

# Top- $k$ data selection via distributed sample quantile inference

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**Editors:** N. Matni, M. Morari, G. J. Pappas

## Abstract

We consider the problem of determining the top- $k$  largest measurements from a dataset distributed among a network of  $n$  agents with noisy communication links. We show that this scenario can be cast as a distributed convex optimization problem called sample quantile inference, which we solve using a two-time-scale stochastic approximation algorithm. Herein, we prove the algorithm's convergence in the almost sure sense to an optimal solution. Moreover, our algorithm handles noise and empirically converges to the correct answer within a small number of iterations.

**Keywords:** Networks, quantile regression, stochastic approximation, convex optimization

## 1. Introduction

The selection of the  $k$  largest quantities among a set of  $n$  real numbers is of great importance in many applications of interest including machine learning (Rejwan and Mansour, 2020), data mining (Tang et al., 2017), and information retrieval (Shi et al., 2019). These are known as *top- $k$*  strategies. Top- $k$  strategies also play a role in remote sensing (Zhang et al., 2022) and selection of the most informative neighbors for distributed optimization (Verma et al., 2021). While this problem is trivial in the centralized case, where all the data is available at a single agent (server), it is a non-trivial problem in applications where the data is distributed over a network of clients. Here, we study the design of a distributed top- $k$  strategy over a network of many agents.

Consider a decentralized system in Fig. 1 where the agents can only communicate with their neighbors over a noisy channels. Each agent has its own measurement and our goal is to select the  $k$  largest ones by learning an optimal threshold. Once this threshold is properly computed, each agent independently declares whether its measurement is above the threshold, and therefore that it has one of the top- $k$  measurements. One possible strategy is to relate the threshold with a sample quantile, which can be cast as quantile inference problem. Remarkably, this problem admits a natural decomposition as a distributed nonsmooth convex optimization problem. Herein, we provide an algorithm based on a two-time-scale subgradient method and perform its corresponding convergence analysis.

There exists a significant literature on the design of top- $k$  strategies in different settings. To the best of our knowledge (Babcock and Olston, 2003) proposed the first distributed algorithm to find the  $k$  largest values in a federated setting using a technique called range caching. Ustebay et al. (2011) provided a top- $k$  algorithm to compute the top- $k$  entries based on selective gossip. Rajagopal et al. (2006) studied the distributed estimation of an arbitrary quantile in a communication-constrained federated scenario. Compared with the above papers, Haeupler et al. (2018) proposed faster optimal gossip algorithms for quantile computation by designing two types of tournament

mechanisms. However, their algorithms are very sensitive to communication noise and do not converge to the exact quantile over noisy channels.

Our work is closely related to the work of Lee et al. (2020), where a two-time-scale algorithm to estimate a sample quantile in a decentralized way is claimed to converge in the mean squared sense. However, due to a technicality, the proof of convergence in (Lee et al., 2020) fails to hold due to the lack of monotonicity of the sequence  $\|\mathbf{E}[\bar{w}(t)] - \theta_p \mathbf{1}\|$ . We avoid this technical difficulty by considering the convergence in the almost sure sense, combining the classical result of Robbins and Siegmund (1971) with a modern result on nonsmooth optimization by Yi and Li (2021).

## 2. Problem setup

We begin our analysis by relating the problem of computing the top- $k$  observations with quantile inference, which is a convex optimization problem. Consider a collection of  $n$  agents,  $[n] = \{1, \dots, n\}$ , interconnected by a time-invariant, undirected, communication graph  $\mathcal{G} = ([n], \mathcal{E})$ , where  $\mathcal{E} \subset [n] \times [n]$  denotes the set of edges between nodes. Each agent holds a non-negative real number, which may represent a sensor measurement, a belief, or an opinion. Throughout this paper, we will simply refer to them as *data*. Let  $z_i \in \mathbb{R}$  be the data of the  $i$ -th agent. The goal of the team of agents is to determine in a distributed fashion the  $k$  agents holding the top- $k$  largest data. We want to do this efficiently. Moreover, the communication among agents occurs in the presence of additive noise.

At first, one may be inclined to consider the following strategy: Each agent keeps a list of  $k$  entries in its memory. At time  $t$  each agent sends this list to its neighbors. At time  $t + 1$ , every agent updates its list with by selecting the top- $k$  received data and discarding the rest. Each agent sorts its list and repeats. While this simple scheme converges to the top- $k$  results in finite time, it has two main drawbacks. First, this scheme requires noiseless communication channels of  $k$  real numbers per channel use. Even the slightest amount of noise will cause the algorithm to diverge. Second, it requires a memory with size  $k$ . If  $k \sim \mathcal{O}(n)$ , the communication and storage requirements will quickly turn the problem of finding the top- $k$  observations across the network prohibitive.

On the other hand, this problem can be conveniently cast into the framework of distributed convex optimization, and admits an implementation where a single real number is exchanged and a single unit of memory is updated at each time. Furthermore, this algorithm is robust to the presence of noise. Consider the problem of inferring the sample quantile from the dataset containing all of the agents' individual data points  $\mathcal{D} \stackrel{\text{def}}{=} \{z_i\}_{i=1}^n$ . Let  $\hat{F}(\xi; \mathcal{D})$  denote the empirical cumulative

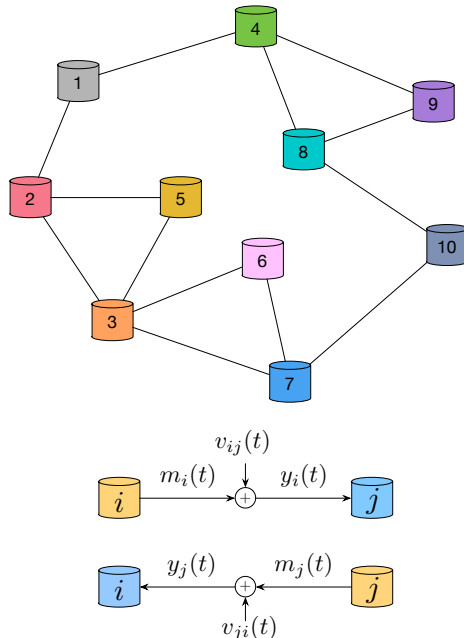
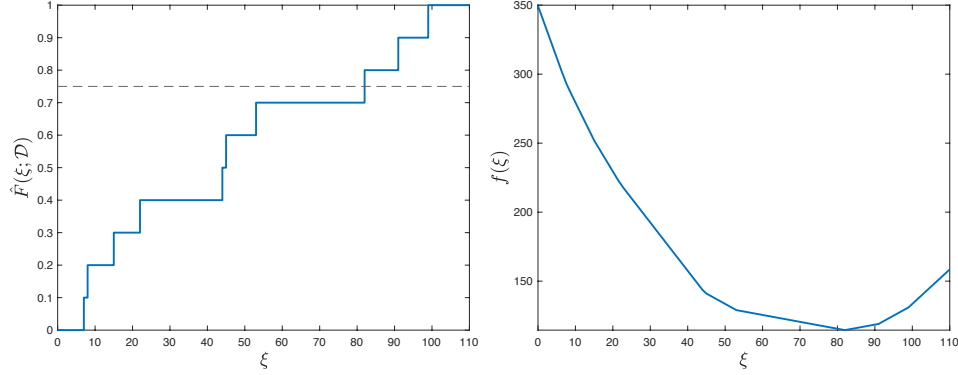


Figure 1: System model – A dataset is distributed across multiple clients. Two clients can communicate if there is a link in between them. The communication links are noisy.


 Figure 2: Empirical CDF and aggregate score function for  $k = 3$ .

distribution function of the data set  $\mathcal{D}$ , defined as:

$$\widehat{F}(\xi; \mathcal{D}) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n \mathbb{1}(z_i \leq \xi). \quad (1)$$

Let  $p \in (0, 1)$ . The (sample)  $p$ -quantile is defined as

$$\theta_p \stackrel{\text{def}}{=} \inf \left\{ \xi \mid \widehat{F}(\xi; \mathcal{D}) \geq p \right\}, \quad (2)$$

A classic result in quantile regression relates Eq. (2) to the solution of the following optimization problem (Koenker, 2005, Section 1.3, pp. 7–9):

$$\theta_p = \arg \min_{\xi \in \mathbb{R}} \sum_{i=1}^n \rho_p(z_i - \xi), \quad \text{where } \rho_p(x) \stackrel{\text{def}}{=} \begin{cases} (p-1)x & \text{if } x < 0 \\ px & \text{if } x \geq 0. \end{cases} \quad (3)$$

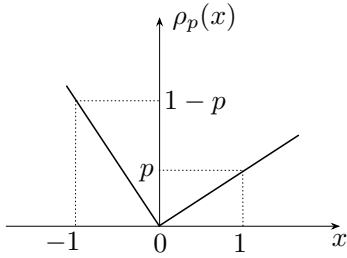


Figure 3: Score function

Let the local functions of the  $i$ -th node be defined as  $f_i(\xi) \stackrel{\text{def}}{=} \rho_p(z_i - \xi)$ , which are called the score functions (see Fig. 3), and the objective be defined as the aggregate score function  $f(\xi) \stackrel{\text{def}}{=} \sum_{i=1}^n f_i(\xi)$ , then the sample quantile is the solution of the following distributed optimization problem:

$$\theta_p = \arg \min_{\xi \in \mathbb{R}} f(\xi) = \arg \min_{\xi \in \mathbb{R}} \sum_{i=1}^n f_i(\xi). \quad (4)$$

A few noteworthy aspects of Eq. (4) are: (1) This is a convex optimization problem; (2) The objective function is nonsmooth; (3) The local functions have bounded subgradients:

$$|g_i(\xi)| \leq \max\{p, 1-p\} \leq 1, \quad g_i \in \partial f_i; \quad (5)$$

and (4) the  $p$ -quantile  $\theta_p$  belongs to the dataset  $\mathcal{D}$ , for any parameter  $p \in \mathcal{P}$ , where

$$\mathcal{P} \stackrel{\text{def}}{=} (0, 1) \setminus \left\{ \frac{1}{n}, \dots, \frac{n-1}{n} \right\}. \quad (6)$$

This framework can be used to compute many statistics of interest. For example, to compute the maximum ( $k = 1$ ), let  $p \in (1 - 1/n, 1)$ . To compute the minimum ( $k = n$ ), let  $p \in (0, 1/n)$ . Provided the number of samples in  $\mathcal{D}$  is odd, to compute the median, set  $p \in ((n - 1)/2n, (n + 1)/2n)$ . In general, if we would like to find the  $k$ -th largest element of  $\mathcal{D}$ , then

$$p \in \left( \frac{n - k}{n}, \frac{n - k + 1}{n} \right). \quad (7)$$

### 3. Distributed algorithm and main result

Herein, we propose and analyze the convergence of a two-time scale distributed sample quantile estimation algorithm in the presence of noise in the communication links. In particular, given the number of sensors  $n$  and any probability  $p \in \mathcal{P}$ , we prove that the algorithm converges to the sample quantile  $\theta_p$ . The non-smoothness of the empirical cumulative distribution function leads to oscillation around the optimal solution in the process of convergence, which is a difficulty that must be addressed. A novel analysis technique is introduced to tackle this problem.

As for the  $\mathcal{G} = ([n], \mathcal{E})$  with  $n$  nodes, let  $\mathcal{N}_i$  denote the set of neighbors of the  $i$ -th node, and  $d_{\max} = \max_i |\mathcal{N}_i|$ . Let  $L$  denote the graph Laplacian matrix and  $\lambda_1, \lambda_2, \dots, \lambda_n$  denote the eigenvalues of  $L$ , which satisfy  $0 = \lambda_1 < \lambda_2 < \dots < \lambda_n$ . Let  $w_i(t)$  denote the local estimate of  $\theta_p$  computed by node  $i$  at the  $t$ -th iteration,  $m_i(t)$  denotes the message sent by node  $i$  to its neighbors  $\mathcal{N}_i$ , and  $v_{ij}(t)$  be the communication noise on the link between nodes  $i$  and  $j \in \mathcal{N}_i$ .

**Assumption 1** *We assume that the noise sequences of random variables  $\{v_{ij}(t)\}_{t=0}^n$  are independent and identically distributed (i.i.d.) satisfying the following two properties:*

$$\mathbf{E}[v_{ij}(t)] = 0 \text{ and } \mathbf{E}[v_{ij}^2(t)] < \infty, \quad (i, j) \in \mathcal{E}. \quad (8)$$

Let  $\alpha(t)$  and  $\beta(t)$  be two deterministic step-size sequences. We initialize  $w_i(0) = z_i$ ,  $i \in [n]$ . On the  $t$ -th round of local communication we perform the following steps:

**1. Message computation:**

$$m_i(t) = w_i(t) - \alpha(t) (\mathbb{1}(w_i(t) \geq z_i) - p). \quad (9)$$

**2. Local estimate update:**

$$w_i(t+1) = m_i(t) - \beta(t) \sum_{j \in \mathcal{N}_i} (m_i(t) - y_j(t)), \quad (10)$$

where  $y_j(t) = m_j(t) + v_{ji}(t)$ .

**Theorem 1** *For a given set of samples  $\mathcal{D} = \{z_i\}_{i=1}^n$  distributed over  $n$  agents connected by a static undirected network  $\mathcal{G}$ , and any  $p \in \mathcal{P}$ , there exist step-size sequences  $\alpha(t)$  and  $\beta(t)$  satisfying*

$$\sum_{t=0}^{\infty} \alpha(t) = \infty, \quad \sum_{t=0}^{\infty} \alpha^2(t) < \infty, \quad (11)$$

$$\sum_{t=0}^{\infty} \beta(t) = \infty, \quad \sum_{t=0}^{\infty} \beta^2(t) < \infty, \quad (12)$$

$$\sum_{t=0}^{\infty} \frac{\alpha^2(t)}{\beta(t)} < \infty, \quad (13)$$

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**Algorithm 1** Distributed two-time scale sample quantile estimation under communication noise

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**Input:** measurements  $\{z_i\}_{i=1}^n$ , initialization  $w_i(0) = z_i$ ,  $i \in [n]$ , two step-size sequences  $\{\alpha(t)\}, \{\beta(t)\}$ , quantile parameter  $p \in (0, 1)$

**for**  $t = 0, 1, \dots$  **do**

    Local subgradient step:  $m_i(t) = w_i(t) - \alpha(t)(\mathbb{1}(w_i(t) \geq z_i) - p)$

    Send message:  $m_i(t)$  to neighbors  $j \in \mathcal{N}_i$

    Receive messages:  $y_j(t) = m_j(t) + v_{ji}(t)$ ,  $j \in \mathcal{N}_i$

    Estimate update step:  $w_i(t+1) = m_i(t) - \beta(t) \sum_{j \in \mathcal{N}_i} (m_i(t) - y_j(t))$

**end**

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such that the sequence  $w_i(t)$  computed according to Algorithm 1 satisfies  $w_i(t) \xrightarrow{\text{a.s.}} \theta_p$ ,  $i \in [n]$ .

#### 4. Almost sure convergence analysis

Our proof relies on a two-time-scale stochastic approximation algorithm. Define the deterministic step-size sequences

$$\alpha(t) \stackrel{\text{def}}{=} \frac{\alpha_0}{(t+1)^{\tau_1}} \quad \text{and} \quad \beta(t) \stackrel{\text{def}}{=} \frac{\beta_0}{(t+1)^{\tau_2}}, \quad \text{where } \beta_0 \leq \frac{2}{\lambda_2 + \lambda_n}, \quad \alpha_0 \geq 1, \quad (14)$$

and  $\tau_1, \tau_2$  satisfy  $0.5 < \tau_2 < \tau_1 \leq 1$  and  $2\tau_1 - \tau_2 > 1$ .

Algorithm 1 can be expressed in vector form as

$$w(t+1) \stackrel{\text{def}}{=} (I - \beta(t)L)(w(t) - \alpha(t)g(t)) + \beta(t)v(t). \quad (15)$$

where

$$w(t) \stackrel{\text{def}}{=} [w_1(t), \dots, w_n(t)]^T, \quad v(t) \stackrel{\text{def}}{=} \left[ \sum_{j \in \mathcal{N}_1} v_{1j}(t), \dots, \sum_{j \in \mathcal{N}_n} v_{nj}(t) \right]^T, \\ g_i(t) \stackrel{\text{def}}{=} \mathbb{1}(w_i(t) \geq z_i) - p, \quad \text{and} \quad g(t) \stackrel{\text{def}}{=} [g_1(t), \dots, g_n(t)]^T. \quad (16)$$

Finally, define the following averages

$$\bar{w}(t) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n w_i(t), \quad \bar{v}(t) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n v_i(t), \quad \text{and} \quad \bar{g}(t) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n g_i(t).$$

Next, we prove the almost surely convergence of  $w_i(t)$  to the quantile  $\theta_p$  defined in the dynamical system in Eq. (15) in term of  $\ell_2$ -norm, where  $p \in \mathcal{P}$ .

**Proof** Start by rewriting  $w_i(t) \xrightarrow{\text{a.s.}} \theta_p$ ,  $i \in [n]$  as

$$\lim_{t \rightarrow +\infty} \|w(t) - \theta_p \mathbf{1}\| = 0 \quad \text{a.s.}, \quad (17)$$

where  $\|\cdot\|$  denotes the  $\ell_2$ -norm. From the triangle inequality, we have  $\|w(t) - \theta_p \mathbf{1}\| \leq \|w(t) - \bar{w}(t) \mathbf{1}\| + \|\bar{w}(t) \mathbf{1} - \theta_p \mathbf{1}\|$ . We will show Eq. (17) holds by first proving that

$$\lim_{t \rightarrow +\infty} \|w(t) - \bar{w}(t) \mathbf{1}\| = 0 \quad \text{a.s.}, \quad \text{and} \quad \lim_{t \rightarrow +\infty} \|\bar{w}(t) \mathbf{1} - \theta_p \mathbf{1}\| = 0 \quad \text{a.s.} \quad (18)$$

**Part 1:** Recall that the Laplacian matrix  $L$  satisfies  $L1 = 0$ . Multiplying both sides of Eq. (15) by  $G \stackrel{\text{def}}{=} \frac{1}{n}11^T$  yields

$$\bar{w}(t+1)1 = G(w(t) - \alpha(t)g(t)) + \beta(t)Gv(t). \quad (19)$$

Let  $B(s) \stackrel{\text{def}}{=} (I - \beta(s)L - G)$ . Since  $B(s)1 = 0$ , by combining Eqs. (15) and (19) we get

$$w(t+1) - \bar{w}(t+1)1 = B(t)(w(t) - \bar{w}(t)1 - \alpha(t)g(t)) + \beta(t)(I - G)v(t). \quad (20)$$

Therefore, after taking the norm and squaring both sides of Eq. (20), we obtain

$$\begin{aligned} \|w(t+1) - \bar{w}(t+1)1\|^2 &= \|B(t)(w(t) - \bar{w}(t)1 - \alpha(t)g(t))\|^2 + \|\beta(t)(I - G)v(t)\|^2 \\ &\quad + 2\langle B(t)(w(t) - \bar{w}(t)1 - \alpha(t)g(t)), \beta(t)(I - G)v(t) \rangle. \end{aligned} \quad (21)$$

Let  $\mathcal{F}_t$  denote the  $\sigma$ -algebra generated by  $\{v(\ell)\}_{\ell=1}^{t-1}$ . Taking the conditional expectation of Eq. (21) with respect to  $\mathcal{F}_t$ , we obtain

$$\mathbf{E}[\|w(t+1) - \bar{w}(t+1)1\|^2 \mid \mathcal{F}_t] = \|B(t)(w(t) - \bar{w}(t)1 - \alpha(t)g(t))\|^2 + \mathbf{E}[\|\beta(t)(I - G)v(t)\|^2], \quad (22)$$

where we used the fact that  $\mathbf{E}[v(t) \mid \mathcal{F}_t] = 0$ . Let  $\eta > 0$ , using Young's inequality<sup>1</sup>, we have

$$\begin{aligned} \mathbf{E}[\|w(t+1) - \bar{w}(t+1)1\|^2 \mid \mathcal{F}_t] &\leq (1 + \eta) \|B(t)(w(t) - \bar{w}(t)1)\|^2 \\ &\quad + (1 + 1/\eta) \|\alpha(t)B(t)g(t)\|^2 + \mathbf{E}[\|\beta(t)(I - G)v(t)\|^2]. \end{aligned} \quad (23)$$

Using the inequality  $\|Ab\| \leq \|A\|\|b\|$ , we obtain

$$\begin{aligned} \mathbf{E}[\|w(t+1) - \bar{w}(t+1)1\|^2 \mid \mathcal{F}_t] &\leq (1 + \eta) \|B(t)\|^2 \|w(t) - \bar{w}(t)1\|^2 \\ &\quad + (1 + 1/\eta) \alpha^2(t) \|B(t)\|^2 \|g(t)\|^2 + \beta^2(t) \|I - G\|^2 \mathbf{E}[\|v(t)\|^2]. \end{aligned} \quad (24)$$

We can also show that

$$\|B(s)\| = \max\{|1 - \lambda_2\beta(s)|, |\lambda_n\beta(s) - 1|\} = 1 - \lambda_2\beta(s). \quad (25)$$

Incorporating Eq. (25) and  $\|I - G\| = 1$  into Eq. (24), the following inequality holds for any  $\eta > 0$ :

$$\begin{aligned} \mathbf{E}[\|w(t+1) - \bar{w}(t+1)1\|^2 \mid \mathcal{F}_t] &\leq (1 + \eta)(1 - \lambda_2\beta(t))^2 \|w(t) - \bar{w}(t)1\|^2 \\ &\quad + (1 + 1/\eta) \alpha^2(t)(1 - \lambda_2\beta(t))^2 \|g(t)\|^2 + \beta^2(t) \mathbf{E}[\|v(t)\|^2]. \end{aligned} \quad (26)$$

Choosing  $\eta = \lambda_2\beta(t)$ , we get

$$\begin{aligned} \mathbf{E}[\|w(t+1) - \bar{w}(t+1)1\|^2 \mid \mathcal{F}_t] &\leq \|w(t) - \bar{w}(t)1\|^2 - \lambda_2\beta(t) \|w(t) - \bar{w}(t)1\|^2 \\ &\quad + \frac{\alpha^2(t)}{\lambda_2\beta(t)} \|g(t)\|^2 + \beta^2(t) \mathbf{E}[\|v(t)\|^2]. \end{aligned} \quad (27)$$

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1.  $(a + b)^2 \leq (1 + \eta)a^2 + (1 + 1/\eta)b^2$ ,  $\eta > 0$ .

The subgradients of the local functions for the distributed quantile computation problem satisfy  $\|g_i(t)\| < 1$ . Thus, we have  $\|g(t)\| \leq \sqrt{n}$ . Together with Eqs. (8), (12) and (13), we obtain

$$\mathbf{E} \left[ \sum_{t=1}^{\infty} \left( \frac{\alpha^2(t)}{\lambda_2 \beta(t)} \|g(t)\|^2 + \beta^2(t) \|v(t)\|^2 \right) \right] < \infty. \quad (28)$$

Therefore, from Lemma 2 (Robbins-Siegmund Theorem (Robbins and Siegmund, 1971)), we conclude that  $\|w(t) - \bar{w}(t)\mathbf{1}\|$  converges almost surely to zero, and  $\sum_{t=1}^{\infty} \beta(t) \|w(t) - \bar{w}(t)\mathbf{1}\|^2 < \infty$ . If  $\tau_2 \leq 1$ , the sequence  $\beta(t)$  is not summable. Therefore,

$$\lim_{t \rightarrow +\infty} \|w(t) - \bar{w}(t)\mathbf{1}\| = 0 \text{ a.s.} \quad (29)$$

**Part 2:** Multiplying both sides of Eq. (15) by  $\frac{1}{n}\mathbf{1}^T$  and subtracting  $\theta_p$ , we have

$$\bar{w}(t+1) - \theta_p = \bar{w}(t) - \theta_p - \alpha(t)\bar{g}(t) + \beta(t)\bar{v}(t). \quad (30)$$

Squaring both sides of Eq. (30) yields

$$|\bar{w}(t+1) - \theta_p|^2 = |\bar{w}(t) - \theta_p - \alpha(t)\bar{g}(t)|^2 + \beta^2(t)|\bar{v}(t)|^2 + 2\beta(t)[\bar{w}(t) - \theta_p - \alpha(t)\bar{g}(t)]\bar{v}(t). \quad (31)$$

Taking conditional expectation with respect to  $\mathcal{F}_t$  and using the fact that  $\mathbf{E}[\bar{v}(t) | \mathcal{F}_t] = 0$ , yields

$$\mathbf{E}[|\bar{w}(t+1) - \theta_p|^2 | \mathcal{F}_t] = |\bar{w}(t) - \theta_p - \alpha(t)\bar{g}(t)|^2 + \beta^2(t)\mathbf{E}[\bar{v}^2(t)]. \quad (32)$$

Defining  $\varphi(t) \stackrel{\text{def}}{=} |\bar{w}(t) - \theta_p - \alpha(t)\bar{g}(t)|^2$ , we have

$$\varphi(t) = |\bar{w}(t) - \theta_p|^2 + \alpha^2(t)|\bar{g}(t)|^2 - 2\alpha(t)(\bar{w}(t) - \theta_p)\bar{g}(t). \quad (33)$$

For each  $g_i(t)$ , the following chain of inequalities holds:

$$(\bar{w}(t) - \theta_p)g_i(t) = (w_i(t) - \theta_p)g_i(t) + (\bar{w}(t) - w_i(t))g_i(t) \quad (34)$$

$$\stackrel{(a)}{\geq} f_i(w_i(t)) - f_i(\theta_p) + (\bar{w}(t) - w_i(t))g_i(t) \quad (35)$$

$$\stackrel{(b)}{=} f_i(w_i(t)) - f_i(\bar{w}(t)) + f_i(\bar{w}(t)) - f_i(\theta_p) + (\bar{w}(t) - w_i(t))g_i(t) \quad (36)$$

$$\stackrel{(c)}{\geq} -2|\bar{w}(t) - w_i(t)| + f_i(\bar{w}(t)) - f_i(\theta_p), \quad (37)$$

where (a) follows the convexity of  $f_i(x)$ , (b) follows from adding and subtracting  $f_i(\bar{w}(t))$ , and (c) follows the 1-Lipschitz condition that  $|g_i(t)| < 1$  and  $|f_i(x) - f_i(y)| \leq |x - y|$ . Therefore,

$$(\bar{w}(t) - \theta_p)\bar{g}(t) = \frac{1}{n} \sum_{i=1}^n (\bar{w}(t) - \theta_p)g_i(t) \geq -\frac{2}{n} \sum_{i=1}^n |\bar{w}(t) - w_i(t)| + \frac{1}{n} \left( f(\bar{w}(t)) - f(\theta_p) \right), \quad (38)$$

and

$$\varphi(t) \leq |\bar{w}(t) - \theta_p|^2 - \frac{2\alpha(t)}{n} \left( f(\bar{w}(t)) - f(\theta_p) \right) + \frac{4\alpha(t)}{n} \sum_{i=1}^n |\bar{w}(t) - w_i(t)| + \alpha^2(t)|\bar{g}(t)|^2. \quad (39)$$

Incorporating Eq. (39) into Eq. (32), we get

$$\begin{aligned} \mathbf{E} \left[ |\bar{w}(t+1) - \theta_p|^2 \mid \mathcal{F}_t \right] &\leq |\bar{w}(t) - \theta_p|^2 - \frac{2\alpha(t)}{n} (f(\bar{w}(t)) - f(\theta_p)) \\ &\quad + \frac{4\alpha(t)}{n} \sum_{i=1}^n |\bar{w}(t) - w_i(t)| + \alpha^2(t) |\bar{g}(t)|^2 + \beta^2(t) \mathbf{E}[\bar{v}^2(t)]. \end{aligned} \quad (40)$$

Since  $\theta_p$  is the minimizer of  $f(\xi)$ , we have  $f(\bar{w}(t)) - f(\theta_p) \geq 0$ . Therefore,

$$\mathbf{E} \left[ \sum_{t=1}^{\infty} \alpha^2(t) |\bar{g}(t)|^2 \right] < \infty \quad \text{and} \quad \mathbf{E} \left[ \sum_{t=1}^{\infty} \beta^2(t) |\bar{v}(t)|^2 \right] < \infty. \quad (41)$$

To apply Lemma 2, we must show that

$$\mathbf{E} \left[ \sum_{t=0}^{\infty} \alpha(t+1) |\bar{w}(t+1) - w_i(t+1)| \right] < \infty. \quad (42)$$

Since  $|\bar{w}(s) - w_i(s)| \leq \|w(s) - \bar{w}(s)\mathbf{1}\|$ , using Eq. (20) the following inequalities hold

$$\begin{aligned} \sum_{t=0}^{\infty} \alpha(t+1) |\bar{w}(t+1) - w_i(t+1)| &\leq \sum_{t=0}^{\infty} \alpha(t+1) \|w(t+1) - \bar{w}(t+1)\mathbf{1}\| \\ &\leq \sum_{t=0}^{\infty} \alpha(t+1) \prod_{s=0}^t \|B(s)\| \|w(0)\| + \sum_{t=0}^{\infty} \alpha(t+1) \sum_{l=0}^t \alpha(l) \prod_{s=l}^t \|B(s)\| \|g(l)\| \\ &\quad + \sum_{t=0}^{\infty} \alpha(t+1) \beta(t) \|I - G\| \|v(t)\| + \sum_{t=0}^{\infty} \alpha(t+1) \sum_{l=0}^t \beta(l) \prod_{s=l+1}^t \|B(s)\| \|I - G\| \|v(l)\|. \end{aligned} \quad (43)$$

By using Lemma 3 and  $\|B(s)\| = 1 - \beta(s)\lambda_2 < \exp(-\beta(s)\lambda_2)$ , we get

$$\sum_{t=0}^{\infty} \alpha(t+1) \prod_{s=0}^t \|B(s)\| < \infty, \quad \sum_{t=0}^{\infty} \alpha(t+1) \sum_{l=0}^t \alpha(l) \prod_{s=l}^t \|B(s)\| < \infty, \quad (44)$$

$$\sum_{t=0}^{\infty} \alpha(t+1) \sum_{l=0}^t \beta(l) \prod_{s=l+1}^t \|B(s)\| < \infty. \quad (45)$$

From Assumption 1, using Jensen's inequality we get  $\mathbf{E}[\|v(l)\|] < \sqrt{\mathbf{E}[\|v(l)\|^2]} < \infty$ . The step-size sequences  $\alpha(t)$  and  $\beta(t)$  satisfy  $\sum_{t=0}^{\infty} \alpha(t+1)\beta(t) < \infty$ . Therefore,

$$\mathbf{E} \left[ \sum_{t=1}^{\infty} \alpha(t) |\bar{w}(t) - w_i(t)| \right] < \infty. \quad (46)$$

Thus, Lemma 2 implies that  $|\bar{w}(t) - \theta_p|$  converges almost surely and

$$\sum_{t=1}^{\infty} \alpha(t) (f(\bar{w}(t)) - f(\theta_p)) < \infty. \quad (47)$$



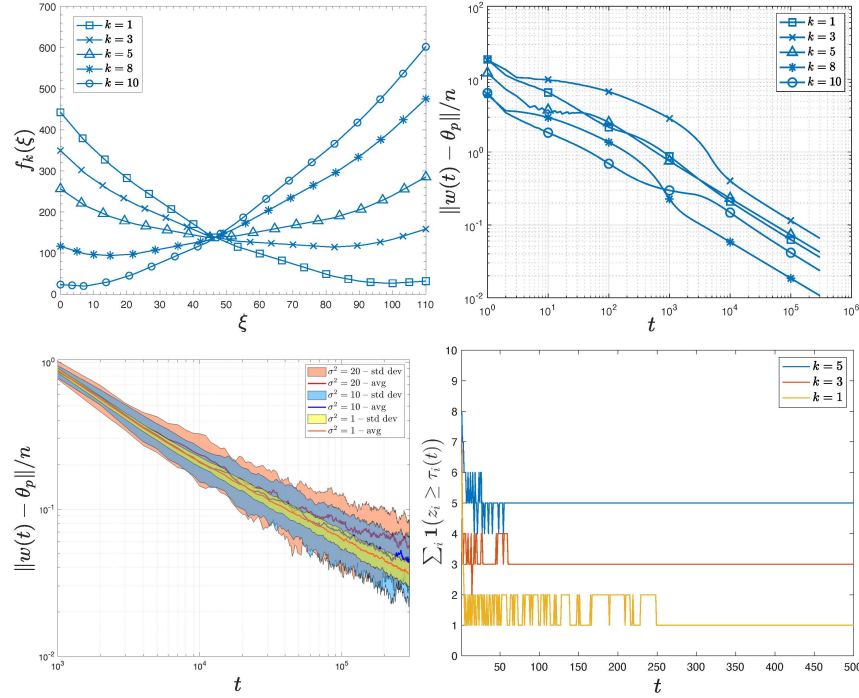


Figure 4: Aggregate score functions (top left) and noiseless distributed sample quantile estimation for different  $k$ ,  $k = 1, 3, 5, 8, 10$  (top right). Convergence of distributed sample quantile inference under different communication noise with  $\sigma^2 = 1, 10, 20$  and  $k = 1$  (bottom left). Number of agents with measurements greater than their corresponding thresholds as a function of iterations, for  $k = 1, 3, 5$  and  $\sigma^2 = 10$  (bottom right).

Since  $\alpha(t)$  is not summable, we have  $\liminf_{t \rightarrow \infty} f(\bar{w}(t)) = f(\theta_p) = 0$  almost surely. Thus, there exists a subsequence  $\{\bar{w}(t_j)\}$  such that

$$\lim_{j \rightarrow \infty} f(\bar{w}(t_j)) = \liminf_{t \rightarrow \infty} f(\bar{w}(t)) = f(\theta_p) = 0 \text{ a.s.} \quad (48)$$

From (Koenker, 2005, Section 1.3, pp. 7–9), since  $np$  is not an integer,  $\theta_p$  is the unique minimum of Eq. (3). Furthermore, since  $f(\xi)$  is continuous, we have  $\lim_{j \rightarrow \infty} |\bar{w}(t_j) - \theta_p| = 0$  almost surely. Together with the fact that  $|\bar{w}(t) - \theta_p|$  converges almost surely, we have

$$\lim_{t \rightarrow \infty} |\bar{w}(t) - \theta_p| = 0 \text{ a.s.} \quad (49)$$

■

## 5. Numerical results

In this section, we provide some numerical results that display the convergence of the distributed sample quantile inference algorithm and its application to top- $k$  data selection. Consider a system with  $n = 10$  agents interconnected by the graph in Fig. 1. The data samples  $\mathcal{D} = (45, 8, 22, 91, 15, 82, 53, 7, 44, 99)$  were drawn from an i.i.d. uniform distribution on  $\{1, \dots, 100\}$ , whose empirical CDF is shown in Fig. 2 (left). The communication noises are zero mean i.i.d. Gaussian

random variables with variance  $\sigma^2$ . The parameters of two time-scale sequences are set to  $\alpha_0 = 80$ ,  $\beta_0 = 2/(\lambda_2 + \lambda_n)$ ,  $\tau_1 = 1$  and  $\tau_2 = 0.505$ .

The panel shown in Fig. 4 displays several aspects that support our theoretical analysis. First, Fig. 4 (top left) shows the aggregate score functions for different values of  $k$ , and we can see how the curvature varies with the value of  $k$  leading to some values of  $\theta_p$  being faster to compute than others. The convergence of distributed sample quantile inference with noiseless links for different  $k$  is shown in Fig. 4 (top right), which shows that the asymptotic convergence rate (slope of the error curves) are approximately constant. However, the offset among curves depends on  $k$ , the dataset, and the graph connectivity and is difficult to characterize.

The convergence of distributed sample quantile estimation and top- $k$  selection under noisy links is shown in Fig. 4 (bottom left). The variance of the noise is set to be  $\sigma^2 = 1, 10, 20$ , and  $k = 1$ . We have generated 100 sample paths for each value of  $\sigma^2$ , and the figure shows the asymptotic convergence of the average of the sample paths. Finally, Fig. 4 (bottom right) shows the convergence of the number of agent with data samples greater than their corresponding thresholds, for  $k = 1, 3, 5$  with  $\sigma^2 = 10$ . The threshold at each agent is set to be  $\tau_i(t) = w_i(t) - 0.5$ . This correction needs to be done to avoid oscillation of the agent holding the  $k$ -th largest data sample. When the number of iterations is larger than 55, 61 and 250, the system can always find the top-5, top-3, top-1 data points, respectively. Therefore, even though the convergence time in the simulations is in the order of  $10^4$  or  $10^5$ , the convergence to the right decision whether an agent is holding one of the top- $k$  data points happen within a much smaller number of iterations, which is a desirable feature<sup>2</sup>.

## 6. Conclusion

In this work, we have studied networked top- $k$  data selection by using distributed sample quantile inference over noisy channels. We have provided the analysis for the almost sure convergence of a two-time-scale distributed subgradient method. Numerical results have demonstrated the convergence of the distributed algorithm and the corresponding data selection. There are two interesting directions for future work. One is to analyze the convergence rate for the distributed sample quantile estimation. The other is to provide new algorithms to speed up the convergence.

## 7. Auxiliary Lemmas

**Lemma 2 (Theorem 1, Robbins and Siegmund (1971))** *Let  $\{v(t), \mathcal{F}_t\}$ ,  $\{d(t), \mathcal{F}_t\}$ , and  $\{a(t), \mathcal{F}_t\}$  be three nonnegative adapted sequences. If*

$$\mathbf{E}[v(t+1) \mid \mathcal{F}_t] \leq v(t) + a(t) - d(t) \quad \text{and} \quad \mathbf{E}\left[\sum_{t=1}^{\infty} a(t)\right] < \infty \quad (50)$$

*then  $\sum_{t=1}^{\infty} d(t) < \infty$  almost surely, and  $v(t)$  converges almost surely.*

**Lemma 3 (Lemma 4.2, Yi and Li (2021))** *Suppose that  $\alpha(t)$  and  $\beta(t)$  satisfy Eqs. (11) to (13). Then for any  $c > 0$ , we have*

$$\sum_{t=0}^{\infty} \alpha(t+1) \exp\left(-c \sum_{s=0}^t \beta(s)\right) < \infty, \quad \text{and} \quad \sum_{t=0}^{\infty} \alpha(t+1) \sum_{l=0}^t \alpha(l) \exp\left(-c \sum_{s=l}^t \beta(s)\right) < \infty. \quad (51)$$

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2. All the code used to obtain the numerical results contained herein can be found at <https://github.com/mullervasconcelos/L4DC23.git>

## Acknowledgments

The authors would like to thank the anonymous reviewers for useful comments which have helped to improve the presentation of this paper. X. Zhang was partially supported by China National Postdoctoral Program for Innovative Talents (No. BX2021346), China Postdoctoral Science Foundation (No. 2022M713316), and National Natural Science Foundation of China (No. 12288201). M. M. Vasconcelos was partially supported by the Commonwealth Cyber Initiative.

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