Proof of Theorem 3. Let $\mathbb{E}_{\mathcal{E}} \coloneqq \{E \in \mathbb{E}_{\mathcal{D}} : \mathcal{E}^*(E) < +\infty\}$. Since $\mathcal{E}$ is downward continuous, we know from Lemma 2 that every linear expectation $E \in \mathbb{E}_{\mathcal{E}}$ is downward continuous. Consequently, if follows from the Daniell–Stone Theorem that for all $E \in \mathbb{E}_{\mathcal{E}}$, $E = \hat{E}|_{\mathcal{D}}$ with

$$\hat{E} : \mathcal{M} \to \overline{\mathbb{R}} : g \mapsto \int g \mathrm{d}P_E.$$

It follows immediately from this and Lemma 1 that $\hat{\mathcal{E}}$ is well defined and extends $\mathcal{E}$.

On several occasions, we will need that for all $f \in \mathcal{M}(\mathcal{D})$ and $E \in \mathbb{E}_{\mathcal{E}}$, $\mathcal{E}^*(E) \in \mathbb{R}$ (due to Lemma 1) and

$$\hat{E}(f) \leq \hat{\mathcal{E}}(f) + \mathcal{E}^*(E). \tag{12}$$

Next, we show that $\hat{\mathcal{E}}$ is a convex expectation. The extension $\hat{\mathcal{E}}$ is a nonlinear expectation: (i) $\mathcal{M}(\mathcal{D})$ includes all constant real functions because $\mathcal{D} \subseteq \mathcal{M}(\mathcal{D})$ and $\mathcal{D}$ includes all constant real functions; (ii) $\hat{\mathcal{E}}$ is isotone because the Lebesgue integral is isotone on $\mathcal{M}(\mathcal{D})$ [20, Chapter 8, Theorem 5 (iv)]; and (iii) $\hat{\mathcal{E}}$ is constant preserving because it extends $\mathcal{E}$ and $\mathcal{E}$ is constant preserving. To verify that $\hat{\mathcal{E}}$ is convex, we fix some $f, g \in \mathcal{M}(\mathcal{D})$ and $\lambda \in [0, 1]$ such that $f+g$ is meaningful and in $\mathcal{M}(\mathcal{D})$ and $\lambda\hat{\mathcal{E}}(f)+(1-\lambda)\hat{\mathcal{E}}(g)$ is meaningful. If $\lambda = 0$ or $\lambda = 1$, clearly $\hat{\mathcal{E}}(\lambda f+(1-\lambda)f) = \lambda\hat{\mathcal{E}}(f) + (1-\lambda)\hat{\mathcal{E}}(g)$; hence, without loss of generality we may assume that $0 < \lambda < 1$. Due to symmetry, and because $\lambda\hat{\mathcal{E}}(f) + (1-\lambda)\hat{\mathcal{E}}(g)$ is meaningful, we need to distinguish three cases: (i) $\hat{\mathcal{E}}(f) = +\infty$ and $\hat{\mathcal{E}}(g) > -\infty$; (ii) $\hat{\mathcal{E}}(f)$ and $\hat{\mathcal{E}}(g)$ both real; and (iii) $\hat{\mathcal{E}}(f) = -\infty$ and $\hat{\mathcal{E}}(g) < +\infty$. In the first case, the required inequality holds trivially. In the second case, it follows from Eqn. (12) that for all $E \in \mathbb{E}_{\mathcal{E}}$, $\hat{E}(f) < +\infty$ and $\hat{E}(g) < +\infty$, so $\lambda\hat{E}(f) + (1-\lambda)\hat{E}(g)$ is meaningful and, due to the linearity of $\hat{E}$ [20, Chapter 8, Theorem 5 (i)], equal to $\hat{E}(\lambda f + (1-\lambda)g)$. Similarly, in the third case, it follows from Eqn. (12) that for all $E \in \mathbb{E}_{\mathcal{E}}$, $\hat{E}(f) = -\infty$ and $\hat{E}(g) < +\infty$, so $\lambda\hat{E}(f) + (1-\lambda)\hat{E}(g)$ is meaningful and, due to the linearity of $\hat{E}$, equal to $\hat{E}(\lambda f + (1-\lambda)g)$. Consequently, in the last two cases,

$$\begin{aligned}
&\hat{\mathcal{E}}(\lambda f + (1-\lambda)g) \\
&\quad = \sup\bigl\{\hat{E}(\lambda f + (1-\lambda)g) - \mathcal{E}^*(E) : E \in \mathbb{E}_{\mathcal{E}}\bigr\} \\
&\quad = \sup\bigl\{\hat{E}(\lambda f) + (1-\lambda)\hat{E}(g) - \mathcal{E}^*(E) : E \in \mathbb{E}_{\mathcal{E}}\bigr\} \\
&\quad \leq \lambda \sup\bigl\{\hat{E}(f) - \mathcal{E}^*(E) : E \in \mathbb{E}_{\mathcal{E}}\bigr\} \\
&\qquad\quad + (1-\lambda) \sup\bigl\{\hat{E}(g) - \mathcal{E}^*(E) : E \in \mathbb{E}_{\mathcal{E}}\bigr\} \\
&\quad = \lambda\hat{\mathcal{E}}(f) + (1-\lambda)\hat{\mathcal{E}}(g),
\end{aligned}$$

as required.

Denk et al. [10, Theorem 3.10] show that the restriction of $\hat{\mathcal{E}}$ to $\mathcal{M}(\mathcal{D}) \cap \mathcal{L}(\mathcal{Y}) \supseteq \mathcal{D}_{\delta,\mathrm{b}}$ is downward continuous on $\mathcal{D}_{\delta,\mathrm{b}}$, so clearly $\hat{\mathcal{E}}$ is downward continuous on $\mathcal{D}_{\delta,\mathrm{b}}$ too.

Proving the upward continuity on $\mathcal{M}_{\mathrm{b}}(\mathcal{D})$ is straightforward. Fix any $(\mathcal{M}_{\mathrm{b}})^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \nearrow f \in \mathcal{M}_{\mathrm{b}}(\mathcal{D})$. For all $E \in \mathbb{E}_{\mathcal{E}}$, $\hat{E}$ is upward continuous on $\mathcal{M}_{\mathrm{b}}$—due to the Monotone Convergence Theorem, see for example [35, Theorem 12.1]—and therefore $\lim_{n\to+\infty} \hat{E}(f_n) = \sup_{n\in\mathbb{N}} \hat{E}(f_n) = \hat{E}(f)$. From this and the isotonicity of $\hat{\mathcal{E}}$, it follows that

$$\begin{aligned}
\lim_{n\to+\infty} \hat{\mathcal{E}}(f_n) &= \sup\bigl\{\hat{\mathcal{E}}(f_n) : n \in \mathbb{N}\bigr\} \\
&= \sup\bigl\{\sup\bigl\{\hat{E}(f_n) - \mathcal{E}^*(E) : E \in \mathbb{E}_{\mathcal{E}}\bigr\} : n \in \mathbb{N}\bigr\} \\
&= \sup\bigl\{\sup\bigl\{\hat{E}(f_n) - \mathcal{E}^*(E) : n \in \mathbb{N}\bigr\} : E \in \mathbb{E}_{\mathcal{E}}\bigr\} \\
&= \sup\bigl\{\hat{E}(f) - \mathcal{E}^*(E) : E \in \mathbb{E}_{\mathcal{E}}\bigr\} \\
&= \hat{\mathcal{E}}(f),
\end{aligned}$$

as required.

To prove the second part of the statement, we assume that $\mathcal{E}$ is an upper expectation. Recall from Lemma 1 that $\mathcal{E}^*(E) = 0$ for all $E \in \mathbb{E}_{\mathcal{E}}$ and that $\mathbb{E}_{\mathcal{E}}$ is the set of dominated linear expectations (on $\mathcal{D}$). Hence, to see that $\hat{\mathcal{E}}$ is positively homogeneous, it suffices to realise that for all $E \in \mathbb{E}_{\mathcal{E}}$ (i) $\mathcal{E}^*(E) = 0$ due to Lemma 1; and (ii) $\hat{E}$ is homogeneous [20, Chapter 8, Theorem 5 (i)]. That $\hat{\mathcal{E}}$ is subadditve follows from a similar argument as the one we used to prove that $\hat{\mathcal{E}}$ is convex. ∎

Proof of Corollary 4. From Theorem 3.10 in [10]—or the functional version of Choquet's Capacitibility Theorem, see [3, Proposition 2.1]—it follows that for all $f \in \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathcal{D}) = \mathcal{M}(\mathcal{D}) \cap \mathcal{L}(\mathcal{Y})$,

$$\hat{\mathcal{E}}(f) = \sup\Bigl\{\lim_{n\to+\infty} \hat{\mathcal{E}}(f_n) : \mathcal{D}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow_{\leq} f\Bigr\}. \tag{13}$$

It remains for us to prove the equality in the statement for all $f \in \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \setminus \mathcal{M}^{\mathrm{b}}(\mathcal{D})$, so let us fix any such f. Then $(f \wedge k)_{k\in\mathbb{N}}$ is an increasing sequence in $\mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathcal{D})$ that converges pointwise to f, and therefore

$$\hat{\mathcal{E}}(f) = \lim_{k\to+\infty} \hat{\mathcal{E}}(f \wedge k) = \sup\bigl\{\hat{\mathcal{E}}(f \wedge k) : k \in \mathbb{N}\bigr\}.$$

Because $f \wedge k \in \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathcal{D})$ for all $k \in \mathbb{N}$, it follows from this equality and Eqn. (13) that

$$\begin{aligned}
\hat{\mathcal{E}}(f) &= \sup\Bigl\{\lim_{n\to+\infty} \hat{\mathcal{E}}(f_n) : k \in \mathbb{N}, \mathcal{D}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow_{\leq} f \wedge k\Bigr\} \\
&= \sup\Bigl\{\lim_{n\to+\infty} \hat{\mathcal{E}}(f_n) : \mathcal{D}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow_{\leq} f\Bigr\},
\end{aligned}$$

as required. ∎

Proof of Equation (2). Due to Lemma 8.1 (and Lemma 8.3) in [35], $\sigma(\mathscr{D})$ is generated by the collection of level sets

$$C \coloneqq \bigl\{\{\omega \in \Omega : f(\omega) \geq \alpha\} : f \in \mathscr{D}, \alpha \in \mathbb{R}\bigr\}.$$

Hence, it follows from Eqn. (1) that every cylinder $F \in \mathscr{F}$ belongs to C, and therefore also to $\sigma(\mathscr{D})$. Consequently, $\sigma(\mathscr{F}) \subseteq \sigma(\mathscr{D})$.

To prove that $\sigma(\mathscr{D}) \subseteq \sigma(\mathscr{F})$, it suffices to verify that any level set in C is a cylinder. To this end, we fix any $f \in \mathscr{D}$ and $\alpha \in \mathbb{R}$. By definition of $\mathscr{D}$, there are some $U \in \mathscr{U}$ and $g \in \mathcal{L}(\mathscr{X}^U)$ such that $f = g \circ \pi_U$. Let $A := \{x \in \mathscr{X}^U : g(x) \geq \alpha\}$. Then clearly

$$\{\omega \in \Omega : f(\omega) \geq \alpha\} = \{\omega \in \Omega : \pi_U(\omega) \in A\},$$

so this level set is indeed a cylinder. ■

Proof of Lemma 9. That R_E is finitely additive with $R_E(\Omega) = 1$ follows immediately because E is a linear expectation. Hence, we focus on the second part of the statement.

First, we assume that E is downward continuous. Then it follows immediately from the Daniell–Stone Theorem that $R_E = P_E|_{\mathscr{F}}$, and therefore R_E is countably additive.

Second, we assume that R_E is countably additive. Then it is well known, see for example Proposition 9 in [20, Chapter 7] or Lemma 4.3 in [33, Chapter II], that for any decreasing $(F_n)_{n\in\mathbb{N}} \in \mathscr{F}^{\mathbb{N}}$—meaning that $F_n \supseteq F_{n+1}$ for all $n \in \mathbb{N}$—with $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$,

$$\lim_{n\to+\infty} R_E(F_n) = 0. \tag{14}$$

To show that E is downward continuous, we fix any $f \in \mathscr{D}$ and any decreasing sequence $(f_n)_{n\in\mathbb{N}} \in \mathscr{D}^{\mathbb{N}}$ that converges pointwise to f. Then

$$E(f_n) - E(f) = E(f_n - f) \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{15}$$

Obviously, $(f_n - f)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathscr{D}$ that converges pointwise to 0.

Fix any $\epsilon \in \mathbb{R}_{>0}$, and let $\beta := \|f_1 - f\| = \sup f_1 - f$. Then for all $n \in \mathbb{N}$, we let $F_n := \{\omega \in \Omega : f_n(\omega) - f(\omega) > \epsilon\}$; it is a bit laborious to verify that $F_n \in \mathscr{F}$, so we leave this as an exercise to the reader. This way, $(F_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathscr{F}$ with $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$, and for all $n \in \mathbb{N}$, $f_n - f \leq \epsilon + \beta \mathbb{I}_{F_n}$ and therefore

$$E(f_n - f) \leq \epsilon + E(\mathbb{I}_{F_n}) = \epsilon + R_E(F_n).$$

It follows from this and Eqn. (14) that

$$\lim_{n\to+\infty} E(f_n - f) \leq \lim_{n\to+\infty} \epsilon + \beta R_E(F_n) = \epsilon.$$

Since this inequality holds for any strictly positive real number ϵ, we infer from it and the one in Eqn. (15) that

$$\lim_{n\to+\infty} E(f_n) = E(f),$$

as required. ■

Proof of Theorem 7. To prove that $\overline{E}$ is downward continuous, we recall from Proposition 6 that $\overline{E}$ is an upper expectation. By Lemmas 1 and 2, it suffices to verify that every dominated linear expectation E in

$$\mathbb{E}_{\overline{E}} := \{E \in \mathbb{E}_{\mathscr{D}} : (\forall f \in \mathscr{D})\ E(f) \leq \overline{E}(f)\}$$

is downward continuous. So fix any $E \in \mathbb{E}_{\overline{E}}$, and let

$$R_E : \mathscr{F} \to [0,1] : F \mapsto E(\mathbb{I}_F).$$

We know from Lemma 9 that R_E is finitely additive with $R_E(\Omega) = 1$, and that E is downward continuous if and only if R_E is countably additive. Hence, it suffices to show that R_E is countably additive, and we will do so by checking that the conditions in Lemma 8 are satisfied.

First, fix any $U \in \mathscr{U}$, and let

$$R_E^U : \wp(\mathscr{X}^U) \to [0,1] : A \mapsto R_E\big(\pi_U^{-1}(A)\big) = E\big(\mathbb{I}_{\pi_U^{-1}(A)}\big).$$

Clearly, R_E^U is a non-negative set function with $R_E^U(\mathscr{X}^U) = R_E(\Omega) = 1$ that is finitely additive. By a standard result in measure theory—see for example Proposition 9 in [20, Chapter 7] or Lemma 4.3 in [33, Chapter II]—R_E^U is countably additive, and therefore a probability measure, if and only if for any decreasing sequence $(A_k)_{k\in\mathbb{N}}$ in $\mathscr{X}^U$ with $\bigcap_{k\in\mathbb{N}} A_k = \emptyset$, $\lim_{k\to+\infty} R_E^U(A_k) = 0$. For any such sequence $(A_k)_{k\in\mathbb{N}}$, the corresponding sequence of indicators $(\mathbb{I}_{\pi_U^{-1}(A_k)})_{k\in\mathbb{N}} \in \mathscr{D}^{\mathbb{N}}$ clearly decreases to 0, and therefore

$$0 \leq \lim_{k\to+\infty} R_E^U(A_k) \leq \lim_{k\to+\infty} \overline{E}\big(\mathbb{I}_{\pi_U^{-1}(A_k)}\big) = \lim_{k\to+\infty} \overline{E}_U(\mathbb{I}_{A_k}) = 0,$$

where for the final equality we used that $\overline{E}_U$ is downward continuous and constant preserving.

Next, fix some $n \in \mathbb{N}$ and $t \in [0,n]$. Then for all $s \in \mathbb{R}_{\geq 0} \setminus \{t\}$,

$$R_E^{\{t,s\}}(D^{\neq}_{\{t,s\}}) \leq \overline{E}_{\{s,t\}}(d^{\neq}_{\{t,s\}}).$$

Hence,

$$\limsup_{s\to t} \frac{R_E^{\{t,s\}}(D^{\neq}_{\{t,s\}})}{|s-t|} \leq \limsup_{s\to t} \frac{\overline{E}_{\{t,s\}}(d^{\neq}_{\{t,s\}})}{|s-t|} \leq \lambda_n,$$

as required. ■

Proof of Proposition 13. We have already established that $(\overline{\mathrm{M}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is a semigroup of upper transition operators, so it remains for us to verify (i) that $\overline{\mathrm{M}}_t$ is downward continuous for all $t \in \mathbb{R}_{>0}$, and (ii) that $(\overline{\mathrm{M}}_t)_{t\in\mathbb{R}_{\geq 0}}$ has uniformly bounded rate.

To verify that $\overline{\mathrm{M}}_t$ is downward continuous for all $t \in \mathbb{R}_{>0}$, we fix some $t \in \mathbb{R}_{>0}$ and $z \in \mathbb{Z}_{\geq 0}$, and consider any $\mathscr{L}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow f \in \mathscr{L}$. On the one hand, since $\overline{\mathrm{M}}_t$ is isotone, $([\overline{\mathrm{M}}_t f_n](z))_{n\in\mathbb{N}}$ decreases, with $\lim_{n\to+\infty}[\overline{\mathrm{M}}_t f_n](z) \geq [\overline{\mathrm{M}}_t f](z)$. On the other hand, for all $n \in \mathbb{N}$, it follows from the subadditivity of $\overline{\mathrm{M}}_t$ that

$$[\overline{\mathrm{M}}_t f_n](z) \leq [\overline{\mathrm{M}}_t (f_n - f)](z) + [\overline{\mathrm{M}}_t f](z).$$

Hence, it suffices for us to show that

$$\lim_{n\to+\infty}[\overline{\mathrm{M}}_t (f_n - f)](z) \leq 0. \tag{16}$$

For all $n \in \mathbb{N}$, let

$$\tilde{f}_n \colon \mathbb{Z}_{\geq 0} \to \mathbb{R} \colon x \mapsto \max\left\{f_n(y) - f(y) \colon y \in \mathbb{Z}_{\geq 0}, y \leq x\right\}.$$

It is easy to verify that for all $n \in \mathbb{N}$, $\tilde{f}_n$ is a bounded function that dominates $f_n - f$, so it follows from the isotonicity of $\overline{\mathrm{M}}_t$ that

$$[\overline{\mathrm{M}}_t (f_n - f)](z) \leq [\overline{\mathrm{M}}_t \tilde{f}_n](z).$$

Moreover, since $\tilde{f}_n$ is increasing (in the sense that $\tilde{f}_n(z) \leq \tilde{f}_n(y)$ whenever $z \leq y$), it follows from Theorem 15, Proposition 16 and Eqn. (18) in [15] that

$$[\overline{\mathrm{M}}_t (f_n - f)](z) \leq \sum_{y=z}^{+\infty} \tilde{f}_n(y)\psi_{\overline{\lambda}t}(\{y-z\}) = \int \tilde{f}_n(z+\bullet)\mathrm{d}\psi_{\overline{\lambda}t},$$

where $\psi_{\overline{\lambda}t} \colon \wp(\mathbb{Z}_{\geq 0}) \to [0,1]$ is the probability measure corresponding to the Poisson distribution with parameter $\overline{\lambda}t$. Finally, it is easy to verify that $(\tilde{f}_n)_{n\in\mathbb{N}}$ is monotone and decreases pointwise to 0, so a straightforward application of the Monotone Convergence Theorem yields

$$\lim_{n\to+\infty}\int \tilde{f}_n(z+\bullet)\mathrm{d}\psi_{\overline{\lambda}t} = 0.$$

Eqn. (16) follows from this equality and the previous inequality, and this finalises our proof for the downward continuity.

Finally, we verify that the sublinear Markov semigroup $(\overline{\mathrm{M}}_t)_{t\in\mathbb{R}_{\geq 0}}$ has uniformly bounded rate—so satisfies Eqn. (4). First, note that due to constant additivity,

$$\begin{aligned}
&\limsup_{t\searrow 0}\frac{1}{t}\sup\left\{[\overline{\mathrm{M}}_t(1-\mathbb{I}_x)](x) \colon x \in \mathcal{X}\right\} \\
&\quad= \limsup_{t\searrow 0}\sup\left\{\frac{[\overline{\mathrm{M}}_t(-\mathbb{I}_x)](x) - (-\mathbb{I}_x(x))}{t} \colon x \in \mathcal{X}\right\}.
\end{aligned}$$

It follows from this, the definition of the norms $\|\bullet\|$ and $\|\bullet\|_{\mathrm{op}}^0$ and Eqn. (11) that

$$\begin{aligned}
&\limsup_{t\searrow 0}\frac{1}{t}\sup\left\{[\overline{\mathrm{M}}_t(1-\mathbb{I}_x)](x) \colon x \in \mathcal{X}\right\} \\
&\quad\leq \limsup_{t\searrow 0}\left\{\left\|\frac{\overline{\mathrm{M}}(-\mathbb{I}_x) - \mathrm{I}(-\mathbb{I}_x)}{t}\right\| \colon x \in \mathcal{X}\right\} \\
&\quad\leq \lim_{t\searrow 0}\left\|\frac{\overline{\mathrm{M}}_t - \mathrm{I}}{t}\right\|_{\mathrm{op}}^0 = \|\overline{\mathrm{L}}\|_{\mathrm{op}}^0 < +\infty,
\end{aligned}$$

where the strict inequality holds because $\overline{\mathrm{L}}$ is a bounded operator. ■