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Vertical Barrier Models as Unified Distortions

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Abstract

Vertical Barrier Models (VBM) are a family of imprecise probability models that generalise a number of well known distortion/neighbourhood models (such as the Pari-Mutuel Model, the Linear-Vacuous Model, and others) while still being relatively simple. Several of their properties were established by Pelessoni, Vicig, and Corsato. In this paper we explore, in a finite framework, further facets of these models: their interpretation as neighbourhood models, the structure of their credal set in terms of maximum number of its extreme points, the result of merging operations with VBMs, conditions for VBMs to be belief functions or possibility measures.

Keywords: vertical barrier models, distortion models, neighbourhood models, 2-monotonicity, belief functions, possibility measures.

1. Introduction

Several imprecise probability models originate from a given precise probability P_0. While taking account of the agent's possible uncertainty on P_0 being the 'true' representation of a certain situation, these models are generally computationally tractable and easy to explain to non-experts. In fact, they may be obtained as functions of P_0, therefore applying a *distortion* to P_0.

The Vertical Barrier Model (VBM) is an interesting instance of such *distortion models*. It was introduced in [4] within the larger family of *Nearly-linear* (NL) models. NL models perform a linear affine transformation to P_0; given $a \in \mathbb{R}, b > 0$, they determine a lower probability given by $\underline{P}(A) = bP_0(A) + a$ whenever this value belongs to $[0, 1]$, $\underline{P}(A) = 0$ if $bP_0(A)+a < 0$ and $\underline{P}(A) = 1$ if $bP_0(A)+a > 1$. Vertical Barrier Models add some constraints to parameters a, b. They are the most prominent subfamily of NL models, because *(a)* they are always (coherent and) 2-monotone, and *(b)* they include the Pari Mutuel Model, the Linear Vacuous Model and other remarkable distortion models as special cases [4]. Several of their properties have been investigated in [4, 14, 15, 16], in the frame of NL models: formulae for their natural extensions are given in [16], their being stable with conditioning (i.e. conditioning a VBM on an event B returns a VBM) is assessed in [14], their dilation properties are established in [14, 15]. In this paper we investigate further features of VBMs. In doing this, we also relate to previous work discussing these aspects for partly overlapping models in [9, 10, 11].

After recalling essential preliminary notions in Section 2, we investigate in Section 3 how VBMs may be viewed as neighbourhood models. For this, a distorting function d_{VBM} is introduced in Section 3.1, demonstrating some of its properties and proving that the lower probability of a VBM can be obtained as the lower envelope of a neighbourhood of P_0, defined by means of d_{VBM}. Section 3.2 shows that some relevant neighbourhood models, such as the Constant Odds Ratio model, are not included into VBMs. The structure of the credal set of a VBM is investigated in Section 4. The main result of this section achieves a strict bound on the maximum number of extreme points of the credal set. Section 5 explores the behaviour of VBMs under the merging procedures of disjunction, conjunction and convex mixture. Section 6 discusses special VBMs. In Section 6.1, necessary and/or sufficient conditions are seeked for a VBM to be (or not to be) a belief function, while Section 6.2 characterises those VBMs that are maxitive (or minitive). Section 7 concludes the paper.

2. Preliminary Properties

Consider an arbitrary possibility space Ω. Let $\mathbb{P}(\Omega)$ denote the set of all probability measures defined on the power set $\mathcal{P}(\Omega)$. We shall use $\subseteq$ to denote inclusion and $\subset$ to denote strict inclusion between events.

Vertical Barrier Models are defined as follows:

Definition 1 *Consider $P_0 \in \mathbb{P}(\Omega)$ and two parameters a, b with $a \le 0, b > 0$ and $a+b \in [0, 1]$. The corresponding* Vertical Barrier Model *(VBM) is identified by (P_0, a, b); its lower probability $\underline{P}$ is given by*

$$\underline{P}(A) = \max\{bP_0(A) + a, 0\}\ \forall A \subset \Omega \text{ and } \underline{P}(\Omega) = 1.$$

The above definition implies that $\underline{P}(\emptyset) = 0$. Note that $P_0(\{\omega\})$ is not required to be strictly positive for every ω; in this paper, the positivity assumption shall only occasionally be imposed, notably in most of Section 6.

VBMs are superior to other subfamilies of nearly-linear uncertainty measures in that they satisfy 2-monotonicity:

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B) \ \forall A, B \subseteq \Omega,$$

and as a consequence also coherence. We will use $\mathcal{M}(\underline{P})$ to denote the *credal set* associated with $\underline{P}$, given by

$$\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}(\Omega) \mid P(A) \geq \underline{P}(A) \ \forall A \subseteq \Omega\},$$

and $\overline{P}$ for the conjugate *upper probability*, given by $\overline{P}(A) = 1 - \underline{P}(A^c)$ for every $A \subseteq \Omega$. One way to define coherence of $\underline{P}$, $\overline{P}$ is to require that $\underline{P}(A) = \min_{P \in \mathcal{M}(\underline{P})} P(A)$ and $\overline{P}(A) = \max_{P \in \mathcal{M}(\underline{P})} P(A)$ for every $A \subseteq \Omega$ [19].

As shown in [16], Vertical Barrier Models include as particular cases some of the most important distortion models in the literature:

- When $a < 0$ and $a + b = 1$, they correspond to the *Pari-Mutuel Model* (PMM) [19], whose lower probability is often written as $\underline{P}_{PMM}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\}$, with $\delta > 0$, so that $a = -\delta$, $b = 1 + \delta$.
- when $a = 0$ and $b < 1$, they boil down to *Linear-Vacuous mixtures* (LV) [7];
- finally, when $b = 1$ and $a \in (-1, 0)$ they correspond to the *Total Variation model* (TV) [6].

It was established in [16, Section 3] that the lower (or upper) probability of any VBM can be expressed as a convex combination of a PMM and a vacuous probability. Specifically, denote by $\underline{P}$ (the lower probability of) a VBM (P_0, a, b). If $a + b > 0$, it holds that

$$\underline{P}(A) = (a+b)\underline{P}_{PMM}(A) + (1-(a+b))\underline{P}_V(A) \ \forall A, \quad (1)$$

where $\underline{P}_{PMM}$ is used to denote the PMM determined by $P_0, \delta = -\frac{a}{a+b}$ and $\underline{P}_V$ denotes the vacuous lower probability, given by $\underline{P}_V(A) = 0$ for every $A \subset \Omega$ and $\underline{P}_V(\Omega) = 1$. On the other hand, when $a + b = 0$ the VBM $\underline{P}$ is equal to $\underline{P}_V$.

It is not difficult to establish that the converse also holds:

Lemma 2 *Let $\underline{P}_{PMM}$ be the PMM associated with a probability measure P_0 and $\delta > 0$, and consider $\epsilon \in (0, 1)$. Then the convex mixture $\underline{P} := (1-\epsilon)\underline{P}_{PMM} + \epsilon \underline{P}_V$ is a VBM.*

Proof Clearly, $\underline{P}(\Omega) = 1$. For $A \neq \Omega$, we have

$$\begin{aligned}\underline{P}(A) &= (1-\epsilon)\max\{(1+\delta)P_0(A) - \delta, 0\} + \epsilon \underline{P}_V(A) \\ &= \max\{(1-\epsilon)(1+\delta)P_0(A) - (1-\epsilon)\delta, 0\},\end{aligned}$$

which identifies a VBM (P_0, a, b) with $a = -(1-\epsilon)\delta < 0$, $b = (1-\epsilon)(1+\delta) > 0$ and $a + b = 1 - \epsilon \in (0, 1)$. ∎

In the sequel, we shall assume that the possibility space Ω is *finite*.

3. VBMs as Neighbourhood Models

Distortion models appear in the literature in two forms: either as a transformation of a probability measure by means of some function [1, 2, 3] or as lower envelopes of neighbourhoods of a probability measure [6, 7, 17]. A unified procedure was presented in [10], showing that the first type of models can be embedded into the second by considering the neighbourhood determined by a suitable premetric. In this section, we show how this can be done for VBMs.

3.1. Expression in Terms of a Distorting Function

Given a distorting function $d : \mathbb{P}(\Omega) \times \mathbb{P}(\Omega) \to [0, \infty)$, a probability measure $P_0 \in \mathbb{P}(\Omega)$ and $\delta > 0$, we define the set

$$B_d^\delta(P_0) = \{P \in \mathbb{P}(\Omega) \mid d(P, P_0) \leq \delta\},$$

and call it the *distortion model* on P_0 associated with d, δ.

In this section, we shall prove that the credal set $\mathcal{M}(\underline{P})$ of a VBM can be obtained as the distortion model associated with some distorting function. Since the LV and PMM were already dealt with in [10], here we shall consider parameters $a < 0 < b$ such that $a + b < 1$. Given such parameters, let us define the function $d_{VBM_{(a,b)}}$ on $\mathbb{P}(\Omega) \times \mathbb{P}(\Omega)$ by

$$d_{VBM_{(a,b)}}(P, P_0) = \max_{A \subseteq \Omega} \frac{P_0(A) - P(A)}{(1-b)P_0(A) - a}. \quad (2)$$

In order to alleviate the notation, we shall denote $d_{VBM_{(a,b)}}$ as d_{VBM} in the sequel, because the properties we shall establish will hold irrespective of the a, b that we fix.

Lemma 3 *d_{VBM} is well-defined and takes values in $[0, +\infty)$.*

Proof To see that d_{VBM} is well-defined, observe that $(1-b)P_0(A) - a > 0$ for every $A \subseteq \Omega$:

- $P_0(A) = 0 \Rightarrow (1-b)P_0(A) - a = -a > 0$;
- $P_0(A) = 1 \Rightarrow (1-b)P_0(A) - a = 1 - b - a > 0$;
- $P_0(A) \in (0, 1) \Rightarrow 0 < a(P_0(A) - 1) = aP_0(A) - a < (1-b)P_0(A) - a$;

on the other hand, $d_{VBM}(P, P_0) \geq \frac{P_0(\emptyset) - P(\emptyset)}{(1-b)P_0(\emptyset) - a} = 0$ and moreover $d_{VBM}(P, P_0)$ is bounded using that Ω is finite. ∎

Let us state some properties of d_{VBM}:

Proposition 4 *Let $a < 0$, $b > 0$ with $a + b < 1$ and let d_{VBM} be given by Equation (2).*

(a) $d_{VBM}(P, P_0) = 0 \Leftrightarrow P = P_0$. [definiteness]

(b) *For every* $\alpha \in [0,1]$, $d_{VBM}(\alpha P_1 + (1-\alpha)P_2, P_0) \leq \max\{d_{VBM}(P_1,P_0), d_{VBM}(P_2,P_0)\}$. *[quasiconvexity]*

(c) $\forall P_0, P_1, P_2 \in \mathbb{P}(\Omega), \forall \epsilon > 0, \exists \delta > 0$ *such that if* $||P_1 - P_2|| < \delta$ *then* $|d_{VBM}(P_1,P_0) - d_{VBM}(P_2,P_0)| < \epsilon$, *where* $||\cdot||$ *is the supremum norm, given by* $||P_1 - P_2|| = \max_{A\subseteq\Omega} |P_1(A) - P_2(A)|$. *[continuity]*

Proof

(a) Trivially, $P = P_0$ implies that $d_{VBM}(P,P_0) = 0$. Conversely, the equality $d_{VBM}(P,P_0) = 0$ implies that $P_0(A) \leq P(A)\ \forall A \subseteq \Omega$, which, taking into account that Ω is finite, is equivalent to $P_0(A) = P(A)\ \forall A \subseteq \Omega$.

(b) When $\alpha \in \{0,1\}$ the thesis is trivial. Let $\alpha \in (0,1)$. Observe that for any event A

$$\frac{P_0(A) - (\alpha P_1(A) + (1-\alpha)P_2(A))}{(1-b)P_0(A) - a} = \alpha \frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} + (1-\alpha)\frac{P_0(A) - P_2(A)}{(1-b)P_0(A) - a},$$

and this sum is bounded by the maximum of $\left\{\frac{P_0(A)-P_1(A)}{(1-b)P_0(A)-a}, \frac{P_0(A)-P_2(A)}{(1-b)P_0(A)-a}\right\}$. From this the thesis follows.

(c) Define $m = \min_{A\subseteq\Omega}\{(1-b)P_0(A) - a\}$. By Lemma 3 and the finiteness of Ω, it is $m > 0$. Fix now $\epsilon > 0$ and take $\delta := m\epsilon$. Then

$$\begin{aligned}
&|d_{VBM}(P_1,P_0) - d_{VBM}(P_2,P_0)| \\
&= \left|\max_{A\subseteq\Omega} \frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} - \max_{A\subseteq\Omega} \frac{P_0(A) - P_2(A)}{(1-b)P_0(A) - a}\right| \\
&= \left|\max_{A\subseteq\Omega} \frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} + \min_{A\subseteq\Omega} \frac{P_2(A) - P_0(A)}{(1-b)P_0(A) - a}\right| \\
&= \left|\max_{A\subseteq\Omega} \left(\frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} + \min_{A\subseteq\Omega} \frac{P_2(A) - P_0(A)}{(1-b)P_0(A) - a}\right)\right| \\
&\leq \left|\max_{A\subseteq\Omega} \left(\frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} + \frac{P_2(A) - P_0(A)}{(1-b)P_0(A) - a}\right)\right| \\
&= \left|\max_{A\subseteq\Omega} \frac{P_2(A) - P_1(A)}{(1-b)P_0(A) - a}\right| \\
&\leq \frac{||P_1 - P_2||}{m} < \frac{m\epsilon}{m} = \epsilon,
\end{aligned}$$

from which the thesis follows.

■

As we have said, VBMs include as particular cases the PMM, LV and TV distortion models. The distorting functions associated with these models were investigated in [10, 11] for the particular case where $P_0(\{\omega\}) > 0$ for every ω. Let us study the connection between those functions and the function d_{VBM} given in Equation (2), if this is extended to the case where $a = 0$ or $a + b = 1$ by taking the maximum on those events A where the denominator is different from zero:

- If $a+b = 1$ and $P_0(\{\omega\}) > 0\ \forall\omega$, then $d_{VBM}(P,P_0) = \max_{A\subset\Omega} \frac{P_0(A)-P(A)}{-a(1-P_0(A))} = \max_{A\subset\Omega} \frac{P_0(A)-P(A)}{\delta(1-P_0(A))}$, which corresponds to $\frac{d_{PMM}}{\delta}$ for $\delta = -a$ and d_{PMM} the distorting function of a PMM in [10, Section 4.1].
- If $a = 0$ and $P_0(\{\omega\}) > 0\ \forall\omega$, then $d_{VBM}(P,P_0) = \max_{\emptyset\neq A\subseteq\Omega} \frac{P_0(A)-P(A)}{(1-b)P_0(A)}$ which corresponds to $\frac{d_{LV}}{\delta}$ for $\delta = 1-b$ and d_{LV} the distorting function of a LV in [10, Section 4.2].
- And finally, if $b = 1$, we obtain $d_{VBM}(P,P_0) = \max_{A\subset\Omega} \frac{P_0(A)-P(A)}{-a}$, that is a scalar transformation of the total variation distance d_{TV}.

Taking into account these connections, it is not difficult to establish that d_{VBM} is not symmetric nor it satisfies the triangle inequality in general: we simply need to refer to the counterexamples for d_{PMM} in [10, Proposition 4.1 (b)].

Next, we prove that d_{VBM} is indeed the distorting function associated with the VBMs. An alternative, more complex proof could be made using similar arguments to those in [10, 11] together with Proposition 4.

Theorem 5 *Let $\underline{P}$ be a VBM associated with a probability measure P_0 and with parameters $a < 0, b > 0$ such that $a + b < 1$. Then $\mathcal{M}(\underline{P}) = B^1_{d_{VBM}}(P_0)$.*

Proof Consider $P \in \mathbb{P}(\Omega)$. Then $P \in B^1_{d_{VBM}}(P_0)$ iff

$$\begin{aligned}
d_{VBM}(P,P_0) = \max_{A\subset\Omega} &\frac{P_0(A) - P(A)}{(1-b)P_0(A) - a} \leq 1 \\
&\Leftrightarrow P_0(A) - P(A) \leq (1-b)P_0(A) - a\ \forall A \subset \Omega \\
&\Leftrightarrow P(A) \geq bP_0(A) + a\ \forall A \subset \Omega.
\end{aligned}$$

Since $P(A) \geq 0\ \forall A \subseteq \Omega$ and $P(\Omega) = \underline{P}(\Omega) = 1$, this is equivalent to $P(A) \geq \underline{P}(A)\ \forall A \subseteq \Omega$, which in turn is equivalent to $P \in \mathcal{M}(\underline{P})$. ■

Observe that the radius of the neighbourhood above is always equal to 1, and seemingly does not depend on the parameters a, b; this is because a, b are incorporated in the definition of d_{VBM}.

3.2. Relationship with Other Distortion Models

As mentioned earlier, VBMs include as particular cases some of the most prominent distortion models considered in the literature. It is not hard to show that they do not include some other important models, such as the Constant Odds Ratio (COR), Kolmogorov and L_1-models discussed in [10, 11]. In the case of the L_1-models, it suffices to observe that L_1-models do not satisfy the property of 2-monotonicity [11] while VBMs do. To see that they do not include the COR or Kolmogorov models either [1], note that any VBM satisfies

$$\underline{P}(A \cup B) - \underline{P}(A) - \underline{P}(B) = -a \tag{3}$$

for every $A, B \subseteq \Omega$ such that $A \cap B = \emptyset$ and $\min\{\underline{P}(A), \underline{P}(B)\} > 0$. This is the basis of the following counterexample:

Example 1 *Consider $\Omega = \{\omega_1, \omega_2, \omega_3\}$, P_0 the probability measure associated with the mass function* $(0.5, 0.3, 0.2)$ *and* $\delta = 0.1$. *From [10, Example 6.1], the corresponding COR model satisfies*

$$\underline{P}_{COR}(\{\omega_1, \omega_2\}) - \underline{P}_{COR}(\{\omega_1\}) - \underline{P}_{COR}(\{\omega_2\}) = 0.0305$$

and

$$\underline{P}_{COR}(\{\omega_1, \omega_3\}) - \underline{P}_{COR}(\{\omega_1\}) - \underline{P}_{COR}(\{\omega_3\}) = 0.02,$$

whence $\underline{P}_{COR}$ is not a VBM by Equation (3).

On the other hand, the Kolmogorov distortion model induced by (P_0, δ) satisfies

$$\underline{P}_K(\{\omega_1, \omega_2\}) - \underline{P}_K(\{\omega_1\}) - \underline{P}_K(\{\omega_2\}) = 0.2$$

and

$$\underline{P}_K(\{\omega_1, \omega_3\}) - \underline{P}_K(\{\omega_1\}) - \underline{P}_K(\{\omega_3\}) = 0,$$

whence $\underline{P}_K$ is not a VBM either. ♦

4. Structure of the Credal Set

Next we investigate the complexity of the credal set associated with a VBM, in terms of the maximal number of its extreme points. Recall that given a credal set $\mathcal{M}$, an element $P \in \mathcal{M}$ is an *extreme point* when $P = \alpha P_1 + (1-\alpha) P_2$ for $\alpha \in (0,1), P_1, P_2 \in \mathcal{M}$ implies that $P = P_1 = P_2$.

Since any VBM is 2-monotone [4, Proposition 4.1], the extreme points of $\mathcal{M}(\underline{P})$ are determined by the permutations of Ω. Denote by S_n the set of permutations of $\{1, \ldots, n\}$. Then, given $\sigma \in S_n$, defining the probability measure P_σ by means of the equalities

$$P_\sigma(\{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(j)}\}) = \overline{P}(\{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(j)}\})$$

for $j = 1, \ldots, n$, it is $ext(\mathcal{M}(\underline{P})) = \{P_\sigma : \sigma \in S_n\}$.

This allows us to bound above the number of extreme points of $\mathcal{M}(\underline{P})$ by $n!$. As we shall show next, this bound can be tightened. For this, let us express $\overline{P}$ as a convex combination of $\overline{P}_{PMM}$ (the upper probability that is conjugate to $\underline{P}_{PMM}$) and the vacuous upper probability $\overline{P}_V$, that is given by

$$\overline{P}_V(A) = 1 \; \forall A \neq \emptyset \text{ and } \overline{P}_V(\emptyset) = 0; \tag{4}$$

from Equation (1), it is $\overline{P} = (1-\alpha)\overline{P}_{PMM} + \alpha \overline{P}_V$, with $\alpha = 1 - (a + b)$.

Let $j_\sigma = \min\{i : \overline{P}(\{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(i)}\}) = 1\}$. If $j_\sigma \geq 2$, this produces

$$\begin{aligned}
P_\sigma(\{\omega_{\sigma(1)}\}) &= (1-\alpha)\overline{P}_{PMM}(\{\omega_{\sigma(1)}\}) + \alpha \\
P_\sigma(\{\omega_{\sigma(i)}\}) &= \overline{P}(\{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(i)}\}) \\
&\quad - \overline{P}(\{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(i-1)}\}) \\
&= (1-\alpha)\overline{P}_{PMM}(\{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(i)}\}) + \alpha \\
&\quad - (1-\alpha)\overline{P}_{PMM}(\{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(i-1)}\}) - \alpha \\
&= (1-\alpha)\overline{P}_{PMM}(\{\omega_{\sigma(i)}\}) \\
&\quad \text{for } i = 2, \ldots, j_\sigma - 1 \\
P_\sigma(\{\omega_{\sigma(j_\sigma)}\}) &= (1-\alpha)\underline{P}_{PMM}(\{\omega_{\sigma(j_\sigma)}, \ldots, \omega_{\sigma(n)}\}) \\
P_\sigma(\{\omega_{\sigma(k)}\}) &= 0 \quad \text{for } k = j_\sigma + 1, \ldots, n.
\end{aligned}$$

This means that, when $j_\sigma \geq 2$, the same extreme point shall be induced by $(j_\sigma - 2)!(n - j_\sigma)!$ permutations: the ones with the same elements in the positions $\{2, \ldots, j_\sigma - 1\}$ and in positions $\{j_\sigma + 1, \ldots, n\}$.

Define now, for a real positive x, $\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \leq x\}$, $\lceil x \rceil = \min\{n \in \mathbb{N} : n \geq x\}$.

We shall make use of the following lemma. Its proof is elementary and therefore omitted.

Lemma 6 *Given $n, k \in \mathbb{N}$ with $n \geq k$,*

$$k!(n-k)! \geq \left\lfloor \frac{n}{2} \right\rfloor! \left\lceil \frac{n}{2} \right\rceil! \tag{5}$$

As a consequence, for any $j \geq 2$, it holds that

$$(j-2)!(n-j)! \geq \left\lfloor \frac{n}{2} - 1 \right\rfloor! \left\lceil \frac{n}{2} - 1 \right\rceil!$$

Proposition 7 *Given a VBM $\underline{P}$ on a space Ω of cardinality n, any vertex of the credal set $\mathcal{M}(\underline{P})$ is obtained from at least $\lfloor \frac{n}{2} - 1 \rfloor! \lceil \frac{n}{2} - 1 \rceil!$ different permutations of Ω.*

Proof Consider a permutation σ. When $j_\sigma = \min\{i : \overline{P}(\{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(i)}\}) = 1\} \geq 2$, there are $(j_\sigma - 2)!(n - j_\sigma)!$ different permutations that originate the same vertex, and by Lemma 6 this value is not smaller than $\lfloor \frac{n}{2} - 1 \rfloor! \lceil \frac{n}{2} - 1 \rceil!$.

On the other hand, if $j_\sigma = 1$ and consequently $\overline{P}(\{\omega_{\sigma(1)}\}) = 1$, then all the permutations σ' satisfying

[1] We refer to [10, 11] for the expression of the lower probabilities of these two models and a deeper account of their properties.

$\sigma'(1) = \sigma(1)$ produce the same extreme point, and there are

$$(n-1)! \geq \left\lfloor \frac{n}{2} - 1 \right\rfloor ! \left\lceil \frac{n}{2} - 1 \right\rceil !$$

such permutations. ■

This allows us to establish the main result in this section:

Theorem 8 *Given a VBM $\underline{P}$ on a space Ω of cardinality n, the maximum number of extreme points of $\mathcal{M}(\underline{P})$ is*

$$\frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !}. \tag{6}$$

Proof That $\frac{n!}{\lfloor \frac{n}{2}-1 \rfloor ! \lceil \frac{n}{2}-1 \rceil !}$ is an upper bound to the number of extreme points is a consequence of Proposition 7.

But since TV models are a particular case of VBM, the maximal number of extreme points must be at least as large as the maximal number of extreme points for TV models, that was established in [11, Proposition A.1] to be

$$\frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor ! (n - \lfloor \frac{n}{2} \rfloor - 1)!} = \frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !}.$$

The double inequality establishes the result. ■

If we look at other particular cases of VBM, we observe that the maximal number of extreme points in Equation (6) is strictly larger than the other particular cases discussed in [10], namely:

- n in the case of LV models;
- $\frac{n!}{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2}-1 \rfloor ! \lceil \frac{n}{2}+1 \rceil !}$ in the case of the PMM.

In the following situation the upper bound (6) is notably lowered:

Proposition 9 *Given a VBM $\underline{P}$ on a space Ω of cardinality n, if $\underline{P}(\{\omega\}) > 0, \forall \omega \in \Omega$, the maximum number of extreme points of $\mathcal{M}(\underline{P})$ is $n(n-1)$.*

Proof By monotonicity of $\underline{P}$, we have that, $\forall A \in \mathcal{P}(\Omega), A \neq \emptyset, A \neq \Omega$, $\underline{P}(A) > 0$ and therefore $\overline{P}(A) < 1$. Hence, $j_\sigma = n$ for any permutation in the proof of Proposition 7, meaning that any vertex is originated by $(n-2)!$ different permutations. Thus, the bound (6) reduces to

$$\frac{n!}{(n-2)!} = n(n-1).$$

■

We recall that the same bound $n(n-1)$ applies to TV models too, when $\underline{P}$ is strictly positive, as shown in [11, Proposition 2.5]. Note also that the condition $\underline{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$ in the above result is equivalent to

$$-\frac{a}{b} < \min_{\omega \in \Omega} P_0(\{\omega\}).$$

5. Processing Vertical Barrier Models

Next we investigate the behaviour of the family of VBMs under a number of merging operations. By *merging*, we refer to a procedure aggregating several belief models, defined on the same domain Ω, into a unique one. These models may arise as the opinion of different experts or from several data sources, for instance. The problem of aggregating imprecise beliefs has been analysed from the axiomatic point of view by Walley in [18]. Other relevant works on this topic are [12, 13].

In this paper, we shall focus on the three most fundamental merging procedures: those of disjunction, conjunction and convex mixture.

Definition 10 *Given two credal sets $\mathcal{M}_1, \mathcal{M}_2$ on Ω, their* disjunction *is given by $\mathcal{M}_1 \cup \mathcal{M}_2$.*

If we interpret $\mathcal{M}_1, \mathcal{M}_2$ as the sets of models that are considered acceptable by two different experts, the disjunction $\mathcal{M}_1 \cup \mathcal{M}_2$ considers acceptable those models that are acceptable for at least one of them.

If we denote by $\underline{P}_1, \underline{P}_2, \underline{P}^\cup$ the lower probabilities obtained as lower envelopes of $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_1 \cup \mathcal{M}_2$, respectively, it holds that

$$\underline{P}^\cup(A) = \min\{\underline{P}_1(A), \underline{P}_2(A)\} \; \forall A.$$

The disjunction $\mathcal{M}_1 \cup \mathcal{M}_2$ is not convex in general; it is not difficult to show that $\underline{P}^\cup$ is also the lower envelope of the convex hull $ch(\mathcal{M}_1 \cup \mathcal{M}_2)$.

It was shown in [5, Example 2] that the disjunction of two PMMs does not produce a PMM in general. The same example can be used to establish that the family of VBMs is not closed under disjunction:

Example 2 *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the probability measures P_0^1, P_0^2 on $\mathcal{P}(\Omega)$ associated with* $(0.5, 0.3, 0.2)$ *and* $(0.3, 0.5, 0.2)$*, respectively. Let $\delta_1 = \delta_2 = 0.1$, and denote by $\underline{P}_1, \underline{P}_2$ the PMMs determined by (P_0^1, δ_1) and (P_0^2, δ_2), respectively. Then $\underline{P}_1, \underline{P}_2$ and their disjunction $\underline{P}^\cup$ are given in the following table:*

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\underline{P}_1(A)$	0.45	0.23	0.12	0.78	0.67	0.45
$\underline{P}_2(A)$	0.23	0.45	0.12	0.78	0.45	0.67
$\underline{P}^\cup(A)$	0.23	0.23	0.12	0.78	0.45	0.45

To see that $\underline{P}^\cup$ is not the lower probability associated with a VBM, observe that

$$\underline{P}^\cup(\{\omega_1, \omega_2\}) - \underline{P}^\cup(\{\omega_1\}) - \underline{P}^\cup(\{\omega_2\}) = 0.32$$
$$\neq 0.1 = \underline{P}^\cup(\{\omega_1, \omega_3\}) - \underline{P}^\cup(\{\omega_1\}) - \underline{P}^\cup(\{\omega_3\}),$$

a contradiction with (3). ♦

The second merging operation we analyse in this paper is that of conjunction:

Definition 11 *Given two credal sets $\mathcal{M}_1, \mathcal{M}_2$ on Ω, their* conjunction *is given by $\mathcal{M}_1 \cap \mathcal{M}_2$.*

If we interpret $\mathcal{M}_1, \mathcal{M}_2$ as the sets of models that are considered acceptable by two different experts, the conjunction $\mathcal{M}_1 \cap \mathcal{M}_2$ only considers acceptable those models that are acceptable for both of them. Unlike disjunction, the process of conjunction always produces a convex (though possibly empty) credal set. Note that, if we denote by $\underline{P}_1, \underline{P}_2, \underline{P}^{\cap}$ the lower envelopes of $\mathcal{M}_1, \mathcal{M}_2$ and the conjunction $\mathcal{M}_1 \cap \mathcal{M}_2$, it will hold that

$$\underline{P}^{\cap}(A) \geq \max\{\underline{P}_1(A), \underline{P}_2(A)\} \, \forall A,$$

with the inequality being possibly strict on some events. A sufficient condition for the equality $\underline{P}^{\cap} = \max\{\underline{P}_1, \underline{P}_2\}$ is precisely the convexity of $\mathcal{M}_1 \cup \mathcal{M}_2$, as shown in [20, Theorem 6]. The equality was investigated in the case of possibility measures in [8].

It was shown in [5, Example 5] that the family of TV models is not closed under conjunction. Using that example, we can easily establish that the family of VBMs is not closed under conjunction either:

Example 3 *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the probability measures P_0^1, P_0^2 on $\mathcal{P}(\Omega)$ associated with $(0.41, 0.37, 0.22)$ and $(0.37, 0.41, 0.22)$, respectively. Let $\delta_1 = \delta_2 = 0.12$, and denote by $\underline{P}_1, \underline{P}_2$ the TV models determined by (P_0^1, δ_1) and (P_0^2, δ_2), respectively. Then $\underline{P}_1, \underline{P}_2$ and their conjunction $\underline{P}^{\cap}$ are given in the following table:*

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\underline{P}_1(A)$	0.29	0.25	0.1	0.66	0.51	0.47
$\underline{P}_2(A)$	0.25	0.29	0.1	0.66	0.47	0.51
$\underline{P}^{\cap}(A)$	0.29	0.29	0.1	0.66	0.51	0.51

Again, to see that $\underline{P}^{\cap}$ is not the lower probability associated with a VBM, we may observe that

$$\underline{P}^{\cap}(\{\omega_1, \omega_2\}) - \underline{P}^{\cap}(\{\omega_1\}) - \underline{P}^{\cap}(\{\omega_2\}) = 0.08$$
$$\neq 0.12 = \underline{P}^{\cap}(\{\omega_1, \omega_3\}) - \underline{P}^{\cap}(\{\omega_1\}) - \underline{P}^{\cap}(\{\omega_3\}),$$

a contradiction with (3). ♦

The third and last merging operation we consider in this paper is that of mixture:

Definition 12 *Let $\mathcal{M}_1, \mathcal{M}_2$ be two credal sets on Ω and consider $\alpha \in (0, 1)$. Their* mixture *corresponds to the credal set $\alpha\mathcal{M}_1 + (1-\alpha)\mathcal{M}_2 := \{P : \exists P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2$ such that $P = \alpha P_1 + (1-\alpha) P_2\}$.*

The mixture aggregation procedure is an intermediate solution between the disjunction and conjunction operations, and can be seen as giving a weight to the opinion of each expert. It is easy to see that the mixture $\alpha\mathcal{M}_1 + (1-\alpha)\mathcal{M}_2$ produces a convex credal set. If we denote by $\underline{P}_1, \underline{P}_2$ and $\underline{P}^{\alpha}$ the lower probabilities that are the lower envelopes of the credal sets $\mathcal{M}_1, \mathcal{M}_2$ and of their mixture $\alpha\mathcal{M}_1 + (1-\alpha)\mathcal{M}_2$, it holds that

$$\underline{P}^{\alpha} = \alpha\underline{P}_1 + (1-\alpha)\underline{P}_2.$$

It was shown in [5] that the majority of distortion models of interest are closed under mixtures. Let us show how the same applies to VBMs:

Proposition 13 *Let $\underline{P}_1, \underline{P}_2$ be the lower probabilities associated with two VBMs, and let $\mathcal{N}_i := \{A \in \mathcal{P}(\Omega) : \underline{P}_i(A) = 0\}$. If $\mathcal{N}_1 = \mathcal{N}_2$, then for any $\alpha \in (0, 1)$, the mixture*

$$\underline{P}_{\alpha} := \alpha\underline{P}_1 + (1-\alpha)\underline{P}_2 \tag{7}$$

is also the lower probability of a VBM.

Proof Let us denote $\mathcal{N} := \mathcal{N}_1 = \mathcal{N}_2$. Given $A = \Omega$, it follows from Definition 1 and Equation (7) that $\underline{P}_{\alpha}(A) = 1$. For any $A \notin \mathcal{N}$, $A \neq \Omega$, it holds that $\underline{P}_i(A) = b_i P_0^i(A) + a_i$ for $i = 1, 2$, whence

$$\begin{aligned}
\underline{P}_{\alpha}(A) &= \alpha[b_1 P_0^1(A) + a_1] + (1-\alpha)[b_2 P_0^2(A) + a_2] \\
&= (\alpha b_1 P_0^1 + (1-\alpha) b_2 P_0^2)(A) + [\alpha a_1 + (1-\alpha) a_2] \\
&= (\alpha b_1 + (1-\alpha) b_2) \cdot \frac{\alpha b_1 P_0^1(A) + (1-\alpha) b_2 P_0^2(A)}{\alpha b_1 + (1-\alpha) b_2} \\
&\quad + [\alpha a_1 + (1-\alpha) a_2] \\
&:= b^{\alpha} P_0^{\alpha}(A) + a^{\alpha},
\end{aligned}$$

where $b^{\alpha} = \alpha b_1 + (1-\alpha) b_2$, $a^{\alpha} = \alpha a_1 + (1-\alpha) a_2$, $P_0^{\alpha}(A) = \frac{\alpha b_1 P_0^1(A) + (1-\alpha) b_2 P_0^2(A)}{\alpha b_1 + (1-\alpha) b_2}$.

On the other hand, if $A \in \mathcal{N}$, then $\underline{P}_i(A) = 0$ for $i = 1, 2$, whence $b_i P_0^i(A) + a_i \leq 0$ for $i = 1, 2$ and from the above reasoning also $b^{\alpha} P_0^{\alpha}(A) + a^{\alpha} \leq 0$.

This implies that $\underline{P}_{\alpha}$ is the VBM $(P_0^{\alpha}, a^{\alpha}, b^{\alpha})$, noting also that by construction $b^{\alpha} > 0, a^{\alpha} \leq 0, a^{\alpha} + b^{\alpha} \leq 1$ and P_0^{α} is a probability measure. ■

6. Special Vertical Barrier Models

Next, we investigate the connection between VBMs and other families of imprecise probability models. As established in [4, Proposition 4.1], the lower probability $\underline{P}$ of a VBM is always 2-monotone; it was also characterised in [4, Proposition 7.3] under which conditions it corresponds to a probability interval. In this section, we shall analyse under which conditions it corresponds to other particular cases of 2-monotone models: belief functions and minitive measures.

Recall that a lower probability $\underline{P}$ is a belief function if and only if for every $p \in \mathbb{N}$ and every $A_1, \ldots, A_p \in \mathcal{P}(\Omega)$,

$$\underline{P}(\cup_{i=1}^{p} A_i) \geq \sum_{J \subseteq \{1,\ldots,p\}} (-1)^{|J|-1} \underline{P}(\cap_{i\in J} A_i).$$

Let an agent assign a VBM (P_0, a, b), with P_0 defined on Ω. Throughout this section, with the exception of Proposition 17, we shall assume that $P_0(\{\omega\}) > 0$ for every $\omega \in \Omega$.

6.1. Belief Functions

We shall first of all give sufficient (and, in some cases, necessary) conditions for the lower probability of a VBM to be a belief function, and then give sufficient conditions for this lower probability not to be a belief function.

With respect to the first problem, we begin recalling that a necessary and sufficient condition for $\underline{P}$ to be a belief function is that its mass function m, given by

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}(B), \tag{8}$$

is non-negative. In this respect, recall from Equation (1) that any VBM $\underline{P}$ satisfies $\underline{P}(A) = (a+b)\underline{P}_{PMM}(A)$ for every $A \neq \Omega$, where $\underline{P}_{PMM}$ is the PMM associated with the probability measure P_0 and the parameter $\delta = -\frac{a}{a+b}$. We shall refer to this $\underline{P}_{PMM}$ as the PMM *associated with* the VBM $\underline{P}$.

When $a + b = 0$, it follows that $\underline{P}$ is the vacuous lower probability, that is a belief function. Similarly, the case $a = 0$ gives a LV model that is also a belief function. As a consequence, in the remainder of this section we shall assume that $a + b > 0 > a$. The case $a + b = 1$ corresponds to the PMM, for which the connection with belief functions was established in [9].

From the correspondence (1) between VBM and PMM, it is immediate to establish that:

Lemma 14

(a) For every $A \subseteq \Omega$, $\underline{P}(A) > 0 \Leftrightarrow \underline{P}_{PMM}(A) > 0$, and for $A \neq \Omega$, $m(A) = (a+b)m_{PMM}(A) \gtreqqless 0 \Leftrightarrow m_{PMM}(A) \gtreqqless 0$.

(b) $m(\Omega) = 1 - (a+b)(1 - m_{PMM}(\Omega))$.

(c) $m_{PMM}(\Omega) \geq 0 \Leftrightarrow m(\Omega) \geq 1 - (a+b)$.

Proof

(a) Trivial.

(b) This is a consequence of the first statement together with the equality $1 = \sum_{B\subseteq\Omega} m(B) = m(\Omega) + \sum_{B\subset\Omega} m(B) = m(\Omega) + (a+b)(1 - m_{PMM}(\Omega))$.

(c) The second statement together with $a + b > 0$ implies that $m_{PMM}(\Omega) = \frac{m(\Omega)-(1-(a+b))}{a+b}$, from which the result follows.

■

Following [9], we define the *non-vacuity* index of $\underline{P}$ by

$$k = \min\{|A| : \underline{P}(A) > 0\}.$$

We shall denote $\mathcal{P}^* = \{A \subseteq \Omega : \underline{P}(A) > 0\} = \{A \subseteq \Omega : P_0(A) > -\frac{a}{b}\}$ and $\mathcal{P}^*_{PMM} = \{A \subseteq \Omega : \underline{P}_{PMM}(A) > 0\}$, where $\underline{P}_{PMM}$ is the PMM associated with $\underline{P}$. Using this notion and the connection between VBM and PMM, we can give sufficient and necessary conditions for $\underline{P}$ to be a belief function:

Theorem 15

(a) The following are sufficient conditions for the lower probability $\underline{P}$ of a VBM to be a belief function:

(i) $k = n - 1$ and $\sum_{\omega\in\Omega} \underline{P}(\{\omega\}^c) \leq 1$.

(ii) There exists a unique event B such that $|B| = k < n - 1$ and $\mathcal{P}^* = \{A \subseteq \Omega : A \supseteq B\}$.

(iii) There exists a unique event B such that $|B| = k - 1 < n - 2$, $bP_0(B) + a = 0$ and $\mathcal{P}^* = \{A \subseteq \Omega : A \supset B\}$.

(b) If $m(\Omega) \geq 1 - (a+b)$, then it is necessary that one of (i)–(iii) holds for $\underline{P}$ to be a belief function.

Proof

(a) Let us show that each of the conditions (i)–(iii) is sufficient for $\underline{P}$ to be a belief function.

(i) In this case it is $m(A) = 0$ for every A with $|A| < n - 1$, $m(A) = \underline{P}(A)$ for every A with $|A| = n-1$ and $m(\Omega) = 1 - \sum_{\omega\in\Omega} m(\{\omega\}^c) \geq 0$, so $\underline{P}$ is a belief function.

(ii) Let $\underline{P}_{PMM}$ be the PMM associated with $\underline{P}$. It follows that $\underline{P}_{PMM}$ satisfies the conditions of [9, Proposition 8], whence $m_{PMM}(A) \geq 0$ for every A and $m_{PMM}(\Omega) = 0$. Applying Lemma 14, we deduce that $m(A) \geq 0$ for every $A \neq \Omega$ and $m(\Omega) = 1 - (a+b) > 0$.

(iii) The condition $bP_0(B) + a = 0$ is equivalent to the following condition in the associated PMM:

$$\delta = -\frac{a}{a+b} = \frac{bP_0(B)}{-bP_0(B) + b} = \frac{P_0(B)}{1 - P_0(B)}.$$

It follows that $\underline{P}_{PMM}$ satisfies the conditions in [9, Proposition 9], whence $m_{PMM}(A) \geq 0$ for every A and $m_{PMM}(\Omega) = 0$. Applying Lemma 14, we deduce again that $m(A) \geq 0$ for every $A \neq \Omega$ and $m(\Omega) = 1 - (a + b) > 0$.

(b) Since $\underline{P}$ is a non-vacuous belief function, its non-vacuity index k is strictly smaller than n. There are two possibilities:

- If $k = n-1$, then it holds that $\sum_{\omega\in\Omega} \underline{P}(\{\omega\}^c) = \sum_{\omega\in\Omega} m(\{\omega\}^c) \leq 1$, taking into account that $\sum_{A\subseteq\Omega} m(A) = 1$ and that $m(A) \geq 0$ for every A. Thus, condition (i) holds.
- Assume next that $k < n-1$. Since $\underline{P}$ is a belief function, we have that $m_{PMM}(A) \geq 0, \forall A \neq \Omega$, by Lemma 14 (a). Further, the assumption $m(\Omega) \geq 1-(a+b)$ implies that $m_{PMM}(\Omega) \geq 0$ by Lemma 14 (c). Therefore, we conclude that $\underline{P}_{PMM}$ is also a belief function. But since from Lemma 14 (a) the non-vacuity index k is the same for the VBM $\underline{P}$ and its associated PMM $\underline{P}_{PMM}$, it follows from [9, Theorem 2] that, if $\underline{P}_{PMM}$ is a belief function and $k < n-1$, either there exists a unique B with $|B| = k$ and such that $\mathcal{P}^*_{PMM} = \{A \subseteq \Omega : A \supseteq B\} = \mathcal{P}^*$, and we are in case (ii); or there is a unique B with $|B| = k-1$ and such that $\mathcal{P}^*_{PMM} = \{A \subseteq \Omega : A \supset B\} = \mathcal{P}$. In that case, [9, Theorem 2] implies that $\delta = -\frac{a}{a+b} = \frac{P_0(B)}{1-P_0(B)}$, which is equivalent to $bP_0(B) + a = 0$. Therefore, we are in case (iii).

∎

Note that the necessity part in this proposition depends on the assumption $m(\Omega) \geq 1-(a+b)$, which in turn is equivalent to $m_{PMM}(\Omega) \geq 0$. It is not difficult to find VBMs that are belief functions for which the associated PMM is not:

Example 4 *Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}, b = 0.7, a = -0.3$ and P_0 the probability measure determined by the mass function* $(0.24, 0.26, 0.25, 0.25)$. *The associated VBM is given by:*

$$\underline{P}(\{\omega_i\}) = 0 \ \forall i = 1, \ldots, 4$$
$$\underline{P}(\{\omega_1, \omega_3\}) = \underline{P}(\{\omega_1, \omega_4\}) = 0.043$$
$$\underline{P}(\{\omega_1, \omega_2\}) = \underline{P}(\{\omega_3, \omega_4\}) = 0.05$$
$$\underline{P}(\{\omega_2, \omega_3\}) = \underline{P}(\{\omega_2, \omega_4\}) = 0.057$$
$$\underline{P}(\{\omega_1, \omega_2, \omega_3\}) = \underline{P}(\{\omega_1, \omega_2, \omega_4\}) = 0.225$$
$$\underline{P}(\{\omega_1, \omega_3, \omega_4\}) = 0.218, \quad \underline{P}(\{\omega_2, \omega_3, \omega_4\}) = 0.232.$$

From these values and Equation (8) we deduce that the mass function m is given by:

$$m(\{\omega_i\}) = 0 \ \forall i = 1, \ldots, 4$$
$$m(\{\omega_i, \omega_j\}) = \underline{P}(\{\omega_i, \omega_j\}) \geq 0 \ \forall i \neq j$$
$$m(\{\omega_1, \omega_2, \omega_3\}) = m(\{\omega_1, \omega_2, \omega_4\}) = 0.075$$
$$m(\{\omega_1, \omega_3, \omega_4\}) = 0.082, \quad m(\{\omega_2, \omega_3, \omega_4\}) = 0.068$$
$$m(\Omega) = 0.4.$$

Thus, $\underline{P}$ is a belief function. However, if we consider the associated PMM it is $m_{PMM}(\Omega) = -0.5$, meaning that the latter is not a belief function; and indeed we observe that none of the conditions (i)–(iii) holds. ♦

In the case of cardinality three, it is not difficult to find a sufficient condition for a VBM to induce a belief function:

Proposition 16 *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider a VBM on $\mathcal{P}(\Omega)$ associated with a probability measure P_0 and two parameters $a \leq 0, b > 0$ such that $a + b \in [0, 1]$. Then $\underline{P}$ is a belief function if $b \leq 1$.*

Proof We shall proceed by determining the mass function of $\underline{P}$ in the different alternatives:

1) $\underline{P}(\{\omega_i\}) > 0$ for $i = 1, 2, 3$. In that case, we get:
   - $m(\{\omega_i\}) = bP_0(\{\omega_i\}) + a$ $(i = 1, 2, 3)$.
   - $m(\{\omega_i, \omega_j\}) = -a \geq 0 \ \forall i \neq j$.
   - $m(\Omega) = 1 - b \geq 0 \Leftrightarrow b \leq 1$.
2) $\exists! \, \omega_i : \underline{P}(\{\omega_i\}) = 0$. Assume w.l.o.g. $i = 1$. Then the mass function of $\underline{P}$ is:
   - $m(\{\omega_1\}) = 0, m(\{\omega_j\}) = bP_0(\{\omega_j\}) + a$ $(j = 2, 3)$.
   - $m(\{\omega_1, \omega_2\}) = m(\{\omega_1, \omega_3\}) = bP_0(\{\omega_1\})$, $m(\{\omega_2, \omega_3\}) = -a \geq 0$.
   - $m(\Omega) = 1 - (b + bP_0(\{\omega_1\}) + a) \geq 1 - b \geq 0$ if $b \leq 1$.
3) $\exists! \, \omega_i : \underline{P}(\{\omega_i\}) > 0$. Assume w.l.o.g. $i = 3$. Then the mass function of $\underline{P}$ is:
   - $m(\{\omega_1\}) = m(\{\omega_2\}) = 0$ and $m(\{\omega_3\}) = bP_0(\{\omega_3\}) + a$.
   - $m(\{\omega_1, \omega_3\}) = bP_0(\{\omega_1\}), m(\{\omega_2, \omega_3\}) = bP_0(\{\omega_2\}), m(\{\omega_1, \omega_2\}) = \underline{P}(\{\omega_1, \omega_2\}) \geq 0$.
   - $m(\Omega) = 1 - (b + a + \underline{P}(\{\omega_1, \omega_2\})) \geq 1 - b \geq 0$ if $b \leq 1$. The first inequality is immediate if $\underline{P}(\{\omega_1, \omega_2\}) = 0$, otherwise $1 - b - a - \underline{P}(\{\omega_1, \omega_2\}) = 1 - b - a - bP_0(\{\omega_1\}) - bP_0(\{\omega_2\}) - a \geq 1 - b$, because $-a - bP_0(\{\omega_i\}) \geq 0$ for $i = 1, 2$.

4) Finally, if $\underline{P}(\{\omega_i\}) = 0$ for $i = 1, 2, 3$, we get:
   - $m(\{\omega_i\}) = 0$ $(i = 1, 2, 3)$.
   - $m(\{\omega_i, \omega_j\}) = \underline{P}(\{\omega_i, \omega_j\}) \geq 0\ \forall i \neq j$.
   - $m(\Omega) = 1 - \sum_{i \neq j} \underline{P}(\{\omega_i, \omega_j\}) \geq 1 - b \geq 0$ if $b \leq 1$. Here, the first inequality can be verified by considering all the possible cases, and reasoning analogously to case 3.

■

To see that this sufficient condition is not necessary, observe that there are cases with cardinality 3 where the PMM, that is a particular case of VBM with $b > 1$, induces a belief function.

To investigate further the role of belief functions within VBMs, we give next a sufficient condition for $\underline{P}$ *not* to be a belief function. Note the following proposition holds even if $P_0(\{\omega\}) = 0$ for some $\omega \in \Omega$.

Proposition 17 *Let $\underline{P}$ be a VBM on $\mathcal{P}(\Omega)$ such that P_0 is an arbitrary probability (not necessarily strictly positive). If there exist $A_1, A_2, A_3 \in \mathcal{P}(\Omega)$ such that $A_i \cap A_j = \emptyset$, $\forall i \neq j$, $\underline{P}(A_i) > 0$, $i = 1, 2, 3$, $A = A_1 \cup A_2 \cup A_3 \neq \Omega$, then $\underline{P}$ is not a belief function.*

Proof Consider a partition Ω_c coarser than Ω, such that A_1, A_2, A_3 are atomic events of Ω_c, and let $\underline{P}_c$ be the restriction of $\underline{P}$ on $\mathcal{P}(\Omega_c)$ and m_c its mass function.

Clearly, if $\underline{P}$ is a belief function on $\mathcal{P}(\Omega)$ also $\underline{P}_c$ is on $\mathcal{P}(\Omega_c)$. Thus, we shall show that $\underline{P}_c$ is not a belief function to prove the thesis. Indeed

$$\begin{aligned} &m_c(A_1 \cup A_2 \cup A_3) = bP_0(A_1 \cup A_2 \cup A_3) + a \\ &- \sum_{1 \leq i < j \leq 3} (bP_0(A_i \cup A_j) + a) + \sum_{i=1}^{3} (bP_0(A_i) + a) \\ &= a < 0 \end{aligned}$$

Hence, $\underline{P}_c$ is not a belief function and so neither is $\underline{P}$. ■

In terms of (P_0, a, b), the condition in this proposition implies that $P_0(A_i) > -\frac{a}{b}$ for $i = 1, 2, 3$, from which it follows that it can only hold when $-\frac{3a}{b} < 1$. Proposition 17 also shows us that

(a) the sufficient condition for belief functions $b < 1$ in Proposition 16 does not extend to cardinalities higher than 3.

(b) Belief functions are characterized when $m(\Omega) \geq 1 - (a + b)$ in Theorem 15. When $k < n - 1$, the set $\mathcal{P}^* = \{A \in \mathcal{P}(\Omega) : \underline{P}(A) > 0\}$ must be either (Theorem 15 (ii)) a filter generated by B or (Theorem 15 (iii)) 'nearly', in the sense that B does not belong to the filter.

When $m(\Omega) < 1 - (a + b)$, these constraints do not necessarily apply, as Example 4 shows. Yet, the path to obtain a belief function remains narrow: Proposition 17 requires (implicitly) that $n \geq 4$, but its other hypotheses are rather mild: it suffices that three disjoint, non-exhaustive events are given positive lower probability.

6.2. Possibility Measures

Next we investigate under which conditions the lower probability of the VBM is minitive, that is, when it satisfies

$$\underline{P}(A_1 \cap A_2) = \min\{\underline{P}(A_1), \underline{P}(A_2)\}\ \forall A_1, A_2 \subseteq \Omega.$$

This is equivalent to showing when the conjugate upper probability $\overline{P}$ of $\underline{P}$ is maxitive, i.e., whether

$$\overline{P}(A_1 \cup A_2) = \max\{\overline{P}(A_1), \overline{P}(A_2)\}\ \forall A_1, A_2 \subseteq \Omega;$$

we shall focus on this second condition in this section. Note that, since in this paper we are focusing on finite possibility spaces, minitivity is equivalent to being a necessity measure and maxitivity is equivalent to being a possibility measure.

The upper probability of a VBM can be computed as [4]

$$\overline{P}(A) = \min\{bP_0(A) + c, 1\}\ \forall A \neq \emptyset, \overline{P}(\emptyset) = 0, \quad (9)$$

where $c = 1 - (a + b)$.

Recall that we are assuming in this section that $P_0(\{\omega\}) > 0$ for every $\omega \in \Omega$. Then, it is easy to show that $\overline{P}$ is only maxitive on very special cases:

Proposition 18 *Let $\overline{P}$ be the upper probability of a VBM where $P_0(\{\omega\}) > 0$ for every $\omega \in \Omega$. Then*

$$\overline{P} \text{ maxitive} \Leftrightarrow |\{\omega : \overline{P}(\{\omega\}) < 1\}| \leq 1.$$

Proof

($\Rightarrow$) Let $\overline{P}$ be maxitive and assume ex-absurdo the existence of different ω_1, ω_2 in Ω satisfying $\overline{P}(\{\omega_1\}), \overline{P}(\{\omega_2\}) < 1$. Then

$$\begin{aligned} &\max\{\overline{P}(\{\omega_1\}), \overline{P}(\{\omega_2\})\} \\ &= \max\{bP_0(\{\omega_1\}) + c, bP_0(\{\omega_2\}) + c\} \\ &< bP_0(\{\omega_1, \omega_2\}) + c, \end{aligned}$$

taking into account that $P_0(\{\omega_1\}), P_0(\{\omega_2\}) > 0$ by assumption. Since on the other hand we also have that $\max\{\overline{P}(\{\omega_1\}), \overline{P}(\{\omega_2\})\} < 1$, it follows that

$$\begin{aligned} &\max\{\overline{P}(\{\omega_1\}), \overline{P}(\{\omega_2\})\} \\ &< \min\{bP_0(\{\omega_1, \omega_2\}) + c, 1\} = \overline{P}(\{\omega_1, \omega_2\}), \end{aligned}$$

a contradiction with the maxitivity of $\overline{P}$.

($\Leftarrow$) If $\overline{P}(\{\omega\}) = 1$ for every $\omega \in \Omega$, then $\overline{P}$ coincides with the vacuous upper probability given by Equation (4), that is maxitive. On the other hand, if there is a unique ω' with $\overline{P}(\{\omega'\}) < 1$, then it follows by monotonicity that

$$\overline{P}(A) = \begin{cases} \overline{P}(\{\omega\}) & \text{if } A = \{\omega\} \\ 1 & \text{otherwise,} \end{cases}$$

that is also maxitive.

■

Note that, by (9), the condition in this last proposition can be equivalently expressed as $\overline{P}(\{\omega\}) \in (0,1) \Leftrightarrow \overline{P}(\{\omega\}) \in (c,1) \Leftrightarrow bP_0(\{\omega\}) + c < 1 \Leftrightarrow P_0(\{\omega\}) < \frac{1-c}{b} = \frac{a+b}{b}$.

The limited relevance of possibility measures within VBMs is anyway already patent from the statement of Proposition 18: a VBM is a possibility iff it is vacuous or nearly, i.e. with non-vacuous imprecise probability assessment on at most one event.

7. Conclusions

Vertical Barrier Models have been originally introduced as distortion models, i.e. as functions of a given probability P_0. We have seen in this paper how they can be interpreted as neighbourhood models, originated from a neighbourhood of P_0 by means of a suitable distorting function. The complexity of the credal set of a VBM has been discussed in terms of the maximum number of its extreme points. Perhaps surprisingly, the bound we found is the same as for TV models, a proper subfamily of VBMs. Mixtures of VBMs are, under some conditions, still VBMs, while disjunction and conjunction operations do not retain this closure property. Several features of VBMs that are (or are not) belief functions have been detected. Although VBMs that are belief functions are certainly not the rule, there is anyway some more flexibility with respect to the already well known subcase of Pari-Mutuel Models.

The following tables summarise some of our results and establish a comparison with the properties of the PMM, LV and TV models:

	Conj.	Disj	Mixt.	Belief?
PMM	YES	NO	YES*	[9, Thm. 2] ($\Leftrightarrow$)
LV	YES	NO	YES	YES
TV	NO	NO	YES	NO
VBM	NO	NO	YES*	Thm. 15 (sufficient)

In the case of mixtures, the assumption $\mathcal{N}_1 = \mathcal{N}_2$ is needed. We make this assumption in Proposition 13 and it holds in particular if the two lower probabilities to be combined are strictly positive on all non-impossible events. More generally, the interpretation of the assumption is that the opinions of two experts may differ on all events but those given null lower probability, in order for the mixture to be a VBM again.

Concerning the maximum number of extreme points of $\mathcal{M}(\underline{P})$ in terms of $n = |\Omega|$, the bounds are as follows:

	Number of extreme points of $\mathcal{M}(\underline{P})$
PMM	$\frac{n!}{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} + 1 \rceil !}$
LV	n
TV	$\frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !}$
VBM	$\frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !}$

A complete characterisation of belief functions within VBMs has however still to be determined. A related issue regards exploring the structure of the mass function of a VBM. A further topic for future work could be the role of VBMs in outer or inner approximations of coherent imprecise probabilities.

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Author Contributions

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