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The Logic FP(Ł, Ł) and Two-Sorted Equational States

Serafina Lapenta slapenta@unisa.it
Sebastiano Napolitano s.napolitano40@studenti.unisa.it
Luca Spada lspada@unisa.it
Department of Mathematics, University of Salerno, Italy.

Abstract

The logic FP(Ł,Ł) was introduced by L. Godo and T. Flaminio as an expansion of Łukasiewicz logic with a modality, to reason about the probability of vague events. We prove that FP(Ł,Ł) is complete with respect to a class of two-sorted algebras, called *equational states*. They are an equational presentation of the well-known theory of states over lattice ordered groups.

Keywords: Łukasiewicz logic, state, completeness, two-sorted algebras

1. Introduction

Łukasiewicz logic, denoted by Ł, has been introduced in the Thirties as a three-valued logic by J. Łukasiewicz and subsequently extended by J. Łukasiewicz and A. Tarski allowing for *infinitely* many degrees of truth. Nowadays, Ł is understood in the more general framework of mathematical fuzzy logic. This framework has its origin in the work of Lotfi Zadeh, that in 1965 introduced the notion of a fuzzy set, laying the groundwork for the birth of fuzzy logics as we know them. Soon after fuzzy sets were defined, researchers started to wonder whether a relation between fuzzy sets and probability theory existed. Zadeh himself argued that probability theory and fuzzy logic are complementary rather than competitive, highlighting the fact that using only classical logic to deal with probability has limitations, while the use of fuzzy logic can provide a more useful framework, see e.g. [10, 11].

Over the years, the merging of logic and probability has been approached in several ways; the most relevant for our work are to be found in [6, 4]. In [6], P. Hajek, L. Godo and F. Esteva defined a two-layer logic, called FP(Ł), to reason formally about probabilities of classical events, using Łukasiewicz logic. In FP(Ł) the atomic components are formulas that read as φ *is probable*. In [4], T. Flaminio and L. Godo extended FP(Ł) to the logic FP(Ł, Ł), in which one can formally reason about the probabilities of *vague* events, using Łukasiewicz logic. Also in FP(Ł, Ł) the atomic formulas are interpreted as φ *is probable*, but now φ is a Łukasiewicz formula that codifies a vague event.

In the same [6], the authors give a semantics for FP(Ł) based on measurable sets and probabilities and they prove completeness of the logic. Similarly, in [4] the authors define a semantics for FP(Ł, Ł) based on the notion of *state* [8] with the goal of interpreting probability measures and expectations in Łukasiewicz logic. States are, essentially, additive and normalized $[0, 1]$-valued functions defined upon appropriate abstract algebras.

At the time of [4], the authors did not obtain a completeness result for FP(Ł, Ł). In [2], Flaminio proved a *non-standard* completeness with respect to a different class of models, in which states are allowed to take values in an ultrapower of $[0, 1]$. Recently, in [3], the author finally obtained a *standard completeness* theorem for FP(Ł, Ł), i.e., completeness with respect to models in which states take values in $[0, 1]$. This shows that FP(Ł, Ł) is indeed the logic of real-valued states on Łukasiewicz events.

In this work, we propose a different semantics for FP(Ł, Ł) that is purely *algebraic*, rather than relying on the primitive notion of state and it is based on the equational presentation of states given by T. Kroupa and V. Marra in [7]. After briefly recalling some preliminary notions in Section 2, we prove the completeness of the FP(Ł, Ł) with respect to our semantics in Section 3. Finally, in Section 4 we prove that the ad-hoc model used to prove completeness is the free algebra in the two-sorted variety of equational states. We conclude the paper by remarking that our semantics is not the *equivalent algebraic semantics* of FP(Ł, Ł) in classical sense of Blok and Pigozzi and we discuss some consequences.

2. Preliminaries

2.1. Łukasiewicz Logic and MV-Algebras

We start by recalling some basic facts in the theory of Łukasiewicz logic and MV-algebras. For further details we refer the reader to [1]. The syntax of Łukasiewicz logic Ł is built as in classical logic starting from an infinite set of propositional variables Var and the *primitive connectives* $\{\to, \neg\}$. The axioms of Ł are a subset of the axioms of propositional classical logic, precisely:

(Ł1) $\varphi \to (\psi \to \varphi)$;

(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;

(Ł3) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$;

(Ł4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$.

The only deduction rule is *Modus Ponens*, i.e., from φ and $\varphi \to \psi$ it is possible to deduce ψ. We denote with Fm the set of all formulas of Ł and we use the notation $\Gamma \vdash_{Ł} \varphi$ to indicate that there exists a deduction of φ from Γ. This means that φ is obtained from instances of the axioms (Ł1)–(Ł4) and the formulas in Γ using Modus Ponens. A formula φ is a *theorem* if it can be deduced by (Ł1)–(Ł4) without further hypothesis. If φ is a theorem, we write $\vdash_{Ł} \varphi$. By $\Gamma \vdash_{Ł} \varphi \leftrightarrow \psi$ we mean that both $\Gamma \vdash_{Ł} \varphi \to \psi$ and $\Gamma \vdash_{Ł} \psi \to \varphi$ hold.

The *equivalent algebraic semantics* of Ł is given by the variety of MV-algebras, defined by C.C. Chang in 1958. An MV-algebra is an algebra $\mathbf{A} := (A, \oplus, \neg, 0)$ such that $(A, \oplus, 0)$ is a commutative monoid; $\neg$ is involutive, meaning that for any $x \in A$, $\neg\neg x = x$; and such for any $x, y \in A$, the following equality holds

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

In this work we use the following derived operations:

$$x \odot y := \neg(\neg x \oplus \neg y), \quad x \ominus y := x \odot \neg y,$$
$$x \to y := \neg x \oplus y, \qquad 1 := \neg 0.$$

Each MV-algebra has an underlying lattice order defined by setting $x \leq y$ if and only if $x \to y = 1$. The joins and meets of this order are definable as follows:

$$x \wedge y := x \odot (x \to y) \text{ and } x \vee y := \neg(\neg x \oplus y) \oplus y.$$

We also use the above abbreviations to define additional Łukasiewicz connectives. We call *MV-chain* any totally ordered MV-algebra. The standard example of an MV-algebra is $([0, 1], \oplus, \neg, 0)$ where the operations are defined, for any $x, y \in [0, 1]$, as

$$x \oplus y := \min(x + y, 1) \text{ and } \neg x := 1 - x.$$

Homomorphisms are defined, as usual, as functions that preserve all operations and constants.

For any MV-algebra $\mathbf{A}$, we call $\mathbf{A}$-*valuation* of the variables Var any function $v \colon \mathsf{Var} \to \mathbf{A}$. Notice that v is readily extended to all formulas of Ł by setting $v(\star(p_1, \dots, p_n)) := \star(v(p_1), \dots, v(p_n))$, with $p_1, \dots, p_n \in \mathsf{Var}$ and $\star$ being any connective of Ł.

Theorem 1 (Completeness) *The following conditions are equivalent for every $\varphi \in \mathsf{Fm}$:*

1. $\vdash_{Ł} \varphi$;
2. *for any MV-algebra* $\mathbf{A}$ *and any* $\mathbf{A}$*-valuation* v, $v(\varphi) = 1_A$;

To prove the previous theorem, one builds the so-called *Lindenbaum-Tarski* algebra as follows. Let Γ be a set of formulas and define on Fm the following relation

$$\varphi \equiv_{\Gamma} \psi \text{ if and only if } \Gamma \vdash_{Ł} \varphi \leftrightarrow \psi.$$

The relation $\equiv_{\Gamma}$ is an equivalence relation and the quotient $\mathsf{Fm}/{\equiv_{\Gamma}}$ can be endowed with the structure of an MV-algebra, that we denote by $\mathsf{LT}_{\Gamma}(\mathsf{Var})$. When $\Gamma = \emptyset$, we simply write $\mathsf{LT}(\mathsf{Var})$. Its elements are denoted simply by $[\varphi]$ for φ a formula. Notice that $\mathbf{A}$-evaluations are in bijection with homomorphisms from $\mathsf{LT}(\mathsf{Var})$ to $\mathbf{A}$.

2.2. The Logic FP(Ł, Ł)

The system FP(Ł, Ł), defined in [4], is obtained by expanding Łukasiewicz logic with a modality $\Box$. The intended meaning of the formula $\Box(\varphi)$ is "φ is probable". The set of all FP(Ł, Ł) formulas, denoted by PFm, is the union of the disjoint sets EFm and MFm, defined as follows:

- EFm is the set of formulas of Łukasiewicz logic, called *event formulas*. We explicitly add to the language the logical constants $\top_1$ and $\bot_1$ and we denote again by Var the set of propositional variables used to defined the formulas in EFm;
- MFm is the smallest set containing all formulas $\Box(\varphi)$, for any $\varphi \in \mathsf{EFm}$, the logical constants $\top_2$ and $\bot_2$ and closed under the connectives of Łukasiewicz logic. The formulas in MFm are called *modal formulas*, the ones of the form $\Box(\varphi)$ are called *atomic* modal formulas.

The axioms and rules of the logic are:

- all axiom schemes of Łukasiewicz (for both modal and non-modal formulas);
- the modal axioms:

(FPL1) $\Box(\neg\varphi) \leftrightarrow \neg\Box(\varphi)$;

(FPL2) $\Box(\varphi \to \psi) \to \Box(\varphi) \to \Box(\psi)$;

(FPL3) $\Box(\varphi \oplus \psi) \leftrightarrow ((\Box(\varphi) \to \Box(\varphi \odot \psi)) \to \Box(\psi))$;

- and the rules:

(MP) *Modus Ponens*: from Φ and $\Phi \to \Psi$ derives Ψ.

(N) *Necessitation*: from φ derives $\Box(\varphi)$.

Deductions in FP(Ł, Ł) are defined similarly to Ł. When a formula Φ has a deduction in FP(Ł, Ł) we write $\vdash_{FP(Ł,Ł)} \Phi$ and we call Φ a *theorem* of FP(Ł, Ł).

In the next proposition we collect some theorems of FP(Ł, Ł) which will be useful in Section 3 to prove the completeness of our algebraic semantics.

Proposition 2

(T1) $\vdash_{FP(Ł,Ł)} \Box(\bot_1) \leftrightarrow \bot_2$;

(T2) $\vdash_{FP(Ł,Ł)} \Box(\top_1) \leftrightarrow \top_2$;

(T3) $\vdash_{FP(Ł,Ł)} \Box(\varphi \leftrightarrow \psi) \to \Box(\varphi) \leftrightarrow \Box(\psi)$;

(T4) $\varphi \leftrightarrow \psi \vdash_{FP(Ł,Ł)} \Box(\varphi) \leftrightarrow \Box(\psi)$;

(T5) $\vdash_{FP(Ł,Ł)} \Box(\varphi \oplus \psi) \leftrightarrow \Box(\varphi) \oplus \Box(\psi \wedge \neg\varphi)$.

Proof For (T1), (T2) see [5, Proposition 15]. For (T3) it is sufficient to apply (FPL2) twice. (T4) follows from (T3) and Modus Ponens. It remains to prove (T5). Observe that $\varphi \oplus \psi \leftrightarrow \varphi \oplus (\psi \wedge \neg\varphi)$ is a theorem in Łukasiewicz logic, therefore using (T4):

$$\vdash_{FP(Ł,Ł)} \Box(\varphi \oplus \psi) \leftrightarrow \Box(\varphi \oplus (\psi \wedge \neg\varphi)),$$

and using (FPL3):

$$\Box(\varphi\oplus(\psi\wedge\neg\varphi)) \leftrightarrow (\Box(\varphi) \to \Box(\varphi\odot(\psi\wedge\neg\varphi))) \to \Box(\psi\wedge\neg\varphi).$$

It is easy to prove that $\varphi \odot (\psi \wedge \neg\varphi) \leftrightarrow \bot_1$ is also a theorem of Ł, therefore

$$\begin{aligned}(\Box(\varphi) \to \Box(\varphi \odot (\psi \wedge \neg\varphi))) \to \Box(\psi \wedge \neg\varphi) \leftrightarrow \\ (\Box(\varphi) \to \Box(\bot_1)) \to \Box(\psi \wedge \neg\varphi) \leftrightarrow \\ (\Box(\varphi) \to \bot_2) \to \Box(\psi \wedge \neg\varphi) \leftrightarrow \\ \big(\neg\Box(\varphi) \to \Box(\psi \wedge \neg\varphi)\big) \leftrightarrow \big(\Box(\varphi) \oplus \Box(\psi \wedge \neg\varphi)\big).\end{aligned}$$

Combining the derivations above we obtain the claim. ■

In [3, Definition 3.6 and Theorem 4.1] two semantics for FP(Ł, Ł) are presented. Both make use of the concept of state on MV-algebras. Below, in Definition 5, we propose an algebraic semantics.

2.3. Equational States on MV-Algebras

Equational states are defined in [7] with the aim of providing an equational description of states. Since equational states are two-sorted algebras, we briefly recall some basic facts on many-sorted algebras. For further details see e.g. [9, Section 2.1 and 2.2].

Let S be a set, whose elements will be called *sorts*. An *S-sorted set* is a family of sets indexed by S, in symbols $X := \{X_s \mid s \in S\}$. We write $x \in X$ as a shorthand for $x \in X_s$ for some $s \in S$. If $X = \{X_s \mid s \in S\}$ and $Y = \{Y_s \mid s \in S\}$ are S-sorted sets, an *S-sorted function* from X into Y is a family of functions $\{f_s : X_s \to Y_s \mid s \in S\}$.

Standard constructions like subsets, unions, equivalence relations, and quotients straightforwardly generalise to S-sorted sets.

Definition 3 *A* many-sorted signature *is a triple* $\Sigma = (S, \Omega, a)$, *where:*

- *S is a set (called the* set of sorts *of Σ);*
- *Ω is a set (called the* set of functions symbols *of Σ);*
- *a is a function $a : \Omega \to S^* \times S$, where S^* is the set of all finite sequences of elements of S.*

Given a many-sorted signature $\Sigma = (S, \Omega, a)$, a Σ-algebra A *consists of an S-sorted set A and, for any $f \in \Omega$ with $a(f) = (s_1, \ldots, s_n, s)$, an S-sorted function $f^{\mathsf{A}} : A_{s_1} \times \cdots \times A_{s_n} \to A_s$.*

Example 1 *Since an algebraic language must contain only function symbols, graphs, i.e., sets with a binary relation, are not algebraic structures in a 1-sorted language. However, they can be presented in the 2-sorted algebraic signature* $(\{v, e\}, \{\mathrm{dom}, \mathrm{cod}\}, a)$, *where* $a(\mathrm{dom}) = (e, v)$ *and* $a(\mathrm{cod}) = (e, v)$. *Therefore, a graph in this language is a pair of sets (E, V), understood as the set of vertices and the set on edges, and the interpretations of* dom *and* cod *are the functions giving, for any edge, its starting and arriving vertex, respectively.*

The equational states of our interested can be presented in the 2-sorted signature (which we fix henceforth) $\underline{\Sigma} := (\underline{S}, \underline{\Omega}, \underline{a})$, where $\underline{S} := \{1, 2\}$, $\underline{\Omega} := \{\oplus_1, \neg_1, 0_1, \oplus_2, \neg_2, 0_2, s\}$, and $\underline{a}(s) = (1, 2)$ —the other values of $\underline{a}$ being the obvious ones.

Definition 4 ([7, Definition 3.1]) *An* equational state *is a $\underline{\Sigma}$-algebra*

$$\mathsf{A} = \langle (A_1, A_2), \oplus_1^{\mathsf{A}}, \neg_1^{\mathsf{A}}, 0_1^{\mathsf{A}}, \oplus_2^{\mathsf{A}}, \neg_2^{\mathsf{A}}, 0_2^{\mathsf{A}}, s^{\mathsf{A}} \rangle$$

such that:

1. $\mathbf{A}_1 = (A_1, \oplus_1^{\mathsf{A}}, \neg_1^{\mathsf{A}}, 0_1^{\mathsf{A}})$ *is an MV-algebra;*
2. $\mathbf{A}_2 = (A_2, \oplus_2^{\mathsf{A}}, \neg_2^{\mathsf{A}}, 0_2^{\mathsf{A}})$ *is an MV-algebra;*
3. $s^{\mathsf{A}} : A_1 \to A_2$ *is a unary operation, called* state-operation, *such that, for each $a, b \in A_1$:*

   (S1) $s^{\mathsf{A}}(0_1^{\mathsf{A}}) = 0_2^{\mathsf{A}}$;

   (S2) $s^{\mathsf{A}}(\neg_1^{\mathsf{A}} a) = \neg_2^{\mathsf{A}} s^{\mathsf{A}}(a)$;

   (S3) $s^{\mathsf{A}}(a \oplus_1^{\mathsf{A}} b) = s^{\mathsf{A}}(a) \oplus_2^{\mathsf{A}} s^{\mathsf{A}}(b \wedge_1^{\mathsf{A}} (\neg_1^{\mathsf{A}} a))$.

When the context suffices to disambiguate we simply write $\mathsf{A} := (\mathbf{A}_1, \mathbf{A}_2)$ *to indicate an equational state as above.*

By [7, Proposition 3.1] the unary operation s is order-preserving.

Example 2 *Let* $Ł_n$ *be the n-element MV-chain on*

$$\left\{\frac{i}{n-1} \mid i \in \mathbb{N} \text{ and } i \leq n-1\right\},$$

where the operations are inherited from the MV-algebra $[0,1]$*. The algebra* $\mathsf{A} = (Ł_3{}^2, Ł_5)$*, where* s^{A} *is defined by:*

$$\begin{aligned} s^{\mathsf{A}}(0,0) &:= 0 \\ s^{\mathsf{A}}(1/2,0) &:= s(0,1/2) = 1/4 \\ s^{\mathsf{A}}(1,0) &:= s(0,1) = s(1/2,1/2) = 1/2 \\ s^{\mathsf{A}}(1,1/2) &:= s(1/2,1) = 3/4 \\ s^{\mathsf{A}}(1,1) &:= 1. \end{aligned}$$

is an equational state.

Example 3 *Let* $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ *be a homomorphism of MV-algebras. The structure* $(\mathbf{A}_1, \mathbf{A}_2)$ *with state-operation h is an equational state.*

Example 4 *Let* $\mathbf{A}_1$ *be the MV-algebra of continuous functions from* $[0,1]$ *into* $[0,1]$*, with operations defined pointwise and let* $\mathbf{A}_2$ *be the MV-algebra* $[0,1]$ *defined in Section 2. The algebra* $\mathsf{A} = (\mathbf{A}_1, \mathbf{A}_2)$ *with* s^{A} *defined as the Riemann integral is an equational state.*

3. The Logic FP(Ł, Ł) and Equational States

In this section we prove that equational states provide an algebraic semantics for FP(Ł, Ł).

Definition 5 *A* $\underline{S}$-sorted model *is a pair* $((\mathbf{A}, \mathbf{B}), v)$*, where:*

- $(\mathbf{A}, \mathbf{B})$ *is an equational state,*
- *v is* **A***-valuation of Łukasiewicz logic.*

Let Φ *be a formula of FP(Ł, Ł). A* valuation *in* $((\mathbf{A}, \mathbf{B}), v)$ *of* Φ*, denoted by* $\|\Phi\|$*, is defined as follows:*

- *if* $\Phi = \varphi$ *is an event formula, then* $\|\varphi\| := v(\varphi)$*;*
- *if* $\Phi = \Box(\varphi)$ *is an atomic modal formula, then* $\|\Box(\varphi)\| := s(v(\varphi))$*;*
- *if* $\Phi = \tau[\Box(\varphi_1), \ldots, \Box(\varphi_n)]$*, where* τ *is an MV-term, then* $\|\Phi\| := \tau^{\mathbf{B}}(\|\Box(\varphi_1)\|, \ldots, \|\Box(\varphi_n)\|)$*.*

Finally, we say that Φ *is* valid *in* $((\mathbf{A}, \mathbf{B}), v)$ *if:*

- $\|\Phi\| = 1_A$ *if* Φ *is a event formula;*
- $\|\Phi\| = 1_B$ *if* Φ *is a modal formula.*

We start by proving the soundness of the two-sorted semantics for the logic FP(Ł, Ł).

Proposition 6 (Soundness) *Let be* $\Phi \in \mathsf{PFm}$*. If* $\vdash_{FP(Ł,Ł)} \Phi$*, then every* $\underline{S}$*-sorted model verifies* Φ*.*

Proof It is sufficient to prove soundness of the Necessitation rule and of the axioms (FPL1)–(FPL3). The proofs for (FPL1) and (FPL2) are easy computations. The Necessitation rule is sound because, by the completeness of Ł, every event formula φ such that $\vdash_{\mathrm{FP}(Ł,Ł)} \varphi$ satisfies $v(\varphi) = 1_A$ for every **A**-valuation. Thus

$$\|\Box(\varphi)\| = s(v(\varphi)) = s(1_A) = 1_B.$$

The only case that remains is (FPL3). It is enough to prove that

$$\|\Box(\varphi \oplus \psi)\| = ((\|\Box(\varphi)\| \to \|\Box(\varphi \odot \psi)\|) \to \|\Box(\psi)\|).$$

By straightforward computation, we have

$$\begin{aligned} (\|\Box(\varphi)\| &\to \|\Box(\varphi \odot \psi)\|) \to \|\Box(\psi)\| \\ &= \neg(\neg\|\Box(\varphi)\| \oplus \|\Box(\varphi \odot \psi)\|) \oplus \|\Box(\psi)\| \\ &= \neg(\neg s(v(\varphi)) \oplus s(v(\varphi \odot \psi))) \oplus s(v(\psi)). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} s(\neg v(\varphi)&\oplus v(\varphi \odot \psi)) \\ &= \neg s(v(\varphi)) \oplus s(v(\varphi) \wedge v(\varphi \odot \psi)) \\ &= \neg s(v(\varphi)) \oplus s(v(\varphi \odot \psi)). \end{aligned}$$

Then

$$\begin{aligned} \neg[\neg s(v(\varphi)) &\oplus s(v(\varphi \odot \psi))] \oplus s(v(\psi)) \\ &= \neg[s(\neg v(\varphi) \oplus v(\varphi \odot \psi))] \oplus s(v(\psi)) \\ &= \neg[s(\neg v(\varphi) \oplus (v(\varphi) \odot v(\psi)))] \oplus s(v(\psi)) \\ &= s[v(\varphi) \odot (\neg v(\varphi) \oplus \neg v(\psi))] \oplus s(v(\psi)) \\ &= s(v(\varphi) \wedge \neg v(\psi)) \oplus s(v(\psi)) \\ &= s(v(\varphi) \oplus v(\psi)) = s(v(\varphi \oplus \psi)). \end{aligned}$$

Since $\|\Box(\varphi \oplus \psi)\| = s(v(\varphi \oplus \psi))$, the thesis is proved. ■

A key step to prove completeness is to translate modal formulas into Łukasiewicz formulas, as done in [5, Section 4.1]. We briefly recall here the translation, which is defined inductively on PFm.

- If $\varphi \in \mathsf{EFm}$ then $\varphi^* := \varphi$;
- $(\top_2)^* = \top$;
- For any atomic formula $\Box(\varphi)$ in MFm, we pick a fresh propositional variable p_φ and set $\Box(\varphi)^* := p_\varphi$;

- For any compound modal formula $\Phi := \star(\Phi_1, ..., \Phi_n)$, with $\star$ any connective of Łukasiewicz logic, we set $\Phi^* := \star(\Phi_1^*, ..., \Phi_n^*)$.

Let $\Gamma \subseteq \mathsf{MFm}$, we set

$$\begin{aligned}\mathsf{FPL}^* :=&\{\Psi^* \mid \Psi \text{ is a instance of } (FPL1)\text{–}(FPL3)\}\cup\\ &\cup\{p_\varphi \mid \ \vdash_Ł \varphi\};\\ \Gamma^* :=&\{\Phi^* \mid \Phi \in \Gamma\}.\end{aligned}$$

Theorem 7 ([5, Lemma 18]) *Let be $\Gamma \cup \{\Phi\}$ be a subset of MFm, then*

$$\Gamma \vdash_{FP(Ł,Ł)} \Phi \quad \textit{if and only if} \quad \Gamma^* \cup \mathsf{FPL}^* \vdash_Ł \Phi^*.$$

We are now ready to prove the completeness of FP(Ł, Ł) with respect to two-sorted models.

Theorem 8 (Completeness) *Let be $\Phi \in \mathsf{PFm}$. If every $\underline{S}$-sorted model validates Φ, then $\vdash_{FP(Ł,Ł)} \Phi$.*

Proof We construct an $\underline{S}$-sorted model that validates a formula Ψ if and only if $\mathsf{FPL}^* \vdash_Ł \Psi^*$. Let LT be the Lindenbaum-Tarski MV-algebra of the event formulas. Notice that the set $\mathcal{B} := \{\Psi^* \mid \Psi \in \mathsf{MFm}\}$ is closed under Łukasiewicz connectives, by the definition of the translation *. We define

$$\Phi^* \equiv_{\mathsf{FPL}^*} \Psi^* \text{ if and only if } \mathsf{FPL}^* \vdash_Ł \Phi^* \leftrightarrow \Psi^*.$$

It is easy to see that $\equiv_{\mathsf{FPL}^*}$ is an equivalence relation, which under the standard Lindenbaum-Tarski construction, leads to an MV-algebra $\mathsf{LT}_\Box := \mathcal{B}/\equiv_{\mathsf{FPL}^*}$. We denote a generic element of $\mathsf{LT}_\Box$ by $[\Psi^*]$.

Observe that $\Box$ induces a function $s_\Box \colon \mathsf{LT} \to \mathsf{LT}_\Box$ by setting, for $[\varphi]_{\mathsf{LT}} \in \mathsf{LT}$,

$$s_\Box([\varphi]_{\mathsf{LT}}) := [\Box(\varphi)^*].$$

Claim 3.1 *The function $s_\Box$ is a state-operation from LT into $\mathsf{LT}_\Box$.*

The assignment $s_\Box$ is well-defined: if $\vdash_Ł \varphi \leftrightarrow \psi$ then by Proposition 2(T4) and Theorem 7, $\mathsf{FPL}^* \vdash_Ł \Box(\varphi)^* \leftrightarrow \Box(\psi)^*$. So we conclude

$$s_\Box([\varphi]_{\mathsf{LT}}) = [\Box(\varphi)^*] = [\Box(\psi)^*] = s_\Box([\psi]_{\mathsf{LT}}).$$

Axiom (S1) holds because, by (T2), $s_\Box([\top_1]_{\mathsf{LT}}) = [(\Box(\top_1))^*] = [(\top_2)^*] = [\top]$. To prove axiom (S2), notice that

$$\begin{aligned}s_\Box(\neg[\varphi]_{\mathsf{LT}}) &= s_\Box([\neg\varphi]_{\mathsf{LT}}) && \text{by definition of } \mathsf{LT}\\ &= [\Box(\neg\varphi)^*] && \text{by definition of } s_\Box\\ &= [\neg\Box(\varphi)^*] && \text{using (FPL2)}\\ &= \neg[\Box(\varphi)^*] && \text{by definition of } \mathsf{LT}_\Box\\ &= \neg(s_\Box([\varphi]_{\mathsf{LT}})) && \text{by definition of } s_\Box.\end{aligned}$$

Finally, reasoning as above and using (T5), we prove (S3).

$$\begin{aligned}s_\Box([\varphi]_{\mathsf{LT}} \oplus [\psi]_{\mathsf{LT}}) &= s_\Box([\varphi \oplus \psi]_{\mathsf{LT}})\\ &= [\Box(\varphi \oplus \psi)^*]\\ &= [\Box(\varphi)^*] \oplus [\Box(\psi \wedge \neg\varphi)^*]\\ &= s_\Box([\varphi]_{\mathsf{LT}}) \oplus s_\Box([\psi]_{\mathsf{LT}} \wedge \neg[\varphi]_{\mathsf{LT}}).\end{aligned}$$

This settles the Claim. Now, assuming that LT is generated by the one-sorted set Y, consider the equational state $\mathcal{ES}_Y := (\mathsf{LT}, \mathsf{LT}_\Box)$ with state operation $s_\Box$ and the $\underline{S}$-sorted model $(\mathcal{ES}_Y, v)$, where $v(\varphi) = [\varphi]_{\mathsf{LT}}$. It is easy to see that if φ is a an event formula then $\|\varphi\| = [\varphi]_{\mathsf{LT}}$ and if Ψ is an atomic modal formulas then $\|\Psi\| = [\Psi^*]$. An easy induction completes the proof that a formula Ψ is valid in $(\mathcal{ES}_Y, v)$ if and only if $\vdash_{\mathsf{FP}(Ł,Ł)} \Psi$.

We now prove the statement of the theorem by contraposition: assume $\nvdash_{\mathsf{FP}(Ł,Ł)} \Phi$, for some $\Phi \in \mathsf{PFm}$. By Theorem 7, $\mathsf{FPL}^* \nvdash_Ł \Phi^*$ and hence Φ is not valid in $(\mathcal{ES}_Y, v)$. ■

4. Algebraic Proprieties of $\mathcal{ES}_Y$

The following definitions are adaptations of the definitions in [9, Section 2.2] to the particular case of equational states.

Definition 9 *Let $\mathsf{A} := (\mathbf{A}_1, \mathbf{A}_2)$ and $\mathsf{B} := (\mathbf{B}_1, \mathbf{B}_2)$ be two equational states. An $\underline{S}$-sorted function $h := (h_1, h_2) \colon \mathsf{A} \to \mathsf{B}$ is said to be a* homomorphism of equational states *(or $\underline{\Sigma}$-*homomorphism*) if $h_1 \colon \mathbf{A}_1 \to \mathbf{B}_1$ and $h_2 \colon \mathbf{A}_2 \to \mathbf{B}_2$ are homomorphisms of MV-algebras, and $h_2 \circ s^{\mathsf{A}} = s^{\mathsf{B}} \circ h_1$.*

Definition 10 *Let (S_1, S_2) be an $\underline{S}$-sorted set. An equational state A is said to be the* free state generated by (S_1, S_2) *if it is generated by (S_1, S_2) and for every equational state B, any function $f \colon (S_1, S_2) \to \mathsf{B}$ extends (uniquely) to a $\underline{\Sigma}$-homomorphism $h \colon \mathsf{A} \to \mathsf{B}$.*

Henceforth, we fix a set of variables $X := (X_1, X_2)$. We denote by $T_{\underline{\Sigma}}(X)_1$ the set of MV-terms over the variables in X_1 and by $T_{\underline{\Sigma}}(X)_2$ the set of terms of the form $\tau(t_1, \ldots, t_n)$, where τ is an MV-term and each t_i is either an MV-term in the variables X_2 or one of the form $s(t)$ for $t \in T_{\underline{\Sigma}}(X)_1$. The $\underline{S}$-sorted set of $\underline{\Sigma}$-terms in X is given by $T_{\underline{\Sigma}}(X) := (T_{\underline{\Sigma}}(X)_1, T_{\underline{\Sigma}}(X)_2)$. This set can be naturally endowed with the structure of a $\underline{\Sigma}$-algebra.

Definition 11 *A X-*valuation *in an equational state A is simply an $\underline{S}$-function from X into A. Any X-valuation v extends, in a unique way, to a $\underline{\Sigma}$-homomorphism from $T_{\underline{\Sigma}}(X)$ into A, which we still indicate with v.*

A $\underline{\Sigma}$-*equation* is a pair of terms (t, t') either both belonging to $T_{\underline{\Sigma}}(X)_1$ or to $T_{\underline{\Sigma}}(X)_2$. We indicate equations just by writing $t = t'$.

Definition 12 *Let* A *be an equational state,* $t = t'$ *be a* $\underline{\Sigma}$*-equation and let* X *be the two-sorted set of variables appearing in* t *or* t'*. The equational state* A validates $t = t'$ *if for any* X*-valuation* $v : X \to$ A *the equality* $v(t) = v(t')$ *holds in* A.

In the general setting of many-sorted algebras extra care is needed when defining the notion of equations and validity, as sorts can be empty. However, in the case of equational states, both sorts must be non-empty because there is a constant symbol for each sort. Therefore, we can safely use the standard definitions. We refer to [9] for a detailed discussion on equations in the general many-sorted setting. Finally, notice that by (the many-sorted version of) Birkhoff's Variety Theorem [9, Corollary 3.14] the class of equational states is a variety of many-sorted algebras.

Let us fix a one-sorted set of variables Y. In this section we write LT for the Lindenbaum-Tarski algebra of Ł generated by Y and $\mathsf{LT}_\Box$ for the set obtained from LT as in Theorem 8. We also write $\mathcal{ES}_Y$ for $(\mathsf{LT}, \mathsf{LT}_\Box)$. In this section we prove that $\mathcal{ES}_Y$ is the free equational state generated by $(Y, \emptyset)$.

Lemma 13 *The MV-algebra generated by* $s_\Box(LT)$ *is* $LT_\Box$*; in symbols:* $LT_\Box = \langle s_\Box(LT)\rangle_{MV}$*.*

Proof With the same notation of Theorem 8, if $[\psi] \in \mathsf{LT}$, then $\Box(\psi)^*$ belongs to $\mathcal{B}$ and consequently, $[\Box(\psi)^*]$ belongs to $\mathsf{LT}_\Box$. Hence, $\langle s_\Box(\mathsf{LT})\rangle_{MV} \subseteq \mathsf{LT}_\Box$.

Conversely, assume $[\Phi^*] \in \mathsf{LT}_\Box$. By the definition of $\mathsf{LT}_\Box$, Φ is a modal formula, so there is an MV-term τ and a n-tuple $\varphi_1, \ldots, \varphi_n \in$ EFm such that $\Phi = \tau(\Box(\varphi_1), \ldots, \Box(\varphi_n))$. Therefore, $\Phi^* = \tau(\Box(\varphi_1)^*, \ldots, \Box(\varphi_n)^*)$. Then,

$$\begin{aligned}[\Phi^*] &= [\tau(\Box(\varphi_1)^*, \ldots, \Box(\varphi_n)^*)]\\ &= \tau([\Box(\varphi_1)^*], \ldots, [\Box(\varphi_n)^*])\\ &= \tau(s_\Box([\varphi_1]_{\mathsf{LT}}), ..., s_\Box([\varphi_n]_{\mathsf{LT}})).\end{aligned}$$

Therefore $[\Phi^*] \in \langle s_\Box(\mathsf{LT})\rangle_{MV}$. Consequently, $\mathsf{LT}_\Box \subseteq \langle s_\Box(\mathsf{LT})\rangle_{MV}$. ■

Lemma 14 *Let* Γ *be a set of Łukasiewicz formulas,* **A** *an MV-algebra and* v *an* **A**-*valuation that validates* Γ*. The function* $h : LT_\Gamma(Var) \to \mathbf{A}$ *defined by* $h([\varphi]_\Gamma) = v(\varphi)$ *is an MV-homomorphism.*

Proof For brevity, let us denote the elements of $\mathsf{LT}_\Gamma(\mathsf{Var})$ by $[\varphi]_\Gamma$. We first show that h is well defined. Let φ and ψ be formulas such that $[\varphi]_\Gamma = [\psi]_\Gamma$. It follows that $\Gamma \vdash_Ł \varphi \leftrightarrow \psi$, and since by hypothesis $v(\Gamma) = 1_A$, we have $v(\varphi \leftrightarrow \psi) = 1_A$. Notice that $v(\varphi \leftrightarrow \psi) = 1_A$ if and only if $v(\varphi) = v(\psi)$. Therefore $h([\varphi]_\Gamma) = h([\psi]_\Gamma)$ and h is well defined. Finally, h is a homomorphism because v is a valuation and therefore it preserves all MV-operations. ■

Theorem 15 *Let* Y *be a set of variables. Then* $\mathcal{ES}_Y$ *is the free equational state generated by the* $\underline{S}$*-sorted set* $(Y, \emptyset)$*.*

Proof Since Y generates LT, it follows by Lemma 13 that $\mathcal{ES}_Y$ is generated by $(Y, \emptyset)$. Hence, to settle the claim, we prove that for every function $f = (f_1, f_2) : (Y, \emptyset) \to (\mathbf{A}, \mathbf{B})$ there exists exactly one $\underline{\Sigma}$-homomorphism $h = (h_1, h_2) : (\mathsf{LT}, \mathsf{LT}_\Box) \to (\mathbf{A}, \mathbf{B})$ that extends f. It is easy to observe that there is exactly one h_1 that extends f_1, and every homomorphism from $\mathsf{LT}_\Box$ to $\mathbf{B}$ extends f_2. It remains to be proved that there is a unique homomorphism h_2 such that $h_2(s_\Box([\varphi]_{\mathsf{LT}})) = s(h_1([\varphi]_{\mathsf{LT}}))$. Since $((\mathbf{A}, \mathbf{B}), h_1)$ is an $\underline{S}$-sorted model, it validates (FPL1)–(FPL3) and every formula of the type $\Box(\varphi)$, for $\vdash_Ł \varphi$. Moreover, it is easy to prove that a valuation of Φ is also a valuation of Φ^*. Then we have a valuation that verifies FPL^*. We apply Lemma 14 with $\Gamma := \mathsf{FPL}^*$ to obtain a homomorphism h_2 such that $h_2(s_\Box([\varphi]_{\mathsf{LT}})) = h_2([\Box(\varphi)^*]) = s(h_1([\varphi]_{\mathsf{LT}}))$.

Finally, we need to verify that h_2 is unique with this propriety. Let h' be a homomorphism such that $s(h_1([\varphi]_{\mathsf{LT}})) = h'(s_\Box([\varphi]_{\mathsf{LT}}))$. We prove by induction on the complexity of Φ that $h_2(\Phi) = h'(\Phi)$.
If $\Phi = \Box(\varphi)$, then

$$\begin{aligned}h_2([\Box(\varphi)^*]) &= h_2(s_\Box([\varphi]_{\mathsf{LT}}))\\ &= s(h_1([\varphi]_{\mathsf{LT}}))\\ &= h'(s_\Box([\varphi]_{\mathsf{LT}}))\\ &= h'([\Box(\varphi)^*]).\end{aligned}$$

If $\Phi = \tau(\Box(\varphi_1), \ldots, \Box(\varphi_n))$, then

$$\begin{aligned}h_2([\Phi^*]) &= \tau(h_2([\Box(\varphi_1)^*]), \ldots, h_2([\Box(\varphi_n)^*]))\\ &= \tau(h'([\Box(\varphi_1)^*]), \ldots, h'([\Box(\varphi_n)^*]))\\ &= h'([\Phi^*]).\end{aligned}$$

Therefore $h = h'$. ■

To conclude this paper, we study the relations between $\underline{\Sigma}$-equations and the formulas of FP(Ł, Ł). We have proved in Theorem 15 that $\mathcal{ES}_Y$ is the free equational state generated by $(Y, \emptyset)$. Thus $\mathcal{ES}_Y$ is isomorphic to the equational state of equivalence classes of terms in $T_{\underline{\Sigma}}((Y, \emptyset))$ because free equational states are unique up to isomorphism. Moreover, it is easy to see that an $\underline{S}$-sorted model $((\mathbf{A}, \mathbf{B}), v_1)$ can be seen as a $(Y, \emptyset)$-valuation $v = (v_1, v_\emptyset) : (Y, \emptyset) \to (\mathbf{A}, \mathbf{B})$.

Thus, every formula $\Phi \in$ PFm can be translated in a $\underline{\Sigma}$-term Φ^t. The translation is obtained by structural induction starting from an assignment that maps Var in Y. In particular, formulas in EFm are translated in MV-terms in which the equational state-function s does not occur. If $\Phi(\Box(\varphi_1), \ldots, \Box(\varphi_n))$ is a formula in MFm, its translation has the form $\tau(s(t_1), \ldots, s(t_n))$, where $\tau, t_1, \ldots, t_n$ are MV-terms. We always denote the translation of a formula $\Phi \in$ PFm by Φ^t.

Corollary 16 *Let Φ be a formula and $\Phi^=$ be the equation $\Phi^t = 1$, where 1 is either 1_A or 1_B according to the sort of Φ. The following are equivalent:*

1. $\vdash_{FP(Ł,Ł)} \Phi$;
2. *$\Phi^=$ is valid in any equational state.*

Proof It is a consequence of Theorem 8 and the definition of validity of an equation. ■

Remark 17 (From terms to formulas) *As we mentioned before, any formula in PFm can be translated in a $\underline{\Sigma}$-term. Moreover, if t is a $\underline{\Sigma}$-term of the first sort, there exists $\varphi \in$ EFm such that the translation of t is φ. This follows from the remark that terms in the first sort are MV-terms and formulas in EFm are Łukasiewicz formulas.*

However, a term like $x \oplus s(y)$ —where x is a variable in the second sort and y is a variable in the first sort— does not have a translation in $FP(Ł,Ł)$. Consequently, only some of the $\underline{\Sigma}$-terms in the second sort can be viewed as a valuation $v\colon (Y, \emptyset) \to \mathcal{ES}_Y$. When τ is a term that can be translated in a formula, we denote its translation by Φ_τ.

Remark 17 yields that equational states are not the equivalent algebraic semantics (in the classical sense of Blok and Pigozzi) of FP(Ł, Ł). Loosely speaking, an equivalent algebraic semantics needs also an inverse translation from terms to formulas, which satisfies an analogous of Corollary 16.

Since not every term can be translated into a formula, we restrict to a subset of $T_{\underline{\Sigma}}(X)$. For any $\underline{S}$-sorted set of variables X, we define $T_{\underline{FS}}(X) := (T_{\underline{FS}}(X)_1, T_{\underline{FS}}(X)_2)$ as follows:

- $(T_{\underline{FS}}(X))_1 = (T_{\underline{S}}(X))_1$;
- $(T_{\underline{FS}}(X))_2$ is the set of terms of the form $\tau(s(t_1), \ldots, s(t_n))$, for τ term in Łukasiewicz logic, and $t_1, \ldots t_n \in T_{\underline{FS}}(X)_1$.

By the definitions of PFm and $T_{\underline{FS}}(X)$, it is straightforward to see that belonging to $T_{\underline{FS}}(X)$ is a necessary and sufficient condition for a term to have a translation in a formula of FP(Ł, Ł).

Definition 18 *An equational state $(\mathbf{A}, \mathbf{B})$ such that $\langle s(\mathbf{A})\rangle_{MV} = \mathbf{B}$ is called* full state. *We write* FS *for the class of all full states.*

Theorem 19 *Let $\tau \in T_{\underline{FS}}(X)$ and let $\tau^=$ be the equation $\tau = 1$, where 1 is either 1_A or 1_B according to the sort of τ. The following are equivalent:*

1. $\vdash_{FP(Ł,Ł)} \Phi_\tau$;
2. *$\tau^=$ is valid in any full state.*

Proof Preliminary notice that by Remark 17 we can restrict our attention to terms in $T_{\underline{FS}}(X)_2$. Moreover, by the definition of $T_{\underline{FS}}(X)$, we can translate such terms into formulas. One direction follows from Corollary 16. To prove the other implication, suppose that $\nvdash_{\mathrm{FP}(Ł,Ł)} \Phi_\tau$. Then by Theorem 8, the inequality $[\Phi^*_\tau] \neq [\top]$ holds in $\mathcal{ES}_X$. By Lemma 13, $\mathcal{ES}_X \in$ FS, consequently $\tau = 1$ is not valid in FS, settling the claim. ■

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