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Sample Complexity Bound for Evaluating the Robust Observer's Performance under Coprime Factors Uncertainty

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Abstract

We address the end-to-end sample complexity bound for learning in closed loop the state estimator-based robust $\mathcal{H}_2$ controller for an unknown (possibly unstable) Linear Time Invariant (**LTI**) system, when given a fixed state-feedback gain. We build on the results from Ding et al. (1994) to bridge the gap between the parameterization of all state-estimators and the celebrated Youla parameterization. Refitting the expression of the relevant closed loop allows for the optimal linear observer problem given a fixed state feedback gain to be recast as a convex problem in the Youla parameter. The robust synthesis procedure is performed by considering bounded additive model uncertainty on the coprime factors of the plant, such that a min-max optimization problem is formulated for the robust $\mathcal{H}_2$ controller via an observer approach. The closed-loop identification scheme follows Zhang et al. (2021), where the nominal model of the true plant is identified by constructing a Hankel-like matrix from a single time-series of noisy, finite length input-output data by using the ordinary least squares algorithm from Sarkar et al. (2020). Finally, a $\mathcal{H}_\infty$ bound on the estimated model error is provided, as the robust synthesis procedure requires bounded additive uncertainty on the coprime factors of the model. Reference Zhang et al. (2022b) is the extended version of this paper.

Keywords: Linear Observers, Coprime Factorization, LTI Systems, Sample Complexity.

1. Introduction

In the past few years, significant research efforts have been spent into employing contemporary high dimensional statistics methods from the machine learning framework in order to approach classical Linear Quadratic control problems, see for instance Dean et al. (2018), Boczar et al. (2018), Mania et al. (2019), Dean et al. (2020), Zheng et al. (2020), Wang et al. (2015), Lee and Lamperski (2020), Tsiamis et al. (2020). In this paper we propose a method for the closed-loop *learning* in finite time (from a single time-series of noisy, finite length input-output data) of the optimal state-estimator for an unknown and potentially unstable Linear and Time Invariant (**LTI**) plant. Unlike the

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existing results mentioned above, which deal with parametric uncertainty on the matrices of some state-space realization of the plant, in this paper we consider bounded, additive model uncertainty on the coprime factors of the plant. Stemming from robust control, this is the preeminent method of modelling uncertainty for LTI dynamics. The main advantage incurred is that we explicitly avoid the need for making assumptions on the McMillan degree of the unknown plant or of its learned model, since in practice it can never be determined with accuracy solely from (noisy) input-output measurements.

An end-to-end sample-complexity bound of learning observer-based $\mathcal{H}_2$ controller for an unknown (potentially unstable) LTI plant that stabilizes the true system with high probability is established by incorporating recent advances in finite-time system identification. The resulting sub-optimal gap is bounded as a function of the level of model uncertainty. The end-to-end sample complexity bound for learning the robust observer-based $\mathcal{H}_2$ controller is $\mathcal{O}\left(\frac{\sqrt{\frac{\log T}{T}}}{1-\alpha\sqrt{\frac{\log T}{T}}}\right)$, where T is the time horizon for learning and α is a constant which is lower bounded by performance of the initial controller in the closed loop.

Paper Organization: Reference Zhang et al. (2022b) is the extended version of this paper. The general setup and problem formulation is given in Section II. The robust observer synthesis with uncertainty on the coprime factors is included in Section III. A brief discussion on the sub-optimality guarantees with end-to-end sample complexity results are stated in Section IV. Conclusion and future directions are given in Section V.

2. General Setup and Technical Preliminaries

The notation used in this paper is fairly common in control systems. Upper and lower case boldface letters (e.g. $\mathbf{G}$) are used to denote transfer function matrices, while lower and upper case letters (e.g. z and A) are used to denote vectors and matrices. The enclosed results are valid for discrete-time linear systems, therefore z denotes the complex variable associated with the **Z**-transform for discrete-time systems. A LTI system is *stable* if all the poles of its TFM are situated inside the unit circle for discrete time systems. The TFM of a LTI system is called *unimodular* if it is square, stable and has a stable inverse. For the sake of brevity the z argument after a transfer function may be omitted. $\mathbb{R}(z)$ denotes the set of all real–rational transfer functions and $\mathbb{R}(z)^{p\times m}$ denotes the set of $p \times m$ matrices having all entries in $\mathbb{R}(z)$. The notation $\mathbf{T}^{\ell\varepsilon}$ is used to indicate the mapping from signal ε to signal ℓ after combining all the ways in which ℓ is a function of ε. For example, $\mathbf{T}^{zw}$ is the mapping from the disturbances w to the regulated measurements z.

2.1. The State Estimation Problem

For a discrete-time **LTI** (Linear and Time Invariant) systems driven by Gaussian process and sensor noise, the state-space model is given by:

$$\begin{aligned} x_{k+1} &= Ax_k + B(u_k + w_k) + \delta_k, \\ y_k &= Cx_k + Du_k + \nu_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state of the system, $u_k \in \mathbb{R}^m$ is the control input and $y_k \in \mathbb{R}^p$ is the measured output and $w_k \in \mathbb{R}^m$, $\delta_k \in \mathbb{R}^n$ are the control additive and state additive disturbances, while

$\nu_k \in \mathbb{R}^p$ is the measurement noise, all considered to be Gaussian with zero mean and covariance matrices $\sigma_w^2 I$, $\sigma_\delta^2 I$ and $\sigma_\nu^2 I$ respectively.

A state estimator (observer) for (1) is defined as a system that provides an estimate $\widehat{x}_k$ of the internal state x_k, while having access solely to the control input u and measured output y, with the underlying requirement that the estimation error converges to zero in the steady-state, that is $\lim\limits_{k\to\infty} (x_k - \widehat{x}_k) = 0$. A state estimator is generically of the form:

$$\widehat{x}(z) = \mathbf{\Psi}^u(z)u(z) + \mathbf{\Psi}^y(z)y(z), \tag{2}$$

where $\mathbf{\Psi}^u(z)$ and $\mathbf{\Psi}^y(z)$ are two LTI filters (stable Transfer Function Matrices (**TFM**s)) *for the design of which one needs to know the model* (1) *of the plant*, see for example Ding et al. (1994). The celebrated Kalman Filter, represents the canonical formulation of performance specifications for a state estimator (2) as it minimizes the transfer from the exogenous signals in (1) (e.g. the measurement noise ν_k) to the estimation error $x_k - \widehat{x}_k$ (by using for example norm based costs).

2.2. Output Feedback Stabilizing Controllers

A standard unity feedback configuration is depicted in Figure 1 , where $\mathbf{G} \in \mathbb{R}(z)^{p\times m}$ is a multivariable LTI plant and $\mathbf{K} \in \mathbb{R}(z)^{m\times p}$ is an LTI controller. Here w, ν and r are the input disturbance, sensor noise and reference signal respectively while u, z and y are the controls, regulated signals and measurements vectors, respectively.

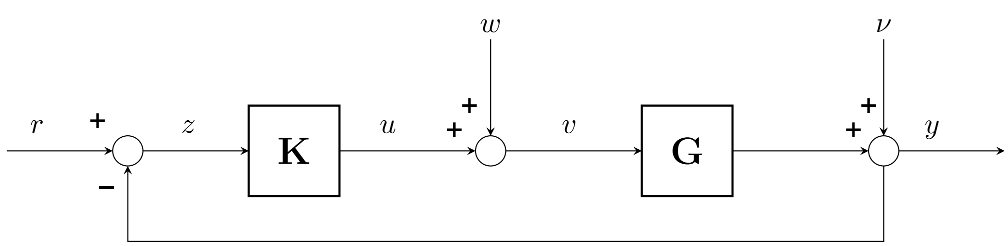

Figure 1: Standard unity feedback loop of the plant $\mathbf{G}$ with the controller $\mathbf{K}$

If all the closed–loop maps from the exogenous signals $[r^T \quad w^T \quad \nu^T\ ]^T$ to any point inside the feedback loop are stable, then $\mathbf{K}$ is said to be an (internally) stabilizing controller of $\mathbf{G}$ or equivalently that $\mathbf{K}$ stabilizes $\mathbf{G}$.

2.3. The Youla-Kuçera Parameterization of All Stabilizing Controllers

Definition 2.1 *(Vidyasagar (1985)) A collection of eight stable TFMs* $\big(\mathbf{M}, \mathbf{N}, \widetilde{\mathbf{M}}, \widetilde{\mathbf{N}}, \mathbf{X}, \mathbf{Y}, \widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}\big)$ *is called a* Doubly Coprime Factorization *(**DCF**) of the plant* $\mathbf{G}$ *if* $\widetilde{\mathbf{M}}$ *and* $\mathbf{M}$ *are invertible, yield the coprime factorizations* $\mathbf{G} = \widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{N}} = \mathbf{N}\mathbf{M}^{-1}$, *and satisfy the following equality (Bézout's identity):*

$$\begin{bmatrix} \widetilde{\mathbf{M}} & \widetilde{\mathbf{N}} \\ -\mathbf{X} & \mathbf{Y} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Y}} & -\mathbf{N} \\ \widetilde{\mathbf{X}} & \mathbf{M} \end{bmatrix} = I_{p+m}, \begin{bmatrix} \widetilde{\mathbf{Y}} & -\mathbf{N} \\ \widetilde{\mathbf{X}} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{M}} & \widetilde{\mathbf{N}} \\ -\mathbf{X} & \mathbf{Y} \end{bmatrix} = I_{p+m}. \tag{3}$$

Theorem 2.2 *(Ding et al. (1994), Vidyasagar (1985)) Given a stabilizable and detectable state-space realization (1) of the plant $\mathbf{G}$, then a DCF as in Definition 2.1 above is given by:*

$$\begin{aligned} \mathbf{M}(z) &= I + F(zI - A_F)^{-1}B, \;\; \mathbf{N}(z) = C_F(zI - A_F)^{-1}B \\ \widetilde{\mathbf{M}}(z) &= I - C(zI - A_L)^{-1}L, \;\; \widetilde{\mathbf{N}}(z) = C(zI - A_L)^{-1}B_L \\ \mathbf{X}(z) &= -F(zI - A_L)^{-1}L, \;\; \mathbf{Y}(z) = I - F(zI - A_L)^{-1}B_L \\ \widetilde{\mathbf{X}}(z) &= -F(zI - A_F)^{-1}L, \;\; \widetilde{\mathbf{Y}}(z) = I + C_F(zI - A_F)^{-1}L \end{aligned} \tag{4}$$

where $A_F \overset{def}{=} A + BF$, $A_L \overset{def}{=} A - LC$, $C_F \overset{def}{=} C + DF$ and $B_L \overset{def}{=} B - LD$, where F and L are stabilizing state-feedback and estimation gains that allocate all eigenvalues of A_F and A_L inside the unit disk.

Remark 1 *Theorem 2.2 above states that the DCF (4) of the plant is essentially equivalent with establishing certain stabilizing state- feedback F and estimation gain L, such that $u_k = F\widehat{x}_k$ in tandem with $\widehat{x}_{k+1} = A\widehat{x}_k + Bu_k + L(y_k - C\widehat{x}_k)$ is the output stabilizing controller $\mathbf{K} = \mathbf{Y}^{-1}\mathbf{X}$.*

Theorem 2.3 **(Youla-Kučera)** *(Vidyasagar, 1985, Ch.5) Let $\left(\mathbf{M}, \mathbf{N}, \widetilde{\mathbf{M}}, \widetilde{\mathbf{N}}, \mathbf{X}, \mathbf{Y}, \widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}\right)$ be a doubly coprime factorization of $\mathbf{G}$. Any controller $\mathbf{K_Q}$ stabilizing the plant $\mathbf{G}$, can be written as*

$$\mathbf{K_Q} = \mathbf{Y_Q}^{-1}\mathbf{X_Q} = \widetilde{\mathbf{X}}_\mathbf{Q}\widetilde{\mathbf{Y}}_\mathbf{Q}^{-1}, \tag{5}$$

where $\mathbf{X_Q}$, $\widetilde{\mathbf{X}}_\mathbf{Q}$, $\mathbf{Y_Q}$ and $\widetilde{\mathbf{Y}}_\mathbf{Q}$ are defined as: $\mathbf{X_Q} \overset{def}{=} \mathbf{X} + \mathbf{Q}\widetilde{\mathbf{M}}$, $\widetilde{\mathbf{X}}_\mathbf{Q} \overset{def}{=} \widetilde{\mathbf{X}} + \mathbf{M}\mathbf{Q}$, $\mathbf{Y_Q} \overset{def}{=} \mathbf{Y} - \mathbf{Q}\widetilde{\mathbf{N}}$, and $\widetilde{\mathbf{Y}}_\mathbf{Q} \overset{def}{=} \widetilde{\mathbf{Y}} - \mathbf{N}\mathbf{Q}$, for some stable $\mathbf{Q}$ in $\mathbb{R}(z)^{m\times p}$. It also holds that $\mathbf{K_Q}$ from (5) stabilizes $\mathbf{G}$, for any stable $\mathbf{Q}$.

2.4. Parameterization of All State Estimators

The following results provides the parameterization of *all* state observers of a given LTI system.

Theorem 2.4 *(Ding et al. (1994)) Given stabilizing state-feedback F and estimation gain L, or equivalently, given a DCF (4) of the LTI plant (1) (see also Remark 1), let us denote $\boldsymbol{P}(z) \overset{def}{=} (zI - A_F)^{-1}B$. Then:* **(A)** *the pair of filters $(\mathbf{\Psi}^u; \mathbf{\Psi}^y)$ generate a state estimator (2) for the system in (1) if and only if*

$$\mathbf{\Psi}^u(z)\mathbf{M}(z) + \mathbf{\Psi}^y(z)\mathbf{N}(z) = \mathbf{P}(z). \tag{6}$$

(B) *Furthermore,* any *state estimator for (1) can be written as*

$$\widehat{x}(z) = \mathbf{\Psi}^u_\mathbf{S}(z)u(z) + \mathbf{\Psi}^y_\mathbf{S}(z)y(z), \tag{7}$$

where

$$\mathbf{\Psi}^u_\mathbf{S}(z) \overset{def}{=} \mathbf{P}(z)\mathbf{Y}(z) + \mathbf{S}(z)\widetilde{\mathbf{N}}(z), \quad \mathbf{\Psi}^y_\mathbf{S}(z) \overset{def}{=} \mathbf{P}(z)\mathbf{X}(z) - \mathbf{S}(z)\widetilde{\mathbf{M}}(z) \tag{8}$$

for some stable $\boldsymbol{S}(z) \in \mathbb{R}(z)^{n\times p}$. Conversely, for any stable $\boldsymbol{S}(z)$ it holds that (7), with $(\mathbf{\Psi}^u_\mathbf{S}, \mathbf{\Psi}^y_\mathbf{S})$ as in (8), is a state estimator for (1).

Remark 2 *The intrinsic connections of Theorem 2.4 with output feedback stabilization are apparent. Furthermore the affine parameterization (8) of all state-estimators is akin to the Youla parameterization of Theorem 2.3. Finally, towards the scope of this paper, is important to note that Theorem 2.4 holds just the same if the plant (1) is in open loop or if the plant is in a feedback interconnection with a stabilizing controller $\mathbf{K}$. However, the maps from the exogenous to the estimation error differ greatly. In this paper we are interested in "learning" the optimal state estimator of an unknown plant* in closed feedback loop. *To this end the following two results (for the closed-loop scenario) will be instrumental towards the main result and surprisingly enough, they cannot be found in the original work from Ding et al. (1994).*

Theorem 2.5 *Consider the the LTI plant (1) in feedback interconnection with the controller given by: $u_k = F\widehat{x}_k$ in tandem with any state-estimator of the form $\widehat{x} = \mathbf{\Psi}^u_{\mathbf{S}}(z)u(z) + \mathbf{\Psi}^y_{\mathbf{S}}(z)y(z)$. The closed loop maps from the disturbances w and measurement noise ν to the estimation error $(x - \widehat{x})$ are affine functions of the $\mathbf{S}$ parameter from Theorem 2.4* **(B)***, moreover:*

$$\mathbf{T}_{\mathbf{S}}^{(x-\widehat{x})w} = \mathbf{\Psi}^u_{\mathbf{S}}(z) \ \text{ and } \ \mathbf{T}_{\mathbf{S}}^{(x-\widehat{x})\nu} = -\mathbf{\Psi}^y_{\mathbf{S}}(z), \ \text{ respectively.} \tag{9}$$

Theorem 2.6 *Given a DCF (4) of the LTI plant (1) and its subsequent stabilizing state-feedback gain $u_k = F\widehat{x}_k$, let us assume that the $F \in \mathbb{R}^{m\times n}$ matrix is* onto *(*i.e. *it has full row rank). Then any stabilizing output feedback controller $\mathbf{K}_{\mathbf{Q}}$ from (5) can be realized as: $u_k = F\widehat{x}_k$ in tandem with the state-estimator $\widehat{x} = \mathbf{\Psi}^u_{\mathbf{S}}(z)u(z) + \mathbf{\Psi}^y_{\mathbf{S}}(z)y(z)$ from (7), where*

$$F\mathbf{S}(z) = \mathbf{Q}(z) + \widetilde{\mathbf{X}}(z) \tag{10}$$

Remark 3 *The two theorems above clarify the fact that the two filters that realize any state estimator (7)* in closed-loop *are actually the closed loop maps from the exogenous signals to the estimation error. Furthermore, the assumption that the state-feedback gain matrix F has a right inverse allows us to rephrase parameterization (8) of all state observers which is affine in $\mathbf{S}$, to a parameterization $(\mathbf{\Psi}^u_{\mathbf{Q}}; \mathbf{\Psi}^y_{\mathbf{Q}})$ affine in the Youla parameter, thus bridging the gap between any stabilizing controller $\mathbf{K}_{\mathbf{Q}}$ from (5) and its realization via: a* fixed *state-feedback gain F in tandem with the dynamic state estimator $(\mathbf{\Psi}^u_{\mathbf{Q}}; \mathbf{\Psi}^y_{\mathbf{Q}})$. The fixed state-feedback gain F belongs to the initial stabilizing controller in the closed loop, since $\widetilde{\mathbf{X}}(z) = -F(zI - A_F)^{-1}L$ is neither a function of $\mathbf{Q}$, nor a function of $\mathbf{S}$.*

2.5. A First Glimpse into the Separation Principle

We summarize below the fact that any stabilizing controller can be realized either as a fixed state-feedback gain in tandem with a dynamic state-estimator or as a fixed estimation gain in tandem with dynamic state feedback. Both parameterizations are affine in the Youla parameter.

Ding et al. (1994) and Subsection 2.4	Alazard and Apkarian (1999)
Any stabilizing $\mathbf{K}_{\mathbf{Q}}$ from (5) can be realized via the static state-feedback gain F in tandem with the dynamic state estimator $(\mathbf{\Psi}^u_{\mathbf{Q}}; \mathbf{\Psi}^y_{\mathbf{Q}})$.	*Any stabilizing $\mathbf{K}_{\mathbf{Q}}$ from (5) can be realized via the static estimation gain L in tandem with the dynamic feedback $\mathbf{Q}$.*
$u_k = F\widehat{x}_k$ $\widehat{x} = \mathbf{\Psi}^u_{\mathbf{Q}}u + \mathbf{\Psi}^y_{\mathbf{Q}}y$	$\widehat{x}_{k+1} = A\widehat{x}_k + Bu_k + L(y_k - C\widehat{x}_k)$ $u = F\widehat{x} + \mathbf{Q}(y - C\widehat{x})$

2.6. The Optimal State Estimator

THE PROBLEM: In this paper we consider the *unknown* plant (1) in feedback interconnection with some *known* stabilizing controller $\mathbf{K}$, controller that is realized as: a fixed state-feedback gain F considered to be immutable (namely $u_k = F\widehat{x}_k$), in tandem with some state-estimator $\widehat{x} = \mathbf{\Psi}^u u + \mathbf{\Psi}^y y$. Initially, we must learn the unknown system with high probability, in finite time, from a single trajectory in the closed loop. Subsequently, we must design the optimal state-observer that in tandem with the state-feedback gain $u_k = F\widehat{x}_k$ yields the optimal LQ performance.

The canonical formulation of performance specifications for a state estimator is to minimize the transfer from the exogenous signals to the estimation error. However, as repeatedly stated above, the declared scope of this work is to design a state-estimator specifically tailored to work in tandem with the fixed state-feedback gain $u_k = F\widehat{x}_k$. In this context, the choice of the optimality criterion is essential, as clarified below.

Proposition 2.7 *We define the* Optimal Observer Evaluation Problem *as follows: given a fixed state-feedback gain F with $u = F\widehat{x}$, minimize the norm of the TFM below after all Youla parameters* $\mathbf{Q}$*:*

$$\min_{\mathbf{Q}\ \text{stable}} \left\| \begin{bmatrix} F\mathbf{\Psi}^u_{\mathbf{Q}} & -F\mathbf{\Psi}^y_{\mathbf{Q}} \end{bmatrix} \right\|_{\mathcal{H}_2} \tag{11}$$

which in turn is equivalent with:

$$\min_{\mathbf{Q}\ \text{stable}} \left\| \Big[I_m - \mathbf{Y_Q}(z) + \big(I_m - \mathbf{M}(z)\big)\mathbf{Q}(z)\widetilde{\mathbf{N}}(z) \quad \mathbf{X_Q}(z) + \big(I_m - \mathbf{M}(z)\big)\mathbf{Q}(z)\widetilde{\mathbf{M}}(z) \Big] \right\|_{\mathcal{H}_2} \tag{12}$$

Remark 4 *(Optimality) The reason behind choosing* (11) *for the observer design is the fact that the model of the plant can never be learned with absolute accuracy. Consequently, in the presence of frequency-domain uncertainties, the problem of designing a state observer is not well-posed, since not even the dimension of the state vector of the true plant can be known. Proposition* 2.7 *defines optimality in the following sense: the objective function from* (11) *pertains to the difference in $\mathcal{H}_2$ performance in the closed loop between the state-feedback control $u = Fx$ (with direct access to the state) and any output feedback controller $\mathbf{K_Q}$. The thorough reasoning for this and all other underlying implications are deferred to* Zhang et al. *(2022b).*

3. Robust Controller Synthesis: An Observer Based Approach

The outcome of the "learning" of the true plant $\mathbf{G}$ from closed-loop measurements comes in the form of a left coprime factorization of what we have dubbed *the nominal model*[1], namely $\mathbf{G}^{\mathtt{md}} = (\widetilde{\mathbf{M}}^{\mathtt{md}})^{-1}\widetilde{\mathbf{N}}^{\mathtt{md}} = \mathbf{N}^{\mathtt{md}}(\mathbf{M}^{\mathtt{md}})^{-1}$. For the detailed description of the learning algorithm we refer to the Appendix G from Zhang et al. (2021). In order to evaluate the discrepancy between the learned $\mathbf{G}^{\mathtt{md}}$ and the true plant, we make a recourse to the preeminent method for modelling uncertainty for LTI systems (stemming from classical robust control), specifically via additive perturbations on the coprime factors.

1. An alternative name might as well have been “the learned model”.

With the DCF of the nominal model of the plant $\mathbf{G}^{\mathtt{md}} = (\widetilde{\mathbf{M}}^{\mathtt{md}})^{-1}\widetilde{\mathbf{N}}^{\mathtt{md}} = \mathbf{N}^{\mathtt{md}}(\mathbf{M}^{\mathtt{md}})^{-1}$, we can write the Bézout's identity that incorporates the coprime factorization of the initial, known stabilizing controller[2] $\mathbf{K}^{\mathtt{md}} = (\mathbf{Y}^{\mathtt{md}})^{-1}\mathbf{X}^{\mathtt{md}} = \widetilde{\mathbf{X}}^{\mathtt{md}}(\widetilde{\mathbf{Y}}^{\mathtt{md}})^{-1}$, specifically:

$$\begin{bmatrix} \widetilde{\mathbf{M}}^{\mathtt{md}} & \widetilde{\mathbf{N}}^{\mathtt{md}} \\ -\mathbf{X}^{\mathtt{md}} & \mathbf{Y}^{\mathtt{md}} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Y}}^{\mathtt{md}} & -\mathbf{N}^{\mathtt{md}} \\ \widetilde{\mathbf{X}}^{\mathtt{md}} & \mathbf{M}^{\mathtt{md}} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix}. \tag{13}$$

Definition 3.1 (Model Uncertainty Set) *The γ-radius* model uncertainty *set for the nominal plant $\mathbf{G}^{\mathtt{md}}$ with $\Delta_{\widetilde{\mathbf{M}}}$, $\Delta_{\widetilde{\mathbf{N}}}$ both stable is defined as:*

$$\mathcal{G}_\gamma \stackrel{\text{def}}{=} \{\mathbf{G} = \widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{N}} \,|\, \widetilde{\mathbf{M}} = (\widetilde{\mathbf{M}}^{\mathtt{md}} + \Delta_{\widetilde{\mathbf{M}}}), \widetilde{\mathbf{N}} = (\widetilde{\mathbf{N}}^{\mathtt{md}} + \Delta_{\widetilde{\mathbf{N}}}); \quad \left\| \begin{bmatrix} \Delta_{\widetilde{\mathbf{M}}} & \Delta_{\widetilde{\mathbf{N}}} \end{bmatrix} \right\|_\infty < \gamma\} \tag{14}$$

Definition 3.2 (γ-Robustly Stabilizing) *A fixed stabilizing controller $\mathbf{K}$ of the nominal plant is said to be γ-robustly stabilizing iff $\mathbf{K}$ stabilizes not only $\mathbf{G}^{\mathtt{md}}$ but also all plants $\mathbf{G} \in \mathcal{G}_\gamma$.*

Assumption 1 *It is assumed that the true plant, denoted by $\mathbf{G}^{\mathtt{pt}}$, belongs to the model uncertainty set introduced in Definition 3.1, i.e. that there exist stable $\Delta_{\widetilde{\mathbf{M}}}$, $\Delta_{\widetilde{\mathbf{N}}}$ with $\left\| \begin{bmatrix} \Delta_{\widetilde{\mathbf{M}}} & \Delta_{\widetilde{\mathbf{N}}} \end{bmatrix} \right\|_\infty < \gamma$ for which $\mathbf{G}^{\mathtt{pt}} = (\widetilde{\mathbf{M}}^{\mathtt{md}} + \Delta_{\widetilde{\mathbf{M}}})^{-1}(\widetilde{\mathbf{N}}^{\mathtt{md}} + \Delta_{\widetilde{\mathbf{N}}})$.*

In the presence of additive uncertainty on the coprime factors the Bézout's identity in (13) no longer holds, however, the following holds for *certain stable* $\Delta_{\mathbf{M}}$, $\Delta_{\mathbf{N}}$ *factors*:

$$\begin{bmatrix} (\widetilde{\mathbf{M}}^{\mathtt{md}} + \Delta_{\widetilde{\mathbf{M}}}) & (\widetilde{\mathbf{N}}^{\mathtt{md}} + \Delta_{\widetilde{\mathbf{N}}}) \\ -\mathbf{X}^{\mathtt{md}}_{\mathbf{Q}} & \mathbf{Y}^{\mathtt{md}}_{\mathbf{Q}} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Y}}^{\mathtt{md}}_{\mathbf{Q}} & -(\mathbf{N}^{\mathtt{md}} + \Delta_{\mathbf{N}}) \\ \widetilde{\mathbf{X}}^{\mathtt{md}}_{\mathbf{Q}} & (\mathbf{M}^{\mathtt{md}} + \Delta_{\mathbf{M}}) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{11}(\mathbf{Q}) & O \\ O & \mathbf{\Phi}_{22}(\mathbf{Q}) \end{bmatrix}. \tag{15}$$

The block diagonal structure of the right hand side term in (15) is due to the fact that $\mathbf{G}^{\mathtt{pt}} = (\widetilde{\mathbf{M}}^{\mathtt{md}} + \Delta_{\widetilde{\mathbf{M}}})^{-1}(\widetilde{\mathbf{N}}^{\mathtt{md}} + \Delta_{\widetilde{\mathbf{N}}}) = (\mathbf{N}^{\mathtt{md}} + \Delta_{\mathbf{N}})(\mathbf{M}^{\mathtt{md}} + \Delta_{\mathbf{M}})^{-1}$ for the stable $\Delta_{\mathbf{M}}$, $\Delta_{\mathbf{N}}$ factors from Assumption 1.

Lemma 3.3 *A stabilizing controller of the nominal plant $\mathbf{K}^{\mathtt{md}}_{\mathbf{Q}} = (\mathbf{Y}^{\mathtt{md}}_{\mathbf{Q}})^{-1}\mathbf{X}^{\mathtt{md}}_{\mathbf{Q}} = \widetilde{\mathbf{X}}^{\mathtt{md}}_{\mathbf{Q}}(\widetilde{\mathbf{Y}}^{\mathtt{md}}_{\mathbf{Q}})^{-1}$ is γ-robustly stabilizing iff for any stable model perturbations $\Delta_{\widetilde{\mathbf{M}}}, \Delta_{\widetilde{\mathbf{N}}}$ with $\left\| [\Delta_{\widetilde{\mathbf{M}}} \quad \Delta_{\widetilde{\mathbf{N}}}] \right\|_\infty < \gamma$ the TFM*

$$\mathbf{\Phi}_{11}(\mathbf{Q}) = I_p + \begin{bmatrix} \Delta_{\widetilde{\mathbf{M}}} & \Delta_{\widetilde{\mathbf{N}}} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Y}}^{\mathtt{md}}_{\mathbf{Q}} \\ \widetilde{\mathbf{X}}^{\mathtt{md}}_{\mathbf{Q}} \end{bmatrix}, \tag{16}$$

from (15) *is unimodular* i.e. *it is square, stable and has a stable inverse.*

Theorem 3.4 *The Youla parameterization yields a γ-robustly stabilizing controller $\mathbf{K}_{\mathbf{Q}}$ iff its corresponding right coprime factors satisfy $\left\| \begin{bmatrix} \widetilde{\mathbf{Y}}_{\mathbf{Q}} \\ \widetilde{\mathbf{X}}_{\mathbf{Q}} \end{bmatrix} \right\|_\infty \leq \dfrac{1}{\gamma}$, where $\mathbf{Q}$ denotes as the Youla parameter.*

As an intermediary result, by employing Theorem 3.4 and the standard inequality from Appendix A of Zhang et al. (2022b) it is concluded that:

2. The controller with which the closed-loop learning is being performed is assumed to be known.

$$\left\| \begin{bmatrix} -\Delta_{\widetilde{\mathbf{M}}} & -\Delta_{\widetilde{\mathbf{N}}} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Y}}_{\mathbf{Q}} \\ \widetilde{\mathbf{X}}_{\mathbf{Q}} \end{bmatrix} \right\|_\infty \leq \left\| \begin{bmatrix} \Delta_{\widetilde{\mathbf{M}}} & \Delta_{\widetilde{\mathbf{N}}} \end{bmatrix} \right\|_\infty \left\| \begin{bmatrix} \widetilde{\mathbf{Y}}_{\mathbf{Q}} \\ \widetilde{\mathbf{X}}_{\mathbf{Q}} \end{bmatrix} \right\|_\infty < \gamma \times \frac{1}{\gamma} = 1.$$

Starting from the left coprime factorization of the true plant, known to be of the form $\mathbf{G}^{\mathrm{pt}} = (\widetilde{\mathbf{M}}^{\mathrm{md}} + \Delta_{\widetilde{\mathbf{M}}})^{-1}(\widetilde{\mathbf{N}}^{\mathrm{md}} + \Delta_{\widetilde{\mathbf{N}}})$, one can always obtain a DCF of the true plant by redefining $\widetilde{\mathbf{M}}^{\mathrm{pt}} \overset{def}{=} \mathbf{\Phi}_{11}^{-1}\big(\widetilde{\mathbf{M}}^{\mathrm{md}} + \Delta_{\widetilde{\mathbf{M}}}\big), \widetilde{\mathbf{N}}^{\mathrm{pt}} \overset{def}{=} \mathbf{\Phi}_{11}^{-1}\big(\widetilde{\mathbf{N}}^{\mathrm{md}} + \Delta_{\widetilde{\mathbf{N}}}\big), \mathbf{M}^{\mathrm{pt}} \overset{def}{=} \big(\mathbf{M}^{\mathrm{md}} + \Delta_{\mathbf{M}}\big)\mathbf{\Phi}_{22}^{-1}$, and $\mathbf{N}^{\mathrm{pt}} \overset{def}{=} \big(\mathbf{N}^{\mathrm{md}} + \Delta_{\mathbf{N}}\big)\mathbf{\Phi}_{22}^{-1}$, such that the Bézout identity holds with the $\mathbf{X}_{\mathbf{Q}}^{\mathrm{md}}$, $\mathbf{Y}_{\mathbf{Q}}^{\mathrm{md}}$, $\widetilde{\mathbf{X}}_{\mathbf{Q}}^{\mathrm{md}}$ and $\widetilde{\mathbf{Y}}_{\mathbf{Q}}^{\mathrm{md}}$ factors available from the known controller. Here, $\mathbf{\Phi}_{11}$, $\mathbf{\Phi}_{22}$ are as in (15). By re-establishing the Bézout identity we are able to formulate the robust version of (12) as:

Theorem 3.5 *The* Robust Linear Observer Evaluation Problem *given a fixed state feedback gain F, with $u = F\widehat{x}$ is defined as :*

$$\begin{aligned} \min_{\mathbf{Q}\ \textit{stable}} \max_{\left\| \begin{bmatrix} \Delta_{\widetilde{\mathbf{M}}} & \Delta_{\widetilde{\mathbf{N}}} \end{bmatrix} \right\|_\infty < \gamma} & \left\| \begin{bmatrix} I_m - \mathbf{Y}_{\mathbf{Q}}^{\mathrm{md}} + (I_m - \mathbf{M}^{\mathrm{md}})\mathbf{Q}\mathbf{\Phi}_{11}^{-1}\big(\widetilde{\mathbf{N}}^{\mathrm{md}} + \Delta_{\widetilde{\mathbf{N}}}\big) \\ \mathbf{X}_{\mathbf{Q}}^{\mathrm{md}} + (I_m - \mathbf{M}^{\mathrm{md}})\mathbf{Q}\mathbf{\Phi}_{11}^{-1}\big(\widetilde{\mathbf{M}}^{\mathrm{md}} + \Delta_{\widetilde{\mathbf{M}}}\big) \end{bmatrix}^T \right\|_{\mathcal{H}_2} \\ \textit{s.t.} \quad & \left\| \begin{bmatrix} \widetilde{\mathbf{Y}}_{\mathbf{Q}} \\ \widetilde{\mathbf{X}}_{\mathbf{Q}} \end{bmatrix} \right\|_\infty \leq \frac{1}{\gamma}. \end{aligned} \tag{17}$$

Remark 5 *Aside from being able to cope with learning unstable plants (in closed loop), this method of modelling uncertainty explicitly avoids the need of knowing* apriori *the McMillan degree (*i.e. *the state dimension of a minimal state-space realization) of the unknown plant, which can never be determined in practice. Since the learned nominal model $\mathbf{G}^{\mathrm{md}}$ and the true plant will not even have the same McMillan degree, it is impossible to retrieve anything about the state representation* (1) *of the true plant solely from the knowledge of $\mathbf{G}^{\mathrm{md}}$. However, by considering a fixed state feedback gain (belonging to the initial controller in the closed loop), it is possible to evaluate the performance of different observers with respect to the Youla parameter $\mathbf{Q}$. The optimal $\mathbf{Q}^*$ from problem* (17) *will yield a robust controller having the same state feedback gain as the initial one, in tandem with an observer that achieves the best $\mathcal{H}_2$ performance when compared to the controller which has full access to the state, namely $u = Fx$.*

It can be seen that (17) is actually phrased in terms of the coprime factors of the true plant, which can never be "learned" in practice. The non-convexity of the standard min-max formulation from Theorem 3.5 (for the robust observer evaluation) is caused by the fact that $\mathbf{\Phi}_{11}^{-1}$ is no longer an affine function in $\mathbf{Q}$, therefore the nonzero duality gap makes it impossible to solve (17) by merely flipping min and max. In order to circumvent this, an upper bound on the cost functional will be derived and we will formulate the robust observer evaluation problem in a quasi-convex form.

Proposition 3.6 *(Quasi-Convex Formulation) For the true plant, $\mathbf{G}^{\mathrm{pt}} \in \mathcal{G}_\gamma$ the robust observer evaluation problem in* (17) *admits the following upper bound:*

$$\begin{aligned} \min_{\alpha \in [0, 1/\gamma)} \frac{1}{1-\gamma\alpha} \min_{\mathbf{Q}\ \textit{stable}} & (1-\gamma\alpha) \left\| \begin{bmatrix} \big(I_m - \mathbf{Y}_{\mathbf{Q}}^{\mathrm{md}}\big) \\ \mathbf{X}_{\mathbf{Q}}^{\mathrm{md}} \end{bmatrix}^T \right\|_{\mathcal{H}_2} + \left\| \begin{bmatrix} I_m - \mathbf{M}^{\mathrm{md}} \end{bmatrix} \right\|_\infty \|\mathbf{Q}\|_{\mathcal{H}_2} \Big(\left\| \begin{bmatrix} \widetilde{\mathbf{N}}^{\mathrm{md}} \\ \widetilde{\mathbf{M}}^{\mathrm{md}} \end{bmatrix}^T \right\|_\infty + \gamma \Big) \\ \textit{s.t.} \quad & \left\| \begin{bmatrix} \widetilde{\mathbf{Y}}_{\mathbf{Q}}^{\mathrm{md}} \\ \widetilde{\mathbf{X}}_{\mathbf{Q}}^{\mathrm{md}} \end{bmatrix} \right\|_\infty \leq \alpha. \end{aligned} \tag{18}$$

The inner objective function in (18) is affine in $\mathbf{Q}$, hence the inner optimization problem in Proposition 3.6 is convex for each fixed α.

Remark 6 *(Validation of Constraints) The quasi-convex formulation from* (18) *is not equivalent with* (17)*. It trades optimality for feasibility in the following sense: for a chosen positive constant* $\alpha < \dfrac{1}{\gamma}$*, the feasible set shrinks to* $\{\mathbf{Q} \in \mathbb{R}(z)^{m\times p}$ *stable* $\mid \left\| \begin{bmatrix} \widetilde{\mathbf{Y}}^{\mathtt{md}}_{\mathbf{Q}} \\ \widetilde{\mathbf{X}}^{\mathtt{md}}_{\mathbf{Q}} \end{bmatrix} \right\|_\infty \leq \alpha\}$ *in order to convexify the inner objective function. Note that as the initial controller is a stablizing one, necessarily* $\mathbf{Q} = \mathbf{0}^{m\times p}$ *should be feasible. This implies that* α *cannot be smaller than* $\left\| \begin{bmatrix} \widetilde{\mathbf{Y}}^{\mathtt{md}} \\ \widetilde{\mathbf{X}}^{\mathtt{md}} \end{bmatrix} \right\|_\infty$.

Remark 7 *The quasi-convex problem in Proposition 3.6 is formulated in frequency domain. To solve it in practice, we need to perform a Finite-Impulse Response (FIR) truncation on Markov parameters of these systems. After the FIR truncation, for each fixed* $\alpha \in [0, 1/\gamma)$*, An equivalent Semi Definite Program (SDP) can be formulated for the inner optimization problem, which would give us the vectorization of Markov parameters of the optimal* $\mathbf{Q}^*$ *to Proposition 3.6. Details on the SDP formulation are given in Appendix B of Zhang et al. (2022b).*

4. Analysis of End-to-End Performance

The performance of the Robust Linear Observer from (17) working in tandem with the fixed state feedback gain F, such that $\widehat{u} = F\widehat{x}$, will be discussed in this section. Denote the $\mathcal{H}_2$-cost of applying the control inputs $\widehat{u} = F\widehat{x}$ and $u = Fx$ by $J_{\widehat{u}}$ and J_u, respectively. Then it is shown in Appendix C of Zhang et al. (2022b) that:

$$J_{\widehat{u}} - J_u \leq \sum_{k=1}^{m} \{ \sum_{t=0}^{\infty} [(Fx_t - F\widehat{x}_t)^T (Fx_t - F\widehat{x}_t)]; \quad w_t = e_k \delta_t \}, \tag{19}$$

where e_k represents the k^{th} standard basis vector in $\mathbb{R}^m$ and δ_t is the discrete Dirac impulse function. Then, by the upper bound from Proposition 3.6, we get that:

$$\begin{aligned} J_{\widehat{u}} - J_u &\leq \|\widehat{u} - u\|_2^2 \\ &\leq \left\| \begin{bmatrix} (I_m - \mathbf{Y}^{\mathtt{md}}_{\mathbf{Q}^*}) & \mathbf{X}^{\mathtt{md}}_{\mathbf{Q}^*} \end{bmatrix} \right\|_{\mathcal{H}_2} + \left\| \begin{bmatrix} I_m - \mathbf{M}^{\mathtt{md}} \end{bmatrix} \right\|_\infty \|\mathbf{Q}^*\|_{\mathcal{H}_2} \frac{1}{1-\gamma\alpha} \Big(\left\| \begin{bmatrix} \widetilde{\mathbf{N}}^{\mathtt{md}} & \widetilde{\mathbf{M}}^{\mathtt{md}} \end{bmatrix} \right\|_\infty + \gamma \Big), \end{aligned} \tag{20}$$

where $\mathbf{Q}^*$ is the optimal solution to (18). Specifically, if the fixed state feedback gain F happens to be the stabilizing Riccati state-feedback F^{opt}, then by the virtue of separation principle the cost J_u in (20) becomes the optimal $\mathcal{H}_2$-cost. In this case, (20) immediately gives a bound for the difference in $\mathcal{H}_2$-cost between the Robust Linear Controller from (17) and the optimal Linear Quadratic Regulator (LQR) for the true plant. The detailed argumentation is deferred to Zhang et al. (2022b).

Furthermore, from (20), it is evident that $(J_{\widehat{u}} - J_u) \sim \mathcal{O}(\dfrac{\gamma}{1-\gamma\alpha})$ which indicates that the sample complexity relies heavily on the chosen constant α. In practice, it is impossible to examine uncountably many α's in $[0, 1/\gamma)$, therefore one could simply pick the value of α empirically, in order to balance the performance and the feasibility. The following remark indicates that α actually serves as an evaluation parameter for the quality of the initial controller in the closed loop, and consequently it cannot be taken to be arbitrarily small.

Remark 8 *(Feasibility) As $\alpha < \dfrac{1}{\gamma}$ is picked manually each time to formulate a new SDP and the performance of the observer degrades much faster with a larger α, one would like α to be as small as possible. However, as shown in Remark 6, a relatively small α may render the feasible set empty. This implies that the robust observer performance essentially relies on the quality of the initial controller (the one with which the learning preocedure is being performed). A better initial controller would provide not only a better fixed feedback gain, but also a larger feasible set for the inner optimization in Proposition 3.6.*

We integrate the above results with the system identification guarantees of Zhang et al. (2022a), to provide end-to-end sample complexity bounds for learning the linear observers given a fixed feedback gain. Then following the system identification procedure with probability at least $(1-\delta)$ where δ is the failure probability, it holds that

$$\| \begin{bmatrix} -\Delta_{\widetilde{\mathbf{M}}} & -\Delta_{\widetilde{\mathbf{N}}} \end{bmatrix} \|_\infty \leq \| \begin{bmatrix} \mathbf{X}^{\mathtt{md}} & \mathbf{Y}^{\mathtt{md}} \end{bmatrix} \|_\infty 12c\beta\mathcal{R}\left(\sqrt{\frac{m\widehat{d}+p\widehat{d}^2+\widehat{d}\log(T/\delta)}{T}}\right)$$

Combining with the prerequisite for robustness analysis, $\| \begin{bmatrix} -\Delta_{\widetilde{\mathbf{M}}} & -\Delta_{\widetilde{\mathbf{N}}} \end{bmatrix} \|_\infty < \gamma$ as in Assumption 1, it is reasonable to consider that the robustness radius γ is at the level $\mathcal{O}\left(\sqrt{\frac{\log T}{T}}\right)$.

Theorem 4.1 *Define $s = 144\left\| \begin{bmatrix} \mathbf{X}^{\mathtt{md}} & \mathbf{Y}^{\mathtt{md}} \end{bmatrix} \right\|_\infty^2 c^2\beta^2\mathcal{R}^2$. Then, the error in $\mathcal{H}_2$ cost of applying the control laws $\widehat{u} = F\widehat{x}$ and $u = Fx$ is bounded as in (20) with probability at least $(1-\delta)$ provided that $T \geq \max\{T_s, T_*(\delta)\}$. Here, T_s takes the larger value between 0 and the right most zero of $\gamma^2 T - s\widehat{d}\log(T/\delta) - s(m\widehat{d}+p\widehat{d}^2)$, and $T_*(\delta) = \inf\{T | d_*(T,\delta) \in \mathcal{D}(T), d_*(T,\delta) \leq 2d_*(\frac{T}{256},\delta)\}$, where, $d_*(T,\delta) = \inf\{d | 16\beta\mathcal{R}\alpha(d) \geq \left\| \widehat{\mathcal{H}}_{0,d,d} - \widehat{\mathcal{H}}_{0,\infty,\infty} \right\|_2\}$,*
$\mathcal{D}(T) = \{d \in \mathbb{N} | d \leq \dfrac{T}{cm^2\log^3(Tm/\delta)}\}$ and $f(d) = \sqrt{d}.(\sqrt{\dfrac{m+dp+\log(T/\delta)}{T}})$.

Combining Theorem 4.1 with (20), it follows that with high probability the difference $J_{\widehat{u}}$ and J_u behaves as

$$J_{\widehat{u}} - J_u \sim \mathcal{O}\left(\frac{\sqrt{\frac{\log T}{T}}}{1-\alpha\sqrt{\frac{\log T}{T}}}\right)$$

5. Conclusion and Future work

In this paper, we have provided the sample complexity bounds for an observer-based robust LQG regulator synthesis procedure for an unknown plant, where uncertainty is modeled as additive perturbations on the coprime factors. We combined finite-time, non-parametric LTI system identification (Sarkar and Rakhlin (2019)) with the Youla parameterization for observer performance evaluation given a fixed state feedback gain.

As an opened avenue for future research is the online learning of the observer-based LQG controller under the same type of model uncertainty. One possible direction is to work out the sample complexity for online learning for: *(a)* the optimal state feedback (LQR) in tandem with *(b)* the optimal state-observer (Kalman Filter (Tsiamis et al. (2020))) for a potentially unstable system.

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