)} .$$

We are now ready to state our main result.

Theorem 2 *Assume Assumption 1. Then, there exist a rate $\gamma \in (0,1)$ and a positive constant $\mathrm{c_S} > 0$ such that*

$$\begin{aligned} \sup_{x \in \mathbb{T}_L^d} |\varphi^{\diamond n}(x) - \varphi^\star(x)| &\leq L d^{1/2} \mathrm{c_S}\, \gamma^{2n-1} \, \|\psi^0 - \psi^\star\|_{\mathrm{Lip}} \\ \sup_{x \in \mathbb{T}_L^d} |\psi^{\diamond n}(x) - \psi^\star(x)| &\leq L d^{1/2} \mathrm{c_S}\, \gamma^{2n} \, \|\psi^0 - \psi^\star\|_{\mathrm{Lip}} . \end{aligned} \tag{15}$$

Similarly, we get the uniform exponential convergence for the gradients

$$\begin{aligned} \sup_{x \in \mathbb{T}_L^d} |\nabla\varphi^{\diamond n}(x) - \nabla\varphi^\star(x)| &\leq \pi \mathrm{c_S}\, \gamma^{2n-1} \, \|\psi^0 - \psi^\star\|_{\mathrm{Lip}} \\ \sup_{x \in \mathbb{T}_L^d} |\nabla\psi^{\diamond n}(x) - \nabla\psi^\star(x)| &\leq \pi \mathrm{c_S}\, \gamma^{2n} \, \|\psi^0 - \psi^\star\|_{\mathrm{Lip}} . \end{aligned} \tag{16}$$

Moreover, γ and $\mathrm{c_S}$ have an explicit expression that can be computed, depending on the choice of the potential V, see (33).

As detailed in the proof of Theorem 2 given in Section 5, for any potential V satisfying Assumption 1, there exists an explicit rate $\bar{\lambda}_V > 0$ such that the rate γ, given by Theorem 2, can be written as $\gamma = \mathrm{e}^{-\bar{\lambda}_V \pi^2 T}$. In fact, $\bar{\lambda}_V$ corresponds to the ergodicity rate of the controlled diffusion when considering as underlying reference system the diffusion driven by $b_s(x) = -\nabla V(x) + \nabla \log \mathrm{P}_{T-s} e^{-\psi^\star}(x)$, *i.e.*, the Schrödinger Bridge SDE (Föllmer and Gantert, 1997).

3. Explicit convergence rates and discussion

In this section, we provide explicit *estimates* of γ and $\mathrm{c_S}$, defined in Theorem 2, for a potential V which is assumed to be α-semiconvex for some $\alpha \leq 0$[3], *i.e.*, V is satisfies for any $x, y \in \mathbb{T}_L^d$,

$$\sin\left(\frac{\pi}{L}(x-y)\right)^{\mathsf{T}} (\nabla V(x) - \nabla V(y)) \geq \frac{\pi\,\alpha}{2L}\, \delta(x,y)^2 , \tag{17}$$

where the $\sin$ function applied to any vector of $\mathbb{T}_L^d$ as to be understood as a component-wise map applied to a representative in $[-\pi/2, +\pi/2)$. An example of such potential is provided in Appendix C.1. For notations' sake, let us denote with $D = L\, d^{1/2}$ the diameter of the torus $\mathbb{T}_L^d$ and let $\eta_D = \exp(D^2\, |\alpha|/8)$. Then the estimates of γ and $\mathrm{c_S}$ are given by

$$\log \gamma \leq \, -\pi^2 T \frac{|\alpha|/4}{\eta_D - 1} \, \exp\left(-D \frac{\|U_\mu\|_{f_V} \vee \|U_\nu\|_{f_V}}{1 - \exp\left(-\frac{|\alpha|/4}{\eta_D - 1}\, \pi^2\, T\right)} \right) \tag{18}$$

3. Let us point out that we cannot expect $\alpha > 0$ since we work on a compact Riemannian manifold.

and

$$c_S \leq 2\frac{\eta_D}{\sqrt{L\pi}} \exp\left(D\frac{\|U_\mu\|_{f_V} \vee \|U_\nu\|_{f_V}}{1-\exp\left(-\frac{|\alpha|/4}{\eta_D - 1}\pi^2 T\right)}\right), \tag{19}$$

where $f_V : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave and continuous function, employed in our proofs in order to get exponential contractive Lipschitz estimates, whereas the f_V-Lipschitz norm $\|\cdot\|_{f_V}$ is defined for any function $\phi : \mathbb{T}^d_L \to \mathbb{R}$ as

$$\|\phi\|_{f_V} \coloneqq \sup_{x\neq y \in \mathbb{T}^d_L} \frac{|\phi(x)-\phi(y)|}{f_V(\delta(x,y))}.$$

The proof of these bounds is postponed to Appendix C.

Moreover, when considering $V = 0$ we recover the classic setting when considering Brownian motions, which corresponds to the quadratic regularized OT, and the computations in Appendix C show that the rate of convergence γ_0 in the asymptotic regime $T \to 0$ behaves as

$$\log\gamma_0 \sim -\pi^2 D^2_{\mu,\nu} D^4 T^{-1} \exp(-D_{\mu,\nu} D^3 T^{-1}) \quad \text{and} \quad c_S \sim \exp(D_{\mu,\nu} D^3 T^{-1}). \tag{20}$$

where $D_{\mu,\nu} \coloneqq \frac{1}{2\pi^2} \|U_\mu\|_{f_0} \vee \|U_\nu\|_{f_0}$.

The general bounds (18) and (19) (for $\alpha < 0$), in the asymptotic regime $T \to 0$ and $D \to +\infty$, may be reduced to

$$|\log\gamma| = \mathcal{O}\left(T\,\eta_D^{-1}\exp(-\eta_D\, D\, T^{-1})\right),$$
$$c_S = \mathcal{O}(\eta_D \exp(\eta_D\, D\, T^{-1})),$$

where we omitted the constants that do not affect significantly this regime. As expected, $\gamma \to 1$ and $c_S \to \infty$, *i.e.*, the asymptotic regime highly slows down the convergence of Sinkhorn algorithm, especially when d is large.

4. Comparison with existing literature and original contributions

The Sinkhorn algorithm is very well-known and its study has been intensified particularly after Cuturi (2013). Nonetheless, its introduction dates back to Yule (1912), and it is often referred to as Iterative Proportional Fitting Procedure (IPFP). We refer to Peyré and Cuturi (2019) for an extensive overview on Entropic Optimal Transport, on Sinkhorn algorithm, on its generalizations and on their applications

On discrete state spaces, convergence of the Sinkhorn algorithm has been proven for the first time by Sinkhorn (1964) and Sinkhorn and Knopp (1967). In this setting, Franklin and Lorenz (1989) show that Sinkhorn algorithm is equivalent to a sequence of iterations of a contraction in the Hilbert projective metric and prove its geometric (*i.e.*, exponential) convergence by relying on Birkoff's theorem. We refer also to Borwein et al. (1994), who focus on fixed-point problems, in settings more general than the matrix one. In particular, they consider Sinkhorn-type algorithms which turn out to be once again equivalent to iterations of a contraction in the Hilbert projective metric.

The continuous counterpart of the Hilbert metric has already been investigated in Chen et al. (2016) and in Deligiannidis et al. (2021). In the latter, the authors provide quantitative stability estimates of IPFP on compact metric spaces, from which its convergence can be deduced. Even though these original approaches provide also quantitative rates of convergence, they badly scale when applied to a multimarginal Optimal Transport setting. Recently, new ideas from convex theory have been introduced in order to tackle the convergence of Sinkhorn algorithm in the multimarginal setting too, for bounded costs (or equivalently compact spaces). Along this line of research, it is worth mentioning Carlier and Laborde (2020) where the authors prove well-posedness of Sinkhorn iterates and their smooth dependence from the marginals. In addition, Di Marino and Gerolin (2020) have proven an L^p (qualitative) convergence of Sinkhorn iterates, and Carlier (2022) which improves the previous results by showing an exponential convergence (with a rate that scales linearly with the number of marginals).

Regarding the primal formulation and the convergence of $(\pi^{n,n})_{n\in\mathbb{N}}$ defined in (12) to the optimal coupling, Nutz and Wiesel (2022) establish qualitative convergence in total variation. Following this work, Eckstein and Nutz (2022) show quantitative (polynomial) convergence in Wasserstein distance. Lastly, Ghosal and Nutz (2022) derive polynomial linear convergence (*i.e.*, of order $O(1/n)$) with respect to a symmetric relative entropy.

Original Contribution. In this paper we provide a new approach in the study of Sinkhorn algorithm on the d-dimensional torus $\mathbb{T}^d_L$ and its main novelties can be summarized as follows.

- Our proofs rely on probabilistic arguments and coupling methods, by exploiting the connection between the Schrödinger potentials and value functions of stochastic optimal control problems. To the best of our knowledge, this is the first paper addressing the problem relying on a (non-trivial) stochastic interpretation, while the existing literature usually relies on convex analysis and/or on the Hilbert metric. Moreover, this probabilistic approach via stochastic optimal control could in principle be carried over to the unbounded case (e.g. in the Euclidean space $\mathbb{R}^d$). Here we specify our results on the torus since it allows us to work on a compact state space while benefiting from its underlying Euclidean structure. However our approach could be extended to smooth compact manifolds without boundaries but at the expense of technicalities, in particular the definition of an appropriate coupling by reflection. Dealing with the torus allows us to reduce these complications at the bare minimum avoid technical details related to general bounded compact state spaces for which topological conditions generally have to be imposed.

- We prove the convergence of Sinkhorn iterates as a corollary of the convergence of their gradients (or equivalently in Lipschitz norms). To the best of our knowledge, our result is the first one addressing the problem directly at the level of the gradients. Moreover, our probabilistic approach provides Lipschitz estimates along solutions of Hamilton-Jacobi-Bellman equations (*i.e.*, for any time $s\in[0,T]$; see (44)).

  Our results should be compared to Deligiannidis et al. (2021) where Lipschitz estimates close to ours are given, but for iterates $(f_n,g_n)_{n\in\mathbb{N}}$ corresponding to $f_n = \mathrm{P}_T\,\mathrm{e}^{-\psi^n}$ and $g_n = \mathrm{P}_T\,\mathrm{e}^{-\varphi^n}$. To show their result, Deligiannidis et al. (2021) rely on Birkoff's theorem for the Hilbert metric since the iterations they consider are then just linear updates. Note that convergence of the gradient of $(\varphi_n,\psi_n)_{n\in\mathbb{N}}$ cannot be deduce from their result since φ_n and ψ_h are non-linear transformation of g_n and f_n respectively.

- We get an exponential rate of convergence $\gamma = \mathrm{e}^{-\bar{\lambda}_V\,\pi^2\,T}$ which converges to 1 as $T\downarrow 0$. This exponential dependence on T is not surprising. Indeed it is well known that convergence of

Sinkhorn algorithm implies quantitative stability (continuous) bounds for Schrödinger problem (and entropically regularized optimal transport, see Eckstein and Nutz (2022)) while on the contrary the optimal transport map is solely $1/2$-Hölder continuous by Gigli (2011).

5. Sketch of the proof

We now introduce the main components of our method to analyse the convergence of the Sinkhorn iterates given by (11). We first introduce the function $\{\mathcal{U}_t^{T,h}\}_{t\in[0,T]}$ defined for any measurable and bounded function $h : \mathbb{T}_L^d \to \mathbb{R}$:

$$\mathcal{U}_t^{T,h} = -\log \mathrm{P}_{T-t}\mathrm{e}^{-h} ,$$

which can be shown, by a direct computation, to correspond to the solution of the Hamilton-Jacobi-Bellman (hereafter HJB) equation defined by

$$\begin{cases} \partial_t u_t + \frac{1}{2}\Delta u_t - \nabla V \cdot \nabla u_t - \frac{1}{2}|\nabla u_t|^2 = 0 \\ u_T = h\,. \end{cases} \tag{21}$$

With these notations, Sinkhorn iterates can be written as

$$\varphi^{n+1} = U_\mu - \mathcal{U}_0^{T,\psi^n} , \qquad \psi^{n+1} = U_\nu - \mathcal{U}_0^{T,\varphi^{n+1}} . \tag{22}$$

To get some bounds on the Lipschitz constant of φ^{n+1} and ψ^{n+1}, we then show that if $h : \mathbb{T}_L^d \to \mathbb{R}$ is Lipschitz, $\mathcal{U}_0^{T,h}$ is also Lipschitz with an explicit bound for its Lipschitz constant.

To do so, we use that $\mathcal{U}_t^{T,h}$ can be represented also as the value function of the SOC problem

$$\begin{aligned} &\mathcal{U}_t^{T,h}(x) = \inf_{q\in\mathcal{A}_{[t,T]}} \mathbb{E}\left[\frac{1}{2}\int_t^T |q_s|^2 \mathrm{d}s + h(X_T^q)\right] \\ &\text{where } \begin{cases} \mathrm{d}X_s^q = (-\nabla V(X_s^q) + q_s)\mathrm{d}s + \mathrm{d}B_s \\ X_t^q = x\,, \end{cases} \end{aligned} \tag{23}$$

where $(B_s)_{s\geq 0}$ is a $(\mathcal{F}_s)_{s\geq 0}$-Brownian motion on $\mathbb{T}_L^d$ defined on the filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0})$ satisfying the usual conditions, and $\mathcal{A}_{[t,T]}$ denotes the set of admissible controls, *i.e.*, $(\mathcal{F}_s)_{s\geq 0}$-progressively measurable processes. We provide a precise statement of this result in Proposition 9 in Appendix A, where we show the optimal control process to be the feedback-process $q_s = -\nabla\mathcal{U}_s^{T,h}(X_s^q)$. Moreover, Proposition 9 in Appendix A provides a non-trivial control of $\|\mathcal{U}_t^{T,h}\|_{\mathrm{Lip}}$ for any function $h \in \mathcal{C}^3(\mathbb{T}_L^d)$. We give here the main ideas of the proof of this result.

We first show that for any pair of stochastic processes $(X_s, Y_s)_{s\in[t,T]}$, starting from $X_t = x$ and $Y_t = y$ respectively, solution of

$$\begin{aligned} \mathrm{d}X_s &= -\nabla V(X_s)\mathrm{d}s - \nabla\mathcal{U}_s^{T,h}(X_s)\mathrm{d}s + \mathrm{d}B_s\,, \\ \mathrm{d}Y_s &= -\nabla V(Y_s)\mathrm{d}s - \nabla\mathcal{U}_s^{T,h}(X_s)\mathrm{d}s + \mathrm{d}\tilde{B}_s\,, \end{aligned} \tag{24}$$

where $(\tilde{B}_s)_{s\geq 0}$ is a $(\mathcal{F}_s)_{s\geq 0}$-Brownian motion on $\mathbb{T}_L^d$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, it holds by (23),

$$\mathcal{U}_t^{T,h}(y) - \mathcal{U}_t^{T,h}(x) \leq \mathbb{E}\Big[h(Y_T) - h(X_T)\Big] .$$

Then, if h is Lipschitz, we consider a particular coupling which is adapted from the usual coupling by reflection for homogeneous diffusion (Wang, 1994; Eberle, 2016) to bound $\mathcal{U}_t^{T,h}(y) - \mathcal{U}_t^{T,h}(x)$. In particular, the novelty of our approach relies in employing coupling by reflection techniques for controlled diffusion processes on the torus, endowed with the distance δ given in (14), that defines a smooth distance on $\mathbb{T}_L^d$ which is equivalent to the Riemannian distance d. An adaptation of the coupling by reflection techniques under this sine-distance is given in Appendix A. Owing to the construction given there, we obtain for any $t \in [0,T]$ and any $h : \mathbb{T}_L^d \to \mathbb{R}$, Lipschitz

$$\|\mathcal{U}_t^{T,h}\|_{f_V} \leq \mathrm{e}^{-\lambda_V \pi^2 (T-t)} \|h\|_{f_V} \tag{25}$$

where the rate $\lambda_V > 0$ and the function $f_V : \mathbb{R}_+ \to \mathbb{R}_+$, which is concave and continuous, are defined in (35) and (36). Moreover, we prove in Proposition 8 in Appendix A that f_V is equivalent to the identity. Therefore, $\|\cdot\|_{f_V}$ is equivalent to the usual Lipschitz norm $\|\cdot\|_{\mathrm{Lip}}$ (*i.e.*, with f_V being the identity and considering the flat-distance) since $\delta(\cdot,\cdot)$ is equivalent to $\mathsf{d}(\cdot,\cdot)$. In particular, we have for any function $\phi : \mathbb{T}_L^d \to \mathbb{R}$

$$\frac{1}{\pi}\|\phi\|_{\mathrm{Lip}} \leq \|\phi\|_{f_V} \leq \frac{C_V^{-1}}{\sqrt{L}\,\pi}\|\phi\|_{\mathrm{Lip}} \, . \tag{26}$$

where C_V is defined in (36). By combining (25) with (26), we are then able to bound $\|\mathcal{U}_t^{T,h}\|_{\mathrm{Lip}}$ as

$$\|\mathcal{U}_t^{T,h}\|_{\mathrm{Lip}} \leq C_V^{-1}\sqrt{\frac{\pi}{L}}\,\mathrm{e}^{-\lambda_V \pi^2 (T-t)}\|h\|_{\mathrm{Lip}}$$

It is then possible studying how the Lipschitzianity propagates along Sinkhorn iterates using the following result.

Lemma 3 *Assume Assumption 1. For all $n \geq 0$ we have*

$$\begin{aligned} \|\varphi^{n+1}\|_{f_V} &\leq \|U_\mu\|_{f_V} + \mathrm{e}^{-\lambda_V \pi^2 T}\|\psi^n\|_{f_V} \\ \|\psi^{n+1}\|_{f_V} &\leq \|U_\nu\|_{f_V} + \mathrm{e}^{-\lambda_V \pi^2 T}\|\varphi^{n+1}\|_{f_V} \end{aligned} \tag{27}$$

Moreover, for all $n \geq 1$ we have

$$\begin{aligned} \|\psi^n\|_{f_V} &\leq \frac{\|U_\nu\|_{f_V} + \exp(-\lambda_V \pi^2 T)\|U_\mu\|_{f_V}}{1-\exp(-2\lambda_V \pi^2 T)} \\ \|\varphi^n\|_{f_V} &\leq \frac{\|U_\mu\|_{f_V} + \exp(-\lambda_V \pi^2 T)\|U_\nu\|_{f_V}}{1-\exp(-2\lambda_V \pi^2 T)}, \end{aligned} \tag{28}$$

Proof As shown in Proposition 9 in Appendix A (see also (25)), the Lipschitz-regularity backward propagates along solutions of HJB equations. Particularly, it holds

$$\|\mathcal{U}_0^{T,\psi^n}\|_{f_V} \leq \|\psi^n\|_{f_V}\,\mathrm{e}^{-\lambda_V \pi^2 T} \, ,$$

which, combined with (22) and an application of the triangular inequality, gives the first claim in (27). The second claim follows by symmetry. Concatenating these two bounds, we obtain

$$\|\psi^{n+1}\|_{f_V} \leq \|U_\nu\|_{f_V} + \mathrm{e}^{-\lambda_V \pi^2 T}\|U_\mu\|_{f_V} + \mathrm{e}^{-2\lambda_V \pi^2 T}\|\psi^n\|_{f_V},$$

from which the first relation in (28) follows by induction. The second relation follows by symmetry. ■

Remark 4 *Let us also point out that the Lipschitz estimates obtained in Lemma 3, as well as the ones proven in Lemma 6 below, hold true also for the normalized Sinkhorn iterates $\varphi^{\diamond n}, \psi^{\diamond n}$ (and for any other trivial additive perturbation of them). Indeed any additive normalization would cancel out when considering Lipschitz norms.*

From the pointwise convergence of the normalized Sinkhorn iterates $\varphi^{\diamond n}, \psi^{\diamond n}$ towards the Schrödinger potentials (which in our compact and smooth setting is guaranteed from the geometric L^p convergence in Di Marino and Gerolin (2020)), the previous regularity result propagates to the potentials and using (26), we obtain the following corollary.

Corollary 5 *Assume Assumption 1. Then it holds*

$$\begin{aligned} \|\psi^\star\|_{f_V} &\leq \frac{\|U_\nu\|_{f_V} + \exp(-\lambda_V\,\pi^2\,T)\|U_\mu\|_{f_V}}{1-\exp(-2\lambda_V\,\pi^2\,T)} \\ \|\varphi^\star\|_{f_V} &\leq \frac{\|U_\mu\|_{f_V} + \exp(-\lambda_V\,\pi^2\,T)\|U_\nu\|_{f_V}}{1-\exp(-2\lambda_V\,\pi^2\,T)} \end{aligned} \tag{29}$$

and therefore the Lipschitz norm of the Schrödinger potentials can be bounded as

$$\begin{aligned} \|\psi^\star\|_{\mathrm{Lip}} &\leq \pi\,\frac{\|U_\nu\|_{f_V} + \exp(-\lambda_V\,\pi^2\,T)\|U_\mu\|_{f_V}}{1-\exp(-2\lambda_V\,\pi^2\,T)} \\ \|\varphi^\star\|_{\mathrm{Lip}} &\leq \pi\,\frac{\|U_\mu\|_{f_V} + \exp(-\lambda_V\,\pi^2\,T)\|U_\nu\|_{f_V}}{1-\exp(-2\lambda_V\,\pi^2\,T)}\,. \end{aligned}$$

We are now ready to prove the key contraction estimates, from which our main result follows. Once again the main idea behind our proof is relying on a stochastic control problem where the Schrödinger potential contributes in the final cost while its gradient drives the controlled SDE. This allows to back-propagate along an HJB equation the Lipschitz regularity of the difference between the Sinkhorn iterates and the target Schrödinger potential. Indeed, if we denote with $\mathcal{D}^n_t := \mathcal{U}^{T,\psi^n}_t - \mathcal{U}^{T,\psi^\star}_t$ (the difference between the evolutions along HJB of ψ^n and $\psi^\star$ respectively) from (21) we deduce that it solves

$$\begin{cases} \partial_t u_t + \frac{1}{2}\Delta u_t + (-\nabla V - \nabla \mathcal{U}^{T,\psi^\star}_t)\cdot\nabla u_t - \frac{1}{2}|\nabla u_t|^2 = 0 \\ u_T = \psi^n - \psi^\star\,, \end{cases}$$

which can be represented (see Proposition 9 in Appendix A) as the value function of the stochastic control problem

$$\begin{aligned} \mathcal{D}^n_t(x) = \inf_{q_\cdot}\mathbb{E}\left[\frac{1}{2}\int_t^T |q_s|^2\mathrm{d}s + \psi^n(X^q_T) - \psi^\star(X^p_T)\right] \\ \text{where}\quad \begin{cases} \mathrm{d}X^q_s = (-\nabla V(X^q_s) - \nabla\mathcal{U}^{T,\psi^\star}_s(X^q_s) + q_s)\mathrm{d}s + \mathrm{d}B_s \\ X^q_t = x\,. \end{cases} \end{aligned} \tag{30}$$

Once the connection with the stochastic optimal control formulation is established, the proof boils down once again in studying how Lipschitz-regularity backward propagates along solutions of HJB equations.

Lemma 6 *Assume Assumption 1. There exist* $\bar{\lambda}_V > 0$*, given by* (46) *in Appendix B, and a continuous concave function* $\bar{f}_V : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\begin{aligned} \|\psi^{n+1} - \psi^\star\|_{\bar{f}_V} &\leq \exp(-\bar{\lambda}_V\, \pi^2\, T)\|\varphi^{n+1} - \varphi^\star\|_{\bar{f}_V} \\ \|\varphi^{n+1} - \varphi^\star\|_{\bar{f}_V} &\leq \exp(-\bar{\lambda}_V\, \pi^2\, T)\|\psi^{n} - \psi^\star\|_{\bar{f}_V}\,. \end{aligned} \tag{31}$$

As a result

$$\begin{aligned} \|\psi^{n+1} - \psi^\star\|_{\bar{f}_V} &\leq \exp(-2\bar{\lambda}_V\, \pi^2\, T)\|\psi^{n} - \psi^\star\|_{\bar{f}_V} \\ \|\varphi^{n+1} - \varphi^\star\|_{\bar{f}_V} &\leq \exp(-2\bar{\lambda}_V\, \pi^2\, T)\|\varphi^{n} - \varphi^\star\|_{\bar{f}_V}\,. \end{aligned} \tag{32}$$

Proof The proof is postponed to Appendix B. ■

We can now complete the proof of Theorem 2.

Proof [Proof of Theorem 2.]
Since $\int \varphi^{\diamond n} \mathrm{d}\mu = \int \varphi^\star \mathrm{d}\nu$ (see (13)), uniformly on $x \in \mathbb{T}^d_L$, it holds

$$\begin{aligned} |\varphi^{\diamond n}(x) - \varphi^\star(x)| &= \left|\varphi^{\diamond n}(x) - \int_{\mathbb{T}^d_L} \varphi^{\diamond n}\, \mathrm{d}\mu - \varphi^\star(x) + \int_{\mathbb{T}^d_L} \varphi^\star\, \mathrm{d}\mu\right| \\ &= \left|\int_{\mathbb{T}^d_L} \Big[(\varphi^n - \varphi^\star)(x) - (\varphi^n - \varphi^\star)(y)\Big] \mathrm{d}\mu(y)\right| \\ &\leq \int_{\mathbb{T}^d_L} \Big|(\varphi^n - \varphi^\star)(x) - (\varphi^n - \varphi^\star)(y)\Big| \mathrm{d}\mu(y) \\ &\leq \|\varphi^n - \varphi^\star\|_{\bar{f}_V} \int_{\mathbb{T}^d_L} \bar{f}_V(\delta(x, y))\, \mathrm{d}\mu(y) \\ &\leq L\, d^{1/2}\, \|\varphi^n - \varphi^\star\|_{\bar{f}_V}\,. \end{aligned}$$

Therefore, by concatenating (32) along n iterates, we end up with

$$\begin{aligned} \sup_{x \in \mathbb{T}^d_L} |\varphi^{\diamond\, n+1}(x) - \varphi^\star(x)| &\leq L\, d^{1/2}\, \exp(-2\, n\, \bar{\lambda}_V\, \pi^2\, T)\|\varphi^1(x) - \varphi^\star(x)\|_{\bar{f}_V} \\ &\leq L\, d^{1/2}\, \exp(-(2\, n + 1)\, \bar{\lambda}_V\, \pi^2\, T)\|\psi^0(x) - \psi^\star(x)\|_{\bar{f}_V}\,. \end{aligned}$$

By reasoning in the same fashion, since $\int \psi^{\diamond n} \mathrm{d}\mu = \int \psi^\star \mathrm{d}\nu$ or simply by relying on (28), we conclude that

$$\sup_{x \in \mathbb{T}^d_L} |\psi^{\diamond n}(x) - \psi^\star(x)| \leq L\, d^{1/2}\, \exp(-2\, n\, \bar{\lambda}_V\, \pi^2\, T)\|\psi^0(x) - \psi^\star(x)\|_{\bar{f}_V}\,.$$

Using (47), we may conclude the proof of (15) by setting

$$\gamma = \exp(-\bar{\lambda}_V\, \pi^2\, T) \text{ and } \mathrm{c_S} = \bar{C}_V^{-1}/\sqrt{L\, \pi}\,, \tag{33}$$

where $\bar{\lambda}_V$ and $\bar{C}_V$ are respectively defined at (46) and (47).

The proof of the convergence of the gradients can be obtained in a similar fashion since

$$\sup_{x \in \mathbb{T}^d_L} |\nabla \varphi^{\diamond n} - \nabla \varphi^\star|(x) = \sup_{x \in \mathbb{T}^d_L} |\nabla \varphi^n - \nabla \varphi^\star|(x) \leq \|\varphi^n - \varphi^\star\|_{\mathrm{Lip}} \leq \pi \|\varphi^n - \varphi^\star\|_{\bar{f}_V}$$

and similarly for $\psi^{\diamond n} - \psi^\star$, from which (16) follows by concatenating the contraction in (32). ■

6. Conclusion

In this paper, we have introduced a new probabilistic approach in the study of Sinkhorn algorithm. We have shown that each iteration is equivalent to solving an Hamilton-Jacobi-Bellman equation, *i.e.*, computing the value function of a stochastic control problem, and showed that the Lipschitz regularity of the previous Sinkhorn iterate propagates to the next one, with a constant dissipative rate. From this contraction estimates we have deduced the exponential convergence of the Sinkhorn iterates and of their gradients. All the dissipative Lipschitz estimates for the value function of the stochastic control problems considered have been deduced via an application of coupling by reflection techniques for controlled diffusion on the torus.

This approach is a complete novelty and could in principle be extended to the non-compact Euclidean case, problem that we address in the follow-up work (Conforti et al., 2023).

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Organisation of the supplementary

In Appendix A, we give an explicit construction for coupling by reflection on the torus, which we adapt for controlled drifts. In Appendix B, we provide the proof of Lemma 6, which is crucial to derive our main result. In Appendix C, we first compute explicit estimates for the convergence rate γ and the constant c_S appearing in Theorem 2, under a weak-semiconvexity assumption, and then give a class of examples of potentials satisfying this condition.

Appendix A. Reflection coupling for HJB estimates on the torus

In this section we adapt the ideas developed in Conforti (2022) for controlled diffusions on $\mathbb{R}^d$ to our compact setting on $\mathbb{T}^d_L$ as (24). Before doing so, let us spend a few sentences on why we cannot rely on the existing coupling by reflection literature. First of all, the SDE that we consider is time-inhomogenoeus which is not usually studied, especially when considering diffusions on a Riemannian manifold such as the torus. Moreover being on the torus brings another issue. As long as Y_s does not belong to the cut-locus of X_s, we may straightforwardly define coupling by reflection as in the flat Euclidean case. However as soon as Y_t belongs to the cut-locus of X_t a more careful analysis is required since we could not apply Ito formula to d^2 anymore. Indeed on any Riemannian manifold M, d^2 fails to be $\mathcal{C}^2$ on the set of conjugates points, *i.e.*, couples $(x, y) \in \mathsf{M} \times \mathsf{M}$ such that y is in the cut-locus of x (Petersen, 2006, Chapter 5.9.1). In order to deal with this issue one could try to restrict the problem to regular domains (submanifolds without conjugate points and with convex boundary) as it is done in Wang (1994), or one could try to combined reflection coupling with different techniques. The latter approach is portrayed in Cranston (1991), where the author alternates coupling by reflection with an independent coupling (in a small-time window, enough to avoid that the coupled diffusion reaches points in the cut-locus). However we could neither rely on this approach because, despite of having chosen independent Brownian motions, in our control setting the process $(Y_t)_{t\geq 0}$ would still depend on $(X_t)_{t\geq 0}$ via the control process (see (42) where we are going to define the reflection coupling by plugging the optimal control for $(X_t)_{t\geq 0}$ in the controlled SDE for $(Y_t)_{t\geq 0}$).

Having in mind our target, we begin by adapting the coupling by reflection technique of Wang (1994); Eberle (2016) to the torus. In this appendix, we are going to consider a time-inhomogeneous SDE on the torus, namely

$$\mathrm{d}X_s = b_s(X_s)\,\mathrm{d}s + \mathrm{d}B_s \quad \text{on } \mathbb{T}^d_L \tag{34}$$

with a time-inhomogeneous drift function $b_s \in \mathcal{C}^{0,1}([0,T) \times \mathbb{T}^d_L)$. Under the condition $b_s \in \mathcal{C}^{0,1}([0,T) \times \mathbb{T}^d_L)$, this SDE admits a unique strong solution and corresponds to a inhomogeneous Markov semigroup that we denote by $(\mathrm{P}^b_{s,t})_{s,t\in[0,T]}$ in the sequel. We also consider an arbitrary modulus of weak-semiconvexity κ_b associated to b, whose definition is provided below.

Definition 7 *A function $\kappa_b : (0, L\,d^{1/2}] \to \mathbb{R}$ is said to be a modulus of weak semi-convexity associated to the drift b if (i) κ_b is continuous on $(0, L\,d^{1/2}]$, (ii) $s \mapsto s\kappa_b(s)$ is integrable on $(0, L\,d^{1/2}]$ and (iii) we have*

$$\kappa_b(r) \leq \inf_{s\in[0,T]} \inf\left\{ -\frac{2L}{\pi} \frac{\sin(\frac{\pi}{L}(x-y))^\intercal (b_s(x) - b_s(y))}{\delta^2(x,y)} \;:\; x \neq y \in \mathbb{T}^d_L \textit{ s.t. } \delta(x,y) = r \right\}.$$

Let us remark that κ_b is always non-positive [4].

By mimicking (Wang, 1994; Eberle, 2016), let us define for any $r \in (0, L\, d^{1/2}]$ the functions

$$f_b(r) := \int_0^r \phi(s)\, g(s)\, \mathrm{d}s \quad \text{where} \quad \phi(r) := \exp\left(\frac{1}{4}\int_0^r s\,\kappa_b(s)\,\mathrm{d}s\right), \\ \Phi(r) := \int_0^r \phi(s)\,\mathrm{d}s \quad \text{and} \quad g(r) := 1 - \frac{\int_0^r \Phi(s)/\phi(s)\,\mathrm{d}s}{2\int_0^{L\,d^{1/2}} \Phi(s)/\phi(s)\,\mathrm{d}s}\,. \tag{35}$$

Let us remark here that the main difference with (Eberle, 2016) is the presence of $L\,d^{1/2}$, *i.e.*the diameter of the torus, as an upper-limit in the integration domain in the definition of g. Finally, consider the multiplicative and rate constants

$$C_b := \frac{\phi(L\,d^{1/2})}{2} = \frac{1}{2}\exp\left(\frac{1}{4}\int_0^{L\,d^{1/2}} s\,\kappa_b(s)\,\mathrm{d}s\right) \\ \text{and} \quad \lambda_b := \left(\int_0^{L\,d^{1/2}} \Phi(s)/\phi(s)\,\mathrm{d}s\right)^{-1}. \tag{36}$$

The key properties of the triplet (C_b, λ_b, f_b) are summarized in the following result.

Proposition 8 *The triplet (C_b, λ_b, f_b) satisfies*

1. *f_b is equivalent to the identity, i.e., for any $r \in [0, L\,d^{1/2}]$,*

$$C_b\, r \leq f_b(r) \leq r \quad \textit{and} \quad C_b \leq f_b^{'}(r) \leq 1$$

2. *for any $r \in [0, L\,d^{1/2}]$, it holds*

$$f_b^{''}(r) - \frac{\kappa_b(r)}{4}\, r\, f_b^{'}(r) \leq -\frac{\lambda_b}{2} f_b(r)\,. \tag{37}$$

Proof

Let us start by simply noticing that the non-positivity of κ_b implies

$$\phi(L\;d^{1/2}) \leq \phi(s) \leq 1\,, \quad \phi(L\;d^{1/2})\, r \leq \Phi(r) \leq r \quad \text{and} \quad \frac{1}{2} \leq g(r) \leq 1\,,$$

which immediately proves the bounds for $f_b^{'}(r) = \phi(r)g(r)$ with $C_b = \phi(L\,d^{1/2})/2$. From the previous bound on g we immediately deduce also

$$\Phi(r)/2 \leq f_b(r) \leq \Phi(r)\,, \tag{38}$$

which combined with the above bound for Φ concludes the proof of the first item.

In order to prove the second item, it is enough to compute

$$f_b^{''}(r) = \phi^{'}(r)g(r) + \phi(r)g^{'}(r) = \frac{\kappa_b(r)}{4}\, r\,\phi(r)g(r) + \phi(r)g^{'}(r)$$

4. This property is inherited from the fact that we work on a compact Riemannian manifold.

which combined with $f_b'(r) = \phi(r)g(r)$, reads as

$$f_b''(r) - \frac{\kappa_b(r)}{4}\, r\, f_b'(r) \le \phi(r)g'(r)\,.$$

Since, for any $r \in [0, L\, d^{1/2}]$, it holds $g'(r) = -\frac{\lambda_b}{2}\,\Phi(r)/\phi(r)$, we deduce

$$f_b''(r) - \frac{\kappa_b(r)}{4}\, r\, f_b'(r) \le -\frac{\lambda_b}{2}\,\Phi(r) \overset{(38)}{\le} -\frac{\lambda_b}{2}\, f_b(r)\,.$$

■

We are now in place to apply the coupling by reflection ideas to the HJB equation. Recall that we consider a filtered probability space $(\Omega, (\mathcal{F}_s)_{s\in[0,T]}, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions and endowed with a Brownian motion on $\mathbb{T}^d_L$.

Recall that for any $h : \mathbb{T}^d_L \to \mathbb{R}$, Lipschitz, its backward evolution along the semigroup associated with (34) is defined as

$$\mathcal{V}^{T,h}_t := -\log \mathrm{P}^b_{t,T}\mathrm{e}^{-h} \quad \forall t \in [0,T]\,.$$

Let us recall that if the SDE is time-homogeneous (e.g., $b_s(x) = V(x)$ for all $s \in [0,T]$) then the previous expression is equivalent to the usual expression $-\log \mathrm{P}_{T-t}\mathrm{e}^{-h}$. As already mentioned in Section 1, this quantity can be seen as the value function of a stochastic optimal control problem, and solves an Hamilton-Jacobi-Bellman equation. We prove this statement in the following result.

Proposition 9 *We recall that $b_s \in \mathcal{C}^{0,1}([0,T) \times \mathbb{T}^d_L)$. Let $h \in \mathcal{C}^3(\mathbb{T}^d_L)$, then*

1. *$\mathcal{V}^{T,h}_t$ is the unique strong solution in $\mathcal{C}^{1,2}([0,T) \times \mathbb{T}^d_L)$ of the HJB equation*

$$\begin{cases} \partial_t u_t + \frac{1}{2}\Delta u_t + b_t(X_t)\cdot\nabla u_t - \frac{1}{2}|\nabla u_t|^2 = 0 \\ u_T = h\,. \end{cases} \tag{39}$$

2. *$\mathcal{V}^{T,h}_t$ is the the value function of the stochastic control problem*

$$\mathcal{J}^{T,h}_t(x) = \inf_{q.\in\mathcal{A}_{[t,T]}} \mathbb{E}\left[\frac{1}{2}\int_t^T |q_s|^2 \mathrm{d}s + h(X^q_T)\right]$$
$$\text{where } \mathbb{P}\text{-a.s. it holds } \begin{cases} \mathrm{d}X^q_s = (b_s(X^q_s) + q_s)\mathrm{d}s + \mathrm{d}B_s \\ X^q_t = x \end{cases}$$

   and $\mathcal{A}_{[t,T]}$ denotes the set of admissible controls, i.e., progressively measurable processes with finite moments on $(\Omega, (\mathcal{F}_s)_{s\in[0,T]}, \mathcal{F}, \mathbb{P})$. Moreover, the optimal control is a feedback-process equal to $-\nabla\mathcal{V}^{T,h}_s(X^q_s)$.

3. *Uniformly on $t \in [0,T]$, it holds*

$$\|\mathcal{V}^{T,h}_t\|_{f_b} \le \mathrm{e}^{-\lambda_b\,\pi^2\,(T-t)}\|h\|_{f_b}$$

   where λ_b, f_b are the quantities defined at (35) and (36).

Moreover, Item 3 can be relaxed to assuming h only Lipschitz.

Proof Let $h \in \mathcal{C}^3(\mathbb{T}^d_L)$. Firstly, let us observe that, under the current assumption, a direct computation shows that $\mathcal{V}^{T,h}_t$ is a classical solution for the HJB associated the generator $L_t = \frac{\Delta}{2} + b_t \cdot \nabla$ and that (39) can be equivalently written as

$$\begin{cases}\partial_t u_t + \frac{1}{2}\Delta u_t + H(x, \nabla u_t) = 0 \\ u_T = h\,;\end{cases} \qquad \text{where } H(x,p) \coloneqq -\inf_{u\in\mathbb{R}^d}\left\{\frac{|u|^2}{2} + b_s(x)\cdot p + \ u\cdot p\right\}\ ,$$

where the above infimum is actually attained at $\omega(x,p) \coloneqq -p$. The uniqueness of the classical solution stated in Item 1 can be deduced via the uniqueness for the HJB equation in the Euclidean space (Conforti, 2022, Proposition 3.1), by meaning of a periodic extension argument (*i.e.*, considering the same HJB where the coefficients are defined on $\mathbb{R}^d$ via periodicity).

Item 2 can be proven via a standard approximation procedure, that we sketch here for readers' convenience. Let us start by introducing for any $M \in \mathbb{N}$ the Hamiltonian $H^M : \mathbb{T}^d_L \times \mathbb{R}^d \to \mathbb{R}$

$$H^M(x,p) \coloneqq -\min_{|u|\le M}\left\{\frac{|u|^2}{2} + b_s(x)\cdot p + \chi_M(|u|)\ u\cdot p\right\}\ ,$$

where χ^M is a smooth function satisfying

$$\chi^M(r) = \begin{cases}1 & \text{if } r \le M \\ 0 & \text{if } r \ge M+1\end{cases} \qquad \text{such that} \quad \sup_{r\ge 0}\left|\chi^{'}_M\right| < +\infty\ .$$

Denote with $\omega^M(x,p)$ the optimal u in the definition of H^M, *i.e.*, the solution of

$$u + \chi_M(|u|)p + |u|\ \chi^{'}_M(|u|)\ p = 0\ ,$$

and notice that for any $|p| < M$, it holds $\omega^M(x,p) = -p$.

Then, for any fixed $M \in \mathbb{N}$, (Zhu, 2011, Theorem 3.1) guarantees the uniqueness of a viscosity solution for the associated HJB equation

$$\begin{cases}\partial_t u_t + \frac{1}{2}\Delta u_t + H^M(x, \nabla u_t(x)) = 0 \\ u_T = h\,,\end{cases} \tag{40}$$

which is equal to the value function u^M of the corresponding stochastic optimal control problem and that it is jointly continuous (in time and space) (cf. (Zhu, 2010, Theorem 3.7 and Remark 2.7)).

Next, we claim that

$$\sup_{x\in\mathbb{T}^d_L\ t\in[0,T]}\left|\nabla\mathcal{V}^{T,h}_t(x)\right| \le K\ , \tag{41}$$

for some positive constant K. By recalling that $\mathcal{V}^{T,h}_t \coloneqq -\log \mathrm{P}^b_{t,T}\mathrm{e}^{-h}$, with $(\mathrm{P}^b_{t,s})_{0\le t\le s\le T}$ being the semigroup associated to (34), we may write

$$\mathcal{V}^{T,h}_t(x) = -\log \mathbb{E}[\exp(-h(X^{t,x}_T))] \quad \text{where } \begin{cases}\mathrm{d}X^{t,x}_s = b_s(X^{t,x}_s)\mathrm{d}s + \mathrm{d}B_s\ , \\ X^{t,x}_t = x\ .\end{cases}$$

Fix $y \in \mathbb{T}^d_L$ and consider a synchronous coupling with the previous process, *i.e.*, write

$$\mathcal{V}^{T,h}_t(y) = -\log \mathbb{E}[\exp(-h(X^{t,y}_T))] \quad \text{where} \begin{cases} \mathrm{d}X^{t,y}_s = b_s(X^{t,y}_s)\mathrm{d}s + \mathrm{d}B_s \ , \\ X^{t,y}_t = y \ , \end{cases}$$

where we have used the same Brownian motion for the two SDEs above. This, the regularity of $h \in \mathcal{C}^3(\mathbb{T}^d_L)$ and the compactness of the torus allow us to write

$$\begin{aligned} \mathcal{V}^{T,h}_t(x) - \mathcal{V}^{T,h}_t(y) = \log\left(1 + \frac{\mathbb{E}[\mathrm{e}^{-h(X^{t,y}_T)} - \mathrm{e}^{-h(X^{t,x}_T)}]}{\mathbb{E}[\exp(-h(X^{t,x}_T))]}\right) &\le K\ \mathbb{E}[\mathrm{e}^{-h(X^{t,y}_T)} - \mathrm{e}^{-h(X^{t,x}_T)}] \\ &\le K\mathbb{E}[\mathsf{d}(X^{t,x}_T, X^{t,y}_T)] \le K\mathsf{d}(x,y)\,. \end{aligned}$$

In the above derivation the constant K differs at each step and we have relied on synchronous coupling technique in order to obtain a closed-form Gronwall-type estimate for $\mathsf{d}(X^{t,x}_T, X^{t,y}_T)$. We omit this explicit derivation since we are going to perform similar computations for the coupling by reflection later, and we are not interested in the magnitude of K. Let us just remark that K depends on the size of the torus, on the Lipschitz norm of the drift and of h and it can be taken uniformly with respect to $t \in [0,T]$. This concludes the proof of the bound claimed in (41).

Since, we already know that $\mathcal{V}^{T,h}_t$ is a classical solution of (39), the uniform estimate (41) implies that $\mathcal{V}^{T,h}_t$ is also a classical solution of the truncated (40) for any $M \ge K$. Since we have uniqueness of viscosity solutions for (40), we finally conclude that for any $M \ge K$, our function $\mathcal{V}^{T,h}_t$ is the unique viscosity solution, *i.e.*the value function of the stochastic control problem associated to H^M. By letting $M \to +\infty$, we conclude the proof of Item 2.

We now show Item 3. Let us start by considering on $(\Omega, (\mathcal{F}_s)_{s\in[0,T]}, \mathcal{F}, \mathbb{P})$ the toroidal SDEs

$$\begin{cases} \mathrm{d}X_s = b_s(X_s)\mathrm{d}s - \nabla\mathcal{V}^{T,h}_s(X_s)\mathrm{d}s + \mathrm{d}B_s \quad \forall s \in [t,T] \\ \mathrm{d}Y_s = b_s(Y_s)\mathrm{d}s - \nabla\mathcal{V}^{T,h}_s(X_s)\mathrm{d}s + \mathrm{d}\hat{B}_s \quad \forall s \in [t,\tau] \quad \text{and } Y_s = X_s \quad \forall s \in [\tau, T] \\ (X_t, Y_t) = (x,y) \in \mathbb{T}^d_L \times \mathbb{T}^d_L \end{cases} \tag{42}$$

where

$$\tau \coloneqq \inf\{s \in [t,T] : X_s = Y_s\} \wedge T\,,$$

and the *reflected* Brownian motion $\hat{B}$ is defined as

$$\mathrm{d}\hat{B}_s \coloneqq (\mathrm{I} - 2\, e_s\, e_s^\mathsf{T}\, \mathbf{1}_{\{s<\tau\}})\, \mathrm{d}B_s \qquad \text{where} \quad e_s \coloneqq \frac{\sin(\frac{\pi}{L}(X_s - Y_s))}{\|\sin(\frac{\pi}{L}(X_s - Y_s))\|}$$

where the $\sin$ function applied to any vector of $\mathbb{T}^d_L$ as to be understood as a component-wise map applied to a representative in $[-\pi/2, +\pi/2)$. In the sequel, we consider the same extension for the $\cos$ function. By Lévy's characterization, $\hat{B}$ is an $\mathcal{F}_s$-adapted Brownian motion on $\mathbb{T}^d_L$. The existence and well-posedness of the above coupling process is well known in literature ; see e.g. Chen and Li (1989), where the authors build such coupling by considering a martingale problem associated to a well-chosen elliptic operator.

Therefore from the sub-optimality of $-\nabla\mathcal{V}^{T,h}_s(X_s)_{s\in[t,T]}$ as a control process in the stochastic optimal control problem over the probability space $\Sigma = (\Omega, (\mathcal{F}_s)_{s\in[t,T]}, \mathbb{P}, (\hat{B}_s)_{s\in[t,T]})$, we deduce that

$$\mathcal{V}^{T,h}_t(y) - \mathcal{V}^{T,h}_t(x) \le \mathbb{E}\Big[h(Y_T) - h(X_T)\Big] \le \|h\|_{f_b}\mathbb{E}[f_b(\delta(X_T, Y_T))]\,. \tag{43}$$

Next we show that $\mathrm{e}^{\lambda s} f_b(\delta(X_s, Y_s))$ is a super-martingale. For notations' sake in what follows we introduce the shortcut $\pi_L := \pi/L$. From a first application of Ito formula for any $i \in [d]$ and for any $s \in [t, \tau)$ we have

$$\begin{aligned}
\mathrm{d}(\sin^2(\pi_L(X^i_s - Y^i_s)) &= \pi_L \sin(2\pi_L(X^i_s - Y^i_s))(b^i_s(X_s) - b^i_s(Y_s))\mathrm{d}s \\
&\quad + 4{\pi_L}^2(\mathrm{e}^i_s)^2 \cos(2\pi_L(X^i_s - Y^i_s))\mathrm{d}s + 2\pi_L\, \sin(2\pi_L(X^i_s - Y^i_s)) \sum_{k=1}^d (e_s\, e_s^\mathsf{T})_{ik}\, \mathrm{d}B^k_s \\
&= \pi_L \sin(2\pi_L(X^i_s - Y^i_s))(b^i_s(X_s) - b^i_s(Y_s))\mathrm{d}s \\
&\qquad + 4{\pi_L}^2 \frac{\sin^2(\pi_L(X^i_s - Y^i_s))}{\|\sin(\pi_L(X_s - Y_s))\|^2} \cos(2\pi_L(X^i_s - Y^i_s))\mathrm{d}s \\
&\qquad + 2\pi_L\, \sin(2\pi_L(X^i_s - Y^i_s)) \sum_{k=1}^d (e_s\, e_s^\mathsf{T})_{ik}\, \mathrm{d}B^k_s\,.
\end{aligned}$$

By summing up we get

$$\begin{aligned}
\mathrm{d}(\|\sin(\pi_L(X_s - Y_s))\|^2) &= \pi_L \sin(2\pi_L(X_s - Y_s))^\mathsf{T}(b_s(X_s) - b_s(Y_s))\mathrm{d}s \\
&\quad + \frac{4{\pi_L}^2}{\|\sin(\pi_L(X_s - Y_s))\|^2} \sum_{i=1}^d \sin^2(\pi_L(X^i_s - Y^i_s)) \cos(2\pi_L(X^i_s - Y^i_s))\mathrm{d}s \\
&\quad + 2\pi_L\, \sin(2\pi_L(X_s - Y_s))^\mathsf{T}\, e_s\, e_s^\mathsf{T}\, \mathrm{d}B_s\,,
\end{aligned}$$

from which we can deduce using $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$,

$$\begin{aligned}
\mathrm{d}\delta(X_s, Y_s) &= L\, \|\sin(\pi_L(X_s - Y_s))\|^{-1} \bigg\{ \frac{\pi_L}{2} \sin(2\pi_L(X_s - Y_s))^\mathsf{T}(b_s(X_s) - b_s(Y_s))\mathrm{d}s \\
&\quad + \frac{2{\pi_L}^2}{\|\sin(\pi_L(X_s - Y_s))\|^2} \sum_{i=1}^d \sin^2(\pi_L(X^i_s - Y^i_s)) \cos(2\pi_L(X^i_s - Y^i_s))\mathrm{d}s \\
&\quad + \pi_L \sin(2\pi_L(X_s - Y_s))^\mathsf{T}\, e_s\, e_s^\mathsf{T}\, \mathrm{d}B_s \bigg\} \\
&\quad - \frac{L\, {\pi_L}^2}{2} \|\sin(\pi_L(X_s - Y_s))\|^{-5} \langle \sin(2\pi_L(X_s - Y_s)), \sin(\pi_L(X_s - Y_s))\rangle^2 \mathrm{d}s \\
&= L\, \|\sin(\pi_L(X_s - Y_s))\|^{-1} \bigg\{ \frac{\pi_L}{2} \sin(2\pi_L(X_s - Y_s))^\mathsf{T}(b_s(X_s) - b_s(Y_s))\mathrm{d}s \\
&\quad + \frac{{\pi_L}^2}{2} \frac{\|\sin(2\pi_L(X_s - Y_s))\|^2}{\|\sin(\pi_L(X_s - Y_s))\|^2}\mathrm{d}s - \frac{2{\pi_L}^2}{\|\sin(\pi_L(X_s - Y_s))\|^2} \sum_{i=1}^d \sin^4(\pi_L(X^i_s - Y^i_s))\mathrm{d}s \\
&\quad + \pi_L \sin(2\pi_L(X_s - Y_s))^\mathsf{T}\, e_s\, e_s^\mathsf{T}\, \mathrm{d}B_s \\
&\quad - \frac{{\pi_L}^2}{2\, \|\sin(\pi_L(X_s - Y_s))\|^4} \langle \sin(2\pi_L(X_s - Y_s)), \sin(\pi_L(X_s - Y_s))\rangle^2 \mathrm{d}s \bigg\}
\end{aligned}$$

or equivalently, by setting $r_s := \delta(X_s, Y_s)$ for notations' sake

$$\mathrm{d}r_s = r_s^{-1}\bigg\{\frac{\pi\, L}{2}\sin(2\pi_L(X_s-Y_s))^\mathsf{T}(b_s(X_s)-b_s(Y_s))\mathrm{d}s + \pi\, L\sin(2\pi_L(X_s-Y_s))^\mathsf{T}\, e_s\, e_s^\mathsf{T}\,\mathrm{d}B_s$$
$$+\frac{(\pi\, L)^2}{2\, r_s^2}\|\sin(2\pi_L(X_s-Y_s))\|^2\mathrm{d}s - \frac{2(\pi\, L)^2}{r_s^2}\sum_{i=1}^d \sin^4(\pi_L(X_s^i-Y_s^i))\mathrm{d}s$$
$$-\frac{\pi^2 L^4}{2\, r_s^4}\langle\sin(2\pi_L(X_s-Y_s)),\sin(\pi_L(X_s-Y_s))\rangle^2\mathrm{d}s\bigg\}\,.$$

By applying Ito formula with f_b (defined at (35)) we deduce

$$\mathrm{d}f_b(r_s) = \frac{f_b^{'}(r_s)}{r_s}\bigg\{\frac{\pi\, L}{2}\sin(2\pi_L(X_s-Y_s))^\mathsf{T}(b_s(X_s)-b_s(Y_s))\mathrm{d}s$$
$$+\pi\, L\sin(2\pi_L(X_s-Y_s))^\mathsf{T}\, e_s\, e_s^\mathsf{T}\,\mathrm{d}B_s + \frac{(\pi\, L)^2}{2\, r_s^2}\|\sin(2\pi_L(X_s-Y_s))\|^2\mathrm{d}s$$
$$-\frac{2(\pi\, L)^2}{r_s^2}\sum_{i=1}^d \sin^4(\pi_L(X_s^i-Y_s^i))\mathrm{d}s - \frac{\pi^2\, L^4}{2\, r_s^4}\langle\sin(2\pi_L(X_s-Y_s)),\sin(\pi_L(X_s-Y_s))\rangle^2\mathrm{d}s\bigg\}$$
$$+\frac{\pi^2\, L^4}{2\, r_s^4}f_b^{''}(r_s)\langle\sin(2\pi_L(X_s-Y_s)),\sin(\pi_L(X_s-Y_s))\rangle^2\mathrm{d}s\,.$$

Now, notice that

$$\frac{\pi\, L}{2}\sin(2\pi_L(X_s-Y_s))^\mathsf{T}(b_s(X_s)-b_s(Y_s))$$
$$=\pi\, L\sum_{i=1}^d \cos(\pi_L(X_s^i-Y_s^i))\sin(\pi_L(X_s^i-Y_s^i))(b_s^i(X_s)-b_s^i(Y_s))$$
$$\le\pi\, L\sum_{i=1}^d \sin(\pi_L(X_s^i-Y_s^i))(b_s^i(X_s)-b_s^i(Y_s)) = \pi\, L\sin\left(\frac{\pi}{L}(X_s-Y_s)\right)^\mathsf{T}(b_s(X_s)-b_s(Y_s))$$
$$\le-\frac{\pi^2}{2}\kappa_b(r_s)\, r_s^2\,,$$

where we have from Definition 7,

$$\kappa_b(r)\le\inf_{s\in[0,T]}\inf\bigg\{-\frac{2L}{\pi}\frac{\sin(\frac{\pi}{L}(x-y))^\mathsf{T}(b_s(x)-b_s(y))}{\delta^2(x,y)}\ :\ x\neq y\in\mathbb{T}_L^d \text{ s.t. } \delta(x,y)=r\bigg\}\,.$$

Therefore we obtain

$$\mathrm{d}f_b(r_s)\le\frac{\pi^2\, L^4}{2\, r_s^4}f_b^{''}(r_s)\langle\sin(2\pi_L(X_s-Y_s)),\sin(\pi_L(X_s-Y_s))\rangle^2\mathrm{d}s$$
$$+\frac{f_b^{'}(r_s)}{r_s}\bigg\{-\frac{\pi^2}{2}\kappa_b(r_s)\, r_s^2\mathrm{d}s + \pi\, L\sin(2\pi_L(X_s-Y_s))^\mathsf{T}\, e_s\, e_s^\mathsf{T}\,\mathrm{d}B_s$$
$$+\frac{(\pi\, L)^2}{2\, r_s^2}\|\sin(2\pi_L(X_s-Y_s))\|^2\mathrm{d}s - \frac{\pi^2\, L^4}{2\, r_s^4}\langle\sin(2\pi_L(X_s-Y_s)),\sin(\pi_L(X_s-Y_s))\rangle^2\mathrm{d}s$$
$$-\frac{2(\pi\, L)^2}{r_s^2}\sum_{i=1}^d \sin^4(\pi_L(X_s^i-Y_s^i))\mathrm{d}s\bigg\}\,.$$

Firstly, notice that the quadratic variation term may be trivially bounded since

$$\begin{aligned}\frac{L^4}{4}&\langle \sin(2\pi_L(X_s-Y_s)), \sin(\pi_L(X_s-Y_s))\rangle^2\\ &=\left(L^2\sum_{i=1}^d \sin^2(\pi_L(X_s^i-Y_s^i))\cos(\pi_L(X_s^i-Y_s^i))\right)^2\le r_s^4\,,\end{aligned}$$

Next, we claim that the summation of final three terms in the above curly bracket is non-positive. Having in mind that, let us start by noticing that for any $\theta=(\theta^i)_{i\in[d]}\in[0,\pi/2)^d$ it holds[5]

$$\begin{aligned}&\|\sin(\theta)\|^2\,\|\sin(2\theta)\|^2-\langle \sin(2\theta),\sin(\theta)\rangle^2-4\|\sin(\theta)\|^2\sum_{i=1}^d \sin^4(\theta^i)\\
&=\sum_{i,j\in[d]}\sin^2(\theta^i)\sin^2(2\theta^j)-\left(\sum_{i=1}^d \sin(\theta^i)\sin(2\theta^i)\right)^2-4\sum_{i,j\in[d]}\sin^2(\theta^i)\sin^4(\theta^j)\\
&=\frac12\sum_{i,j\in[d]}\left(\sin(\theta^i)\sin(2\theta^j)-\sin(2\theta^i)\sin(\theta^j)\right)^2-4\sum_{i,j\in[d]}\sin^2(\theta^i)\sin^4(\theta^j)\\
&=\frac12\sum_{i,j\in[d]}\left(\sin(\theta^i)\sin(2\theta^j)-\sin(2\theta^i)\sin(\theta^j)\right)^2\\
&\qquad\qquad -2\sum_{i,j\in[d]}\sin^2(\theta^i)\sin^2(\theta^j)\left(\sin^2(\theta^i)+\sin^2(\theta^j)\right)\\
&=\sum_{i,j\in[d]}\frac12\left(\sin(\theta^i)\sin(2\theta^j)-\sin(2\theta^i)\sin(\theta^j)\right)^2-2\sin^2(\theta^i)\sin^2(\theta^j)\left(\sin^2(\theta^i)+\sin^2(\theta^j)\right)\end{aligned}$$

We are going to show that each term of the above summation is non-positive. Therefore let $x,y\in[0,\pi/2)$ and, owing to the duplication formula for the sine function, notice that

$$\begin{aligned}\frac12&\left(\sin(x)\sin(2y)-\sin(2x)\sin(y)\right)^2-2\sin^2(x)\sin^2(y)\left(\sin^2(x)+\sin^2(y)\right)\\
&=2\sin^2(x)\sin^2(y)\left[\left(\cos(y)-\cos(x)\right)^2-\sin^2(x)-\sin^2(y)\right]\\
&=4\sin^2(x)\sin^2(y)\left[\cos^2(x)+\cos^2(y)-\cos(x)\cos(y)-1\right]\,.\end{aligned}$$

Therefore our claim follows once we prove that the two-variable function

$$a,b\in[0,1]\mapsto f(a,b):=a^2+b^2-ab-1$$

is non-positive, or equivalently that for any fixed $b\in[0,1]$ the one-variable function

$$a\in[0,1]\mapsto g_b(a):=a^2+b^2-ab-1$$

5. Let us recall that in the definition of $\sin(\pi_L(x-y))$ we always chose the representative such that $\pi_L(x-y)\in[0\pi/2)$, therefore it is enough considering $\theta\in[0,\pi/2)^d$.

is non-positive. This latter claim is always met since (for any fixed $b \in [0,1]$) the above-defined function g_b achieves its maximum value in $a = b/2$ which is negative and reads as

$$g_b(b/2) = \frac{3}{4}\, b^2 - 1 < b^2 - 1 \leq 0 \,.$$

We have therefore proven that

$$\begin{aligned} \mathrm{d} f_b(r_s) \leq 2\pi^2 \Big\{ f_b^{''}(r_s) - f_b^{'}(r_s)\, \frac{\kappa_b(r_s)}{4}\, r_s \Big\} \mathrm{d}s \\ + \qquad \frac{f_b^{'}(r_s)}{r_s}\, \pi\, L\, \sin(2\pi_L(X_s - Y_s))^\intercal\, e_s\, e_s^\intercal\, \mathrm{d}B_s \,, \end{aligned}$$

which combined with the differential property (37) for the function f_b and with $r_s = 0$ on $[\tau, T]$ and $f_b(0) = 0$, reads as

$$\mathrm{d} f_b(r_s) \leq -\lambda_b\, \pi^2\, f_b(r_s)\, \mathrm{d}s + \pi\, L\, \frac{f_b^{'}(r_s)}{r_s}\, \sin\Big(\frac{2\pi}{L} (X_s - Y_s) \Big)^\intercal\, e_s\, e_s^\intercal\, \mathrm{d}B_s \qquad \forall s \in [t, T] \,.$$

By tacking expectation, from Gronwall Lemma we deduce

$$\mathbb{E}\big[f_b(\delta(X_T, Y_T) \big] \leq \mathrm{e}^{-\lambda_b\, \pi^2\, (T-t)} \delta(x,y) \,.$$

Therefore, by (43), for any $x \neq y \in \mathbb{T}^d_L$ we have shown

$$\mathcal{V}_t^{T,h}(y) - \mathcal{V}_t^{T,h}(x) \leq \mathrm{e}^{-\lambda_b\, \pi^2\, (T-t)} \|h\|_{f_b} \delta(x,y) \,,$$

which ends our proof for Item 3.

Finally, it is possible to relax Item 3 to the case $h \in \mathrm{Lip}(\mathbb{T}^d_L)$. This follows by a standard approximation technique, which is detailed in (Conforti, 2022, Lemma 3.1) for the Euclidean case and that applies straightforwardly to ours (by meaning of a periodic extension argument). ■

Appendix B. Proof of Lemma 6

Let us firstly notice that as a byproduct of Proposition 9 in Appendix A and (29), we know that

$$\begin{aligned} \|\mathcal{U}_s^{T,\psi^\star}\|_{\mathrm{Lip}} \leq \pi\, \|\mathcal{U}_s^{T,\psi^\star}\|_{f_V} \leq \pi\, \mathrm{e}^{-\lambda_V\, \pi^2\, (T-s)} \|\psi^\star\|_{f_V} \\ \leq \frac{\pi\, \mathrm{e}^{\lambda_V\, \pi^2\, s}}{2}\, \frac{\|U_\nu\|_{f_V} + \exp(-\lambda_V\, \pi^2\, T) \|U_\mu\|_{f_V}}{\sinh(\lambda_V\, \pi^2\, T)} \,. \end{aligned} \tag{44}$$

Now, if we denote the new drift as $b_s(x) \coloneqq -\nabla V(x) - \nabla \mathcal{U}_s^{T,\psi^\star}(x)$, for any $x \neq y \in \mathbb{T}^d_L$ such that $\delta(x,y) = r$, we have using (44)

$$\begin{aligned} & -\frac{2L}{\pi}\, \frac{(b_s(x) - b_s(y)) \cdot \sin(\frac{\pi}{L}(x-y))}{\delta^2(x,y)} \\ & \quad = \frac{2L}{\pi}\, \frac{(\nabla V(x) - \nabla V(y)) \cdot \sin(\frac{\pi}{L}(x-y))}{\delta^2(x,y)} + \frac{2L}{\pi}\, \frac{(\nabla \mathcal{U}_s^{T,\psi^\star}(x) - \nabla \mathcal{U}_s^{T,\psi^\star}(y)) \cdot \sin(\frac{\pi}{L}(x-y))}{\delta^2(x,y)} \\ & \quad \geq \kappa_V(r) - \frac{4L}{\pi\, r} \|\mathcal{U}_s^{T,\psi^\star}\|_{\mathrm{Lip}} \geq \bar{\kappa}_V(r) \,, \end{aligned}$$

where

$$\kappa_V(r) \coloneqq \inf\left\{ -\frac{2L}{\pi} \frac{(\nabla V(x) - \nabla V(y)) \cdot \sin(\frac{\pi}{L}(x-y))}{\delta^2(x,y)} \;:\; x \neq y \in \mathbb{T}^d_L \quad \text{s.t.} \quad \delta(x,y) = r \right\},$$

and

$$\bar{\kappa}_V(r) \coloneqq \kappa_V(r) - \frac{4}{r} \frac{\|U_\mu\|_{f_V} \vee \|U_\nu\|_{f_V}}{1 - \exp(-\lambda_V\, \pi^2\, T)}\,. \tag{45}$$

By applying Proposition 9 in Appendix A to the SOC problem (30), considering $\bar{\kappa}_V$ as a modulus of weak-semiconvexity for $b_s(x) \coloneqq -\nabla V(x) - \nabla \mathcal{U}^{T,\psi^\star}_s(x)$ (see Definition 7), we end up with

$$\|\mathcal{D}^n_t\|_{\bar{f}_V} \leq \mathrm{e}^{-\bar{\lambda}_V\, \pi^2\,(T-t)} \|\psi^n - \psi^\star\|_{\bar{f}_V}\,,$$

where

$$\bar{\lambda}_V = \left(\int_0^{L\, d^{1/2}} \int_0^r \exp\left(\frac{1}{2}(r^2 - s^2) - \frac{1}{4}\int_s^r t\bar{\kappa}_V(t)\mathrm{d}t \right) \mathrm{d}s\; \mathrm{d}r \right)^{-1} \tag{46}$$

and $\bar{f}_V$ is a concave continuous function satisfying for any function $\phi : \mathbb{T}^d_L \to \mathbb{R}$

$$\frac{1}{\pi}\, \|\phi\|_{\mathrm{Lip}} \leq \|\phi\|_{\bar{f}_V} \leq \frac{\bar{C}_V^{-1}}{\sqrt{L\,\pi}}\, \|\phi\|_{\mathrm{Lip}}\;, \text{ with } \bar{C}_V = \frac{e^{-L^2\, d/2}}{2}\, \exp\left(\frac{1}{4}\int_0^{L\, d^{1/2}} s\, \bar{\kappa}_V(s)\, \mathrm{d}s \right)\;. \tag{47}$$

In order to conclude, it is enough to recall that $\varphi^\star, \psi^\star$ satisfies the Schrödinger system

$$\begin{cases} \varphi = U_\mu - \mathcal{U}^{T,\psi^\star}_0 \\ \psi = U_\nu - \mathcal{U}^{T,\varphi^\star}_0\,, \end{cases}$$

which combined with Sinkhorn algorithm definition gives

$$\|\varphi^{n+1} - \varphi^\star\|_{\bar{f}_V} = \|\mathcal{U}^{T,\psi^\star}_0 - \mathcal{U}^{T,\psi^n}_0\|_{\bar{f}_V} = \|\mathcal{D}^n_0\|_{\bar{f}_V} \leq \mathrm{e}^{-\bar{\lambda}_V\, \pi^2\, T} \|\psi^n - \psi^\star\|_{\bar{f}_V}\,.$$

This proves the second bound in (31). The first one can be proven in the same way, using the very same $\bar{\lambda}_V, \bar{f}_V$ (thanks to the symmetrized definition for $\bar{\kappa}_V$ given at (45)). The estimates in (32) are just the two-step bounds given via the former ones.

Appendix C. Explicit convergence rates provided in Theorem 2

In this appendix, we carry out the computations of the convergence rates presented in Section 3. We recall from the proof of Theorem 2, see (33), that for any potential V, the convergence rate γ and the constant $\mathrm{c_S}$ may be computed as $\gamma = \exp(-\bar{\lambda}_V \pi^2 T)$ and $\mathrm{c_S} = \bar{C}_V^{-1}/\sqrt{L\,\pi}$, where $\bar{\lambda}_V$ and $\bar{C}_V$ are respectively given at (46) and (47), and are associated to the modulus introduced at (45) as

$$\bar{\kappa}_V(r) \coloneqq \kappa_V(r) - \frac{4}{r} \frac{\|U_\mu\|_{f_V} \vee \|U_\nu\|_{f_V}}{1 - \exp(-\lambda_V\, \pi^2\, T)}\,.$$

Let us recall that our final goal here is to give estimates of γ and $\mathrm{c_S}$ when considering a time-homogeneous drift induced by a potential V (i.e, $b_s(x) = -\nabla V(x)$) satisfying a one-sided Lipschitz bound (which can be interpreted as a version of α-semiconvexity on the torus cf. (17)) for any $x \neq y \in \mathbb{T}^d_L$,

$$\sin\Big(\frac{\pi}{L}(x-y)\Big)^{\mathsf{T}}(\nabla V(x)-\nabla V(y)) \geq \frac{\pi\,\alpha}{2L}\,\delta(x,y)^2\ ,$$

for some $\alpha \leq 0$. The above condition trivially implies a constant (and non-positive) modulus of semi-convexity $\kappa_V = \alpha$ to which we associate via equations (35) and (36) the triplet C_V, λ_V, f_V. Particularly, we have

$$\phi_V(r) = \mathrm{e}^{\frac{\alpha}{8}r^2}, \quad C_V = \frac{\mathrm{e}^{\frac{\alpha}{8}L^2\,d}}{2} \quad \text{and}$$
$$\lambda_V^{-1} = \int_0^{Ld^{1/2}} \mathrm{e}^{-\frac{\alpha}{8}r^2}\int_0^r \mathrm{e}^{\frac{\alpha}{8}s^2}\mathrm{d}s\,\mathrm{d}r \leq \int_0^{Ld^{1/2}} r\,\mathrm{e}^{-\frac{\alpha}{8}r^2}\,\mathrm{d}r\,,\ \textit{i.e.}\ \lambda_V \geq \frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2d}-1}\,.$$

Before moving on, let us notice that in the Brownian motion case (*i.e.*when considering $V = 0$) we have $\alpha = \kappa_V = 0$ and therefore

$$\phi_0(r) = 1\,, \quad C_0 = \frac{1}{2}\,, \quad \lambda_0 = \frac{2}{L^2\,d}\,, \text{ and } \quad f_0(r) = r - \frac{L^2\,d}{6}r^3\,,$$

which agrees with the above lowerbounds in the $|\alpha|$ vanishing limit.

Given the above premesis, we are now ready to compute the triplet $(\bar{C}_V, \bar{\lambda}_V, \bar{f}_V)$ associated to $\bar{\kappa}_V$ and, for notations' sake, let us introduce

$$M \coloneqq \frac{\|U_\mu\|_{f_V} \vee \|U_\nu\|_{f_V}}{1-\exp(-\lambda_V\,\pi^2\,T)} \geq 0\,.$$

Then, we immediately have $\bar{\kappa}_V(r) = \alpha - \frac{4}{r}M$. By equations (35) and (36), we deduce that

$$\bar{\phi}(r) = \mathrm{e}^{\frac{\alpha}{8}r^2 - Mr}, \quad \bar{C}_V = \frac{\mathrm{e}^{\frac{\alpha}{8}L^2d - MLd^{1/2}}}{2} \quad \text{and}$$
$$\bar{\lambda}_V^{-1} = \int_0^{Ld^{1/2}} \mathrm{e}^{-\frac{\alpha}{8}r^2+Mr}\int_0^r \mathrm{e}^{\frac{\alpha}{8}s^2-Ms}\mathrm{d}s\,\mathrm{d}r \leq e^{MLd^{1/2}}\int_0^{Ld^{1/2}} r\,\mathrm{e}^{-\frac{\alpha}{8}r^2}\,\mathrm{d}r$$
$$\text{and hence } \bar{\lambda}_V \geq \frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2d}-1}\,\mathrm{e}^{-MLd^{1/2}}\,.$$

In the above estimate we have decided to bound the exponential terms depending from M since in the T vanishing limit (*i.e.* the interesting regime that approximates the optimal transportation problem) the leading order term will be exponential in M, *i.e.*equal to $\mathrm{e}^{-MLd^{1/2}}$ (up to a polynomial in M prefactor). In the Brownian motion case, where there is no factor α, one may want to carry out the exact computations which lead to

$$\bar{\lambda}_0 = \frac{M^2}{e^{Ld^{1/2}M} - (1 + M\,L\,d^{1/2})}\,.$$

Then, we obtain

$$\log\gamma = -\,\pi^2 T\,\bar{\lambda}_V \leq -\pi^2 T\frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2d}-1}\,\mathrm{e}^{-MLd^{1/2}}\ ,$$
$$c_S = \frac{\bar{C}_V^{-1}}{\sqrt{L\,\pi}} = 2\frac{\mathrm{e}^{\frac{|\alpha|}{8}L^2d + MLd^{1/2}}}{\sqrt{L\,\pi}}\ ,$$

whereas in the Brownian case (where we denote by γ_0 the convergence rate) we have the same constant c_S (with $|\alpha| = 0$) and rate of convergence

$$\log\gamma_0 = -\pi^2 T\,\bar\lambda_0 = -\pi^2\,T\,\frac{M^2}{e^{Ld^{1/2}M} - 1 - M L d^{1/2}} \le -\pi^2 T\,M^2\exp(-Ld^{1/2}M)\,.$$

By recalling the definition of M, we end up with

$$\begin{aligned}\log\gamma &\le -\pi^2 T\frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2 d} - 1}\,\exp\left(-L\,d^{1/2}\,\frac{\|U_\mu\|_{f_V}\vee\|U_\nu\|_{f_V}}{1-\exp(-\lambda_V\,\pi^2\,T)}\right)\\ &\le -\pi^2 T\frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2 d} - 1}\,\exp\left(-L\,d^{1/2}\,\frac{\|U_\mu\|_{f_V}\vee\|U_\nu\|_{f_V}}{1-\exp\left(-\frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2 d}-1}\,\pi^2\,T\right)}\right)\end{aligned}$$

and similarly

$$c_S \le 2\frac{\mathrm{e}^{\frac{|\alpha|}{8}L^2 d}}{\sqrt{L\,\pi}}\,\exp\left(L\,d^{1/2}\frac{\|U_\mu\|_{f_V}\vee\|U_\nu\|_{f_V}}{1-\exp\left(-\frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2 d}-1}\,\pi^2\,T\right)}\right)\,,$$

which are the bounds given at (18) and (19). By following the same reasoning for the Brownian case and by recalling that $\lambda_0 = 2/(L^2\,d)$, we end up with

$$\begin{aligned}\log\gamma_0 &\le -\pi^2 T\,\frac{\|U_\mu\|^2_{f_0}\vee\|U_\nu\|^2_{f_0}}{(1-\exp(-2\,\pi^2\,T/L^2 d))^2}\,\exp\left(-L\,d^{1/2}\,\frac{\|U_\mu\|_{f_0}\vee\|U_\nu\|_{f_0}}{1-\exp(-2\,\pi^2\,T/L^2 d)}\right)\\ &\le -L^4\,d^2\,\frac{\|U_\mu\|^2_{f_0}\vee\|U_\nu\|^2_{f_0}}{4\,\pi^2\,T}\,\exp\left(-L\,d^{1/2}\,\frac{\|U_\mu\|_{f_0}\vee\|U_\nu\|_{f_0}}{1-\exp(-2\,\pi^2\,T/L^2 d)}\right)\end{aligned}$$

which in the small-time limit behaves as

$$\log\gamma_0 \sim -\pi^2\,D^2_{\mu,\nu}\,D^4\,T^{-1}\,\exp(-D_{\mu,\nu}\,D^3\,T^{-1})\,,$$

where $D_{\mu,\nu} \coloneqq \frac{1}{2\pi^2}\,\|U_\mu\|_{f_0}\vee\|U_\nu\|_{f_0}$ and $D = L\,d^{1/2}$, which is exactly what claimed at (20).

Using (26), the above bounds may also be written as

$$\log\gamma \le -\pi^2 T\frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2 d} - 1}\,\exp\left(-2\mathrm{e}^{\frac{|\alpha|}{8}L^2 d}\sqrt{\frac{L\,d}{\pi}}\frac{\|U_\mu\|_{\mathrm{Lip}}\vee\|U_\nu\|_{\mathrm{Lip}}}{1-\exp\left(-\frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2 d}-1}\,\pi^2\,T\right)}\right)$$

$$c_S \le 2\frac{\mathrm{e}^{\frac{|\alpha|}{8}L^2 d}}{\sqrt{L\,\pi}}\,\exp\left(2\mathrm{e}^{\frac{|\alpha|}{8}L^2 d}\sqrt{\frac{L\,d}{\pi}}\frac{\|U_\mu\|_{\mathrm{Lip}}\vee\|U_\nu\|_{\mathrm{Lip}}}{1-\exp\left(-\frac{|\alpha|/4}{\mathrm{e}^{\frac{|\alpha|}{8}L^2 d}-1}\,\pi^2\,T\right)}\right)\,.$$

C.1. An example with a trigonometric potential

Here we demonstrate that it is natural to assume α-semiconvexity conditions for potentials on the torus, by showing that (17) is satisfied by a well-defined class of potentials.

Let us started by considering $(\alpha_i, \beta_i)_{i\in[d]} \in \mathbb{R}^{2d}$ and for any $i = 1, ..., d$, define $\sigma_i = \sqrt{\alpha_i^2 + \beta_i^2}$. Without loss of generality, we reorder the dimensions such that $\sigma_1 \geq ... \geq \sigma_d$.

Consider the time-homogeneous drift induced by the potential V defined by

$$V(x) = \frac{L}{8}\sum_{i=1}^{d} \alpha_i \sin\left(\frac{2\pi}{L}x^i + \omega_i\right) + \beta_i \cos\left(\frac{2\pi}{L}x^i + \omega_i\right) \quad \forall x \in \mathbb{T}_L^d,$$

where $\omega_i \in \mathbb{R}$ is a phase-shifter for any $i = 1, ..., d$. Then, for any $(x, y) \in \mathbb{T}_L^d \times \mathbb{T}_L^d$, we have

$$\begin{aligned}&\sin\left(\frac{\pi}{L}(x-y)\right)^{\mathsf{T}}(\nabla V(x) - \nabla V(y)) = \\ &\quad -\frac{\pi}{2}\sum_{i=1}^{d}\sin^2\left(\frac{\pi}{L}(x^i - y^i)\right)\left\{\alpha_i \sin\left(\frac{\pi}{L}(x^i+y^i)+\omega_i\right) + \beta_i\cos\left(\frac{\pi}{L}(x^i+y^i)+\omega_i\right)\right\}.\end{aligned} \tag{48}$$

Let $r \in [0, L\, d^{1/2}]$. To be able to compute the weak-semiconvexity modulus $\kappa_V(r)$, we first expect to solve the following maximization problem

$$\begin{aligned}\max_{(x,y)\in\mathbb{T}_L^d\times\mathbb{T}_L^d} &\sum_{i=1}^{d}\sin^2\left(\frac{\pi}{L}(x^i - y^i)\right)\left\{\alpha_i \sin\left(\frac{\pi}{L}(x^i+y^i)+\omega_i\right) + \beta_i\cos\left(\frac{\pi}{L}(x^i+y^i)+\omega_i\right)\right\} \\ &\text{subject to } \delta(x,y) = r\,,\end{aligned}$$

which, by a change of variable, reads as

$$\begin{aligned}\max_{(u,v)\in\mathbb{T}_L^d\times\mathbb{T}_L^d} &\sum_{i=1}^{d}\sin^2\left(\frac{\pi}{L}u^i\right)\left\{\alpha_i \sin\left(\frac{\pi}{L}(u^i+2v^i)+\omega_i\right) + \beta_i\cos\left(\frac{\pi}{L}(u^i+2v^i)+\omega_i\right)\right\} \\ &\text{subject to } L^2\sum_{i=1}^{d}\sin^2\left(\frac{\pi}{L}u^i\right) = r^2\,.\end{aligned} \tag{49}$$

Since the above constraint does not depend on v, and for any fixed $u^i \in \mathbb{T}_L^1$

$$\max_{v_i\in\mathbb{T}_L^1} \alpha_i \sin\left(\frac{\pi}{L}(u^i+2v^i)+\omega_i\right) + \beta_i\cos\left(\frac{\pi}{L}(u^i+2v^i)+\omega_i\right) = \sigma_i\,,$$

Problem (49) simply reduces to

$$\max_{u\in\mathbb{T}_L^d}\sum_{i=1}^{d}\sigma_i\sin^2\left(\frac{\pi}{L}u^i\right) \quad \text{subject to } \sum_{i=1}^{d}\sin^2\left(\frac{\pi}{L}u^i\right) = (r/L)^2.$$

Because of our $\{\sigma_i\}_{i\in[d]}$ ordering choice (and that $\sigma_i \geq 0$ for any $i \in [d]$) we may finally conclude that the maximum value in (49) is equal to

$$\sum_{i=1}^{d}\sigma_i\min(1, ((r/L)^2 - i + 1)^+). \tag{50}$$

By combining (50) with (48) , we therefore have

$$\kappa_V(r) = -\frac{L}{r^2}\sum_{i=1}^{d}\sigma_i \min(1, ((r/L)^2 - i + 1)^+) \qquad \text{if } r \in (0, L\,d^{1/2}]$$
$$\kappa_V(0) = -\sigma_1/L \qquad \text{(by continuity)}$$

Let us stress out that $s \mapsto s\kappa_V(s)$ is integrable on $(0, L\,d^{1/2}]$ and that for any $r \in [0, L\,d^{1/2}]$, we have $\kappa(r) \leq 0$, as required by Definition 7.