Closure Operators, Classifiers and Desirability

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Abstract

At the core of Bayesian probability theory, or dually desirability theory, lies an assumption of linearity of the scale in which rewards are measured. We revisit two recent papers that extend desirability theory to the nonlinear case by letting the utility scale be represented either by a general closure operator or by a binary general (nonlinear) classifier. By using standard results in logic, we highlight the connection between these two approaches and show that this connection allows us to extend the separating hyper plane theorem (which is at the core of the duality between Bayesian decision theory and desirability theory) to the nonlinear case.

Keywords: closure operators, classifiers, desirability, belief structure

1. Introduction

At the core of Bayesian probability theory, or dually desirability theory, lies an assumption of linearity of the scale in which rewards are measured. It is well-known in utility theory that linearity is a strong assumption, being violated for instance in domains with budget constraints, problems with lack of liquidity, wealth effects and risk-aversion.

There have been few works that studied extensions of linear-desirability theory to account for the above issues [16, 17, 18, 24]. These papers are related to a long line of works that proposed alternative theories that accommodate systematic departures from expected utility, while retaining much of its mathematical properties, a classical reference is [10].

Recently, [15] proposed a unifying theoretical framework to extend linear-desirability theory to the nonlinear case by letting the utility scale be represented by a general closure operator. This framework retains the overall logical structure of linear-desirability theory, which is based on the following axioms: (i) gaining money is desirable; (ii) losing money is undesirable; but replaces the linearity axiom with: (iii) the value of money is measured on a logically consistent utility scale (determined by a general closure operator).

A more operational approach to extend linear-desirability to the nonlinear case was pursued in [6]. This approach starts from the observation that the logical consistency of a set of linearly-desirable gambles can be checked by solving a binary linear classification problem. Then the authors extend desirability to the nonlinear case by instead considering a binary nonlinear classification problem. This framework imposes the logical constraints of desirability theory by forcing the classifier to separate the non-negative gambles (gaining money is desirable) from the negative ones (losing money is undesirable). Moreover, theoretical results and numerical algorithms are provided to learn classifiers from a dataset made of accepted and rejected gambles for three closure operators: conic hull, convex hull and the so-called orthant-hull (or monotonic-hull).

The works [15, 6] show that the previous approaches to nonlinear-desirability [16, 18] can be seen as particular cases of these formulations.

Independently, [11], working with a finite set of alternatives in the framework of [12], defined the concept of preference ordering respecting a classifier and provided a connection between classifiers and closure operators. This result somehow generalises the duality of convex sets and support half-spaces in convex geometry. It is also worth to point out that the approach followed by [11] just exploits the relationship between closure operators and their closed belief models.

In the current manuscript, we extend the work [11] to sets of gambles. The main difference is that we further require the closure operators (equivalently, the classifiers) to satisfy a monotonicity property: if $f \geq g$ (where $\geq$ is the pointwise-order in $\mathbb{R}$) then the gamble f should be more desirable than the gamble g. The appropriateness of monotonicity is justified by the fact that a set of gambles is more than just a set of alternatives, their value is measured in an ordinal scale. This connection allows us to straightforwardly extend the separating hyperplane theorem (which is at the core of the duality between Bayesian decision theory and desirability theory) to the nonlinear case.

Since the connections provided in this manuscript follow by standard basic results in lattice theory and algebraic logic, we formalise these results in the framework of *belief structures* introduced by [9].

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2. Preliminaries

2.1. Linear-Desirability Theory

We briefly review standard linear-desirability theory. A gamble is a bounded function from a space of possibility Ω to $\mathbb{R}$, denoted by $\mathcal{L}$. We denote a set of gambles that a subject, Alice, finds desirable with $A \subseteq \mathcal{L}$. These are the gambles that she is willing to accept and thus commits herself to the corresponding transactions. The crucial question is how to provide a criterion for a set A of gambles, representing an assessment of desirability, to be regarded rational. A general way to formalise rationality is to regard $\mathcal{L}$ as an algebra of formulas on top of which to define a logic. This leads us directly to formulate rationality as logical consistency.

Let $\mathcal{L}^+$ be the subset of the non-negative gambles excluded the zero gamble. Negative gambles are defined by $\mathcal{L}^- := -\mathcal{L}^+$, and we shall also use $\mathcal{L}_0^\star := \mathcal{L}^\star \cup \{0\}$, with $\star = +, -$, and $\mathcal{L}^< := \{f \in \mathcal{L} \mid \sup f < 0\}$.

First of all, since positive gambles may increase Alice's utility without ever decreasing it, we first have that:

(A0) $\mathcal{L}^+$ should always be desirable.

This defines the tautologies of the calculus. Moreover, whenever A are desirable for Alice, then any gamble whose desirability is implied by the gambles in A should also be desirable. This implication corresponds to the deductive closure K of a set of desirable gambles A, given by:

(A1) $K(A) := \text{posi}(\mathcal{L}^+ \cup A)$.

where posi is the conic hull operator.

Finally, deductive coherence is defined as follows:

Definition 1 (Coherence postulate) *A set $\mathcal{D} := K(A)$ of desirable gambles is coherent if and only if*

(A2) $\mathcal{D} \cap \{0\} = \emptyset$.

Axioms (A0–A2) lead to the so-called *theory of desirable gambles* [23],[1] here referred to by *linear-desirability theory*. If we replace $\mathcal{L}^+$ with $\mathcal{L}_0^+$ in A0 and A1, the conic hull operator with its 'topological closure' in A1, and finally $\{0\}$ with $\mathcal{L}^<$ in A2, we obtain the so-called *theory of almost desirable gambles* (or linear-almost-desirability theory).

2.2. Belief Structures

We present and slightly generalise the framework introduced in [9]. The modification is simply due to the fact that we conceptually separate the notion of closure operator and that of consistency.[2]

[1] Given (A0–A1), (A2) is equivalent to $\mathcal{L}_0^- \cap \mathcal{D} = \emptyset$.

[2] In doing so, we can for instance naturally cover the case of linear-desirability.

Let $(\mathcal{S}, \sqsubseteq)$ be a complete lattice over a non-empty set $\mathcal{S}$, whose elements are called belief models. Its supremum is denoted by $1_\mathcal{S}$, and its infimum by $0_\mathcal{S}$. Typically $\mathcal{S}$ is the collection of sets of formulas defined over an appropriate language, and thus $1_\mathcal{S} = \mathcal{S}$ and $0_\mathcal{S} = \emptyset$.

A map $K : \mathcal{S} \to \mathcal{S}$ is a *closure operator* over $\mathcal{S}$ if it satisfies the following properties:

(K1–Extensiveness) $A \sqsubseteq K(A)$;

(K2–Monotonicity) if $A \sqsubseteq A'$ then $K(A) \sqsubseteq K(A')$;

(K3–Idempotency) $K(K(A)) = K(A)$.

for all $A, A' \in \mathcal{S}$. It is immediate to verify that:

Proposition 2 *Let $\mathcal{L}$ be a topological ordered vector space*[3] *and $\mathcal{S} := \wp(\mathcal{L})$. The following are closure operators on $(\mathcal{S}, \subseteq)$:*

$$K_{po}(A) := \text{posi}(A \cup \mathcal{L}^+), \quad (1)$$
$$K_{co}(A) := \text{conv}(A \cup \mathcal{L}^+), \quad (2)$$
$$K_{or}(A) := \{f \in \mathcal{L} \mid (\exists g \in A\ f \geq g) \vee f \in \mathcal{L}^+\}, \quad (3)$$
$$K_{\overline{po}}(A) := \text{cl}\big(\text{posi}(A \cup \mathcal{L}_0^+)\big), \quad (4)$$
$$K_{\overline{co}}(A) := \text{cl}\big(\text{conv}(A \cup \mathcal{L}_0^+)\big), \quad (5)$$
$$K_{\overline{or}}(A) := \text{cl}\{f \in \mathcal{L} \mid (\exists g \in A\ f \geq g) \vee f \in \mathcal{L}_0^+\}. \quad (6)$$

where posi *is the conic hull,* conv *is the convex hull, and* cl *is the topological closure.*

Given two closure operators K_1 and K_2 over the same $\mathcal{S}$, we write $K_1 \leq_\mathcal{S} K_2$ (or simply $K_1 \leq K_2$ when $\mathcal{S}$ is clear from the context) if $K_1(A) \sqsubseteq K_2(A)$, for every $A \in \mathcal{S}$. In such case we say that K_1 is *weaker* than K_2.

A belief model is closed if $K(A) = A$. The family of all closed belief models of $\mathcal{S}$ is denoted by $\mathcal{S}_K$. Hence observe that:

Proposition 3 *Given two closure operators K_1 and K_2 over the same $\mathcal{S}$, the following are equivalent*

1. $K_1 \leq_\mathcal{S} K_2$
2. $\mathcal{S}_{K_2} \subseteq \mathcal{S}_{K_1}$

Moreover, the set of all closure operators over $\mathcal{S}$ forms a complete lattice under $\leq_\mathcal{S}$.

The next Proposition states a well known correspondence between closure operators and so-called topped intersection structures (or closure systems) (see e.g. [7, Thm 7.3])

[3] From now on we always assume that the topology is the order topology.

Proposition 4 *Let $K : \mathcal{S} \to \mathcal{S}$ be a closure operator on a complete lattice $(\mathcal{S}, \sqsubseteq)$. Then the family of closed belief models $\mathcal{S}_K$ is closed under arbitrary infima and contains $0_{\mathcal{S}}$ and $1_{\mathcal{S}}$. Moreover, if $\mathcal{A}$ is any collection of elements of $\mathcal{S}$ which is closed under arbitrary infima and contains $0_{\mathcal{S}}$ and $1_{\mathcal{S}}$, there is a unique closure operator $K_{\mathcal{A}} : \mathcal{S} \to \mathcal{S}$ such that $\mathcal{S}_{K_{\mathcal{A}}} = \mathcal{A}$.*

A subset (predicate) $C \subset \mathcal{S}_K$ of closed belief models is called a *consistency set* (or predicate) if

(C1) C is closed under arbitrary non-empty infima;

(C2) $1_{\mathcal{S}} \notin C$.

We say that a belief model $A \in \mathcal{S}$ is *consistent* if $K(A) \in C$.

Definition 5 *A* belief structure *is a quadruple* $\mathfrak{B} := (\mathcal{S}, \sqsubseteq, K, C)$ *where*

- $(\mathcal{S}, \sqsubseteq)$ *is a complete lattice*
- $K : \mathcal{S} \to \mathcal{S}$ *is a closure operator*
- C *is a consistency set.*

We say that a belief structure $\mathfrak{B}$ is *classical* whenever $K = K_{\overline{C}}$, where $\overline{C} := C \cup \{1_{\mathcal{S}}\}$, and *paraconsistent*[4] otherwise. The idea is that a belief structure is classical whenever the deductive closure of every inconsistent belief model is trivial. Paraconsistent belief structures are structures in which, for closed models, being inconsistent is not tantamount to being trivial. Stated otherwise, inconsistency is not 'explosive'.

An example of classical belief structure is the linear-almost desirability theory. Linear-desirability is, on the other hand, a paraconsistent belief structure.

The ordered structure $(C, \sqsubseteq)$, where the partial order is inherited from $\mathcal{S}$ is a complete meet-semi-lattice. Hence it is immediate to verify that

Proposition 6 *Let $\mathfrak{B}$ be a belief structure and $A \in \mathcal{S}$. If A is consistent, then*

$$K(A) = \inf\{C \in C \mid A \sqsubseteq C\}$$

Notice however that $(C, \sqsubseteq)$ is not necessarily a complete lattice, and in particular it may have no maximal elements. The (possibly empty) collection of these maximal elements is denoted by $\mathcal{M} := \{M \in C \mid \forall C \in C, (M \sqsubseteq C \Rightarrow M = C\}$.

Finally, $\mathfrak{B}$ is said to be *strong* if $(C, \sqsubseteq)$ is dually atomic, that is if $\mathcal{M} \neq \emptyset$ and for every $C \in C$, $C = \inf\{M \in \mathcal{M} \mid C \sqsubseteq M\}$. Both the theory of almost desirable gambles and of desirable gambles are strong belief structures [23, 3].

The following representation result is immediate

[4]The term comes from *paraconsistent* logics, see e.g. [19].

Proposition 7 *Let $\mathfrak{B}$ be a strong classical belief structure. Then it holds that*

$$K(A) = \inf\{M \in \mathcal{M} \mid A \sqsubseteq M\},$$

for every $A \in \mathcal{S}$.

2.3. Morphisms

In this Section, we introduce the appropriate concept of morphism to compare belief structures. As noticed in [9], what matters in a belief structure is its closure operator, and thus the complete meet-semi lattice given by its closed belief models, and its consistency set. We therefore define the following.

Definition 8 *Given two belief structures $\mathfrak{B}_0$ and $\mathfrak{B}_1$, a* c-homomorphism *is an order-preserving function $f : \mathcal{S}_0 \to \mathcal{S}_1$ with the additional properties*

(H1) $f(K_0(A)) = K_1(f(A))$*;*

(H2) *if $B \in C_0$ then $f(B) \in C_1$;*

(H3) $f(\inf(\mathcal{A})) = \inf(\{f(A) : A \in \mathcal{A}\})$*;*

for every belief model $A \in \mathcal{S}_0$, every closed model $B \in \mathcal{S}_{K_0}$, and every family of closed belief models $\mathcal{A} \subseteq \mathcal{S}_{K_0}$.

A c-homomorphism f is called a *c-embedding* if, in addition, for every closed $A, B \in \mathcal{S}_{K_0}$ it satisfies

(H4) $A \sqsubseteq_0 B$ iff $f(A) \sqsubseteq_1 f(B)$;[5]

(H5) $A \in C_0$ iff $f(A) \in C_1$.

A c-embedding that is onto when restricted to $\mathcal{S}_{K_1}$ is called a *c-isomorphism*.

2.4. Structures on Sets

In what follows we will be using $\mathcal{S} := \wp(\mathcal{L})$, for some language $\mathcal{L}$, meaning that the partial order $\sqsubseteq$ coincides with the subset relation $\subseteq$.

First, notice that whenever $\mathfrak{B}$ is a strong classical belief structure, in this case Proposition 7 readily implies the following separation property: $a \notin K(A)$ iff $a \notin M$ for some $M \in \mathcal{M}$ such that $A \subseteq M$.

Let $\mathfrak{B}$ be a belief structure. A gamble belonging to $\mathsf{T} := K(\emptyset)$ is called a *tautology*. A gamble $\bot \in \mathcal{L}$ for which the equivalence

$$\bot \in K(A) \Leftrightarrow K(A) \notin C$$

holds for every $A \in \mathcal{S}$ is called a *falsum*. The set of all falsa of $\mathfrak{B}$ is denoted by F. In general we say that

[5]By reflexivity and anti-symmetry of $\sqsubseteq$, condition (H4) yields that f is injective on $\mathcal{S}_{K_0}$.

consistency is definable for a belief structure $\mathfrak{B}$ whenever $\mathsf{F} \neq \emptyset$. Consistency is definable in the theory of desirable gambles, and $\mathsf{F} = \{0\}$. Consistency is also definable in the theory of almost desirable gambles but $\mathsf{F} = \{f \in \mathcal{L} \mid \sup f < 0\}$. Contrary to the theory of desirable gambles, the theory of almost desirable gambles is not paraconsistent; it is classical and all its falsa $\bot \in \mathsf{F}$ satisfy the following explosion law

$$\bot \in K(A) \Leftrightarrow K(A) = 1_{\mathcal{S}},$$

where $1_{\mathcal{S}} = \mathcal{L}$.
Let $a, b \in \mathcal{L}$, and $A \in \mathcal{S}$. We define the following properties (rules).

(C-rule) $a \in K(A) \Rightarrow K(A \cup \{b\}) \notin C$;

(I-rule) $K(A \cup \{a\}) \notin C \Rightarrow b \in K(A)$.

When (C-rule) holds between $a, b \in \mathcal{L}$ for every $A \in \mathcal{S}$, we call b a *quasi-negation* of a.[6] If both (C-rule) and (I-rule) hold for every $A \in \mathcal{S}$, we call b a *negation* of a. We say that a belief structure $\mathfrak{B}$ for which consistency is definable is *quasi-negative* if every $a \in \mathcal{L}$ which is neither a tautology nor a falsum has a quasi-negation which is not a falsum.[7] Structure $\mathfrak{B}$ is said to be *negative*, if every $a \in \mathcal{L}$ has a negation. Let $\mathfrak{B}$ be a (quasi-)negative belief structure, then we say that $A \in C$ is *nega-complete* if, for every $a \in \mathcal{L}$, either a or some of its (quasi-)negation belongs to A.

The theory of desirable gambles is a negative belief structure: for a given gamble f, one of its negation coincides with $-f$. Moreover all its maximally consistent elements are nega-complete.

3. Generalised Desirability Theories

In this section we focus on the case when the underlying language is given by a topological ordered vector space. For simplicity, we actually take $\mathcal{L} := \mathbb{R}^{|\Omega|}$, with finite Ω with dimension n and equipped with the usual order, and $\mathcal{S} := \wp(\mathcal{L})$.[8]

Definition 9 *Let $\mathfrak{B} := (\mathcal{S}, \subseteq, K, C)$ be a belief structure over $\mathcal{L}$. We call it a* generalised desirability theory *(GDT for short) whenever the following properties are satisfied:*

(G1) *$K(\mathcal{L}^+) = K(0_{\mathcal{S}}) = \mathcal{L}^+$ is the minimal element of its consistency set C, and in particular $K(A) \supseteq \mathcal{L}^+$, for every $A \in \mathcal{S}$*

(G2) *$K_{or} \leq K$,*

(G3) *$K(A) \in C$ if and only if $K(A) \cap \{0\} = \emptyset$.*

A generalised almost desirability theory (GADT for short) is defined analogously, simply changing properties (G1–G3) similar to what is done in linear-desirability theory. The 'nonlinear (almost) desirability theories' introduced in [6, 15] are belief structures that are generalised (almost) desirability theories.

Finally, following [20, Sections 1.3.3-4], we introduce two operations on generalised desirability theories: conditioning and marginalisation. Such operations generalise the classical operations baring the same name so to cover also the definitions of conditioning and marginalisation for the nonlinear desirability theory discussed by [15].

Let $\Delta \subset \Omega$, $\mathcal{L}' := \mathbb{R}^{|\Delta|}$, $\mathcal{S}' := \wp(\mathcal{L}')$. Consider the map $\lceil_{\Delta} \colon \mathcal{L}' \to \mathcal{L}$ defined as

$$\lceil_{\Delta}(g)(x) := \begin{cases} g(x) & \text{if } x \in \Delta \\ 0 & \text{else.} \end{cases}$$

We then straightforwardly lift this map to the following order preserving function $\lceil_{\Delta} \colon \mathcal{S}' \to \mathcal{S}$ simply by setting $\lceil_{\Delta}(A) := \{\lceil_{\Delta}(g) \in \mathcal{L} \mid g \in A\}$.

Conversely, consider the map $\rfloor_{\Delta} : \mathcal{S} \to \mathcal{S}'$ defined as $\rfloor_{\Delta}(A) := \{g \in \mathcal{L}' \mid \lceil_{\Delta}(g) \in A\}$. Notice that $\lceil_{\Delta}(\rfloor_{\Delta}(A)) = \{f \in A \mid f = fI_{\Delta}\} \subseteq A$.

Definition 10 *Let $\mathfrak{B} := (\mathcal{S}, \subseteq, K, C)$ be a GDT over $\mathcal{L}$ and $\mathfrak{B}' := (\mathcal{S}', \subseteq, K', C')$ a GDT over $\mathcal{L}'$. Assume that $\rfloor_{\Delta}$ maps closed belief models to closed belief models and that $K \circ \lceil_{\Delta}$ is a c-embedding such that*

$$A = \rfloor_{\Delta}(K(\lceil_{\Delta}(A))),$$

for every closed belief model $A \in \mathcal{S}'_{K'}$. Then we call $\mathfrak{B}'$ a conditioning belief structure *of $\mathfrak{B}$ with respect to Δ, and $\rfloor_{\Delta}$ the corresponding* conditioning *operator.*

If $\mathfrak{B}'$ is a conditioning belief structure of $\mathfrak{B}$ with respect to Δ, then by definition every closed belief model of $\mathfrak{B}'$ is the conditional set of a closed belief model of $\mathfrak{B}$. Moreover, the following holds.

Proposition 11 *The conditional set of a consistent closed belief model of $\mathfrak{B}$ is a consistent closed belief model of $\mathfrak{B}'$.*

Marginalisation and its properties can be introduced and discussed analogously. Let $\gamma : \Omega \to \Delta$ be a surjective map. We can define the γ-lifting $\uparrow_{\gamma}^{\Omega} \colon \mathcal{L}' \to \mathcal{L}$ as $\uparrow_{\gamma}^{\Omega}(g) = g \circ \gamma$, and extend it to sets of gambles in the obvious way by setting $\uparrow_{\gamma}^{\Omega}(A) := \{\uparrow_{\gamma}^{\Omega}(g) \in \mathcal{L} \mid g \in A\}$. As γ induces a partition of Ω, by $\mathcal{L}_{\gamma}$ we denote gambles that are constant on each partition. Obviously $\uparrow_{\gamma}^{\Omega}$ defines an isomorphism between $\mathcal{L}_{\gamma}$ and $\mathcal{L}'$. Consider $\downarrow_{\Delta}^{\gamma} \colon \mathcal{S} \to \mathcal{S}'$ defined as $\downarrow_{\Delta}^{\gamma}(A) := (\uparrow_{\gamma}^{\Omega})^{-1}(A \cap \mathcal{L}_{\gamma})$.

[6] Clearly if b is a quasi-negation of a, a is also a quasi-negation of b.

[7] Notice that in a quasi-negative belief structure, b is a quasi-negation of a if and only if $\mathsf{F} \subseteq K(\{a, b\})$, and a falsum is the negation of a tautology (and vice-versa).

[8] In Section 2.1, we said that gambles are bounded real-valued functions on Ω. Note that, for finite Ω, any gamble g can be written as a vector of its n possible values (one for each $\omega \in \Omega$). Also notice that for $\mathbb{R}^n$, the order topology and the standard topology are equivalent.

Definition 12 *Let $\mathfrak{B} := (\mathcal{S}, \subseteq, K, C)$ be a GDT over $\mathcal{L}$ and $\mathfrak{B}' := (\mathcal{S}', \subseteq, K', C')$ a GDT over $\mathcal{L}'$. Assume that $\downarrow^{\gamma}_{\Delta}$ maps closed belief models to closed belief models and that $K \circ \uparrow^{\Omega}_{\gamma}$ is a c-embedding. Then we call $\mathfrak{B}'$ the* marginal belief structure *of $\mathfrak{B}$ with respect to γ, and $\downarrow^{\gamma}_{\Delta}$ the corresponding* marginalisation *operator.*

The γ-marginal set of $A \in \mathcal{S}$ is $\downarrow^{\gamma}_{\Delta} (A)$, and it is sometimes identified with $A \cap \mathcal{L}_{\gamma}$. If $\mathfrak{B}'$ is a marginal belief structure of $\mathfrak{B}$ with respect to γ, then by definition every closed belief model of $\mathfrak{B}'$ is the marginal set of a closed belief model of $\mathfrak{B}$. Moreover, by reasoning exactly as for Proposition 11, one can verify the following.

Proposition 13 *The marginal set of a consistent closed belief model of $\mathfrak{B}$ is a consistent closed belief model of $\mathfrak{B}'$.*

4. On Closures and Weak Orders

In this section we are going to connect closure operators and classifiers as well as derive a 'general' version of the separating hyperplane theorem, with in mind an application to GADTs.

Analogously to what done before, for simplicity we will focus on $\mathcal{L} := \mathbb{R}^n$, whose natural order is denoted by $\leq$, and $\mathcal{S} := \wp(\mathbb{R}^n)$. By $\mathbb{1}_n \in \mathcal{L}$ we denote the constant gamble $(1, \ldots, 1)$.

4.1. Dominance and Continuity

We first introduce some useful definitions.

Definition 14 *Let K be a closure operator over $\mathcal{S}$.*

- *A set $A \in \mathcal{S}$ satisfies dominance if, whenever $g \in A$ and $f \geq g$, it holds that $f \in A$; hence when every closed belief model $A \in \mathcal{S}_K$ satisfies dominance, we say that K satisfies dominance.*
- *We say that $A \in \mathcal{S}$ satisfies continuity (or is continuous) whenever the fact that $g + \epsilon \mathbb{1}_n \in A$ holds for every $\epsilon > 0$ implies that $g \in A$;[9] hence if K satisfies dominance, we say that K also satisfies continuity whenever every closed belief model $A \in \mathcal{S}_K$ is continuous.*

Notice that all closure operators in Proposition 2 satisfy dominance, whereas continuity is satisfied by the closure operators (4-6). Moreover, by definition every GDT and GADT satisfies dominance.

By Proposition 3, the collection of closure operators over $\mathcal{S}$ constitutes a complete lattice. It actually turns out that the intersection of any family of closure operators preserves both dominance and continuity. More specifically, the following result holds.

Proposition 15 *Let $\{K_i \mid i \in I\}$ be a family of closure operators over $\mathcal{S}$ satisfying dominance (resp. continuity). Then $K := \bigcap_{i \in I} K_i$ also satisfies dominance (resp. continuity). Stated otherwise, given a closure operator K satisfying dominance (resp. continuity), K is generated by $\{K_i \mid i \in I\}$ if and only if:*

1. *$K \leq K_i$, for all $i \in I$.*
2. *if $A \in \mathcal{S}_K$ and if $g \notin A$ then there exists $i \in I$ such that $g \notin K_i(A)$.*

In Proposition 15, neither *consistency* nor belief structures are mentioned. This representation result can be readily extended to the case of classical belief structures as follows:

Corollary 16 *Let $\mathfrak{B}$ be a classical belief structure (whose closure operator satisfies dominance, resp. continuity). Then a family $\{K_i \mid i \in I\}$ of closure operators (satisfying dominance, resp. continuity) generate K if and only if*

1. *$K \leq K_i$, for all $i \in I$.*
2. *if $A \in C$, then for every $g \notin A$ there exists $i \in I$ such that $g \notin K_i(A)$.*

Whenever $\mathfrak{B}$ is paraconsistent, the restriction of the second condition to consistent sets only in the proposition above is in general obviously not enough to enforce that K may be generated by the family $\{K_i \mid i \in I\}$, the counter example being, again, linear-desirability.

4.2. A Representation Result

A *weak order* $\succeq$ is a binary relation on a set $\mathcal{L}$ which is *transitive* and *total*[10]. Whenever $x \succeq y$ and $y \succeq x$, we write $x \approx y$ and say that x and y are order equivalent.

Definition 17 *We say that a weak order is* (weakly) order-preserving *if $f \geq g$ implies $f \succeq g$. An order-preserving weak order satisfies* order-continuity *whenever $f + \epsilon \mathbb{1}_n \succeq g$ for all $\epsilon > 0$ implies $f \succeq g$.*

We define the *support function* of a set $A \subseteq \mathcal{L}$ with respect to the order $\succeq$ simply as the collection of all order-equivalence infima of A with respect to $\succeq$:

$$s_{\succeq}(A) := \{h \in \underline{A} \mid h \succeq g, \forall g \in \underline{A}\}, \tag{7}$$

where $\underline{A} := \{g \in \mathcal{L} \mid \ f \succeq g, \ \ \forall f \in A\}$. We then define the *support half-space* of the set $A \neq \emptyset$ as

$$S_{\succeq}(A) := \{g \in \mathcal{L} \mid \ \ g \succeq f \ , \ \forall f \in s_{\succeq}(A)\}. \tag{8}$$

and set $S_{\succeq}(\emptyset) := \emptyset$. In particular notice that $S_{\succeq}(A) = \mathcal{L}$ whenever $s_{\succeq}(A) = \emptyset$, for $A \neq \emptyset$.

[9] To say that $A \in \mathcal{S}$ is continuous is tantamount to say that A is closed according to the order topology.

[10] $(\forall x, y \in \mathcal{L}$ either $x \succeq y$ or $y \succeq x$).

Remark 18 *In Section 5 we provide examples supporting the concept of a half-space. To briefly illustrate the intuition, we can consider standard almost linear desirability. In this case, whenever $0 \in A$, and thus $\mathcal{L}_0^+ \subseteq S_{\geq}(A)$, $S_{\geq}(A)$ may be considered equivalent to a maximal set defined by a total order.*

We can now prove the following result.

Proposition 19 *Let K be a closure operator over $\mathcal{S}$, which satisfies dominance (resp. continuity). Then there exists a family of order-preserving (resp. order-continuous) weak-orders such that:*

$$K(A) = \bigcap_{i \in \mathcal{I}} (S{\geq}_i(A) \cup \mathsf{T}), \tag{9}$$

for all $A \subseteq \mathcal{L}$. Conversely, for any family $\{\geq_i \mid i \in \mathcal{I}\}$ of order-preserving (resp. order-continuous) weak-orders, and a sequence $(X_i : i \in \mathcal{I})$ where each X_i satisfies dominance (and is continuous), the map $\kappa : \mathcal{S} \to \mathcal{S}$ defined as

$$\kappa(A) := \bigcap_{i \in \mathcal{I}} (S_{\geq_i}(A) \cup X_i) \tag{10}$$

is a closure operator that satisfies dominance (resp. continuity) and such that $\mathsf{T} = \bigcap_{i \in \mathcal{I}} X_i$.

Notice that, as for Proposition 15, in Proposition 19 neither *consistency* nor belief structures are mentioned.

Equations (9) and (10) of Proposition 19 tell us that any closure operator (satisfying dominance, resp. continuity) can be represented by a family of weak orders (satisfying dominance, resp. continuity), and that any family of weak orders (satisfying dominance, resp. continuity) generates a closure operator (satisfying dominance, resp. continuity), meaning in particular that any weak order (satisfying dominance, resp. continuity) can be equivalently seen as a closure operator (satisfying dominance, resp. continuity) via Equation (8).

As an immediate corollary of Propositions 15 and 19, we have that, given a closure operator K over $\mathcal{S}$ which satisfies dominance (continuity), a closed belief model $A \in \mathcal{S}_K$ and $g \notin A$, there exists a order-preserving (order-continuous) weak-order $\geq$ such that $g \notin S_{\geq}(A)$. As we will clarify in the next sections with some examples, this result might be understood as a 'generalisation' of the standard separating hyperplane theorem from convex geometry. For this reason, we call the closure operator $S_{\geq}(A)$ defined by the weak-order $\geq$, as in (8), a **binary (nonlinear) classifier**.

5. Particular Cases

The second part of Proposition 19 provides a way to construct a closure operator essentially as intersection of support half-spaces defined by a set of order-preserving weak-orders $\geq_i$. We will now discuss some particular cases focusing on GADTs, and therefore closure operators which satisfy both dominance and continuity, and such that sets T and X_i from Proposition 19 always coincide with $\mathcal{L}_0^+$. In particular, we will consider GADTs whose underlying closure operators are (4)–(6) from Proposition 2. Also, for simplicity in our examples we always assume that $0 \in A$. We first provide some useful definitions and results which allow us to connect weak-orders to utility functions.

Definition 20 *A utility function $u : \mathcal{L} \to \mathbb{R}$ is* non-decreasing *if $u(f) \geq u(g)$ for each $f \geq g$. It is said to be* order-continuous *whenever $u(f + \epsilon \mathbb{1}_n) \geq u(g)$ for each $\epsilon > 0$ implies $u(f) \geq u(g)$.*

Proposition 21 *Let $\geq$ be an order-preserving order-continuous weak-order on $\mathcal{L}$. Then there is a non-decreasing order-continuous utility function $u : \mathcal{L} \to \mathbb{R}$ that represents $\geq$ and vice versa, that is*

$$f \geq g \quad \textit{iff} \quad u(f) \geq u(g). \tag{11}$$

Proposition 21 provides a representation of $\geq_i$ via a utility function. Let u_i be the utility that represents $\geq_i$, we then note that (7) can be rewritten as:

$$s_{\geq_i}(A) := \sup_{h \in \underline{A}} u_i(h),$$

where $\underline{A} = \{g \in \mathcal{L} \mid \; u_i(f) \geq u_i(g), \;\; \forall f \in A\}$ and we set $\sup_{h \in \emptyset} u_i(h) = -\infty$, leading to the following equivalent definition of support half-space:

$$S_{\geq_i}(A) = \left\{ g \in \mathcal{L} \mid \;\; u_i(g) \geq \sup_{h \in \underline{A}} u_i(h) \right\}. \tag{12}$$

Therefore, we can build a closure operator by defining a suitable class of utility functions.

5.1. Linear Utility

A linear utility is a linear function of g. In particular, we will consider linear utility functions defined as

$$u_i(g) := p_i^\top g, \tag{13}$$

where $p_i \in \mathbb{R}^n$ is a probability vector. It is immediate to notice that, with this definition of the coefficients p_i, u is non-decreasing and, moreover, (order-)continuous.

The support half-space defined by u_i is:

$$S_{\geq_i}(A) := \left\{ g \in \mathcal{L} \mid \;\; u_i(g) \geq \sup_{f \in \underline{A}} u_i(f) \right\}. \tag{14}$$

It is worth noticing that when $c = \sup_{f \in \underline{A}} u_i(f) \leq 0$, then $\mathcal{L}_0^+ \subset \{g \in \mathcal{L} \mid \;\; u_i(g) \geq c\}$, meaning that the union of $S_{\geq_i}(A)$ as defined in (14) with $\mathcal{L}_0^+$ (as in the equations of

Proposition 19) is redundant. In particular, whenever $0 \in A$, $S_{\geq_i}(A)$ is equal to either a half-space in $\mathbb{R}^n$ or to $\mathbb{R}^n$.

From a standard result from convex geometry, we know that any closed convex set is generated by the intersection of the half-spaces that contain it. This is a particular case of Proposition 15 and holds for any closure operator whose closed belief models are convex.

About consistency, note the following, whose proof is trivial.

Lemma 22 *Given a set $A \subseteq \mathcal{L}$, the closed belief model $K(A) := \bigcap_{i \in \mathcal{I}} (S_{\geq_i}(A) \cup \mathcal{L}_0^+)$, with $S_{\geq_i}(A)$ defined by (14), is consistent if $\sup_{f \in \underline{A}} u_i(f) \geq 0$, for some $i \in I$.*

Example 1 *Consider the set of gambles $A = \{g_1 = (0,0), g_2 = (-1,2), g_3 = (-0.5,3), g_4 = (2,-1)\}$. Note that, since A is finite, $\sup_{h \in \underline{A}} p_i^\top f = \min_{h \in A} p_i^\top h$. Figure 1(left) shows the closure of A computed using $K_{\overline{co}}$. This closure operator is generated by the intersection of (14) for all the possible p_i as depicted in Figure 1(right) (the figure only shows 20 lines $\{g : p_i^\top g - \min_{f \in A} p_i^\top f = 0\}$). The intersection of the support half-spaces includes the non-negative gambles and excludes the negative ones. Note that, there exists a minimal number of support half-spaces that generate $K_{\overline{co}}(A)$, an algorithm to compute it is provided in [6].*

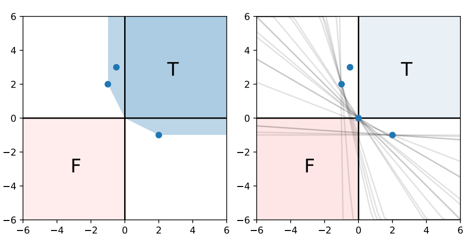

Figure 1: $K_{\overline{co}}(A)$ (left, blue region, including the border) and generating utilities (right). The coloured regions labelled T and F, represent the non-negative gambles (tautologies) and, respectively, negative ones (falsa).

Any closed belief model for $K_{\overline{po}}$ is a closed convex cone, which is convex and, therefore, generated by (14) as we will show in the following example.

Example 2 *Consider the set of gambles A in blue in Figure 2(left), which is a closed belief model for $K_{\overline{po}}$ (and $K_{\overline{co}}$). Figure 2(right) shows the support half-spaces that generate it for all the possible p_i (the figure shows the same 30 p_i as in the previous example). There are less than 30 lines, because for any support half-space in (14) such that $p_i^T g < 0$ for some $g \in A$, we have that $\sup_{h \in \underline{A}} p_i^T h = -\infty$ and the relative $S_{\geq_i}(A) = \mathbb{R}^n$.*

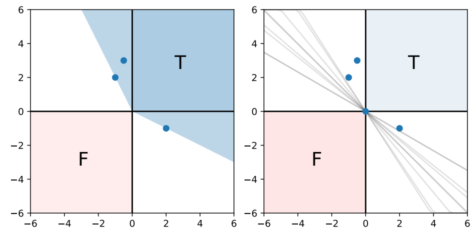

Figure 2: $K_{\overline{po}}(A)$ (left, blue region, including the border) and generating utilities (right).

As shown above, both $K_{\overline{po}}, K_{\overline{co}}$ can be generated by linear utilities. For $K_{\overline{co}}$, the intercept term in the half-space $\{g : p_i^\top g \geq \sup_{f \in \underline{A}} p_i^\top f\}$ allows us to consider situations in which a subject judges rewards of gambles having limited financial resources. The maximum $|\sup_{f \in \underline{A}} p_i^\top f|$ for $p = [1,0]$ or $p = [0,1]$ provides a measure of the budget of the subject [16, 6] (for $K_{\overline{po}}$ this term is equal to ∞ and indeed there are no budget constraints in linear-desirability).

5.2. Chebyshev Utility

The Chebyshev utility [14] is defined as

$$u_i(g) := \max_{j=1,\ldots,n} p_{ij}(g_j - c_{ij}), \tag{15}$$

with $p_i, c_i \in \mathbb{R}^n$, $p_i \geq 0$, $\sum_{j=1}^n p_{ij} = 1$. As in the previous Section 5.1, the support half-space defined by u_i is then given by Equation (14). As with Lemma 22, it is immediate to verify the following.

Lemma 23 *Given a set $A \subseteq \mathcal{L}$, the closed belief model $K(A) := \bigcap_{i \in \mathcal{I}} (S_{\geq_i}(A) \cup \mathcal{L}_0^+)$, with $S_{\geq_i}(A)$ defined by Equation (14), is consistent if $A \cap \mathcal{L}^< = \emptyset$.*

We are going to show with an example that, for finite sets A, the closed belief model $K_{\overline{or}}(A)$ is generated by Equation (14) for all the possible values of p_i, c_i. This is a known result in multi-objective optimisation, see for instance [14]. Note in fact that, the border of $K_{\overline{or}}(A)$ is the Pareto frontier defined by A.

Example 3 *Consider again the set of gambles $A = \{g_1 = (0,0), g_2 = (-1,2), g_3 = (-0.5,3), g_4 = (2,-1)\}$. Figure 3(top-left) shows the closure of A computed using $K_{\overline{or}}$. This closure operator is generated by the intersection of Equation (14) for all the possible p_i, c_i as depicted in Figure 3(top-right). The figure only shows the zero level curve of 45 'support half-spaces'. Their intersection includes the non-negative gambles and excludes the negative ones. An example of a zero-level curve for one of these support half-spaces is shown in Figure 3(bottom).*

Note that, Proposition 15 does not hold for $K_{\overline{or}}$ if we consider the support half-spaces (14), because $K_{\overline{or}}(A)$ are in general non-convex sets, as depicted in Figure 4.

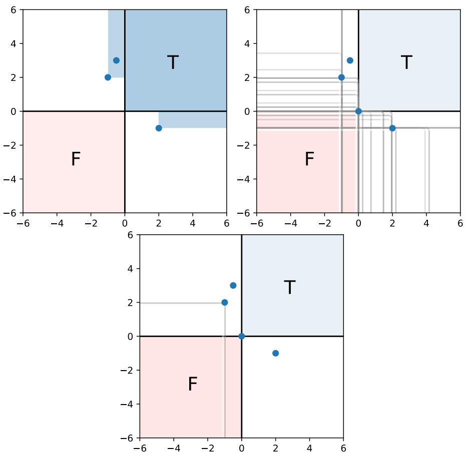

Figure 3: $K_{\overline{or}}(A)$ (left, blue region, including the border) and generating utilities (right). An example of a line defining a support half-space (bottom).

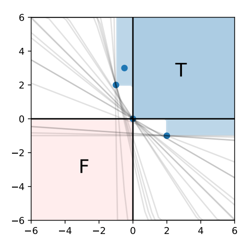

Figure 4: $K_{\overline{or}}(A)$ (blue region including the border) and linear utilities.

5.3. L_d^d Utility

In the previous two cases, we defined utility functions whose support half-spaces generate known closure operators. We can instead use directly (10) to define novel closure operators such as the ones defined via L_d^d utility functions:

$$u_i(g) := p_i^\top (g - c_i)^d, \tag{16}$$

for $d = 1, 3, 5, \ldots$, where $p_i \in \mathbb{R}^n$ is a probability vector and $c_i \in \mathbb{R}^n$ is a reference point.

Proposition 24 *The utility in (16) is non-decreasing and order-continuous.*

Example 4 *Consider the same set of gambles A as in the previous examples. We generated 10 linear utilities by randomly sampling p_i, c_i. Figure 5 shows the lines $\{g : p_i^\top(g-c_i)^d = \min_{f\in A} p_i^\top(f-c_i)^d = \sup_{f\in \underline{A}} p_i^\top(f-c_i)^d\}$ for $d = 3$ and, respectively, $d = 31$. The intersection of the support half-spaces defined by these lines includes the non-negative gambles and excludes the negative ones.*

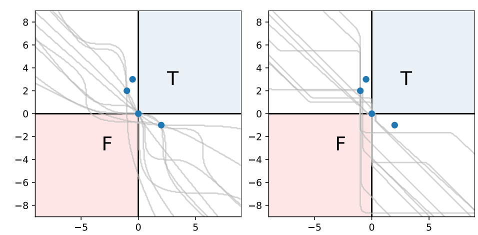

Figure 5: Closure operator defined by L_d^d utilities $d = 3$ (left) and $d = 31$ (right).

6. Utility Defined by Lower (Upper) Previsions

Any closure operator satisfying dominance defines a utility function as follows:

$$u(g) := \{\sup \lambda \mid g - \lambda \mathbb{1}_n \in K(A)\}.$$

In the theory of desirable gambles, the above utility function is usually denoted as $u(g) = \underline{P}_{K(A)}(g)$ and called the *lower-prevision* of g defined by $K(A)$. If K satisfies continuity, it holds for the support half-space that:

$$\left\{g \in \mathcal{L} \mid \underline{P}_{K(A)}(g) \geq \inf_{f\in K(A)} \underline{P}_{K(A)}(f)\right\} = K(A), \tag{17}$$

see [15, Prop. 10]. The support half-space (17) generates $K(A)$. For finite A, as show in [6], (17) can be represented by a piecewise-affine binary classifier for $K_{\overline{po}}, K_{\overline{co}}, K_{\overline{or}}$. What is the relationship between the lower previsions generated by $K_{\overline{po}}, K_{\overline{co}}, K_{\overline{or}}$ and the linear and Chebyshev utilities discussed previously?

The relationship is similar to that between credal sets and lower previsions in linear-almost desirability. For instance, if we consider $K_{\overline{or}}$, then $\{g \mid \underline{P}_{K(A)}(g) = 0\}$ represents the Pareto frontier defined by A (this line defines the level-curves of the lower-prevision utility). Each $S_{\geq_i}(A)$, which generates $K(A)$ as the intersection of (14), can instead be thought of as a scalarisation of the Pareto front, that is a different attempt of the subject to compare the gambles in $K(A)$.

As discussed previously, linear-desirability theory is a negative belief structure, where a negation for a given gamble g coincides with $-g$ (or $-g - \epsilon\mathbb{1}_n$ with $\epsilon > 0$ for almost desirability). It is immediate to verify the following.

Proposition 25 *Let K be a closure operator whose closed belief models are convex. Then $-g - \epsilon\mathbb{1}_n$ is a quasi-negation of g.*

A result connected to the above proposition was proven in [15, Prop.8]. In linear almost-desirability theory, the gamble $-g - \epsilon\mathbb{1}_n$ is always a quasi-negation of g. This is not the

case for GADT (and also GDT for $\epsilon = 0$). For instance, the set $\mathcal{L} \setminus \mathcal{L}^{<}$ (which is not convex) is a consistent set for the GADT defined by $K_{\overline{or}}$ and includes $-g - \epsilon \mathbb{1}_n, g$, therefore, violating the (C-rule). This means that for instance upper previsions, which in linear-desirability theory are defined using a negation, and thus a fortiori a quasi-negation, of a gamble $\overline{P}(g) = \{\inf \lambda \mid \lambda \mathbb{1}_n - g \in K(A)\}$, are in general meaningless in GADT (and GDT). Indeed, in GADT/GDT, it may hold that $\overline{P}(g) < \underline{P}(g)$ [15, Sec.3].
A way to address this issue would be to only consider closure operators for which $-g - \epsilon \mathbb{1}_n$ is at least a quasi-negation of g (e.g., the convex ones) meaning that, for any gamble g, the gambles $-g(-\epsilon \mathbb{1}_n), g$ cannot be both in $K(A) \in C$.

7. Discussions

General desirability theory for objects. In Section 3, we defined three properties that the closure operator of a GADT over a topological ordered vector space must satisfy: (i) $\mathsf{T} := \mathcal{L}_0^+$; (ii) $\mathsf{F} := \mathcal{L}^{<}$; (iii) K satisfies dominance. Conditions (i) and (iii) are properties of the closure operators whereas condition (ii) concerns the definability of consistency, and all three may be motivated by the considered underlying ordered structure. However, results such as Propositions 15 and 19 might remain valid if we disregard part of the specific structure of the underlying language and drop/change some (all) of these three properties.[11] This is the setting discussed in [11], where we have simply a *closure operator on a set of objects (alternatives)*. This setting was originally studied by [12] for the representation of preferences over menus, and generalised to convex closure operators, that is closure operators that satisfy the general anti-exchange property, by [21]. An alternative approach that aligns with this direction is [8], which focuses on desirability over objects ('things').

Preferences and desirability. In [15], the authors extend the notion of preference to GDT in two ways: f is strictly preferred to g if (i) $\underline{P}(f - g) > 0$; (ii) using a decision criterion such as Γ-maximin ($\underline{P}(f) > \underline{P}(g)$), Γ-maximax ($\overline{P}(f) > \overline{P}(g)$), interval dominance ($\underline{P}(f) > \overline{P}(g)$) etc.

Approach (ii) defines transitive preferences (and so linear) whenever the upper probability is well-defined ($-g$ is the negation of g). Therefore, it cannot capture nontransitive preferences. Approach (i) only considers the difference gamble $f - g$ and, therefore, cannot distinguish pairs of gambles f_1, g_1 and f_2, g_2 such that $f_1 - g_1 = f_2 - g_2$.

For this reason, these two approaches cannot model *context based preferences*, where the preference depends nonlinearly on the pair of gambles to be compared. This means that nonlinear preference theory and general (nonlinear) desirability theory cannot be derivable from each other in general. We believe that a consistent way to model nonlinear preferences is to define a closure operator directly on pair of gambles – this generalises the approach derived by [22, 10] to model non-transitive preferences using a two-argument function over a pair of alternatives (note that, their function corresponds to a binary classifier). We believe that general (nonlinear) choice functions should be defined in the same way, that is directly via a closure operator. This is indeed one of the applications of the approach recently proposed by [8].

Robustness and complexity. Inference in linear-almost-desirability can be formulated as a linear programming problem. This then determines the computational complexity in strong and epistemic credal networks [13]. Can suitable choices of closure operators decrease the computational complexity of these inferences (and provide an approximation)? In algorithm rationality, for polynomial gambles, we showed that $K_{\overline{po}} \leq K_{SOS}$ (for the closed belief models belonging to the consistent set of $K_{\overline{po}}$) [4] and used K_{SOS} to derive a **P**-time approximation of inferences in linear-almost-desirability.

Finally, since GADT is useful to model nonlinear utilities in economics, it would be interesting to investigate the application of GADT to dynamical systems [1, 2].

Appendix A. Proofs

Proof of Proposition 3. The proofs of both claims are exactly the same as for Proposition 1.5.1 (A) and Corollary 1.5.4 in [25].

Proof of Proposition 11. Let $A \in C$ and assume $A' := \rfloor_\Delta(A) \notin C'$. We know that $A' \in \mathcal{S}'_{K'}$. Since $K \circ \lceil_\Delta$ is a c-embedding, $K(\lceil_\Delta(A')) = K(\lceil_\Delta(\rfloor_\Delta(A))) \notin C$. However we have that $\lceil_\Delta(\rfloor_\Delta(A)) \subseteq A$, and therefore $K(\lceil_\Delta(A')) \subseteq A$, which means that $A \notin C$, a contradiction.

Proof of Proposition 15. It is immediate to verify that dominance (resp. continuity) is preserved under arbitrary intersection. We conclude by noticing that the second claim is just a rephrasing of the first one.

Proof of Proposition 19. Given a weak-order $\geq$ and a set $A \in \mathcal{S}$, $s_{\geq}(A)$ is an equivalence class: $f \approx_C g$, for every $f, g \in s_{\geq}(A)$. Given this, from now on, depending on the context, we sometimes treat it as a set or as a 'prototypical' element. That is, when we write $f \geq s_{\geq}(A)$, what we 'really' mean is that $f \geq g$, for every $g \in s_{\geq}(A)$.

We adapt the proof in [11, Prop. 2]. The main differences are due to the different definitions (7) and (8). First, we prove that if K is a closure operator which satisfies dominance (and continuity), then there exists a set of order-preserving

[11] We can simply remove the dominance (resp. continuity) condition and we can consider a weak-order/utility which is not order-preserving/non-decreasing. [5] includes an example where the dominance property is dropped.

(and order-continuous) weak-orders that generate it. Let C be a closed belief model in $\mathcal{S}_K$, we can associate to C an order-preserving (and order-continuous) weak-order $\geq_C$ as follows:

$$\begin{aligned} &f \approx_C g \text{ if } f, g \in C,\\ &f \approx_C g \text{ if } f, g \notin C, \\ &f >_C g \text{ if } f \in C, g \notin C. \end{aligned} \tag{18}$$

Obviously $\geq_C$ is transitive and total. It can be verified that it is also order preserving. In fact, let $f \geq g$. Assume $f \notin C$. Since K satisfies dominance, it holds that $g \notin C$, meaning that by definition $f \approx_C g$ and thus $f \geq_C g$. Assume $f \in C$. If $g \in C$, then as before $f \geq_C g$. If $g \notin C$, then by definition $f >_C g$ and thus $f \geq_C g$. Similarly one can verify continuity. Assume first $f + \epsilon \mathbb{1}_n \approx_C g$, with $f + \epsilon \mathbb{1}_n, g \in C$, for every $\epsilon > 0$. Since K is continuous, we have that $f \in C$ and therefore $f \approx_C g$. Consider now the case when $f + \epsilon \mathbb{1}_n >_C g$, with $f + \epsilon \mathbb{1}_n \in C$ for every $\epsilon > 0$, but $g \notin C$. Again, since K is continuous, we have that $f \in C$ and therefore $f >_C g$. Finally, assume $f + \epsilon \mathbb{1}_n \approx_C g$, with $f + \epsilon \mathbb{1}_n, g \notin C$, for every $\epsilon > 0$. Suppose $f \in C$. Since $f + \epsilon \mathbb{1}_n > f$, for $\epsilon > 0$ and K satisfies dominance, $f + \epsilon \mathbb{1}_n \in C$. A contradiction. Therefore $f \notin C$ and $f \approx_C g$.

Given $A \in \mathcal{S}$, we write $T_{\geq_C}(A) := S_{\geq_C}(A) \cup \mathsf{T}$. Then, notice that for every $A \in \mathcal{S}$ we have that

$$T_{\geq_C}(A) = \begin{cases} \mathsf{T} & \text{if } A = \emptyset \\ C & \text{if } \emptyset \subsetneq A \subseteq C \\ \mathcal{L} & \text{otherwise.} \end{cases} \tag{19}$$

By definition $T_{\geq_C}(A) \supseteq \mathsf{T}$, for every $A \in \mathcal{S}$. Moreover, whenever $A = \emptyset$ it holds that $K(A) = \mathsf{T} = T_{\geq_C}(A)$. Assume $\emptyset \subsetneq A \subseteq C$. Then $K(A) \subseteq K(C) = C$, and, therefore, $K(A) \subseteq T_{\geq_C}(A) = C$. Finally, assuming $A \nsubseteq C$ yields $K(A) \subseteq T_{\geq_C}(A) = \mathcal{L}$.
Therefore, we have proven that $K(A) \subseteq \bigcap_{C \in \mathcal{S}_K} T_{\geq_C}(A)$, where the index set I is the set of closed belief models of K. Let $\geq_{K(A)}$ be the weak-order defined by the closed belief model $K(A)$. Since $A \subseteq K(A)$ from (19) we derive that $K_{\geq_{K(A)}}(A) = K(A)$. Thus, $\bigcap_{C \in \mathcal{S}_K} K_{\geq_C}(A) \subseteq K_{\geq_{K(A)}}(A) = K(A)$. Therefore, $K(A) = \bigcap_{C \in \mathcal{S}_K} K_{\geq_C}(A)$.

Let $\{\geq_i \mid i \in \mathcal{I}\}$ be a family of order-preserving (resp. order-continuous) weak-orders with X_i satisfying dominance for each $i \in \mathcal{I}$ (and resp. X_i is continuous). To prove (10), we have to show that κ, defined as $\kappa(A) := \bigcap_{i \in \mathcal{I}} (S_{\geq_i}(A) \cup X_i)$ for $A \in \mathcal{S}$, is a closure operator. As before, given $i \in \mathcal{I}$ and $A \in \mathcal{S}$, we write $T_{\geq_i}(A) := S_{\geq_i}(A) \cup X_i$. By definition, for every A and index i, we have that $A \subseteq T_{\geq_i}(A)$. Therefore, κ is extensive. Let $A \subseteq B$. If $A = \emptyset$, then $T_{\geq_i}(A) = X_i \subseteq T_{\geq_i}(B)$. Hence, let $A \neq \emptyset$. Assume that $s_{\geq_i}(B) \neq \emptyset$. Then $s_{\geq_i}(A) \neq \emptyset$ and it holds that $s_{\geq_i}(A) \geq_i s_{\geq_i}(B)$, meaning that $T_{\geq_i}(A) \subseteq T_{\geq_i}(B)$. Assume that $s_{\geq_i}(B) = \emptyset$. This means that $T_{\geq_i}(B) = \mathcal{L}$ and $T_{\geq_i}(A) \subseteq T_{\geq_i}(B)$. Therefore, κ is monotonic.

In order to prove idempotence we have to show that $\kappa(\kappa(A)) \subseteq \kappa(A)$ (because we already know that κ is monotonic). Note that if $g \in \kappa(\kappa(A))$, by definition of κ, then $g \in T_{\geq_i}(\kappa(A))$ for every i. Therefore, $g \geq_i f$ for all $f \geq_i s_{\geq_i}(\kappa(A))$, for $i \in \mathcal{I}$.

Assume that $f \in s_{\geq_i}(\kappa(A))$, this implies that $f \in \kappa(A) = \bigcap_{i \in \mathcal{I}} T_{\geq_i}(A) \subseteq T_{\geq_i}(A)$. Therefore, since $g \geq_i f$ and $f \in T_{\geq_i}(A)$, then we have that $g \in T_{\geq_i}(A)$. Hence, $g \in \bigcap_{i \in \mathcal{I}} T_{\geq_i}(A) = \kappa(A)$.
We verify that κ satisfies dominance. To do so, since it is immediate to verify that dominance is preserved under arbitrary intersections, it is enough to check that for every i and every $A \subset \mathcal{L}$, whenever $g \in T_{\geq_i}(A)$ and $f \geq g$, then $f \in T_{\geq_i}(A)$. Hence, let us fix an index i and a set A. Remember that we assume that X_i satisfies dominance and is continuous. Let $f \in \mathcal{L}$ such that $f \geq g$, for some $g \in T_{\geq_i}(A)$. By order-preservation, $f \geq_i g$. We have some cases to consider. Assume $g \in X_i$, then since X_i satisfies dominance it holds that $f \in X_i$. Let us assume now that $g \in S_{\geq_i}(A)$. First notice that whenever $s_{\geq_i}(A) = \emptyset$, it holds that $f \in \mathcal{L} = S_{\geq_i}(A)$. Assume $s_{\geq_i}(A) \neq \emptyset$. Since $g \in S_{\geq_i}(A)$, it holds that $g \geq_i s_{\geq_i}(A)$. Hence $f \geq_i g \geq_i s_{\geq_i}(A)$, and therefore $f \in S_{\geq_i}(A)$. Finally, since each $\geq_i$ is order-continuous, κ satisfies continuity.

Proof of Proposition 21. Let u be an a non-decreasing order-continuous utility function, define the order $f \geq_u g$ if $u(f) \geq u(g)$, then $\geq_u$ is clearly an order-preserving continuous weak order. For the other direction, let $\geq$ be an order-preserving weak-order and let $D = \{\alpha \mathbb{1}_n \mid \alpha \in \mathbb{R}\}$ (the diagonal of the positive and negative orthant in $\mathbb{R}^n$). Define $U_g = \{d \in D \mid d \geq g\}$ and $L_g = \{d \in D \mid g \geq d\}$. Note that we have $L_g \cup U_g = D$, because $\geq$ is complete. Since $\geq$ is order-preserving, then $U_g, L_g \neq \emptyset$. Moreover, since D is connected, it must exist $\alpha_g \in \mathbb{R}$ such that $\alpha_g \mathbb{1}_n \in U_g \cap L_g$, that is $\alpha_g \mathbb{1}_n \equiv g$. It is easy to see that if we define $u(g) = u(\alpha_g \mathbb{1}_n) = \alpha_g$ then u represents $\geq$. Indeed, if $f \geq g$ if and only if $\alpha_f \mathbb{1}_n \sim f \geq g \sim \alpha_g \mathbb{1}_n$ and $u(f) = \alpha_f \geq \alpha_g = u(g)$. Moreover, it is easy to prove that if $\geq$ is also order-continuous then u is order continuous because $\alpha_g \mathbb{1}_n + \epsilon \mathbb{1}_n \geq \alpha_g \mathbb{1}_n$.

Proof of Proposition 25. Since any closed belief model C is convex, if $g, -g - \epsilon \mathbb{1}_n$ are both in it, then for $w = 0.5$ we have that $wg + (1 - w)(-g - \epsilon \mathbb{1}_n) = -0.5\epsilon \mathbb{1}_n \in \mathcal{L}^<$ showing that A is not consistent.

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