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A Theory of Desirable Things

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Abstract

Inspired by the theory of desirable gambles that is used to model uncertainty in the field of imprecise probabilities, I present a theory of desirable things. Its aim is to model a subject's beliefs about which things are desirable. What the things are is not important, nor is what it means for them to be desirable. It can be applied to gambles, calling them desirable if a subject accepts them, but also to pizzas, calling them desirable if my friend Arthur likes to eat them. Regardless of the particular things that are considered, inference rules are imposed by means of an abstract closure operator, and models that adhere to these rules are called coherent. I consider two types of models, each of which can capture a subject's beliefs about which things are desirable: sets of desirable things and sets of desirable sets of things. A crucial result is that the latter type can be represented by a set of the former.

Keywords: desirable things, coherence, finite coherence, representation theorem, closure operator

1. Introduction

The theory of imprecise probabilities [2, 24] is often thought of as a theory of partially specified probabilities, which involves manipulating sets of probabilities and their lower and upper expectations. Its mathematical underpinnings, however, are provided by an underlying theory of sets of desirable gambles [24, 26, 25, 4, 19]: sets of gambles—rewards with an uncertain payoff—that a subject finds desirable, in the sense that she prefers those gambles to the status quo. In particular, well known imprecise probability models such as credal sets, lower and upper expectations, partial preference oderings, belief functions and lower and upper probabilities, all correspond to special cases [25]. Other models, such as choice functions, have even more expressive power [21], but these too have recently been incorporated into the theory of desirable gambles, through their equivalence with sets of desirable sets of gambles [9, 10, 23], a new type of desirability model that extends the theory of sets of desirable gambles to allow for a notion of disjunction.

The main contribution of this paper is to show that many of the central ideas behind the theory of desirable gambles, both in its original and generalised form, are not constrained to the context of gambles. We do this by putting forward a theory of desirable things. As we will see, these things can be, quite literally, anything, even pizzas. All that is needed to develop the theory is an inference mechanism that allows us to infer the desirability of things from that of other things; we do this by means of an abstract closure operator. In the case of desirable gambles, this would typically be closure with respect to positive linear combinations. The point of the paper, however, is that any other closure operator can be used instead, and that the gambles can instead by arbitrary things. This allows us to generalise the notion of a coherent set of desirable gambles to that of a coherent set of desirable things, and similarly for coherent sets of desirable sets of gambles and coherent sets of desirable sets of things. Furthermore, and perhaps most surprisingly, we show that the connection between these two types of models also extends from gambles to things.

Proofs of all results are available in an extended online version [7], which also contains more context and discussion and at times slightly more general results, but otherwise essentially presents the same material.

2. Desirable Things

Let $\mathcal{T}$ be a set containing all things whose desirability we wish to model. We'll use $\mathcal{P}(\mathcal{T})$ to denote its powerset: the set of all sets of things. A crucial feature of our framework is that it does not matter what the things in $\mathcal{T}$ are, nor what it means for them to be desirable. Desirability is simply an abstract feature that these things may or may not have, and the goal is to model a subject's beliefs about which of the things in $\mathcal{T}$ are desirable. In particular, we put forward two different types of models that can be used for this purpose.

The first—and most simple—model is a *set of desirable things* $D \subseteq \mathcal{T}$, which we will sometimes refer to as an SDT. As the terminology suggests, this is simply a set containing things that our subject deems desirable. No exhaustivity claim is made though: the model does not claim that the things in D are the only desirable ones; it simply says that every thing t in D is desirable.

Example 1 *Let $\mathcal{T}$ be the set of all types of pizza, and call a pizza desirable if my friend Arthur likes to eat it. Let the subject who's beliefs we are modelling be myself. A set of desirable things—in this case, a set of desirable pizzas—is* *then simply a set $D \subseteq \mathcal{T}$ of pizzas that I think Arthur likes to eat. I might for example provide the set of desirable pizzas $D = \{\text{peperoni}, \text{meatballs}\}$, which expresses that I think that Arthur likes to eat pizza peperoni and pizza meatballs.*

The second model is a *set of desirable sets of things* $K \subseteq \mathcal{P}(\mathcal{T})$. Rather than use desirable things as its elements, this model instead considers *desirable sets of things*, which are *sets that contain at least one desirable thing*. For ease of reference, we will often simply speak of *desirable sets*, leaving it implicit that they are actually desirable sets of things, and then refer to K as a set of desirable sets[1] or, even shorter, as an SDS. Again, no exhaustivity claim is made: there might be desirable sets that are not in K. The model simply—and only—says that every set T in K is deemed desirable by the subject who's beliefs we are modelling, meaning that he thinks that every set $T \in K$ contains at least one desirable thing. This second type of model is more complex, but also has more expressive power. The following two examples provide a first illustration.

Example 2 *Continuing with the pizza example, consider a situation where I remember that Arthur very much likes either pizza* peperoni and olives*, or pizza* chicken and olives*, but that I have forgotten which of these two he likes. Using our abstract terminology, this means that at least one of these two pizzas is desirable. I would like to include this piece of information in my model. With sets of desirable things, this is not possible. Sets of desirable sets, however, can. It suffices to assess that the set* $T = \{\text{peperoni and olives}, \text{chicken and olives}\}$ *is desirable, meaning that it contains at least one desirable pizza. I could for example put forward the set of desirable sets*

$$K = \big\{\{\text{peperoni}\}, \{\text{meatballs}\}, \{\text{peperoni and olives}, \text{chicken and olives}\}\big\},$$

which contains three desirable sets. The first two express the same information as that in Example 1, while the second adds the new disjunctive statement.

Example 3 *Let $\mathcal{T}$ be some set of propositions and let desirable propositions be propositions that are true. Consider now three propositions p_1, p_2 and p_3 in $\mathcal{T}$. I furthermore know that at least two of these propositions are true, but I don't know which two are true. I can model this information with the following set of desirable sets (of propositions):* $K = \big\{\{p_1, p_2\}, \{p_2, p_3\}, \{p_1, p_3\}\big\}$. *It expresses that if I remove any of the three considered propositions, then—since at least two of the propositions are true—the set of the remaining two propositions will contain at least one proposition that is true.*

[1] If the things are sets themselves, this is a bad idea, as the distinction between sets of desirable things and sets of desirabe sets then becomes unclear. In such a case, it seems preferable to not use the shorthand sets of desirable sets, and to always speak of sets of desirable sets of things.

3. Coherence

For many sets of things $\mathcal{T}$, and many notions of desirability, it makes sense to impose rules that statements about desirability should adhere to. One might think of them as rationality constraints. We will consider three such rules here. The first rule puts forward a set $A_{\text{not}} \subseteq \mathcal{T}$ of things that should never—on grounds of rationality—be desirable:

R_{not}. The things in A_{not} are not desirable.

The second puts forward a set $A_{\text{des}} \subseteq \mathcal{T}$ of things that should always be desirable:

R_{des}. The things in A_{des} are desirable.

Example 4 *Continuing with the pizza example, we might want to impose, on grounds of rationality, that pizza Hawaii should never be desirable and that pizza margherita should always be desirable. To do so, it suffices to let* $A_{\text{not}} = \{\text{Hawaii}\}$ *and* $A_{\text{des}} = \{\text{margherita}\}$

Our third rule builds in an inference mechanism, expressed by means of a closure operator cl.

Definition 1 *A map* $\text{cl}\colon \mathcal{P}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$ *is a closure operator*[2] *if*

cl_1. $A \subseteq \text{cl}(A)$ *[extensive]*
cl_2. *if* $A \subseteq B$, *then* $\text{cl}(A) \subseteq \text{cl}(B)$ *[monotone]*
cl_3. $\text{cl}(\text{cl}(A)) = \text{cl}(A)$ *[idempotent]*
cl_4. $\text{cl}(\emptyset) = \emptyset$

These conditions are slightly stronger than what is usually required of a closure operator. In particular, the last condition, stating that the closure of the empty set should be empty, is typically not imposed. We choose to impose it anyway because it better corresponds to the interpretation that we want to attach to such a closure operator, which goes as follows: for every set of things A, we take $\text{cl}(A)$ to contain all the things whose desirability can be inferred from the desirability of the things in A:

R_{cl}. If the things in $A \subseteq \mathcal{T}$ are desirable, then so are the things in $\text{cl}(A)$.

I invite the reader to observe that for an operator that has this interpretation, the properties in Definition 1 make perfect sense. In particular, cl_4 imposes that an empty assessment does not lead to meaningfull inferences.[3]

[2] Similar operators are also used in abstract logic. An operator that satisfies cl_1–cl_3 is then called a consequence operator [28], and a consequence operator that additionally satisfies cl_4 is called axiomless [31].

[3] If we were to drop axiom cl_4, the theory that we are about to develop would still work [15], provided that axiom K_5 further on is imposed for $\mathcal{A} = \emptyset$ as well. We find this less intuitive though. It furthermore would not yield a more expressive theory, since drawing inferences from nothing—like from the empty set—is already built into the theory as it is: that is exactly what axioms D_2 and K_4 are for. We do not consider this to be a form of inference though, and therefore prefer to separate this from the axioms involving the closure operator cl, by requiring that $\text{cl}(\emptyset) = \emptyset$.

Example 5 *Going back to our pizza example, a possible inference mechanism could be that if pizza* cheese *is deemed desirable, then every desirable pizza will continue to be desirable if we replace its crust with a cheese crust. This corresponds to the closure operator* cl *that, for every set* $A \subseteq \mathcal{T}$ *of pizzas returns the closure*

$$\mathrm{cl}(A) := A \cup \{\text{pizza with cheese crust}\colon \text{pizza} \in A\}$$

if $\text{cheese} \in A$ *and* $\mathrm{cl}(A) := A$ *otherwise. The closure of* $A = \{\text{meatballs}, \text{cheese}\}$ *would then be* $\mathrm{cl}(A) = A \cup \{\text{meatballs with cheese crust}, \text{ cheese with cheese crust}\}$.

Example 6 *For another example, let* $\mathcal{O}$ *be some set of options, let* $\mathcal{T} = \mathcal{T}_{\mathcal{O}} := \{(o_1, o_2)\colon o_1, o_2 \in \mathcal{O}\}$ *be the set of all ordered pairs of such options, and call a pair* $(o_1, o_2) \in \mathcal{T}$ *desirable if our subject prefers* o_1 *over* o_2*. In that context, one might want to impose transitivity. To do so, we can let* cl *to be the operator* trans *that, for every set of preferences* $A \subseteq \mathcal{T}$*, returns the closure*

$$\mathrm{trans}(A) := \{(o_0, o_n)\colon o_i \in \mathcal{O} \text{ for all } 0 \leq i \leq n, \\ (o_{i-1}, o_i) \in A \text{ for all } 1 \leq i \leq n\} \quad (1)$$

of A *with respect to transitivity. So* $\mathrm{trans}(A)$ *contains the preferences that can be inferred from the ones in* A *by applying transitivity. The closure of* $A = \{(o_1, o_2), (o_2, o_3)\}$*, with* o_1, o_2, o_3 *all different, would then for example be* $\mathrm{trans}(A) = A \cup \{(o_1, o_3)\}$.

We will call a model coherent if it adheres to the rules $\mathrm{R_{not}}$, $\mathrm{R_{des}}$ and $\mathrm{R_{cl}}$—and, in the case of sets of desirable sets, if it is furthermore compatible with our interpretation for a desirable set. For sets of desirable things, this is straightforward and self-explanatory.

Definition 2 *A set of desirable things* D *is* coherent *if*

D_1. $A_{\mathrm{not}} \cap D = \emptyset$

D_2. $A_{\mathrm{des}} \subseteq D$

D_3. $\mathrm{cl}(D) = D$

We let $\mathbf{D}$ *be the set of all coherent sets of desirable things.*

For sets of desirable sets, we require a new piece of machinery: for any subset $\mathcal{A}$ of $\mathcal{P}(\mathcal{T})$, we let

$$\mathcal{S}_{\mathcal{A}} := \big\{\{t_A\colon A \in \mathcal{A}\}\colon t_A \in A \text{ for all } A \in \mathcal{A}\big\}$$

be the set of all selections from $\mathcal{A}$. Any such selection $S \in \mathcal{S}_{\mathcal{A}}$ is a set obtained by selecting from each $A \in \mathcal{A}$ a single thing t_A; that is, $S = \{t_A\colon A \in \mathcal{A}\}$. The set $\mathcal{S}_{\mathcal{A}}$ is simply the set of all sets S that can be obtained in this way.

Example 7 *Going back to the pizza example, let* $\mathcal{A} = \{A_1, A_2\}$*, with* $A_1 = \{\text{peperoni}, \text{meatballs}\}$ *and* $A_2 = \{\text{peperoni}, \text{cheese}\}$*. In that case,* $t_{A_1} \in A_1$ *can be either* peperoni *or* meatballs*, whereas* $t_{A_2} \in A_2$ *can be* peperoni *or* cheese*. Each of the four possible combinations corresponds to a selection, so we find that the set of selections that corresponds to* $\mathcal{A}$ *is*

$$\mathcal{S}_{\mathcal{A}} = \big\{\{\text{peperoni}\}, \{\text{peperoni}, \text{cheese}\}, \\ \{\text{meatballs}, \text{peperoni}\}, \{\text{meatballs}, \text{cheese}\}\big\}.$$

Using this notion of selections, we now define coherence for sets of desirable sets as follows.

Definition 3 *A set of desirable sets (of things)* K *is* coherent *if*

K_1. $\emptyset \notin K$

K_2. *if* $A \subseteq B$ *and* $A \in K$*, then also* $B \in K$

K_3. *if* $A \in K$ *then also* $A \setminus A_{\mathrm{not}} \in K$

K_4. $\{t\} \in K$ *for all* $t \in A_{\mathrm{des}}$

K_5. *if* $\emptyset \neq \mathcal{A} \subseteq K$ *and, for all* $S \in \mathcal{S}_{\mathcal{A}}$, $t_S \in \mathrm{cl}(S)$*, then* $\{t_S\colon S \in \mathcal{S}_{\mathcal{A}}\} \in K$

We let $\mathbf{K}$ *be the set of all coherent sets of desirable sets.*

The first two of these axioms follow from the fact that K consists of desirable sets. Indeed, since a set is desirable if it contains at least one desirable thing, it follows at once that the empty set cannot be desirable (K_1) and that supersets of desirable sets should be desirable as well (K_2).

The last three of these axioms implement the rules $\mathrm{R_{not}}$, $\mathrm{R_{des}}$ and $\mathrm{R_{cl}}$, again taking into account that K consists of desirable sets. The third axiom (K_3) implements $\mathrm{R_{not}}$, which says that the things in A_{not} cannot be desirable. If a set A contains at least one desirable thing, this clearly implies that $A \setminus A_{\mathrm{not}}$ should also contain at least one desirable thing. The fourth axiom (K_4) implements $\mathrm{R_{des}}$, which simply states that every thing in A_{des} should be desirable. The fifth axiom (K_5), finally, implements $\mathrm{R_{cl}}$, which says that if all the things in a set A are desirable then the same is true for $\mathrm{cl}(A)$. To see how this indeed leads to K_5, the crucial observation is that $\mathcal{S}_{\mathcal{A}}$ contains at least one selection S whose elements are all desirable. The reason is that every $A \in \mathcal{A} \subseteq K$ contains at least one desirable thing. So if, for each $A \in \mathcal{A}$, we let $t_A \in A$ be that desirable thing, then $S = \{t_A\colon A \in \mathcal{A}\} \in \mathcal{S}_{\mathcal{A}}$ consists of desirable things only. For that particular $S \in \mathcal{S}_{\mathcal{A}}$, $\mathrm{R_{cl}}$ therefore imposes that $\mathrm{cl}(S)$ consists of desirable things only, and thus in particular, that $t_S \in \mathrm{cl}(S)$ is desirable. It follows that $\{t_S\colon S \in \mathcal{S}_{\mathcal{A}}\}$ contains at least one desirable thing, making it a desirable set. For that reason, it makes sense to require that it belongs to K, as K_5 does.

Example 8 *To illustrate* K$_5$*, consider the closure operator* cl *of Example 5 and the assessment* $\mathcal{A} = \{A_1, A_2\}$ *of Example 7. For each of the three* $S \in \mathcal{S}_{\mathcal{A}}$ *that contain* peperoni*, let* $t_S :=$ peperoni $\in S \subseteq \mathrm{cl}\, S$*. For* $S = \{\text{meatballs, cheese}\}$*, let* $t_S :=$ meatballs with cheese crust $\in \mathrm{cl}(S)$*. If* A_1 *and* A_2 *both belong to* K*, it then follows from* K$_5$ *that*

$$\{t_S \colon S \in \mathcal{S}_{\mathcal{A}}\} = \{\text{peperoni, meatballs with cheese crust}\}$$

belongs to K *as well. To understand why this is reasonable, recall that* $A_1, A_2 \in K$ *means that* A_1 *and* A_2 *are both desirable: they each contain at least one desirable thing. In case* peperoni *is not desirable, this implies that* meatballs *and* cheese *are both desirable, and therefore also, by applying* cl*, that* meatballs with cheese crust *is desirable. So we indeed find that* peperoni *or* meatballs with cheese crust *should be desirable.*

An important feature of the rules and axioms above is that they can accommodate a wide range of different context, by considering various sets of things $\mathcal{T}$ and various choices of A_{not}, A_{des} and cl. The examples above served as a first simple illustration of this range; more complex examples will be given further on. It is also not necessary to impose all of the rules; each of them is optional. Not imposing R$_{\text{not}}$ or R$_{\text{des}}$ corresponds to setting $A_{\text{not}} = \emptyset$ and $A_{\text{des}} = \emptyset$, respectively, whereas not imposing R$_{\text{cl}}$ corresponds to letting cl be equal to the identity operator id, defined for each $A \subseteq \mathcal{T}$ by $\mathrm{id}(A) := A$. That the corresponding axioms are in those cases indeed redundant is easy to see, except for axiom K$_5$, for which it is not obvious that it has no implications in case cl = id. Nevertheless, the following result shows that it indeed does not.

Proposition 4 *If* cl = id*, then any set of desirable sets* $K \subseteq \mathcal{P}(\mathcal{T})$ *that satisfies* K$_2$ *will also satisfy* K$_5$*.*

We end this section by addressing the question whether it is at all possible to be coherent; that is, whether the proposed axioms are compatible. The following result presents a simple intuitive condition that is both necessary and sufficient for this to be the case.

Proposition 5 **D** *is non-empty—so there is at least one coherent SDT—if and only if* $\mathrm{cl}(A_{\text{des}}) \cap A_{\text{not}} = \emptyset$*. Similarly,* **K** *is non-empty—so there is at least one coherent SDS—if and only if* $\mathrm{cl}(A_{\text{des}}) \cap A_{\text{not}} = \emptyset$*.*

4. Representation

An essential feature of the theory of desirable things here presented, is the connection between the two frameworks it consists of. We start by showing that sets of desirable sets (of things) are more expressive than sets of desirable things. The reason, quite simply, is that every set of desirable things D has a corresponding set of desirable sets

$$K_D := \{A \in \mathcal{P}(\mathcal{T}) \colon A \cap D \neq \emptyset\}. \tag{2}$$

It is furthermore coherent if and only if D is.

Proposition 6 *Consider a set of desirable things* D *and its corresponding set of desirable sets* K_D*. Then* D *is coherent if and only if* K_D *is.*

More involved sets of desirable sets can for example be obtained by taking intersections of these basic ones. In particular, with any non-empty set $\mathcal{D}$ of sets of desirable things, we can associate a set of desirable sets

$$K_{\mathcal{D}} := \textstyle\bigcap_{D \in \mathcal{D}} K_D.$$

This kind of model expresses that at least one of the SDTs in $\mathcal{D}$ is representative, in the sense that a set of things is deemed desirable according to $K_{\mathcal{D}}$ if and only if it is desirable according to all $D \in \mathcal{D}$. If every set of desirable things in $\mathcal{D}$ is coherent, the resulting set of desirable sets $K_{\mathcal{D}}$ will furthermore be coherent as well. This follows from Proposition 6 and the fact that coherence is preserved under intersections.

Proposition 7 *For every non-empty set* $\mathcal{K} \subseteq \mathbf{K}$ *of coherent sets of desirable sets,* $\bigcap \mathcal{K}$ *is a coherent set of desirable sets as well. Similarly, for every non-empty set* $\mathcal{D} \subseteq \mathbf{D}$ *of coherent sets of desirable things,* $\bigcap \mathcal{D}$ *is a coherent set of desirable things.*

Corollary 8 *For any non-empty set* $\mathcal{D} \subseteq \mathbf{D}$ *of coherent sets of desirable things,* $K_{\mathcal{D}}$ *is a coherent set of desirable sets.*

What is much more surprising, and a central result of this report, is that the converse is true as well: every coherent set of desirable sets K corresponds to a set $\mathcal{D}$ of coherent sets of desirable things.

Theorem 9 *A set of desirable sets (of things)* K *is coherent if and only if there is a non-empty set* $\mathcal{D} \subseteq \mathbf{D}$ *of coherent sets of desirable things such that* $K = K_{\mathcal{D}}$*.*

As a first serious illustration of the proposed framework, and of Theorem 9 in particular, we apply it to the case of desirable gambles, leading to variations on existing results for so-called coherent sets of desirable gambles, as well as for generalisations thereof where instead of gambles, arbitrary vectors are considered.

Example 9 *Let* $\mathcal{X}$ *be a set of states, typically regarded as the possible outcomes of some experiment, and let* $\mathcal{G}(\mathcal{X})$ *be the set of all bounded real functions on* $\mathcal{X}$*, called gambles. A* *gamble f in $\mathcal{G}(\mathcal{X})$ can then be interpreted as an uncertain reward $f(x)$ whose value depends on the unkown state x. Several notions of desirability can be considered here, but one that is particulary common is to call such a gamble desirable if our subject (strictly) prefers this gamble—so receiving the reward $f(x)$ after the state x has been determined—to the status quo—so to not gambling at all.*

The definitions of coherence that are commonly used for sets of desirable gambles then correspond to particular cases of Definition 2 [19, 4]. One popular choice, that makes sense if desirability is defined as above, is to let $A_{\text{not}} := \{f \in \mathcal{G}(\mathcal{X})\colon f \leq 0\}$ and

$$A_{\text{des}} := \{f \in \mathcal{G}(\mathcal{X})\colon f \geq 0 \text{ and } (\exists x \in \mathcal{X}) f(x) > 0\}, \tag{3}$$

and to let $\text{cl} := \text{posi}$ be the closure operator with respect to positive linear combinations, defined for all $A \subseteq \mathcal{G}(\mathcal{X})$ by

$$\text{posi}(A) := \{\textstyle\sum_{i=1}^{n} \lambda_i f_i \colon n > 0, f_i \in A, \lambda_i > 0\}.$$

In this case, A_{not} expresses that a gamble that only yields negative rewards is not desirable, A_{des} expresses that a non-negative but possibly positive reward should always be desirable, and cl corresponds to an assumption that the awards are expressed in units of some linear utility scale. Alternative versions of this concept of a coherent set of desirable gambles are typically very similar, and also correspond to special case of our framework, but with small differences in the choice of A_{des} and A_{not}; $A_{\text{des}} := \{f \in \mathcal{G}(\mathcal{X})\colon \inf(f) > 0\}$ is for example often considered as well. Regardless of the particular choice, Definition 3 will lead to a corresponding notion of coherent sets of desirable sets of gambles, and Theorem 9 shows that the latter can always be represented by a set of coherent sets of desirable gambles. A similar result can for example be found in Reference [9], but the desirable sets (of gambles) can only be finite there, and the version of axiom K_5 used there, with cl = posi, is therefore simpler. Such simplifications are also possible in our general framework of desirable things; we will get to that in Section 5.

Rather than focus on gambles, completely similar concepts and results can also be obtained for vectors in some arbitrary vector space $\mathcal{V}$. The choice of A_{des} and A_{not} will then be different, and will depend on the particular vector space and context, but the inference mechanism that is expressed by cl = posi then still makes sense (because linear combinations do), and all the results continue to apply. In that context, a version of Theorem 9 is given in Reference [10], be it again for the case where desirable sets can only be finite. Examples of things in a vector space for which the notion of desirability has proven useful include, for example, desirable polynomials [14] and desirable matrices [3].

The main determining feature of the cases considered in the example above, is that the inference mechanism consists in taking positive linear combinations. However, this is not always defensible. In the case of desirable gambles, for example, one might want to drop the assumption of linear utility that justifies the use of the posi operator. The framework of desirable things makes this easy; one can easily use a different closure operator such as, for example, the convex hull.

Example 10 *Let $\mathcal{T}$ be the set $\mathcal{G}(\mathcal{X})$ of all gambles, as in Example 9. With $A_{\text{not}} := \{f \in \mathcal{G}(\mathcal{X})\colon f \leq 0\}$ and A_{des} as in Equation (3). Rather than choose cl := posi, however, we can drop the assumption of linear utility and use any other closure operator instead. For coherent sets of desirable gambles, this corresponds to the theory of nonlinear desirability of Miranda and Zaffalon [17, 30], who provide several examples for such closure operators. Another example of such a closure operator, which was essentially already put forward by Quaeghebeur [20], is to let cl := CH be the convex hull operator, defined for all $A \subseteq \mathcal{G}(\mathcal{X})$ by*

$$\text{CH}(A) := \{\textstyle\sum_{i=1}^{n} \lambda_i f_i \colon n > 0, f_i \in A, \lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1\},$$

which can easily be seen to be a closure operator. In any case, regardless of the particular choice of cl, it follows from our results that it not only leads to a notion of coherent sets of desirable gambles, as in Reference [17], but also to a notion of coherent sets of desirable sets of gambles, as well as a connection between both as provided by Theorem 9.

The framework of desirable things is also not restricted to the vector spaces—such as sets of gambles—to which it has typically been applied in the past: moving away from the posi operator also opens up the possibility of moving away from vector spaces. For example, if inference is expressed by means of the convex hull operator, $\mathcal{T}$ can be any convex space. Interestingly, this makes it possible to directly deal with probabilities and horse lotteries. In contrast, previous attempts at applying the ideas behind desirability to horse lotteries consisted in embedding horse lotteries in a vector space and working with positive linear rather than convex combinations in that vector space [29, 22, 12].

Example 11 *Let $\mathcal{T}$ be the set of all probability mass functions on some finite set $\mathcal{R}$ that consists of 'prizes'. Any such mass function is called a lottery, and provides a probability for winning each of the prizes in $\mathcal{R}$. We could then call a lottery desirable if our subject is willing to let this lottery determine which prize he will get from $\mathcal{R}$. The set of things $\mathcal{T}$ is now a convex space, but not a vector space. Nevertheless, the theory we developed can still be applied. Here too, we can for example consider CH as our closure operator, which builds in an assumption that convex mixtures of desirable lotteries should be desirable as* *well. A_{not} and A_{des} can be chosen freely, but will typically depend on the particular set of prizes $\mathcal{R}$. For any given such set-up, the theory we developed provides a notion of sets of desirable lotteries, of sets of desirable sets of lotteries, as well as a connection between both types of models.*

The example of lotteries on a finite set $\mathcal{R}$ is furthermore completely arbitrary; we can consider any other convex space instead. $\mathcal{R}$ could for example be infinite, and $\mathcal{T}$ could then be the set of all finitely additive probability measures on $\mathcal{P}(\mathcal{R})$, or if we endow $\mathcal{R}$ with a sigma algebra, the set of all countably additive probability measures on that algebra. As another example, $\mathcal{T}$ could be the set of all horse loteries [1] on a state space $\mathcal{X}$ with a finite set of prizes $\mathcal{R}$:

$$\mathcal{T} = \{h \in \mathbb{R}_{\geq 0}^{\mathcal{X}\times\mathcal{R}} \colon \textstyle\sum_{r\in\mathcal{R}} h(x,r) = 1 \text{ for all } x \in \mathcal{X}\}. \tag{4}$$

These horse lotteries are typically interpreted as gambles for which the reward associated with every $x \in \mathcal{X}$ is not numeric, but rather a lottery on $\mathcal{R}$. Here too, $\mathcal{R}$ could also be infinite; the lottery associated with each $x \in \mathcal{X}$ will then be a (finitely or countably additve) probability measure on $\mathcal{R}$ instead of a probability mass function.

5. Finite Desirable Sets and Simpler Axioms

For sets of desirable sets, the axiom that imposes the effects of cl is quite demanding: it requires us to combine the effects of infinitely many desirable sets in $\mathcal{A}$ to draw conclusions about the desirability of a new set $\{t_S \colon S \in \mathcal{S}_{\mathcal{A}}\}$. In contrast, in previous studies of sets of desirable sets of *gambles* [9, 10], the axiom that imposes the effects of posi was simpler than K_5, and essentially required K_5 only for the case $|\mathcal{A}| = 2$. We now intend to investigate whether—and if yes, under which conditions—such simplifications can also be obtained in our more general context. We will in particular consider three possible simplifications for K_5. First, a version that imposes K_5 for finite $\mathcal{A}$ only:

$K_{5\text{fin}}$. if $\emptyset \neq \mathcal{A} \subseteq K$ is finite and, for all $S \in \mathcal{S}_{\mathcal{A}}$, $t_S \in \text{cl}(S)$, then $\{t_S \colon S \in \mathcal{S}_{\mathcal{A}}\} \in K$.

Second, mimicking the approach in earlier studies for the case of desirable gambles, a version that essentially focusses on the case $|\mathcal{A}| = 2$:

$K_{5\text{bin}}$. if $A, B \in K$ and, for all $a \in A$ and $b \in B$, $t_{a,b} \in \text{cl}(\{a,b\})$, then $\{t_{a,b} \colon a \in A, b \in B\} \in K$.

And finally, a version that only considers the case $|\mathcal{A}| = 1$:

$K_{5\text{un}}$. if $A \in K$ and, for all $a \in A$, $t_a \in \text{cl}(\{a\})$, then $\{t_a \colon a \in A\} \in K$

Definition 10 *A set of desirable sets $K \subseteq \mathcal{P}(\mathcal{T})$ is called* finitely coherent *if it satisfies K_1–K_4 and $K_{5\text{fin}}$. It is called* 2-coherent *if it satisfies K_1–K_4 and $K_{5\text{bin}}$. It is called* 1-coherent *if it satisfies K_1–K_4 and $K_{5\text{un}}$.*

A common feature of the previous studies that used a simpler version of K_5 [9, 10], resembling $K_{5\text{bin}}$, is that the desirable sets of gambles they consider are all finite: in those studies, sets of desirable sets (of gambles) are subsets of

$$\mathcal{P}_{\text{fin}}(\mathcal{T}) \coloneqq \{A \in \mathcal{P}(\mathcal{T}) \colon |A| < \infty\}$$

rather than $\mathcal{P}(\mathcal{T})$. In such a setting, some of the coherence axioms need to be modified slightly, in such a way that they do not enforce us to add infinite sets of things. In particular, B and $\{t_S \colon S \in \mathcal{A}_{\mathcal{A}}\}$ need to be finite for K_2 and K_5 to make sense, respectively. To enable us to consider this setting as well, we introduce a notion of coherence for sets of desirable sets that explicitly restricts attention to $\mathcal{P}_{\text{fin}}(\mathcal{T})$.

Definition 11 *A set of desirable sets (of things) $K \subseteq \mathcal{P}(\mathcal{T})$ is* coherent in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ *if it satisfies*

K_1^{fin}. *$\emptyset \notin K$*
K_2^{fin}. *if $A \subseteq B \in \mathcal{P}_{\text{fin}}(\mathcal{T})$ and $A \in K$, then also $B \in K$*
K_3^{fin}. *if $A \in K \cap \mathcal{P}_{\text{fin}}(\mathcal{T})$ then also $A \setminus A_{\text{not}} \in K$*
K_4^{fin}. *$\{t\} \in K$ for all $t \in A_{\text{des}}$*
K_5^{fin}. *if $\emptyset \neq \mathcal{A} \subseteq K \cap \mathcal{P}_{\text{fin}}(\mathcal{T})$ and, for all $S \in \mathcal{S}_{\mathcal{A}}$, $t_S \in \text{cl}(S)$, then if $\{t_S \colon S \in \mathcal{S}_{\mathcal{A}}\} \in \mathcal{P}_{\text{fin}}(\mathcal{T})$, also $\{t_S \colon S \in \mathcal{S}_{\mathcal{A}}\} \in K$,*

It is called finitely coherent in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ *if it satisfies K_1^{fin}–K_4^{fin} and*

$K_{5\text{fin}}^{\text{fin}}$. *if $\emptyset \neq \mathcal{A} \subseteq K \cap \mathcal{P}_{\text{fin}}(\mathcal{T})$ is finite and, for all $S \in \mathcal{S}_{\mathcal{A}}$, $t_S \in \text{cl}(S)$, then $\{t_S \colon S \in \mathcal{S}_{\mathcal{A}}\} \in K$.*

It is called 2-coherent in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ *if it satisfies K_1^{fin}–K_4^{fin} and*

$K_{5\text{bin}}^{\text{fin}}$. *if $A, B \in K \cap \mathcal{P}_{\text{fin}}(\mathcal{T})$ and, for all $a \in A$ and $b \in B$, $t_{a,b} \in \text{cl}(\{a,b\})$, then $\{t_{a,b} \colon a \in A, b \in B\} \in K$.*

It is called 1-coherent in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ *if it satisfies K_1^{fin}–K_4^{fin} and*

$K_{5\text{un}}^{\text{fin}}$. *if $A \in K \cap \mathcal{P}_{\text{fin}}(\mathcal{T})$ and, for all $a \in A$, $t_a \in \text{cl}(\{a\})$, then $\{t_a \colon a \in A\} \in K$*

These new versions of the axioms, and the associated notions of coherence, simply amount to imposing the previous ones only to the extent that they involve sets in $\mathcal{P}_{\text{fin}}(\mathcal{T})$. For that reason, coherence trivially implies coherence in $\mathcal{P}_{\text{fin}}(\mathcal{T})$, and similarly for finite coherence (in $\mathcal{P}_{\text{fin}}(\mathcal{T})$), 2-coherence (in $\mathcal{P}_{\text{fin}}(\mathcal{T})$) and 1-coherence (in $\mathcal{P}_{\text{fin}}(\mathcal{T})$).

As is to be expected from their definitions, coherence, finite coherence, 2-coherence and 1-coherence are related as well, and similarly for the versions that focus on $\mathcal{P}_{\text{fin}}(\mathcal{T})$. In that order, earlier notions imply later ones.

Proposition 12 *Let $K \subseteq \mathcal{P}(\mathcal{T})$ be a set of desirable sets. Then coherence implies finite coherence, which implies 2-coherence, which implies 1-coherence. The same is true for the versions that focus on $\mathcal{P}_{\text{fin}}(\mathcal{T})$.*

The more interesting question is whether finite coherence, 2-coherence or 1-coherence implies coherence, and similarly for the versions that focus on $\mathcal{P}_{\text{fin}}(\mathcal{T})$, since that would allow us to replace our axioms with simpler yet equivalent ones. A first important result is that this is always the case provided that the closure operator cl is *unitary*. In that case, without loss of power, coherence (in $\mathcal{P}_{\text{fin}}(\mathcal{T})$) can therefore be replaced by 1-coherence (in $\mathcal{P}_{\text{fin}}(\mathcal{T})$).

Definition 13 *A closure operator* cl *is unitary if*

$$\text{cl}(A) = \bigcup_{t \in A} \text{cl}(\{t\}) \textit{ for all } A \in \mathcal{P}(\mathcal{T}).$$

Proposition 14 *Assume that* cl *is unitary and consider any set of desirable sets* $K \subseteq \mathcal{P}(\mathcal{T})$*. Then coherence is equivalent to finite coherence, to 2-coherence, and to 1-coherence. Similarly, coherence in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$ *is equivalent to finite coherence in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$*, to 2-coherence in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$*, and to 1-coherence in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$*.*

A particular class of unitary closure operators to which this result can be applied, are those that close a set with respect to an equivalence relation. Another are closure operators for which $\text{cl}(A)$ is the upset of A according to some partial order on $\mathcal{T}$. Unfortunately though, being unitary is a very strong property to ask of a closure operator, and only the most simple closure operators will satisfy it. The posi operator that is typically used in the context of desirable gambles—see Example 9—is for example not unitary, nor is the convex hull operator that appeared in Examples 10 and 11 or the transitive closure in Example 6.

Fortunately, we can also obtain similar equivalences—yet not for 1-coherence—for closure operators that are not unitary, provided we make some other assumptions. We start by doing so for the versions that focus on $\mathcal{P}_{\text{fin}}(\mathcal{T})$.

Our first result is that finite coherence in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ is equivalent to coherence in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ provided that the closure operator cl is *finitary*. As can be seen from the following definition, these finitary closure operators are a superset of the unitary ones.

Definition 15 *A closure operator* cl *is finitary if*

$$\text{cl}(A) = \bigcup_{B \subseteq A, |B| < \infty} \text{cl}(B) \tag{5}$$

Corollary 16 *If* cl *is finitary, a set of desirable sets* $K \subseteq \mathcal{P}(\mathcal{T})$ *is finitely coherent in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$ *if and only if it is coherent in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$*.*

Similarly, it is also possible to replace finite coherence in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ with 2-coherence in $\mathcal{P}_{\text{fin}}(\mathcal{T})$. For that result, the closure operator cl needs to be *incremental*. Here too, it can easily be seen that unitary closure operators are always incremental.

Definition 17 *A closure operator is* incremental *if for all* $A \in \mathcal{P}(\mathcal{T})$*,* $a \in \mathcal{T}$ *and* $t \in \text{cl}(A \cup \{a\})$*, there is some* $t_A \in \text{cl}(A)$ *such that* $t \in \text{cl}(\{t_A, a\})$*.*

Proposition 18 *If* cl *is incremental, a set of desirable sets* $K \subseteq \mathcal{P}(\mathcal{T})$ *is finitely coherent in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$ *if and only if it is 2-coherent in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$*.*

Combining the preceding two results, we arrive at a sufficient condition for coherence in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ to be equivalent to finite coherence in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ and 2-coherence in $\mathcal{P}_{\text{fin}}(\mathcal{T})$.

Corollary 19 *If* cl *is finitary and incremental, then coherence in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$ *is equivalent to 2-coherence in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$*, and to finite coherence in* $\mathcal{P}_{\text{fin}}(\mathcal{T})$*.*

Unlike the condition that cl should be unitary, requiring that it is finitary and incremental is not that much to ask. For example, it is easy to verify that the posi operator in Example 9, the convex hull operator in Examples 10 and 11, and the transitive closure in Example 6, are all examples of closure operators that are both finitary and incremental.

We have at this point established several conditions under which coherence in $\mathcal{P}_{\text{fin}}(\mathcal{T})$ can be replaced by a simpler alternative. The first of these results—Proposition 14—also applies to our original notion of coherence—so not in $\mathcal{P}_{\text{fin}}(\mathcal{T})$— but the others do not, which is unfortunate because that is the case to which Theorem 9 applies. To obtain similar results also for this case, we will impose an additional condition, this time on K instead of cl: we will focus on finitary K.

Definition 20 *A set of desirable sets* $K \subseteq \mathcal{P}(\mathcal{T})$ *is called* finitary *if, for all* $A \subseteq \mathcal{T}$*, we have that* $A \in K$ *if and only if there is a finite* $B \subseteq A$ *such that* $B \in K$*.*

Any such finitary set of desirable sets is completely determined by its restriction to $\mathcal{P}_{\text{fin}}(\mathcal{T})$. Furthermore, as we will see further on, the coherence of a finitary set of desirable sets is also determined by its restriction to $\mathcal{P}_{\text{fin}}(\mathcal{T})$, at least if we impose some suitable conditions.

To show this, we start by defining, for any set of desirable sets $K \subseteq \mathcal{P}(\mathcal{T})$, the set of all supersets of its restriction to $\mathcal{P}_{\text{fin}}(\mathcal{T})$:

$$\text{fin}(K) := \{A \in \mathcal{P}(\mathcal{T}) \colon B \in K \cap \mathcal{P}_{\text{fin}}(\mathcal{T}), B \subseteq A\}.$$

It is easy to see that K is finitary if and only if it coincides with $\text{fin}(K)$. In fact, even for K that are not finitary, $\text{fin}(K)$ is always finitary.

Proposition 21 *For any set of desirable sets* $K \subseteq \mathcal{P}(\mathcal{T})$*,* $\text{fin}(K)$ *is a finitary set of desirable sets.*

Proposition 22 *A set of desirable sets* $K \subseteq \mathcal{P}(\mathcal{T})$ *is finitary if and only if* $K = \text{fin}(K)$*.*

Due to this last result, establishing that a finitary set of desirable sets K satisfies—some version of—coherence is equivalent to establishing that $\mathrm{fin}(K)$ does. Our next result provides some ways for doing just that.

Proposition 23 *Consider any set of desirable sets $K \subseteq \mathcal{P}(\mathcal{T})$. If K is finitely coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, then $\mathrm{fin}(K)$ is finitely coherent and $\mathrm{fin}(K) \cap \mathcal{P}_{\mathrm{fin}}(\mathcal{T}) = K \cap \mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, and similarly for 2-coherence and 1-coherence. Furthermore, if K is finitely coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ and cl is finitary, then $\mathrm{fin}(K)$ is coherent.*

As an almost immediate consequence, we find that for finitary sets of desirable sets, the distinguishement between the different notions of coherence we consider, and their restrictions to $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, disappears. For coherence itself, this does require cl to be finitary though.

Proposition 24 *Consider a set of desirable sets $K \subseteq \mathcal{P}(\mathcal{T})$ that is finitary. Then K is finitely coherent if and only if it is finitely coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, and similarly for 2-coherence and 1-coherence. Furthermore, if cl is finitary, then K is coherent if and only if it is coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$.*

Taking into account our earlier equivalences between coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, finite coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ and 2-coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, this directly leads to the following similar results for the case where we do not focus on $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, be it for finitary sets of desirable sets only.

Corollary 25 *If cl is finitary, a finitary set of desirable sets $K \subseteq \mathcal{P}(\mathcal{T})$ is coherent if and only if it is finitely coherent.*

Corollary 26 *If cl is incremental, a finitary set of desirable sets $K \subseteq \mathcal{P}(\mathcal{T})$ is finitely coherent if and only if it is 2-coherent.*

Corollary 27 *If cl is finitary and incremental, a finitary set of desirable sets $K \subseteq \mathcal{P}(\mathcal{T})$ is coherent if and only if it is 2-coherent, and if and only if it is finitely coherent.*

What is nice about these results is that if we combine them with our representation result in Theorem 9, we see that for finitary sets of desirable sets, representation in terms of a set of coherent sets of desirable things can be obtained under weaker conditions than coherence. On the one hand, for finitary incremental closure operators, every set of desirable sets K that is finitary and 2-coherent will be of the form $K_{\mathcal{D}}$, for some non-empty $\mathcal{D} \subseteq \mathbf{D}$. On the other hand, for finitary closure operators, every set of desirable sets K that is finitary and finitely coherent will be of the form $K_{\mathcal{D}}$. Unfortunately though, these results only provide sufficient conditions for such a representation, because even for finitary incremental closure operators, sets of desirable sets of the form $K_{\mathcal{D}}$ need not be finitary.

Example 12 *We consider the case of desirable gambles, as in Example 9, with $\mathrm{cl} = \mathrm{posi}$, $A_{\mathrm{not}} \coloneqq \{f \in \mathcal{G}(\mathcal{X}) \colon f \leq 0\}$ and A_{des} as in Equation (3). We focus on a simple version with two states only: $\mathcal{X} = \{a, b\}$. For any $n \in \mathbb{N}$, we let $f_n \in \mathcal{G}(\mathcal{X})$ be the gamble defined by $f_n(a) \coloneqq -1/n$ and $f_n(b) \coloneqq 1$, and use it to define a set of desirable gambles*

$$D_n \coloneqq \{f \in \mathcal{G}(\mathcal{X}) \colon f \geq \alpha f_n \text{ with } \alpha \in \mathbb{R}_{\geq 0}\} \setminus \{0\}.$$

It is fairly simple to show that each of these sets of desirable gambles is coherent. Now let $A \coloneqq \{f_n \colon n \in \mathbb{N}\}$. Then for all $n \in \mathbb{N}$, since $f_n \in A \cap D_n$, we know that $A \cap D_n \neq \emptyset$ and therefore, that $A \in K_{D_n}$. So we see that $A \in K_{\mathcal{D}}$, with $\mathcal{D} \coloneqq \{D_n \colon n \in \mathbb{N}\}$. However, there is no finite subset B of A such that $B \in K_{\mathcal{D}}$, because $f_m \notin D_n$ for all $n > m$.

The only simpler condition we have seen that is necessary and sufficient for coherence, and hence for a representation in the style of Theorem 9, is 1-coherence for unitary closure operators, because that did not involve K being finitary. Combining Theorem 9 with Proposition 14 indeed immediately yields the following simplification of Theorem 9.

Theorem 28 *If cl is unitary, a set of desirable sets K is 1-coherent if and only if there is a non-empty set $\mathcal{D} \subseteq \mathbf{D}$ of coherent sets of desirable things such that $K = K_{\mathcal{D}}$.*

There is however also another way in which we can obtain a representation in the style of Theorem 9 under conditions that are simpler than—yet equivalent to—coherence, which consists in focussing on finite desirable sets only. That is, by considering sets of desirable sets in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ that are coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$. For those, we have already established several conditions that are simpler yet equivalent to coherence, so simplifying coherence is not the issue here. Instead, what's missing is a representation result for such models. In fact, it is not clear what such a representation would look like, since models of the form $K_{\mathcal{D}}$ are subsets of $\mathcal{P}(\mathcal{T})$ but not of $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$. This is easily fixed though because we can simply consider the restriction to $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$. To that end, with any set of desirable things D, we associate the restriction

$$K_D^{\mathrm{fin}} \coloneqq \{A \in \mathcal{P}_{\mathrm{fin}}(\mathcal{T}) \colon A \cap D \neq \emptyset\} = K_D \cap \mathcal{P}_{\mathrm{fin}}(\mathcal{T})$$

of K_D to $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, and with every non-empty set $\mathcal{D}$ of sets of desirable things, the restriction

$$K_{\mathcal{D}}^{\mathrm{fin}} \coloneqq \bigcap_{D \in \mathcal{D}} K_D^{\mathrm{fin}} = K_{\mathcal{D}} \cap \mathcal{P}_{\mathrm{fin}}(\mathcal{T})$$

of $K_{\mathcal{D}}$ to $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$. The question then is whether a result similar to Theorem 9 can be obtained here as well. That is, for any set of desirable sets in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, is coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ equivalent with having a representation of this form, with $\mathcal{D}$ a set of coherent sets of desirable sets? Our next result shows that this is true for finitary closure operators; coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ can even be replaced by finite coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$. The proof is fairly simple, and essentially consists in applying Theorem 9 to $\mathrm{fin}(K)$, which is coherent due to Proposition 23, and observing that $\mathrm{fin}(K) \cap \mathcal{P}_{\mathrm{fin}}(\mathcal{T}) = K$.

Theorem 29 *If* cl *is finitary, then a set of desirable sets $K \subseteq \mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ is finitely coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ if and only there is a non-empty set $\mathcal{D} \subseteq \mathbf{D}$ of coherent sets of desirable things such that $K = K_{\mathcal{D}}^{\mathrm{fin}}$. The same is true if we replace finite coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ by coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$.*

Combining this result with Proposition 18, we obtain a similar representation theorem for 2-coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, provided that cl is also incremental.

Theorem 30 *If* cl *is finitary and incremental, then a set of desirable sets $K \subseteq \mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ is 2-coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ if and only there is a non-empty set $\mathcal{D} \subseteq \mathbf{D}$ of coherent sets of desirable things such that $K = K_{\mathcal{D}}^{\mathrm{fin}}$.*

Combining Theorem 29 with Proposition 14, finally, yields a similar result for 1-coherence in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$, provided that cl is unitary.

Theorem 31 *If* cl *is unitary, then a set of desirable sets $K \subseteq \mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ is 1-coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T})$ if and only there is a non-empty set $\mathcal{D} \subseteq \mathbf{D}$ of coherent sets of desirable things such that $K = K_{\mathcal{D}}^{\mathrm{fin}}$.*

Since many closure operators are both finitary and incremental, the most important of these three results is arguably Theorem 30; in particular, it allows us to recover and extend some earlier results for the case of vector spaces.

Example 13 *We already explained in Example 9 that representation results in the style of Theorem 9 have already been obtained for the case where $\mathcal{T}$ is a vector space—such as the set $\mathcal{G}(\mathcal{X})$ of all gambles on a state space $\mathcal{X}$—and* cl = posi*, but that these results took desirable sets to be finite, and replaced axiom* K_5 *with a simpler version. We can now be more specific about this: the results we were referring to correspond to a special case of Theorem 30, with* cl = posi*—which is both finitary and incremental—and $\mathcal{T}$ either a vector space or, more specifically, a set of gambles $\mathcal{G}(\mathcal{X})$, and the simpler axiom we were referring to is then of course axiom* $\mathrm{K}_{5\mathrm{bin}}^{\mathrm{fin}}$*. That said, Theorem 30 is much more broadly applicable than the results in References [9, 10]: any set $\mathcal{T}$ will do, and it applies to any* cl *that is finitary and incremental. Since the convex hull operator* CH *is finitary and incremental, this implies that Theorem 30 can for instance be applied to the settings we discussed in Examples 10 and 11. So in all of these settings, we can replace* K_5 *by* $\mathrm{K}_{5\mathrm{bin}}^{\mathrm{fin}}$ *and still obtain a representation in the style of Theorem 9, provided we focus on finite desirable sets only.*

Example 14 *For our final example, we return to the setting of preferences, in Example 6, and show that we can use it to represent sets of desirable sets of preferences in terms of strict total orders. To that end, let* cl = trans *be the transitive closure operator of Example 6 and let $A_{\mathrm{des}} \coloneqq \emptyset$ and $A_{\mathrm{not}} \coloneqq \{(o,o)\colon o \in \mathcal{O}\}$. A set of desirable preferences D is then coherent if and only if it is irreflexive and transitive, here expressed as $(o,o) \notin D$ and $(o_1,o_2),(o_2,o_3) \in D \Rightarrow (o_1,o_3) \in D$. Since* trans *is a finitary incremental closure operator, it therefore follows from Theorem 30 that a set of desirable sets of preferences $K \subseteq \mathcal{P}_{\mathrm{fin}}(\mathcal{T}_{\mathcal{O}})$ is 2-coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T}_{\mathcal{O}})$ if and only if it is of the form $K_{\mathcal{D}}^{\mathrm{fin}}$, where each $D \in \mathcal{D}$ represents a binary relation on $\mathcal{O}$ that is irreflexive and transitive. For the representing binary relations to be strict total orders, they must also be connected, which here translates to the requirement that for any $o_1, o_2 \in \mathcal{O}$ such that $o_1 \neq o_2$, either $(o_1,o_2) \in D$ or $(o_2,o_1) \in D$. If we let*

$$\mathcal{A}_{\mathrm{tot}} \coloneqq \{\{(o_1,o_2),(o_2,o_1)\}\colon o_1,o_2 \in \mathcal{O}, o_1 \neq o_2\},$$

this corresponds to requiring that $A \cap D \neq \emptyset$ for all $A \in \mathcal{A}_{\mathrm{tot}}$ and $D \in \mathcal{D}$, or equivalently, that $\mathcal{A}_{\mathrm{tot}} \subseteq K_{\mathcal{D}}$. So we conclude that a set of desirable sets of preferences $K \subseteq \mathcal{P}_{\mathrm{fin}}(\mathcal{T}_{\mathcal{O}})$ can be represented by a set of total strict orders if and only if it is 2-coherent in $\mathcal{P}_{\mathrm{fin}}(\mathcal{T}_{\mathcal{O}})$ and includes $\mathcal{A}_{\mathrm{tot}}$.

In summary, we see that focussing on finite desirable sets allows us to obtain several necessary and sufficient conditions for representation. All of them require that cl is at least finitary though. If we do allow for infinite desirable sets, Theorem 9 shows that coherence provides a necessary and sufficient condition for non-finitary closure operators as well, but this condition can then not be further simplified—at least not with our results—without giving up necessity.

6. Where To Go From Here

The main contribution of this paper, I hope, has been to show that the theory of desirable gambles can be generalised from gambles to things, without giving up on the main ideas and results. That said, I have only done so up to some extent. I focussed on extending the notion of a coherent set of desirable gambles, and that of a coherent set of desirable sets of gambles, and on the connection between these two.

There is more to the theory of desirable gambles though, and it remains to be studied to which extent these other aspects can be similarly generalised as well. If the theory of desirable things is to be a proper generalisation of that of desirable gambles, as I hope it will, developing these other aspects will be essential. I conclude this paper with a brief overview of these aspects, which I hope can provide some inspiration for others wishing to further extend the framework here proposed. For more details, I refer to the extended online version of this paper [7].

Conservative inference. One of the features of the theory of desirable gambles that makes it important for imprecise probabilities, is that it provides a natural way to do conservative inference: starting from an initial assessment (of either desirable gambles or desirable sets of gambles), if it is at all possible to extend it to a coherent model, then there will always be a unique smallest such coherent extension, called the natural extension [24, 9]. Since we know from Proposition 7 that coherence is preserved under taking intersections, it should be rather straightforward to develop a similar notion of natural extension for desirable things.

Additional axioms. The closure operator that we employ in our definitions of coherence is what gives the proposed framework its generality, but it also comes at the price of having to put all of the specific inference aspects of the considered setting into this single operator. It would therefore be interesting to combine the axioms here proposed with additional ones. Example 14 provided a simple illustration of such an approach, where the addition of an extra condition guaranteed that the representing binary relations were strict total orders. Several more involved examples of such an approach have already been obtained for desirable gambles [10, 5], but it remains to be seen to which extent this is possible for desirable things as well.

Logic. As hinted at in Footnote 2, the idea of representing inference principles with closure operators is used in (abstract) logic as well. Coherent sets of desirable things then essentially correspond to so-called closed theories. Results—or perhaps rather ideas—from abstract logic could therefore usefully be employed in the study of sets of desirable things.[4] Sets of desirable sets of things, on the other hand, do not seem to have an analogue in abstract logic. Nevertheless, sets of desirable sets of things can be given a logical interpretation, which we've recently explored [15].

Desirable preferences. As illustrated in Examples 6 and 14, it is possible to consider preferences—and hence decision making—in the abstract setting of things, simply by directly considering the desirability of preferences. Since in the case of gambles, the connection between desirability, preferences and decision making has proven to be fruitful, it seems worthwile to further explore this connection in our more general context as well.

Choice functions. One of the most general, and arguably also most intuitive ways in which the theory of desirable gambles has been connected with decision making, is through the connection between coherent sets of desirable sets of gambles on the one hand, and coherent choice functions on the other. In fact, these two types of models have been shown to be equivalent [10]. The advantage of this connection is that we can combine the intuitive language of the latter—which involves statements about choices—with the mathematical power of the former, in the form of representation theorems such as the ones presented in this paper. In particular, this has lead to axiomatic characterisations for the use of various types of decision rules, including maximising expected utility, E-admissibility and maximality [10, 5, 21]. The results in this paper therefore beg the question whether a similar connection with choice functions is possible also in our more general setting.

Nonlinear operators. One of the main reasons why coherent sets of desirable gambles have played a foundational role in the field of imprecise probabilities, is because, as explained in the introduction, many well known imprecise probability models correspond to special cases. An important class of such special cases are nonlinear operators, including coherent lower and upper expectations (or previsions) and various types of nonlinear set functions [24, 25].

Given the developments in this paper, it makes sense to wonder if similar connections are possible in our more general context as well, and if yes for which types of things. For the case of gambles with general closure operators, some examples of such connections can already be found in the work of Miranda and Zaffalon [17]. But even for that case, much remains to be explored. If we adopt the convex hull as closure operator, for example, I expect that coherent sets of desirable gambles can be connected to convex lower and upper previsions and convex risk measures [18, 16].

Multivariate models. A final feature of desirable gambles that I would like to point out, is their use in a multivariate context: marginalisation, conditioning and independence, for example, are concepts that have been succesfully applied, not only to sets of desirable gambles [4, 13], but also to sets of desirable sets of gambles [23]. This has for example made it possible to apply sets of desirable gambles in the context of credal networks [8]—imprecise generalisations of Bayesian networks. For sets of desirable gambles with general closure operators, marginalisation and conditioning have already been explored as well [17]. Extending these ideas from gambles to arbitrary things seems impossible, but if the things in question have sufficient structure, this does seem feasible. For example, as one of the reviewers kindly suggested, we could consider uncertain things: mappings from a state space to a set of things; gambles and horse lotteries, for example, are specific instances of such uncertain things. These uncertain things are just special types of things, so the framework here presented can be applied, but they also provide sufficient structure to allow for a multivariate treatment.

[4] Consider for example the well known result that coherent sets of desirable gambles can be represented in terms of maximal sets of desirable gambles. One might wonder to which extent this generalises to the setting of things. In abstract logic, this corresponds to the question of whether closed theories can be represented in terms of so-called maximal theories; a question that has been thoroughly studied in that field.

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