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Robust Possibilistic Production Planning under Temporal Demand Uncertainty with Knowledge on Dependencies

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Abstract

In this paper, we deal with production planning problem under temporal demand uncertainty. More precisely, a demand forecasting for a given period could move backward in time or forward in time. We investigate the case where knowledge on dependencies on demand is available. This knowledge is taken into account through a family of copula function. The aim results of the paper are: (a) this approach do not increase the complexity of production planning problem, (b) limit the conservatism of fuzzy robust approach for production planning problem and evaluates more precisely the necessity that the cost of a production plan does not exceed a certain threshold.

Keywords: production planning, fuzzy robust optimization, possibility theory

1. Introduction

Demand uncertainties are a critical factor to consider in production planning since they induce risks for the manufacturer such as backordering and obsolete inventory (see [11] for a comprehensive review). Nowadays it is often difficult to access to the probability distribution on the demand and approach based on robust optimization or fuzzy optimization give the possibility to deal with a poorer knowledge on the demand. In the literature most of the paper focus on the uncertain demands in periods [1, 10, 13], few on the uncertain demands in periods and orders (MRP) [2]. To our knowledge, the imprecision on the date of the orders has not been studied much in the literature [3, 4, 5] while it is often present and is due to production delays (see [11, 7] for a comprehensive review).

In fuzzy robust optimization some methods have been proposed to moderate the conservatism of the solution. In [9] the moderation is done by aggregated the possibility and the neccessity measure; in [8] the number of deviations (i.e budget) to the most possible value is limited by a parameter called budget and in [6] a parametric copula function to take into account the dependencies is proposed.

In this paper a capacitated production planning problem under temporal demand uncertainty is discussed. To moderate the conservatism of fuzzy robust optimization the approach proposed in [6] is developed. We show that introducing uncertainty on times in the possibilistic setting does not make our problem much computationally harder than its deterministic counterpart. Furthermore, the computational experiments performed suggest that taking additional information about possibility distributions of the cumulative demands given by fuzzy intervals and copula function may lead to better quality over a set of plausible scenarios and reduce the over conservatism of fuzzy the robust optimization approach.

2. Production Planning Problem

In this section we formulate a production planning problem that we examine in the paper. It is a simple version of the well-known *capacitated single-item lot sizing problem with backordering* (see, e.g., [12]) without setup cost. We first state the problem with precise parameters. Then we assume that demands are subject to uncertainty - the rest of the parameters are precisely known.

2.1. Deterministic Problem

We are given T periods, a *demand* d_t in each period $t, t \in [T]$ ($[T]$ denotes the set $\{1, \dots, T\}$), *production*, *inventory* and *backordering costs* and a *selling price*, denoted by c^P, c^I, c^B and b^P, respectively, which do not depend on period t. The problem is to find a feasible *production amount* x_t in each period t, called production plan, subject to the condition of satisfying each demand, which minimizes the total storage and backordering costs minus the benefit from selling the product.

Let us denote by $\mathbb{X}$ the set of *production plans*. We assume that $\mathbb{X}$ is specified by some linear constraints, for example

$$\mathbb{X} = \{\boldsymbol{x} = (x_t)_{t\in[T]} \in \mathbb{R}_+^T : x_t \geq 0, c_t^l \leq x_t \leq c_t^u, \\ K_t^L \leq \sum_{i\in[t]} x_i \leq K_t^U, t \in [T]\},$$

where c_t^l, c_t^u and K_t^L, K_t^U are given *capacity* and *cumulative capacity limits*, respectively.

Our problem can be modeled by the following linear program:

$$\min \sum_{t\in[T]} (c^I I_t + c^B B_t + c^P x_t - b^P s_t) \tag{1}$$

$$\text{s.t. } B_t - I_t = D_t - X_t \qquad t \in [T], \tag{2}$$

$$\sum_{i\in[t]} s_i = D_t - B_t \qquad t \in [T], \tag{3}$$

$$B_t, I_t, s_t \geq 0 \qquad t \in [T], \tag{4}$$

$$\boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^T_+, \tag{5}$$

where $D_t = \sum_{i\in[t]} d_i$ and $X_t = \sum_{i\in[t]} x_i$, D_t and X_t stand for the cumulative demand up to period t and the cumulative production up to period t, respectively. I_t and B_t are respectively the inventory and the backordering à period t

An easy computation shows that (1)-(5) can be rewritten in the following equivalent compact form, which is more convenient to analyze:

$$\min_{\boldsymbol{x}\in\mathbb{X}} C(\boldsymbol{x},\boldsymbol{D}) = \min_{\boldsymbol{x}\in\mathbb{X}} \sum_{t\in[T]} \max\{c^I(X_t - D_t), c^B(D_t - X_t)\} + c^P X_T - b^P \min\{X_T, D_T\}, \tag{6}$$

where $\boldsymbol{D} = (D_t)_{t\in[T]}$ is a T-vector of cumulative demands.

2.2. Model of Demand under Uncertainty

We now admit that demands are subject to temporal uncertainty. This means that a part of demand forecasting for a given period t can be advanced to the previous period $t-1$ or delayed to the next period $t+1$. More formally, let $\hat{d}_t$ be the forecasting demand at period t and $\hat{D}_t = \sum_{i\in[t]} \hat{d}_i$ the cumulative forecasting demand. The manager can give for each period $t \in [T]$ the maximum percent of demand $\delta^-_t \in [0,1]$ that can be advanced to the previous period $t-1$ and maximum percent of demand $\delta^+_t \in [0,1]$ that can be delayed to the next period $t+1$. From this knowledge fuzzy sets on the cumulative demand can be build. Fuzzy sets are interpreted as possibility distribution π. This possibility distribution $\pi_{\widetilde{\boldsymbol{D}}}$ is defined in terms of $\lambda - cut$: $\mathbf{D}^{[\lambda]} = \{\mathbf{D} | \pi_{\widetilde{\boldsymbol{D}}} \geq \lambda\}, \forall \lambda \in [0,1]$ by the following equation 7.

$$\begin{array}{lll} \mathbf{D}^{[\lambda]}: & \{\boldsymbol{D}^{[\boldsymbol{\lambda}]} \in \mathbb{R}^T & \\ s.t. & D^{[\lambda]}_t \geq \hat{D}_t - \beta^+_t \delta^+_t \hat{d}_t, \forall t \in [T] & (a) \\ & D^{[\lambda]}_t \leq \hat{D}_t + \beta^-_{t+1}\delta^-_{t+1}\hat{d}_{t+1}, \forall t \in [T] & (b) \\ & \beta^-_t \leq 1-\lambda, \forall t \in [T] & (c) \\ & \beta^+_t \leq 1-\lambda, \forall t \in [T] & (d) \\ & \beta^+_t\delta^+_t + \beta^-_t\delta^-_t \leq 1, \forall t \in [T] & (e) \\ & \beta^-_t, \beta^+_t \in [0,1], \forall t \in [T]\} & \end{array} \tag{7}$$

Constraints (a).7 and (b).7 mean that the cumulative demand of a period is imprecise due to the quantity that is advanced or delayed. Constriants (c).7 and (d).7 mean that the maximal deviation is bounded by $1-\lambda$. When $\lambda = 1$ the deviation is 0 and the cumulative demand is crisp. When $\lambda = 0$ the maximum deviation is possible. Finally constraints (e).7 mean that it is not possible to advanced to $t-1$ or delayed to $t+1$ more than all the periodic demand $\hat{d}_t$. We can see that $\mathbf{D}^{[\lambda_1]} \subseteq \mathbf{D}^{[\lambda_2]}, \forall \lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 > \lambda_2$. The scenario with possibility $\Pi(\mathbf{D}) = 1$ is the scenario without temporal deviation from the forecasting one $\mathbf{D}^{[1]} = \hat{\mathbf{D}}$.

The problem with time uncertainty can be formulate as a version of (6) with uncertain cumulative demands $\widetilde{\boldsymbol{D}} = (\widetilde{D}_t)_{t\in[T]}$:

$$\widetilde{\min_{\boldsymbol{x}\in\mathbb{X}}} C(\boldsymbol{x}, \widetilde{\boldsymbol{D}}) = \widetilde{\min_{\boldsymbol{x}\in\mathbb{X}}} \sum_{t\in[T]} C_t(x_t, \widetilde{D}_t) = \tag{8}$$

$$\sum_{t\in[T]} \max\{c^I(X_t - \widetilde{D}_t), c^B(\widetilde{D}_t - X_t)\} + c^P X_T - b^P \min\{X_T, \widetilde{D}_T\}. \tag{9}$$

For simplicity we denote by C_t (resp. C) the cost $C_t(x_t, \widetilde{D}_t)$ (resp.$C(\boldsymbol{x}, \widetilde{\boldsymbol{D}})$).

3. Robust Possibilistic Optimization Taking into Account the Dependencies Belief

Robust possibilistic optimization maximize the degree of certainty that a solution cost is lower than a given threshold g.

$$\max_{\boldsymbol{x}\in\mathbb{X}} \quad N(C \leq g) \tag{10}$$

To take into account the belief on dependencies [6] has proposed an approach based on copula function. In this approach the necessity measure is computed from the marginal possibility distribution and a function $H_\alpha : [0,1]^T \to [0,1]$ (eq.11) where $\alpha \in [0,1]$ is the degree of belief that they is negative dependencies, i.e. if $\alpha = 0$ we do not have negative dependencies between uncertainty, if $\alpha = 1$ we know that there is negative dependencies between uncertainty of each periods.

$$H_\alpha(\boldsymbol{u}) = \alpha.W(\boldsymbol{u}) + (1-\alpha)M(\boldsymbol{u}), \alpha \in [0,1] \tag{11}$$

with $W(\boldsymbol{u}) = \max\{\sum_{i=1}^T u_i - T + 1, 0\}$ and $M(\boldsymbol{u}) = \min_{i=1,\dots,T}(u_i)$.
Based on the marginal possibility distribution and a function H_α, the necessity degree to have a vector $\boldsymbol{x} \in \mathbb{R}^T$ lower or equal to a given vector $\boldsymbol{a}$ is:

$$N(\boldsymbol{x} \leq \boldsymbol{a}) = 1 - H_\alpha(\Pi(a_1 < x_1), \dots, \Pi(a_T < x_T)) \tag{12}$$

Hence the problem 10 can be formalized as:

$$\begin{array}{lll} \min_{\boldsymbol{x}\in\mathbb{X}} & \epsilon & \\ & \max_{\{\boldsymbol{c}|H_\alpha(\boldsymbol{c})\geq\epsilon\}} \sum_{t\in[T]} c_t \leq g & \\ & C_t = c_t & \forall t\in[T] \end{array} \quad (13)$$

with $H_\alpha(\boldsymbol{c}) = H_\alpha(\Pi(c_1 < C_1), ..., \Pi(c_T < C_T))$

The problem eq.13 is composed to a maximization problem in the constraints, we call this problem the adversarial problem. Before studying the specificity of the production planning problem let us recall the results on the set $\{\boldsymbol{c}|H_\alpha(\boldsymbol{c}) \geq \epsilon\}$ given in [6].

Theorem 1 *The adversarial problem:* $\max_{\{\boldsymbol{c}|H_\alpha(\boldsymbol{c})\geq\epsilon\}} \boldsymbol{c}^T\boldsymbol{x}, \alpha \in [0,1]$ *is equivalent to:*

- $\max_{\boldsymbol{c}\in C^1} \boldsymbol{c}^T\boldsymbol{x}$ *if* $\alpha = 0$,
- $\max_{\boldsymbol{c}\in C^2} \boldsymbol{c}^T\boldsymbol{x}$ *if* $\alpha = 1$,
- $\max_{\boldsymbol{c}\in C^3} \boldsymbol{c}^T\boldsymbol{x}$ *if* $W^m_{\alpha,\epsilon} \neq 0$ *and* $M^m_{\alpha,\epsilon} \neq 0$
- $\max\{\max_{\boldsymbol{c}\in C^1} \boldsymbol{c}^T\boldsymbol{x}, \max_{\boldsymbol{c}\in C^2} \boldsymbol{c}^T\boldsymbol{x}, \max_{\boldsymbol{c}\in C^3} \boldsymbol{c}^T\boldsymbol{x}\}$ *if* $W^m_{\alpha,\epsilon} = 0$ *and* $M^m_{\alpha,\epsilon} = 0$,
- $\max\{\max_{\boldsymbol{c}\in C^1} \boldsymbol{c}^T\boldsymbol{x}, \max_{\boldsymbol{c}\in C^3} \boldsymbol{c}^T\boldsymbol{x}\}$ *if* $W^m_{\alpha,\epsilon} = 0$ *and* $M^m_{\alpha,\epsilon} \neq 0$,
- $\max\{\max_{\boldsymbol{c}\in C^2} \boldsymbol{c}^T\boldsymbol{x}, \max_{\boldsymbol{c}\in C^3} \boldsymbol{c}^T\boldsymbol{x}\}$ *if* $W^m_{\alpha,\epsilon} \neq 0$ *and* $M^m_{\alpha,\epsilon} = 0$.

$$C^1 = \{\boldsymbol{c} : c_t \leq \overline{C}_t^{[M^M_{\alpha,\epsilon}]}, t \in [T]\}$$

$$\begin{aligned} C^2 = \{\boldsymbol{c} : & \sum_{t\in[T]} u_t - W^M_{\alpha,\epsilon} = T-1, \\ & c_t \leq \overline{C}_t^{[u_t]}, t\in[T], \boldsymbol{u}\in[0,1]^n\}, \end{aligned}$$

$$\begin{aligned} C^3 = \{\boldsymbol{c} : & \sum_{t\in[T]} u_t - (\zeta W^M_{\alpha,\epsilon} + (1-\zeta)W^m_{\alpha,\epsilon}) = T-1, \\ & c_j \leq \overline{C}_t^{[u_j]}, t\in[T] \\ & c_j \leq \overline{C}_t^{[\zeta M^m_{\alpha,\epsilon}+(1-\zeta)M^M_{\alpha,\epsilon}]}, t\in[T] \\ & \boldsymbol{u}\in[0,1]^n, \zeta\in[0,1]\}. \end{aligned}$$

with $W^m_{\alpha,\epsilon}$ (resp. $W^M_{\alpha,\epsilon}$) and $M^m_{\alpha,\epsilon}$ (resp. $M^M_{\alpha,\epsilon}$) the minimal (resp. maximal) value of $W(\boldsymbol{u})$ respectively $M(\boldsymbol{u})$ sucht that $\epsilon = \alpha.W(\boldsymbol{u}) + (1-\alpha)M(\boldsymbol{u})$.

In the context of production planning we do not access directly to marginal distribution π_{C_t} but we start from the marginal possibility distribution on demand $\pi_{\mathcal{D}_t}$. In the next section we will formulate the production planning problem taking into account the dependencies and show how to solve it.

4. Robust Possibilistic Production Planning Taking into Account the Dependencies Belief

4.1. Problem Formulation

The proposition 1 shows that the uncertainty set $\hat{\mathbf{D}}$ is equivalent to a more compact form which is more convenient to analyze. Proposition 2 define the marginal possibility distribution π_{D_t}. For ease of reading, the proves are in appendix.

Proposition 1 $\mathbf{D}^{[\lambda]} = \{\hat{D}_t - (1-\lambda)\delta^+_t\hat{d}_t \leq D_t \leq \hat{D}_t + (1-\lambda)\delta^-_{t+1}\hat{d}_{t+1} | D_t \leq D_{t+1} \forall t \in [T-1]\}$

Proposition 2 *The marginal possibility distribution π_{D_t} is triangular possibility distribution represented by the lower possible value: $\hat{D}_t - \delta^+_t\hat{d}_t$, the most possible value: $\hat{D}_t$ and the higher possible value: $\hat{D}_t + \delta^-_{t+1}\hat{d}_{t+1}$.*

We can now define the marginal possibility distribution on cost from the marginal possibility distribution on D_t, where $D_t^{[\lambda]}$ is the λ-cut of possiblity distribution π_{D_t}.

$$C_t^{[\lambda]} = [\min_{D_t\in D_t^{[\lambda]}} C_t, \max_{D_t\in D_t^{[\lambda]}} C_t] = [\underline{C}_t^{[\lambda]}, \overline{C}_t^{[\lambda]}] \quad (14)$$

Proposition 3 *The possibility distribution π_{C_t} is convex on $x \in [\min_{D_t\in D_t^{[0]}} C_t, \min_{D_t\in D_t^{[1]}} C_t]$ and concave on $x \in [\max_{D_t\in D_t^{[1]}} C_t, \max_{D_t\in D_t^{[0]}} C_t]$.*

The necessity degree of cost under temporal demand uncertainty taking into account dependence with function H_α is:

$$N(\boldsymbol{C} \leq \boldsymbol{c}) = 1 - \begin{cases} H_\alpha(\boldsymbol{c}) \text{ if } \exists \boldsymbol{D} \in \boldsymbol{D}^{[0]} \text{s.t. } C_t = c_t \forall t \in [T] \\ \text{else } 0 \end{cases} \quad (15)$$

The condition $\exists \boldsymbol{D} \in \boldsymbol{D}^{[0]}$ means that the cumulative demand must be nondecreasing.

The problem of Robust possibilistic production planning under temporal uncertainty taking into account dependencies belief is:

$$\begin{array}{ll} \min_{\boldsymbol{x}\in\mathbb{X}} & \epsilon \\ & \max_{\{C=\boldsymbol{c}|H_\alpha(\boldsymbol{c})\geq\epsilon\}} \sum_{t\in[T]} C_t \leq g \\ & \boldsymbol{D} \in \boldsymbol{D}^{[0]} \end{array} \quad (16)$$

4.2. Solving the Adversarial Problem

In this section we focus on the problem :

$$\begin{array}{ll} \max_{\{C=\boldsymbol{c}|H_\alpha(\boldsymbol{c})\geq\epsilon\}} & \sum_{t\in[T]} c_t \\ & \boldsymbol{D} \in \boldsymbol{D}^{[0]} \end{array} \quad (17)$$

Let us present the first important result of the paper

Theorem 2 *The problem 17 can be computed in $O(T^2)$ times*

We will now proceed to develop the proof of this result. From theorem 1 solving the problem 17 is equivalent to solving the problem with constraints $\{\boldsymbol{c} \in C^1, \boldsymbol{D} \in \boldsymbol{D}^{[0]}\}$, $\{\boldsymbol{c} \in C^2, \boldsymbol{D} \in \boldsymbol{D}^{[0]}\}$ and $\{\boldsymbol{c} \in C^3, \boldsymbol{D} \in \boldsymbol{D}^{[0]}\}$. From proposition 4 and to prove theorem 2 we study the vertex of each set of constraints described just before. These analyses are detailed in the following sections.

Proposition 4 *the optimal value of problem 17 is obtain on the vertex of the polyhedron defined by its constraints*

Remark 1 *In [4] it shows that all the vertex of cumulative demand uncertainty set $\boldsymbol{D}$ for a vector $\lambda \in [0,1]^T$ can be represented by a layered graph $G = (V, A)$. The set V is partitioned into $T+2$ disjoint layers $V_0, V_1, ..., V_{T+2}$ in which $V_0 = \{s\}$, $V_{T+1} = \{w\}$ and $V_t = \{D_t^i | D_t^i \in D_t^{[\lambda_t]} \cap \bigcup_{j\in[T]}\{\underline{D}_j^{[\lambda_t]}, \overline{D}_j^{[\lambda_t]}\}$. Let $A = A_1 \cup ... \cup A_{T+1}$. Arc $(u, v) \in A_1$ if $u \in V_0$ and $v \in V_1$, arc $(u, v) \in A_{T+1}$ if $u \in V_T$ and $v \in V_{T+1}$ and $(u, v) \in A_t, \forall t \in [T]$ if $u \in V_{t-1}$, $v \in V_t$ and $u \leq v$. We can associate with each arc a length $l_{u,w}$ in the following way:*

$$l_{u,v} = \begin{cases} C_t \text{ if } (u,v) \in A_t, \forall t \in [T] \\ 0 \text{ if } (u,v) \in A_{T+1} \end{cases} \tag{18}$$

Hence the maximal cost is the longest path of this graph.

4.3. Resolution of Problem 17 with Constraints $\{\boldsymbol{c} \in C^1, \boldsymbol{D} \in \boldsymbol{D}^{[0]}\}$

Since $D_t^{[M_{\alpha,\epsilon}^M]} \subseteq D_t^{[0]}$ the constraints of problem 17 with-constraints $\{\boldsymbol{c} \in C^1, \boldsymbol{D} \in \boldsymbol{D}^{[0]}\}$ can be reformulate as:

$$\begin{aligned} &\max \textstyle\sum_{t\in[T]} c_t \\ &c_t = \max \begin{cases} c^I(X_t - D_t) \\ c^B(D_t - X_t) \end{cases} \quad \forall t \in [T-1] \\ &c_T = \max \begin{cases} c^I(X_T - D_T) - b^P D_T \\ c^B(D_T - X_T) - b^P X_T \end{cases} \\ &D_t \in D_t^{[M_{\alpha,\epsilon}^M]}, \forall t \in [T] \\ &D_t \leq D_{t+1}, \forall t \in [T-1] \end{aligned} \tag{19}$$

From remark 1 we know that solving problem.17 with $\boldsymbol{c} \in C^1$ and $\boldsymbol{D} \in \boldsymbol{D}^{[0]}$ boils down to finding a longest path form s to w so the general complexity is $O(T^3)$. Nevertheless in context of temporal uncertainty we can ameliorate the complexity since we have $\overline{D}_t^{[M_{\alpha,\epsilon}^M]} \leq \underline{D}_{t+2}^{[M_{\alpha,\epsilon}^M]}, \forall t \in [T-2]$ hence $|\{D_t^i | D_t^i \in D_t^{[M_{\alpha,\epsilon}^M]} \cap \bigcup_{j\in[T]}\{\underline{D}_j^{[M_{\alpha,\epsilon}^M]}, \overline{D}_j^{[M_{\alpha,\epsilon}^M]}\}| \leq 6$ so A has $O(T)$ arcs and V has $O(T)$ nodes. So we have the following lemma.

Lemma 1 *The problem 17 with constraints $\{\boldsymbol{c} \in C^1, \boldsymbol{D} \in \boldsymbol{D}^{[0]}\}$ ($\alpha = 0$) can be computed in $O(T)$ times*

4.4. Resolution of Problem 17 with Constraints $\{\boldsymbol{c} \in C^2, \boldsymbol{D} \in \boldsymbol{D}^{[0]}\}$

We formulate the problem eq.20 with the convex combination of the vertex of the constraint $\sum_{t\in[T]} u_t - \epsilon = T - 1$ with $\epsilon \in [0,1]$. This vertex is composed of T vectors such that $\forall i \in [T] : \boldsymbol{u}^i = (u_i = \epsilon, u_j = 1 \; \forall j \in [T] j \neq i)$. So the problem can be reformulate with convex combination of extreme points becomes:

$$\begin{aligned} &\text{s.t.} \\ &(a) c_t = \max \begin{cases} c^I(X_t - D_t) \\ c^B(D_t - X_t) \end{cases} \quad \forall t \in [T-1] \\ &(b) c_T = \max \begin{cases} c^I(X_T - D_T) - b^P D_T \\ c^B(D_T - X_T) - b^P X_T \end{cases} \\ &D_t \in D_t^{[v_t]}, \forall t \in [T] \\ &D_t \leq D_{t+1}, \forall t \in [T-1] \\ &v_t = \textstyle\sum_{\{i\in[T] | i\neq t\}} \gamma_i + \epsilon\gamma_t, \forall t \in [T] \\ &\textstyle\sum_{t\in[T]} \gamma_t = 1 \\ &\gamma_t \in [0,1], \forall t \in [T] \end{aligned} \tag{20}$$

Proposition 5 *The constraints $D_t \leq D_{t+1}, \forall t \in [T-1]$ of problem (20) are always satisfied*

From proposition 5, for each vector $\boldsymbol{u}$ the vertex of $\boldsymbol{D}$ is the products of the extremes points of the intervals: $\prod_{t\in[T]}\{\underline{D}_t^{[\epsilon]}, \overline{D}_t^{[\epsilon]}\}$. So the vertex of problem 20 is composed of $2T$ vectors: $\forall i \in [T] : u_i = \epsilon, \forall j \neq i \in [T] \; u_j = 1$ with $D_i = \underline{D}_t^{[\epsilon]}$ or $D_i = \overline{D}_t^{[\epsilon]}$ and $\forall j \neq i \in [T], D_j = \hat{D}_j$. So we have the following lemma.

Lemma 2 *The problem 17 with constraints $\{\boldsymbol{c} \in C^2, \boldsymbol{D} \in \boldsymbol{D}^{[0]}\}$ ($\alpha = 1$) can be computed on $O(T^2)$*

4.5. Resolution of Problem 17 with Constraints $\{\boldsymbol{c} \in C^3, \boldsymbol{D} \in \boldsymbol{D}^{[0]}\}$

The set C^3 is the intersection of two sets depending on ζ namely $C_1^3(\zeta)$ and $C_2^3(\zeta)$:

$$\begin{aligned} C_1^3(\zeta) = \{\boldsymbol{c} : &\sum_{t\in[T]} u_t - (\zeta W_{\alpha,\epsilon}^M + (1-\zeta) W_{\alpha,\epsilon}^m) = T - 1, \\ &c_j \leq \overline{C}_t^{[u_j]}, t \in [T].\} \\ C_2^3(\zeta) = \{\boldsymbol{c} : & \\ &c_j \leq \overline{C}_t^{[\zeta M_{\alpha,\epsilon}^m + (1-\zeta) M_{\alpha,\epsilon}^M]}, t \in [T]\}. \end{aligned}$$

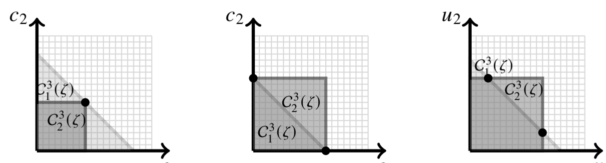
Which satisfied the properties:

Property 1 $\forall \zeta_1, \zeta_2 \in [0,1]$ *if* $\zeta_1 > \zeta_2$ *then* $C_1^3(\zeta_1) \subseteq C_1^3(\zeta_2)$

and

Property 2 $\forall \zeta_1, \zeta_2 \in [0,1]$ *if* $\zeta_1 < \zeta_2$ *then* $C_2^3(\zeta_1) \subseteq C_2^3(\zeta_2)$

Since $C_1^3(\zeta) \subseteq C_2$ and result of section 4.4 the constraints $D_t \leq D_{t+1}, \forall t \in [T-1]$ be can be removed.



(a) case of $\zeta^{\square}$ (b) case of $\zeta^{\triangle} \geq 0$ (c) case of $\zeta^{\triangle} < 0$

Figure 1: Extreme value of ζ

We can see that the polyhedron $C_1^3(\zeta)$ with equality constraints is a hyperplane and the polyhedron $C_2^3(\zeta)$ is a hyperrectangle. We denote by $\zeta^{\square}$ (eq.21) the maximal value of ζ such that $C_1^3(\zeta) \cap C_2^3(\zeta) = C_2^3(\zeta)$ and by $\zeta^{\triangle}$(eq.22) the maximal value of ζ such that $C_1^3(\zeta) \cap C_2^3(\zeta) = C_1^3(\zeta)$. Nevertheless $\zeta^{\triangle}$ can be < 0 in this case the vertex are vertex for $\zeta = 0$. The different cases are represented for 2-dimensional example on figure 1.

$$\zeta^{\square} = \frac{T(1 - M^M_{\alpha,\epsilon}) + W^m_{\alpha,\epsilon} - 1}{T(M^m_{\alpha,\epsilon} - M^M_{\alpha,\epsilon}) + W^m_{\alpha,\epsilon} - W^M_{\alpha,\epsilon}} \tag{21}$$

$$\zeta^{\triangle} = \frac{W^m_{\alpha,\epsilon} - M^M_{\alpha,\epsilon}}{M^m_{\alpha,\epsilon} - M^M_{\alpha,\epsilon} + W^m_{\alpha,\epsilon} - W^M_{\alpha,\epsilon}} \tag{22}$$

The vertex of variables $\boldsymbol{u}$ is composed of $T+1$ vectors:

$$\begin{aligned}
&\forall i \in [T],\\
&\quad \boldsymbol{u}^j = \\
&\qquad \begin{cases} \text{if } \zeta^{\triangle} \geq 0 \begin{cases} u_j = 1, \\ u_i = \zeta^{\triangle} W^M_{\alpha,\epsilon} + (1-\zeta^{\triangle}) W^m_{\alpha,\epsilon}, \\ \forall i \in [T] i \neq j \end{cases} \\ \text{else} \begin{cases} u_j = 1 - (\frac{M^M_{\alpha,\epsilon} - W^m_{\alpha,\epsilon}}{T-1}), \\ u_i = M^M_{\alpha,\epsilon}, \forall i \in [T] i \neq j \end{cases} \end{cases}\\
&\quad \boldsymbol{u}^M = (\zeta^{\square} M^m_{\alpha,\epsilon} + (1-\zeta^{\square}) M^M_{\alpha,\epsilon})_{t \in [T]}
\end{aligned}$$

Since the constraints $D_t \leq D_{t+1}, \forall t \in [T-1]$ are satisfied the vertex for each vertex vectors $\boldsymbol{u}$, $V_t = \{\underline{D}_t^{[u_t]}, \overline{D}_t^{[u_t]}\}$. The longest path for a given vertex vector $\boldsymbol{u}$ is computed in $O(T)$. The number of vertex vector is $T+1$ the complexity is in $O(T^2)$. So we have the following lemma:

Lemma 3 *The problem 17 with constraints C^3 and $\boldsymbol{D} \in \boldsymbol{D}^{[0]}$ can be computed on $O(T^2)$*

5. Solving the Robust Possibilistic Production Planning

Before we propose an algorithm for solving the problem (16), we will focus on the problem (16) for a fixed ϵ. Thus it boils down to checking if constraint $\max_{\boldsymbol{c} \in \mathcal{H}^{\alpha}_{\epsilon}} \boldsymbol{c}^T \boldsymbol{x} \leq g$ is satisfied. For the sake of brevity, we only consider the case when $W^m_{\alpha,\epsilon} \neq 0$ and $M^m_{\alpha,\epsilon} \neq 0$, the other cases can be handled in a similar manner. Consider the following mathematical programming problem:

$$\begin{aligned}
\min_{\boldsymbol{x} \in \mathbb{X}} \quad & h^{\epsilon} \\
\text{s.t.} \quad & \max_{\boldsymbol{c} \in C^1, \boldsymbol{D} \in \boldsymbol{D}^{[0]}} \boldsymbol{c}^T \boldsymbol{x} \leq h^{\epsilon}, && (23)\\
& \max_{\boldsymbol{c} \in C^2, \boldsymbol{D} \in \boldsymbol{D}^{[0]}} \boldsymbol{c}^T \boldsymbol{x} \leq h^{\epsilon}, && (24)\\
& \max_{\boldsymbol{c} \in C^3, \boldsymbol{D} \in \boldsymbol{D}^{[0]}} \boldsymbol{c}^T \boldsymbol{x} \leq h^{\epsilon}, && (25)
\end{aligned}$$

The left hand sides of the constraints (23)-(25) are the longest path problem. Based on the dual formulation of each longest path problem the problem above can be linearized.

We are now ready to propose an algorithm for solving problem (16), which is based on the standard binary search in $[0,1]$ (the interval of possible values of ϵ) due to the fact that h^{ϵ} is nonincreasing function of ϵ. We call it the *binary search based algorithm*. In order to find an optimal (x^*, ϵ^*) with a given error tolerance $\xi > 0$, we seek at each iteration, for a fixed ϵ, a feasible solution $\boldsymbol{x}$ for which $h^{\epsilon} \leq g$ is satisfied. This which boils down to solving the linear programing formulation of the problem above. The running time of the above algorithm is $O(I(T) \log \xi^{-1})$ time, where $I(|T|)$ is the time required for solving the linear program.

6. Illustration on Example and Experimental Results

In this section, we firstly present example illustrating the robust possibilistic approach (see Sec. 4) to the production planning problem 13 with temporal demand uncertainty with knowledge on dependencies (see Sec. 2.2) and compare this approach with the deterministic problem and the classic fuzzy robust one that assumes positive dependencies. Secondly experiments results will be given to discuss more generally the advantage of the proposed approach.

We consider a 10 periods example with $\hat{d} = [892, 276, 725, 832, 782, 485, 603, 560, 699, 613]$, $c^P = 100$, $b^P = 2 \times c^P$, $c^I = 0.01 \times c^P$ and $c^B = 0.2 \times b^P$. We have only upper capacity constraints $c^u = [1074, 1080, 1157, 1152, 941, 953, 1171, 930, 937, 1200]$. The goal $g = 0.2 \times C(\hat{\boldsymbol{x}}^{Opt}) + 0.8 \times C(\boldsymbol{x}^{Opt}_{\alpha=0,\epsilon=0})$ where $\hat{\boldsymbol{x}}^{Opt}$ is the optimal solution of the nominal scenario and

$\boldsymbol{x}^{Opt}_{\alpha=0,\epsilon=0}$ is the optimal solution for dependence parameter $\alpha = 0$ and necessity 1 (the most robust solution). The minimal and maximal cumulative demand with cumulative production of optimal solutions for different knowledge on dependencies are repented on figure 2. In orange the optimal solution for the nominal production. We can see that globally the fuzzy robust approach moves the cumulative production to the left to be more robust to temporal uncertainty. Moreover when α decrease this phenomena increase too. It is due to the fact that when α decrease it is more possible that the demand of all periods deviated simultaneously. Hence the knowledge on dependencies moderate the conservatism of production.

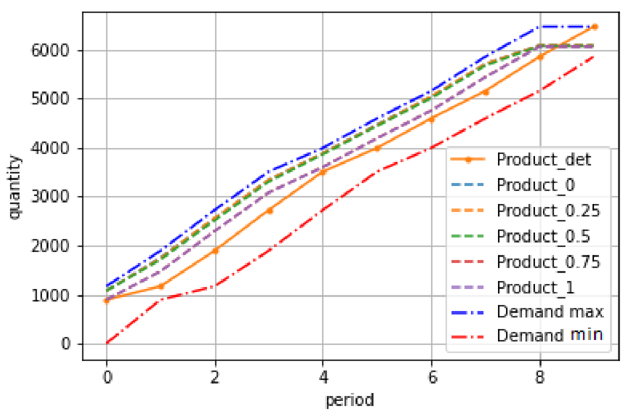


Figure 2: Cumulative production

The necessity degree of violation of $C(\boldsymbol{x}) \leq g$ is represented in Figure 3. First, we can see that deterministic solution (solution for the nominal demand) can be risked event if we have high level of negatives dependencies ($\alpha = 1$) and naturally the risk increase if the dependencies become more positives.

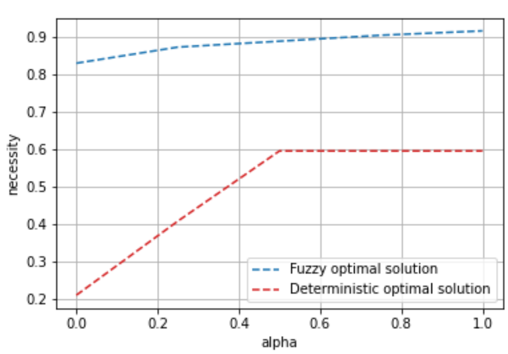


Figure 3: Necessity of $C(\hat{\boldsymbol{x}}^{Opt}) \leq g$ and $C(\boldsymbol{x}^{Opt}_{\alpha}) \leq g$

To study more generally the proposed approach, experimentation are made for 50 random instances with horizon 10 where $d = U(100, 1000), c^u = U(800, 1200), c^I = U(1, 10), c^B = U(10, 100), c^P = U(1, 10), b^P = U(200, 1000)$, $g = 0.2 \times C(\hat{\boldsymbol{x}}^{Opt}) + 0.8 \times C(\boldsymbol{x}^{Opt}_{\alpha=0,\epsilon=0})$. Figures 4–6 below show the box-plot of necessity of $C(\boldsymbol{x}^{Opt}_{\alpha}) \leq g$: *alpha Fuzzy robust*, $C(\hat{\boldsymbol{x}}^{Opt}) \leq g$: *Deterministic* and $C(\boldsymbol{x}^{Opt}_{0}) \leq g$: *Fuzzy robust*, for $\alpha \in \{0.25, 0.5, 1\}$. We can observe that when the dependencies (α parameter) increase the *Fuzzy robust* (solution for $\alpha = 0$) is less robust than the *alpha Fuzzy robust* hence over estimate the dependencies make lead to less robust solution. It is interested to note that with high dependencies the *Deterministic* solution is more robust than the *Fuzzy robust*.

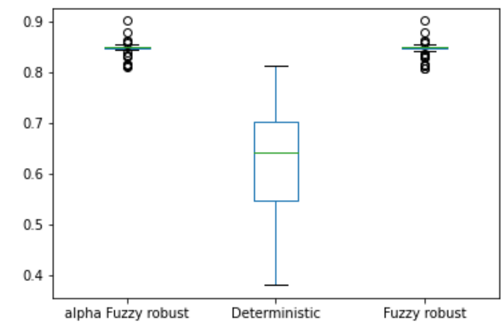


Figure 4: Degree of necessity of different types of optimal solutions for $\alpha = 0.25$

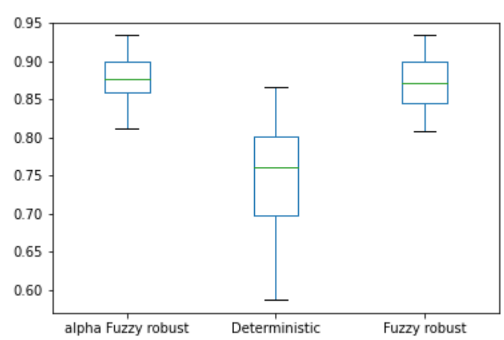


Figure 5: Degree of necessity of different types of optimal solutions for $\alpha = 0.5$

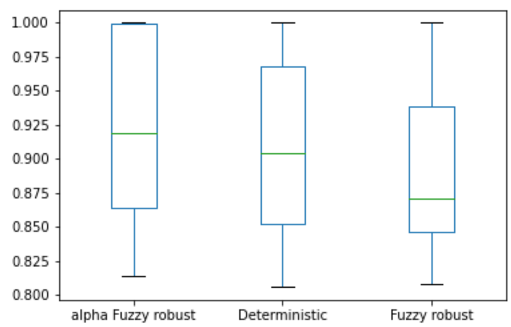


Figure 6: Degree of necessity of different types of optimal solutions for $\alpha = 1$

7. Conclusion

In this paper we show that taking into account knowledge on dependencies for production planning under temporal uncertainty do not increase the complexity. Moreover we show that this knowledge influence in term of chosen production plan and on the quality guaranty of the production plan. As perspective we would like to investigate more local dependencies, in fact in production planning the dependencies positive or negative could be dependent on the periods.

Proofs

Proof prop.1:

- Let us suppose that $\beta_t^- = \beta_t^+ = (1-\lambda)$ and constraint (e).7 is not saturated then $\hat{D}_t - (1-\lambda)\delta_t^+\hat{d}_t \leq D_t$ and $D_{t-1} \leq \hat{D}_{t-1} + (1-\lambda)\delta_t^-\hat{d}_t$.
- Let us suppose that constraint (e).7 is saturated. then $D_t \leq \hat{D}_t + \beta_{t+1}^-\delta_{t+1}^-\hat{d}_{t+1}$, $\hat{D}_{t+1} - \beta_{t+1}^+\delta_{t+1}^+\hat{d}_{t+1} \leq D_{t+1}$ and $\beta_{t+1}^+\delta_{t+1}^+ + \beta_{t+1}^-\delta_{t+1}^- = 1$. so

$$\begin{array}{rl} \hat{D}_{t+1} - \frac{1-\beta_{t+1}^-\delta_{t+1}^-}{\delta_{t+1}^+}\delta_{t+1}^+\hat{d}_{t+1} & \leq D_{t+1} \\ \hat{D}_{t+1} - \hat{d}_{t+1} + \beta_{t+1}^-\delta_{t+1}^-\hat{d}_{t+1} & \leq D_{t+1} \\ \hat{D}_t + \beta_{t+1}^-\delta_{t+1}^-\hat{d}_{t+1} & \leq D_{t+1} \\ D_t^{[\lambda]} & \leq D_{t+1} \end{array} \tag{26}$$

□

Proof prop.2:
From proposition 1 we know that $\hat{D}_t - (1-\lambda)\delta_t^+\hat{d}_t \leq \hat{D}_t \leq \hat{D}_{t+1} - (1-\lambda)\delta_{t+1}^+\hat{d}_{t+1} \forall \lambda \in [0,1]$ so the lower part of the marginal distribution is linear.
Concerning the upper bound, noted that $\hat{D}_t + (1-\lambda)\delta_{t+1}^-\hat{d}_{t+1} \leq \hat{D}_{t+1}, \forall \lambda \in [0,1]$ and $\Pi_t(\hat{D}_t) = 1$ hence $\Pi(\hat{D}_1, .., \hat{D}_t + (1-\lambda)\delta_{t+1}^-\hat{d}_{t+1}, \hat{D}_{t+1}, \hat{D}_T) = \Pi_t(\hat{D}_t + (1-\lambda)\delta_{t+1}^-\hat{d}_{t+1}) = \lambda$ whatever the copula function. So the upper bound of the marginal is linear. □

Proof prop.3:
For the minimum we can see that if $X_t \in D_t^{[\lambda]}$ then $C_t = 0, \forall t \in [T-1]$ and $C_T = -b^P X_T$. we denote by λ^* the highness value of λ such that $X_t \in D_t^{[\lambda]}$. So $\forall \lambda_i \geq \lambda^*$ the cost will be $c^I(X_t - \overline{D_t}^{[\lambda_i]}), \forall t \in [T-1]$ and $\min_{D_T \in D_T^{[\lambda]}} c^I(X_T - D_T) - b^P \min\{X_T, D_T\}$ or $c^B(\underline{D_t}^{[\lambda_i]} - X_t)$ for $t \in [T-1]$ and $\min_{D_T \in D_T^{[\lambda]}} c^B(D_T - X_T) - b^P \min\{X_T, D_T\}$. For the maximal side, we need to note that the function C_t is concave on D_t. □

Proof prop.4:
In production planning problem 17 the function is linear. Since the upper bound of the possibility distribution π_{C_t} is pairwise linear concave function (prop.3) the domain defined by the constraints $C_t = c_t, \forall t \in [T]$ is a concave polyhedron, the set $\{\boldsymbol{c} | H_\alpha(\boldsymbol{c}) \geq \epsilon\}$ is concave polyhedron (th.1) and $\boldsymbol{D} \in \boldsymbol{D}^{[0]}$ is convex polyhedron. □

Proof prop.5:
Those constraints are always satisfied iff:

$$\begin{aligned} &\hat{D}_t + (1 - \sum_{\{i\in[T] | i\neq t\}} \gamma_i + \epsilon\gamma_t)\delta_{t+1}^-\hat{d}_{t+1} \leq \\ &\hat{D}_{t+1} - (1 - \sum_{\{i\in[T] | i\neq t+1\}} \gamma_i + \epsilon\gamma_{t+1})\delta_{t+1}^+\hat{d}_{t+1}, \\ &\qquad \forall \gamma_t \in [0,1], \sum_{i\in[T]} \gamma_i = 1 \end{aligned}$$

For D_t we have:

$$\begin{aligned} &\hat{D}_t + (1 - \sum_{\{i\in[T] | i\neq t\}} \gamma_i + \epsilon\gamma_t)\delta_{t+1}^-\hat{d}_{t+1} \\ &\leq \hat{D}_t + (1 - \sum_{\{i\in[T] | i\neq t\}} \gamma_i + \epsilon\gamma_t)\hat{d}_{t+1} \\ &= \hat{D}_{t+1} - \sum_{\{i\in[T] | i\neq t\}} \gamma_i + \epsilon\gamma_t\hat{d}_{t+1} \\ &= \hat{D}_{t+1} - (1 + \gamma_t(1-\epsilon))\hat{d}_{t+1} \end{aligned}$$

And for the D_{t+1} we have:

$$\begin{aligned} &\hat{D}_{t+1} - (1 - \sum_{\{i\in[T] | i\neq t\}} \gamma_i + \epsilon\gamma_t)\delta_{t+1}^+\hat{d}_{t+1} \\ &\geq \hat{D}_{t+1} - (1 - \sum_{\{i\in[T] | i\neq t\}} \gamma_i + \epsilon\gamma_t)\hat{d}_{t+1} \\ &= \hat{D}_{t+1} - (\epsilon + \gamma_t(1-\epsilon))\hat{d}_{t+1} \end{aligned}$$

And $1 + \gamma_t(1-\epsilon) \geq \epsilon + \gamma_t(1-\epsilon)\square$

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