Scalable and Robust Tensor Ring Decomposition for Large-scale Data (Supplementary Material)

Yicong He[1] **George K. Atia**[1,2]

[1]Department of Electrical and Computer Engineering, University of Central Florida, Orlando, FL, 32816, USA.
[2]Department of Computer Science, University of Central Florida, Orlando, FL, 32816, USA.

Before presenting the proof of the propositions, we need the following definition on Tensor Connect Product (TCP), which computes the tensor core merging.

Definition 1 (Tensor Connect Product (TCP) [Wang et al., 2017])**.** *Let* $\mathcal{Z}_k \in \mathbb{R}^{r_k \times I_k \times r_{k+1}}, k = 1, \ldots, N$ *be* N *3-order tensors. The tensor connect product (TCP) between* $\mathcal{Z}_k$ *and* $\mathcal{Z}_{k+1}$ *is defined as,*

$$\mathcal{Z}^{(k,k+1)} = \text{fold}\left(\mathbf{L}\left(\mathcal{Z}_k\right) \times \mathbf{R}\left(\mathcal{Z}_{k+1}\right)\right)$$

where $\text{fold}(\mathbf{X})$ *denotes the operation of reshaping the unfolding matrix* $\mathbf{X}$ *back to tensor* $\mathcal{X}$ *and*

$$\mathbf{L}(\mathcal{X}) = \left(\mathbf{X}_{(3)}\right)^T \in \mathbb{R}^{(r_k I_k) \times r_{k+1}}$$

$$\mathbf{R}(\mathcal{X}) = \mathbf{X}_{(1)} \in \mathbb{R}^{r_k \times (I_k r_{k+1})} .$$

First, we consider the computation of the Gram matrix using only two core tensors. According to the tensor core merging of two core tensors $\mathcal{Z}_k$ and $\mathcal{Z}_{k+1}$, we establish the following lamma.

Lemma 1. *Let* $\mathcal{Z}_k \in \mathbb{R}^{r_k \times I_k \times r_{k+1}}, k = 1, \ldots, N$*, be 3-rd order tensors. The Gram matrix of* $\mathbf{Z}_{[2]}^{(k,k+1)}$ *can be computed as*

$$\mathbf{G}_{\mathcal{Z}^{(k,k+1)}} = \mathbf{Z}_{[2]}^{(k,k+1),T} \mathbf{Z}_{[2]}^{(k,k+1)} = \Phi(\mathbf{Q}_k \mathbf{Q}_{k+1}) \tag{1}$$

where $\mathbf{Q}_k(:, i \times r_{k+1} + j) = \text{vec}\{(\mathcal{Z}_k(:,:,i))\, \mathcal{Z}_k(:,:,j)^T\}$*, with* $\text{vec}\{.\}$ *denoting the vectorization operation, and* $\Phi(\mathbf{X})$ *is a reshape operation by which* $\mathbf{X} \in \mathbb{R}^{m^2 \times n^2}$ *is first divided into* $m \times n$ *blocks* $\{\mathbf{X}_{ij}\}_{i,j=1}^{m,n} \in \mathbb{R}^{m \times n}$*, then reshaped as*

$$\Phi(\mathbf{X}) = \left[\text{vec}\{\mathbf{X}_{11}^T\}\ \text{vec}\{\mathbf{X}_{21}^T\}\ \ldots\ \text{vec}\{\mathbf{X}_{mn}^T\}\right]^T .$$

1 PROOF OF LEMMA 1

Proof. From TCP in Definition 1, we can express the fiber-wise relation between mode-2 fibers of $\mathcal{Z}_k$, $\mathcal{Z}_{k+1}$ and $\mathcal{Z}^{(k,k+1)}$ as

$$\mathcal{Z}^{(k,k+1)}(i,:,j) = \sum_{m=1}^{r_{k+1}} \mathcal{Z}_{k+1}(m,:,j) \otimes \mathcal{Z}_k(i,:,m) , \tag{2}$$

with $i \in [1, r_k], j \in [1, r_{k+2}]$ and $\otimes$ denotes the kronecker product. Then, the (i,j)-th entry of the Gram matrix of $\mathbf{Z}_{[2]}^{(k,k+1)}$ can be computed as

$$\left[\mathbf{Z}_{[2]}^{(k,k+1),T} \mathbf{Z}_{[2]}^{(k,k+1)}\right]_{i,j} = \left(\mathcal{Z}^{(k,k+1)}(p_i,:,q_i)\right)^T \mathcal{Z}^{(k,k+1)}(p_j,:,q_j) , \tag{3}$$

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where $p_i = \lceil i/r_{k+1} \rceil, q_i = \mod (i-1, r_{k+1}) + 1$. Substituting (2) in (3) and using the property

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}) \,, \tag{4}$$

we have that

$$\left[\mathbf{Z}_{[2]}^{(k,k+1),T} \mathbf{Z}_{[2]}^{(k,k+1)}\right]_{i,j} = \mathbf{v}_1^T \mathbf{v}_2 \,, \tag{5}$$

where

$$\mathbf{v}_1 = \text{vec}\{\mathcal{Z}_k(p_i, :, :)^T \mathcal{Z}_k(p_j, :, :)\}$$
$$\mathbf{v}_2 = \text{vec}\{\mathcal{Z}_{k+1}(:, :, q_i) \mathcal{Z}_{k+1}(:, :, q_j)^T\} \,.$$

Therefore, by defining $\mathbf{Q}_k(:, i \times r_{k+1} + j) = \text{vec}\{(\mathcal{Z}_k(:, :, i)) \, \mathcal{Z}_k(:, :, j)^T\}$, the Gram matrix of $\mathbf{Z}_{[2]}^{(k,k+1)}$ can be computed as

$$\mathbf{G}_{\mathcal{Z}^{(k,k+1)}} = \mathbf{Z}_{[2]}^{(k,k+1),T} \mathbf{Z}_{[2]}^{(k,k+1)} = \Phi(\mathbf{Q}_k \mathbf{Q}_{k+1}) \tag{6}$$

where $\Phi(\mathbf{X})$ is a reshape operation by which $\mathbf{X} \in \mathbb{R}^{m^2 \times n^2}$ is first divided into $m \times n$ blocks $\{\mathbf{X}_{ij}\}_{i,1,1}^{m,n} \in \mathbb{R}^{m \times n}$, then reshaped as

$$\Phi(\mathbf{X}) = \left[\text{vec}\{\mathbf{X}_{11}^T\} \ \text{vec}\{\mathbf{X}_{21}^T\} \ \ldots \ \text{vec}\{\mathbf{X}_{mn}^T\}\right]^T \,.$$

□

2 PROOF OF PROPOSITION 1

Following (1) in the proof of Lemma 1, for $\mathcal{Z}^{\leq c} \in \mathbb{R}^{r_1 \times \prod_{k=1}^{c} I_k \times r_{c+1}}$ which is a subchain obtained by merging c cores $\{\mathcal{Z}_k\}_{k=1}^{c}$, according to TCP, we can express the fiber-wise relation between mode-2 fibers of $\mathcal{Z}_c$ and $\mathcal{Z}^{\leq c-1}$ as

$$\mathcal{Z}^{\leq c}(i, :, j) = \sum_{m=1}^{r_c} \mathcal{Z}_c(m, :, j) \otimes \mathcal{Z}^{\leq c-1}(i, :, m) \,. \tag{7}$$

With the above recursion equation we have

$$\begin{aligned} &\mathcal{Z}^{\leq c}(i, :, j) \\ &= \sum_{m=1}^{r_c} \mathcal{Z}_c(m, :, j) \otimes \left(\sum_{m=1}^{r_{c-1}} \mathcal{Z}_{c-1}(m, :, j) \otimes \ldots \left(\sum_{m=1}^{r_2} \mathcal{Z}_2(m, :, j) \otimes \mathcal{Z}_1(i, :, m) \right) \right) \end{aligned} \tag{8}$$

Again, using the property in (4), we can obtain that

$$\left[\mathbf{Z}_{[2]}^{\leq c,T} \mathbf{Z}_{[2]}^{\leq c}\right]_{i,j} = \mathbf{v}_1^T \mathbf{Q}_2^T \ldots \mathbf{Q}_{c-1}^T \mathbf{v}_2 \,, \tag{9}$$

where

$$\mathbf{v}_1 = \text{vec}\{\mathcal{Z}_1(p_i, :, :)^T \mathcal{Z}_1(p_j, :, :)\}$$
$$\mathbf{v}_2 = \text{vec}\{\mathcal{Z}_c(:, :, q_i) \mathcal{Z}_c(:, :, q_j)^T\}$$
$$\mathbf{Q}_k(:, i \times r_{k+1} + j) = \text{vec}\{(\mathcal{Z}_k(:, :, i)) \, \mathcal{Z}_k(:, :, j)^T\}, k = 2, \ldots, c-1$$
$$p_i = \lceil i/r_{c+1} \rceil, q_i = \mod (i-1, r_{c+1}) + 1$$

Then, the Gram matrix of $\mathbf{Z}_{[2]}^{\leq c}$ can be computed as

$$\mathbf{G}_{\mathcal{Z}^{\leq c}} = \mathbf{Z}_{[2]}^{\leq c,T} \mathbf{Z}_{[2]}^{\leq c} = \Phi\left(\prod_{k=1}^{c} \mathbf{Q}_k\right) \,, \tag{10}$$

where $\mathbf{Q}_k(:, i \times r_{k+1} + j) = \text{vec}\{(\mathcal{Z}_k(:, :, i)) \, \mathcal{Z}_k(:, :, j)^T\}$ for $k > 1$ and

$$\mathbf{Q}_1(:, i \times r_2 + j) = \begin{cases} \text{vec}\{(\mathcal{Z}_1(:, :, i))\mathcal{Z}_1(:, :, j)^T\}, c \text{ is even} \\ \text{vec}\{(\mathcal{Z}_1(:, :, j))\mathcal{Z}_1(:, :, i)^T\}, c \text{ is odd} \end{cases}$$

References

Wenqi Wang, Vaneet Aggarwal, and Shuchin Aeron. Efficient low rank tensor ring completion. In *Proceedings of the IEEE International Conference on Computer Vision*, pages 5697–5705, 2017.