Appendix A. Technical Lemmas and Proofs

Lemma 18 *For any non-degenerate prequential situation $v = (i, w) \in (\mathscr{I}_r \times \mathscr{X})^*$ and any non-negative superfarthingale $F \in \overline{\mathbb{F}}$, $F(v) \leq \prod_{k=1}^{|w|} \frac{1}{(\max \iota_k)^{w_k}(1-\min \iota_k)^{1-w_k}} F(\square)$.*

Proof Consider any non-degenerate prequential situation $vI_rx \in (\mathscr{I}_r \times \mathscr{X})^*$. If $x = 1$ then $0 < \max I_r \leq 1$, and

$$\begin{aligned} &F(vI_rx) \\ &\leq \frac{1}{\max I_r}\big[\max I_r F(vI_r1) + (1-\max I_r)F(vI_r0)\big] \\ &\leq \frac{1}{\max I_r}\overline{E}_{I_r}(F(vI_r\,\cdot)) \leq \frac{1}{\max I_r}F(v). \end{aligned}$$

If $x = 0$ then $0 \leq \min I_r < 1$, and

$$\begin{aligned} &F(vI_rx) \\ &\leq \frac{1}{1-\min I_r}\big[\min I_r F(vI_r1) + (1-\min I_r)F(vI_r0)\big] \\ &\leq \frac{1}{1-\min I_r}\overline{E}_{I_r}(F(vI_r\,\cdot)) \leq \frac{1}{1-\min I_r}F(v). \end{aligned}$$

Above, the first, second and third inequalities follow from the non-negativity of F, Equation (1) and the superfarthingale property, respectively. Hence, $F(vI_rx) \leq \frac{1}{(\max I_r)^x(1-\min I_r)^{1-x}}F(v)$. A simple induction argument now leads to the desired result. ■

Lemma 19 *For every non-degenerate computable forecasting system $\varphi \in \Phi$ there's a recursive natural map $C\colon \mathscr{X}^* \to \mathbb{N}$ such that for every test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$ it holds that $T(w) \leq C(w)$ for all $w \in \mathscr{X}^*$.*

Proof Define the map $C'\colon \mathscr{X}^* \to \mathbb{R}$ by letting $C'(w) := \prod_{k=1}^{|w|} \frac{1}{\overline{\varphi}(w_{1:k-1})^{w_k}(1-\underline{\varphi}(w_{1:k-1}))^{1-w_k}}$ for all $w \in \mathscr{X}^*$. This map is real-valued, since $0 < \overline{\varphi}$ and $\underline{\varphi} < 1$ by the non-degeneracy of φ. Since φ is computable, C' is computable as well. Let's now prove that $T(w) \leq C'(w)$ for all $w \in \mathscr{X}^*$. Fix any situation $w \in \mathscr{X}^*$ and any $x \in \mathscr{X}$. If $x = 1$, then

$$\begin{aligned} T(wx) &\leq \frac{1}{\overline{\varphi}(w)}\big[\overline{\varphi}(w)T(w1) + (1-\overline{\varphi}(w))T(w0)\big] \\ &\leq \frac{1}{\overline{\varphi}(w)}\overline{E}_{\varphi(w)}(T(w\,\cdot)) \leq \frac{1}{\overline{\varphi}(w)}T(w). \end{aligned}$$

If $x = 0$, then

$$\begin{aligned} T(wx) &\leq \frac{1}{1-\underline{\varphi}(w)}\big[\underline{\varphi}(w)T(w1) + (1-\underline{\varphi}(w))T(w0)\big] \\ &\leq \frac{1}{1-\underline{\varphi}(w)}\overline{E}_{\varphi(w)}(T(w\,\cdot)) \leq \frac{1}{1-\underline{\varphi}(w)}T(w). \end{aligned}$$

Above, the first, second and third inequalities follow from the non-negativity of T, Equation (1) and the supermartingale property, respectively. Hence,

$$T(wx) \leq \frac{1}{\overline{\varphi}(w)^x\big(1-\underline{\varphi}(w)\big)^{1-x}}T(w).$$

A simple induction argument now shows that indeed $T(w) \leq C'(w)$ for all $w \in \mathscr{X}^*$.

Since C' is a computable real map, there's a recursive rational map $q\colon \mathscr{X}^* \times \mathbb{N}_0 \to \mathbb{Q}$ such that $|C'(w) - q(w,n)| \leq 2^{-n}$ for all $w \in \mathscr{X}^*$ and $n \in \mathbb{N}_0$. Let $C\colon \mathscr{X}^* \to \mathbb{N}$ be defined as $C(w) := \max\{1, \lceil q(w,1)+1\rceil\}$ for all $w \in \mathscr{X}^*$, with $\lceil\bullet\rceil\colon \mathbb{R} \to \mathbb{Z}$ the ceiling function and $\mathbb{Z}$ the set of integer numbers. It's easy to see that C is natural-valued, positive and recursive. Furthermore, we have that $T(w) \leq C'(w) \leq q(w,1) + 1/2 \leq C(w)$ for all $w \in \mathscr{X}^*$. ■

Lemma 20 *There's a single algorithm that, upon the input of a code for a lower semicomputable map $F\colon (\mathscr{I}_r \times \mathscr{X})^* \to [0,+\infty]$, outputs a code for a lower semicomputable test superfarthingale $F' \in \overline{\mathbb{F}}$ such that*

(i) *$F'(v) = 0$ for all degenerate prequential situations $v \in (\mathscr{I}_r \times \mathscr{X})^*$;*

(ii) *for any rational forecasting system $\varphi_r \in \Phi_r$, $F'(\varphi_r[w], w) = F(\varphi_r[w], w)$ for all $w \in \mathscr{X}^*$ for which $(\varphi_r[w], w)$ is non-degenerate, provided that the map $F(\varphi_r[\bullet], \bullet)\colon \mathscr{X}^* \to \mathbb{R}$ is a positive test supermartingale for φ_r.*

Proof Start from a code for the map $F\colon (\mathscr{I}_r \times \mathscr{X})^* \to [0,+\infty]$ that is lower semicomputable. By Corollary 6, we can invoke a single algorithm that outputs a code $q\colon (\mathscr{I}_r \times \mathscr{X})^* \times \mathbb{N}_0 \to \mathbb{Q}$ for F such that $q(v,\bullet) \nearrow F(v)$ and $q(v,n) < q(v,n+1)$ for all $v \in (\mathscr{I}_r \times \mathscr{X})^*$ and $n \in \mathbb{N}_0$. We'll now use the code q to construct a code q' for a lower semicomputable test superfarthingale $F' \in \overline{\mathbb{F}}$ that satisfies the requirements of the lemma.

Let $q'\colon (\mathscr{I}_r \times \mathscr{X})^* \times \mathbb{N}_0 \to \mathbb{Q}$ be defined by $q'(\square, n) := 1$ and

$$q'(vI_rx, n) := \begin{cases} \max\big(A(v, I_r, x, n) \cup \{0\}\big) & \text{if } vI_rx \text{ is non-degenerate} \\ 0 & \text{if } vI_rx \text{ is degenerate,} \end{cases}$$

for all $v = (i, w) \in (\mathscr{I}_r \times \mathscr{X})^*$, $I_r \in \mathscr{I}_r$, $x \in \mathscr{X}$ and $n \in \mathbb{N}_0$, where $A\colon (\mathscr{I}_r \times \mathscr{X})^* \times \mathscr{I}_r \times \mathscr{X} \times \mathbb{N}_0 \to \{Q \subseteq \mathbb{Q}\colon |Q| < \infty\}$ is defined by

$$\begin{aligned} A(v, I_r, x, n) := \big\{&q(vI_rx, m) \in \mathbb{Q}\colon 0 \leq m \leq n, \\ &0 \leq q(vI_r\,\cdot, m) \text{ and} \end{aligned}$$

$$\overline{E}_{I_r}(q(vI_r\,\cdot, m)) \leq q'(v, n)\big\}. \quad (6)$$

By construction, since the map A outputs finite sequences of rationals, the map q' is well-defined, non-negative and rational. It's not too difficult to see that the map A, and therefore also the map q', is recursive.

The map q' is non-decreasing in its second argument, as we now show by induction on its first argument. We start by observing that $q'(\square, n) \leq q'(\square, n+1)$ for all $n \in \mathbb{N}_0$. For the induction step, fix any $v = (i, w) \in (\mathscr{I}_r \times \mathscr{X})^*$, $I_r \in \mathscr{I}_r$, $x \in \mathscr{X}$ and $n \in \mathbb{N}_0$, and assume that $q'(v, n) \leq q'(v, n+1)$. We then have to show that also $q'(vI_rx, n) \leq q'(vI_rx, n+1)$. This is trivial when vI_rx is degenerate; when vI_rx is non-degenerate, it follows readily from the inequality $A(v, I_r, x, n) \subseteq A(v, I_r, x, n+1)$, which is itself immediate from Equation (6).

For any $n \in \mathbb{N}_0$, the map $q'(\bullet, n)\colon (\mathscr{I}_r \times \mathscr{X})^* \to \mathbb{R}$ is a test superfarthingale. To prove this, we may clearly concentrate on the superfarthingale condition. Fix any $v \in (\mathscr{I}_r \times \mathscr{X})^*$, $I_r \in \mathscr{I}_r$ and $n \in \mathbb{N}_0$, and infer from Equation (6) that $A(v, I_r, 1, n) = \emptyset \Leftrightarrow A(v, I_r, 0, n) = \emptyset$, so we only need to consider two cases. If $A(v, I_r, 1, n) = A(v, I_r, 0, n) = \emptyset$, then $q'(vI_r\,\cdot, n) = 0$, and therefore trivially $\overline{E}_{I_r}(q'(vI_r\,\cdot, n)) = \overline{E}_{I_r}(0) = 0 \leq q'(v, n)$, where the second equality follows from C1. Otherwise, because the map q is increasing in its second argument, there's an $m \in \{0, \dots, n\}$ such that $\overline{E}_{I_r}(q(vI_r\,\cdot, m)) \leq q'(v, n)$, with $q(vI_r\,\cdot, m) = \max(A(v, I_r, \cdot, n) \cup \{0\}) \geq q'(vI_r\,\cdot, n)$, where the last inequality takes into account that there may be some $x \in \mathscr{X}$ such that vI_rx is degenerate. Hence, indeed, in this case also

$$\overline{E}_{I_r}(q'(vI_r\,\cdot, n)) \overset{\text{C5}}{\leq} \overline{E}_{I_r}(q(vI_r\,\cdot, m)) \leq q'(v, n).$$

As a final preliminary step, we infer from Lemma 18 that for every (non-degenerate) prequential situation $v \in (\mathscr{I}_r \times \mathscr{X})^*$ there's some real $B_v \in \mathbb{R}$ such that $q'(v, n) \leq B_v$ for all $n \in \mathbb{N}_0$.

With this set-up phase completed, let F' be defined as $q'(v, \bullet) \nearrow F'(v)$ for all $v \in (\mathscr{I}_r \times \mathscr{X})^*$; note that $F'(\square) = 1$. This map is well-defined, real-valued, non-negative and lower semicomputable due to the non-decreasingness, boundedness, non-negativity and recursiveness of q' respectively, so we only need to check the superfarthingale property explicitly in order to conclude that F' is a lower semicomputable test superfarthingale. To this end, fix any $v \in (\mathscr{I}_r \times \mathscr{X})^*$ and $I_r \in \mathscr{I}_r$. If we recall that the map $q(\bullet, n)\colon (\mathscr{I}_r \times \mathscr{X})^* \to \mathbb{R}$ is a test superfarthingale for every $n \in \mathbb{N}_0$, we immediately infer from C6 and the real-valuedness of F' that $\overline{E}_{I_r}(F'(vI_r\,\cdot)) = \lim_{n\to\infty} \overline{E}_{I_r}(q'(vI_r\,\cdot, n)) \leq \lim_{n\to\infty} q'(v, n) = F'(v)$.

We are done if we can show that F' satisfies the conditions (i) and (ii). For (i), fix any degenerate prequential situation $v \in (\mathscr{I}_r \times \mathscr{X})^*$ and note that then $q'(v, n) = 0$ for all $n \in \mathbb{N}_0$ by construction. Hence, indeed, $F'(v) = 0$.

For (ii), fix any rational forecasting system $\varphi_r \in \Phi_r$, consider the map $T\colon \mathscr{X}^* \to \mathbb{R}$ defined by $T(w) := F(\varphi_r[w], w)$ for all $w \in \mathscr{X}^*$, and assume that T is a positive test supermartingale. We must now show that $F'(\varphi_r[w], w) = T(w)$ for all $w \in \mathscr{X}^*$ for which the prequential situation $(\varphi_r[w], w)$ is non-degenerate.

By construction, $F'(\varphi_r[w], w) \leq F(\varphi_r[w], w) = T(w)$ for all $w \in \mathscr{X}^*$. Assume towards contradiction that there's some $\overline{w} \in \mathscr{X}^*$ for which $(\varphi_r[\overline{w}], \overline{w})$ is non-degenerate and $F'(\varphi_r[\overline{w}], \overline{w}) < T(\overline{w})$, implying that there's some $\epsilon > 0$ such that $q'((\varphi_r[\overline{w}], \overline{w}), n) + \epsilon < T(\overline{w})$ for all $n \in \mathbb{N}_0$. We'll use an induction argument to show that this is impossible.

Since by assumption $q((\varphi_r[\overline{w}], \overline{w}), \bullet) \nearrow T(\overline{w}) > 0$ and $q((\varphi_r[\overline{w}], \overline{w}), n) < q((\varphi_r[\overline{w}], \overline{w}), n+1)$ for all $n \in \mathbb{N}_0$, there are $\epsilon_0, \epsilon_1, \dots, \epsilon_{|\overline{w}|} \in \mathbb{R}$ and $n_0, n_1, \dots, n_{|\overline{w}|} \in \mathbb{N}_0$ such that

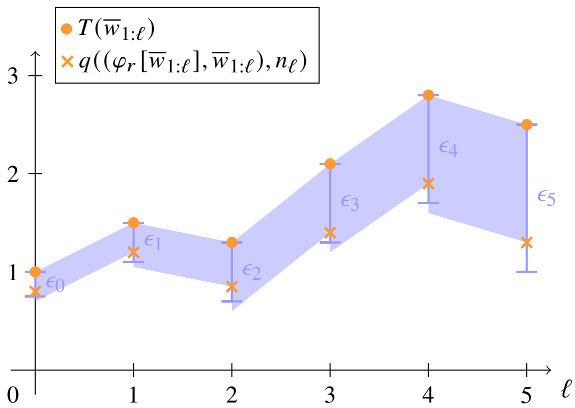
$$0 < \epsilon_0 < \epsilon_1 < \dots < \epsilon_{|\overline{w}|} < \epsilon \quad (7)$$

$$T(\overline{w}_{1:\ell}) < q((\varphi_r[\overline{w}_{1:\ell}], \overline{w}_{1:\ell}), n_\ell) + \epsilon_\ell \quad (8)$$

$$0 \leq q((\varphi_r[\overline{w}_{1:k}]\varphi_r(\overline{w}_{1:k}), \overline{w}_{1:k}\,\cdot), n_{k+1}) \quad (9)$$

$$q((\varphi_r[\overline{w}_{1:k}]\varphi_r(\overline{w}_{1:k}), \overline{w}_{1:k}\,\cdot), n_{k+1}) + \epsilon_k < T(\overline{w}_{1:k}\,\cdot) \quad (10)$$

for all $k \in \{0, 1, \dots, |\overline{w}| - 1\}$ and $\ell \in \{0, 1, \dots, |\overline{w}|\}$. The argument starts with $\ell := |\overline{w}|$ and $k := |\overline{w}| - 1$, finding ϵ_ℓ such that (7) is satisfied, and finding n_{k+1} such that (8) and (9) are satisfied. We then move to $\ell := |\overline{w}| - 1$ and $k := |\overline{w}| - 2$, find ϵ_ℓ such that (7) and (10) are satisfied, and find n_{k+1} such that (8) and (9) are satisfied. And so on …; these conditions are depicted below for a situation $\overline{w} \in \mathscr{X}^*$ for which $|\overline{w}| = 5$.



Now, let $N := \max\{n_0, n_1, \dots, n_{|\overline{w}|}\}$. To start the induction argument, observe that, trivially, $q'(\square, N) = 1 > T(\square) - \epsilon_0$. For the induction step, we fix any $k \in \{0, 1, \dots, |\overline{w}| - 1\}$ and assume that $q'((\varphi_r[\overline{w}_{1:k}], \overline{w}_{1:k}), N) > T(\overline{w}_{1:k}) - \epsilon_k$. It then follows that

$$\overline{E}_{\varphi_r(\overline{w}_{1:k})}\big(q((\varphi_r[\overline{w}_{1:k}]\varphi_r(\overline{w}_{1:k}), \overline{w}_{1:k}\,\cdot), n_{k+1})\big)$$

$$\overset{(10),C5}{\leq} \overline{E}_{\varphi_r(\overline{w}_{1:k})}(T(\overline{w}_{1:k}\,\cdot) - \epsilon_k)$$
$$\overset{C4}{=} \overline{E}_{\varphi_r(\overline{w}_{1:k})}(T(\overline{w}_{1:k}\,\cdot)) - \epsilon_k$$
$$\leq T(\overline{w}_{1:k}) - \epsilon_k$$
$$\leq q'((\varphi_r[\overline{w}_{1:k}], \overline{w}_{1:k}), N),$$

where the penultimate inequality follows form the assumption that T is a supermartingale, and the last inequality from the induction hypothesis. Hence, by Equations (6) and (9),

$$q((\varphi_r[\overline{w}_{1:k+1}], \overline{w}_{1:k+1}), n_{k+1}) \in A((\varphi_r[\overline{w}_{1:k}], \overline{w}_{1:k}), \varphi_r(\overline{w}_{1:k}), \overline{w}_{k+1}, N),$$

which implies that

$$q'((\varphi_r[\overline{w}_{1:k+1}], \overline{w}_{1:k+1}), N) \geq \max A((\varphi_r[\overline{w}_{1:k}], \overline{w}_{1:k}), \varphi_r(\overline{w}_{1:k}), \overline{w}_{k+1}, N)$$
$$\geq q((\varphi_r[\overline{w}_{1:k+1}], \overline{w}_{1:k+1}), n_{k+1})$$
$$\overset{(8)}{>} T(\overline{w}_{1:k+1}) - \epsilon_{k+1}.$$

Repeating this argument until we reach $k = |\overline{w}| - 1$, we eventually find that $q'((\varphi_r[\overline{w}], \overline{w}), N) > T(\overline{w}) - \epsilon_{|\overline{w}|} > T(\overline{w}) - \epsilon$, which is the desired contradiction. ■

The following result is now immediate.

Corollary 21 *There's a single algorithm that, upon the input of a code for a lower semicomputable map $F\colon (\mathscr{I}_r \times \mathscr{X})^* \to [0, +\infty]$, outputs a code for a lower semicomputable test superfarthingale $F' \in \overline{\mathbb{F}}$ such that, for all prequential situations $v \in (\mathscr{I}_r \times \mathscr{X})^*$,*

(i) *$F'(v) = 0$ if v is degenerate;*

(ii) *$F'(v) = F(v)$ if v is non-degenerate and F is a positive test superfarthingale.*

Proof of Theorem 17. We'll give a proof for the first inequality, the proof for the second one is similar. Assume towards contradiction that there's some real number ϵ, with $0 < \epsilon < 1$, such that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\upsilon_{1:k}, \iota_{k+1})[\omega_{k+1} - \min \iota_{k+1}]}{\sum_{k=0}^{n-1} S(\upsilon_{1:k}, \iota_{k+1})} < -\epsilon.$$

Let the map $F := (\mathscr{I}_r \times \mathscr{X})^* \to \mathbb{R}$ be defined by

$$F(v) := \prod_{k=0}^{|v|-1} \left[1 - \frac{\epsilon}{3} S(v_{1:k}, i_{k+1})[w_{k+1} - \min i_{k+1}]\right] \quad \text{for all } v = (i, w) \in (\mathscr{I}_r \times \mathscr{X})^*.$$

We'll now show in a number of steps that F is a lower semicomputable test superfarthingale for which $\limsup_{n\to\infty} F(\upsilon_{1:n}) = \infty$, implying that υ can't be game-random.

Trivially, $F(\square) = 1$, and also $F \geq 0$, since $\epsilon < 1$, $|S| \leq 1$ and $|x - \min I_r| \leq 1$ for all $x \in \mathscr{X}$ and $I_r \in \mathscr{I}_r$. Moreover, for any $v \in (\mathscr{I}_r \times \mathscr{X})^*$ and $I_r \in \mathscr{I}_r$, we have that

$$\overline{E}_{I_r}(F(vI_r\,\cdot)) \overset{C2}{=} F(v)\overline{E}_{I_r}\Big(1 + \frac{\epsilon}{3} S(v, I_r)[\min I_r - X]\Big)$$
$$\overset{C2,C4}{=} F(v)\Big[1 + \frac{\epsilon}{3} S(v, I_r)\overline{E}_{I_r}(\min I_r - X)\Big]$$
$$\overset{C4}{=} F(v)\Big[1 + \frac{\epsilon}{3} S(v, I_r)(\min I_r + \overline{E}_{I_r}(-X))\Big]$$
$$\overset{(2)}{=} F(v),$$

so we find that F is a test superfarthingale. From the recursiveness of S and the rational-valuedness of the forecasts $I_r \in \mathscr{I}_r$ and outcomes $x \in \mathscr{X}$ it follows that F is recursive, and therefore lower semicomputable as well. We conclude that F is a lower semicomputable test superfarthingale.

By assumption, for any $m, M \in \mathbb{N}_0$, there's some $N > m$ such that $\sum_{k=0}^{N-1} S(\upsilon_{1:k}, \iota_{k+1}) \geq M$ and

$$\frac{\sum_{k=0}^{N-1} S(\upsilon_{1:k}, \iota_{k+1})[\omega_{k+1} - \min \iota_{k+1}]}{\sum_{k=0}^{N-1} S(\upsilon_{1:k}, \iota_{k+1})} < -\epsilon. \tag{11}$$

This will allow us to obtain a lower bound for $F(\upsilon_{1:N})$. Since $1 - \frac{\epsilon}{3} S(v, I_r)[x - \min I_r] > 1/2$ for all $v \in (\mathscr{I}_r \times \mathscr{X})^*$, $I_r \in \mathscr{I}_r$ and $x \in \mathscr{X}$, it holds that $F(\upsilon_{1:N}) = \exp(K)$, with

$$K := \sum_{k=0}^{N-1} \ln\Big(1 - \frac{\epsilon}{3} S(\upsilon_{1:k}, \iota_{k+1})[\omega_{k+1} - \min \iota_{k+1}]\Big).$$

Since $\ln(1 + x) \geq x - x^2$ for all $x > -1/2$, we infer that

$$K \geq -\frac{\epsilon}{3} \sum_{k=0}^{N-1} S(\upsilon_{1:k}, \iota_{k+1})[\omega_{k+1} - \min \iota_{k+1}] - \frac{\epsilon^2}{9} \sum_{k=0}^{N-1} S(\upsilon_{1:k}, \iota_{k+1})^2[\omega_{k+1} - \min \iota_{k+1}]^2$$

and, also taking into account Equation (11), $S^2 = S$ and $[\omega_{k+1} - \min \iota_{k+1}]^2 \leq 1$,

$$\geq \frac{\epsilon^2}{3} \sum_{k=0}^{N-1} S(\upsilon_{1:k}, \iota_{k+1}) - \frac{\epsilon^2}{9} \sum_{k=0}^{N-1} S(\upsilon_{1:k}, \iota_{k+1})$$
$$= \frac{2\epsilon^2}{9} \sum_{k=0}^{N-1} S(\upsilon_{1:k}, \iota_{k+1}).$$

Hence,

$$F(\upsilon_{1:N}) \geq \exp\Big(\frac{2\epsilon^2}{9} \sum_{k=0}^{N-1} S(\upsilon_{1:k}, \iota_{k+1})\Big) \geq \exp\Big(\frac{2\epsilon^2}{9} M\Big).$$

After recalling that the inequality above holds for any $M \in \mathbb{N}_0$ and for arbitrarily large well-chosen $N \in \mathbb{N}_0$, we conclude that $\limsup_{n\to\infty} F(\upsilon_{1:n}) = \infty$. ■