Investigating a Generalization of Probabilistic Material Implication and Bayesian Conditionals

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Abstract

Probabilistic "if A then B" rules are typically formalized as Bayesian conditionals $\boldsymbol{P}(B \mid A)$, as many (e.g., Pearl) have argued that Bayesian conditionals are the correct way to think about such rules. However, there are challenges with standard inferences such as modus ponens and modus tollens that might make probabilistic material implication a better candidate at times for rule-based systems employing forward-chaining; and arguably material implication is still suitable when information about prior or conditional probabilities is not available at all. We investigate a generalization of probabilistic material implication and Bayesian conditionals that combines the advantages of both formalisms in a systematic way and prove basic properties of the generalized rule, in particular, for inference chains in graphs.

1 INTRODUCTION

Natural language "if-then" rules cover a wide range of logically distinct conditions such as *analytical truths* ("if x is an even number, x is divisible by two"), *conceptual relationships* ("if x is a human, then x is mortal"), *inductive inferences* ("if the sprinkler is on, the grass will become wet"), *abductive inferences* ("if the light switch is on but the light is off, the light bulb is broken"), *normative constraints* ("if the traffic light is red, you are not allowed to drive"), *hypotheticals* ("if I were to run fast, I would be out of breath"), and *counterfactuals* ("if the driver had been able to brake in time, they would not have killed the deer") (see also Ciardelli and Ommundsen [2022], Crupi and Iacona [2021], Dubois and Prade [1990], Nilsson [1993], Greene [2008]).

Treating conditionals as material implications, however, has long been seen to be problematic (e.g., Adams [1965]) and the widely accepted solution is to view such rules as *probabilistic* and best modeled by the Bayesian conditional $\boldsymbol{P}(B \mid A)$ [e.g., Pearl, 1988]. At the same time, Bayesian conditionals are not appropriate for handling (indicative) conditionals [e.g., see Khoo and Mandelkern, 2018, for a discussion], and cannot be used for inference (e.g., using *modus tollens*) if the probability of the prior is zero: $P(A) = 0$.

Fortunately, there are ways to combine probabilistic logical and Bayesian inference in a way that views probabilistic material implication and Bayesian conditionals as two limit points of a "rule-continuum", $A \xrightarrow{\gamma} B$, expressed by a real-valued parameter $\gamma \in [0, 1]$ (with $\gamma = 1$ for material implication $A \rightarrow B$, and $\gamma = 0$ for the Bayesian conditional $A|B$).

The goal of this paper is to investigate the formal properties of the "generalized rule" $\xrightarrow{\gamma}$ and to develop a framework that can be used for reasoning with it. We start with a brief motivation for the generalization and introduce its mathematical form. Then we start to investigate its various properties: functional relationships among various probabilities comparing material implication, Bayesian conditional, and the generalized rule, as well as their bounds. Then we consider implication chains and prove a result on the bounds for generalized implication chains. We conclude with a brief discussion and summary of our findings.

2 MOTIVATION AND RELATED WORK

Efforts to combine probability and logic go back at least to Leibniz, Jakob Bernoulli, and Boole (e.g., see Hailperin [1996], and also Hailperin [1984] for a formal framework for "probability models"). Mapping logical operators onto set-theoretic operators in the usual way, we get set-theoretic, and hence, probabilistic interpretations of propositional sentences, e.g., from $A \rightarrow B \equiv \neg A \vee B$ we get the standard

$$\boldsymbol{P}(A \rightarrow B) = \boldsymbol{P}(\overline{A}) + \boldsymbol{P}(A \cap B). \quad (1)$$

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It was recognized at least as early as Boole [1916] (see also Hailperin [1984]) that bounds are needed when dealing with probabilities of events (i.e., sets representing propositional sentences). Using linear programming, the existence of lower and upper bound functions for the probability of any propositional logic formula was established in Hailperin [1965]. In Hailperin [1984], these results were extended, and probability bounds on *modus ponens* and *hypothetical syllogism* were given.

For our purposes, the most salient motivation for working with imprecise, i.e., bounded, probabilities is that they are *unavoidable* if we want to use *modus ponens* with probabilistic implication rules: given $a = \boldsymbol{P}(A)$ and $m = \boldsymbol{P}(A \to B)$, applying *modus ponens* to infer $b = \boldsymbol{P}(B)$ only yields bounds on $b \in [x, m]$ where $x = \boldsymbol{P}(A \cap B)$. A formal semantic method to compute the bounds on the probability of a given sentence in the predicate calculus was developed in Nilsson [1986] and extended in Fagin and Halpern [1989]. Note that we can also use the Bayesian conditional $B \mid A$ for inference (e.g., instead of material implication) to get bounds on B which were initially presented in Wagner [2004] (see below for details).

An abstract notion of probabilistic logical implication $A \xrightarrow{\gamma} B$ was introduced in Nguyen et al. [2002], and the probability of such a generalized implication was computed as a function of a parameter γ, $0 \leq \gamma \leq 1$:

$$\begin{aligned} \boldsymbol{P}(A \xrightarrow{\gamma} B) &= 1 - \frac{\boldsymbol{P}(A)(1 - \boldsymbol{P}(B \mid A))}{\gamma + (1-\gamma)\boldsymbol{P}(A)} \\ &= 1 - \frac{a(1-c)}{\gamma + (1-\gamma)a} \end{aligned} \tag{2}$$

Alternatively, we can write this generalized rule as

$$\boldsymbol{P}(A \xrightarrow{\gamma} B) = \frac{\boldsymbol{P}(A \cap B) + \gamma \cdot \boldsymbol{P}(\overline{A})}{\boldsymbol{P}(A) + \gamma \cdot \boldsymbol{P}(\overline{A})}$$

which more clearly shows the form to be some sort of interpolation from none of $\overline{A}$ to all of $\overline{A}$.

It is immediately apparent that $\gamma = 0$ corresponds to the Bayesian conditional, and $\gamma = 1$ corresponds to material implication, hence γ is interpolating between the Bayesian conditional and material implication. Hence, these two interpretations of the phrase "if A then B" are actually different manifestations of the same abstract concept—the two endpoints of a one-parameter family of probabilistic logical implications. Figure 1 indicates how to view (2) in a Venn diagram.

For the rest of the paper, we will use the notational conventions in Definition 1.

Definition 1 (Notation for basic probability variables)**.** Fix A, B, γ and let $X = A \cap B$. Then $a = \boldsymbol{P}(A)$, $b = \boldsymbol{P}(B)$, $x = \boldsymbol{P}(X)$, $m = \boldsymbol{P}(A \to B)$ for *m*aterial implication, $c = \boldsymbol{P}(B \mid A)$ for Bayesian *c*onditional, $g = \boldsymbol{P}(A \xrightarrow{\gamma} B)$ for *g*eneralized implication.

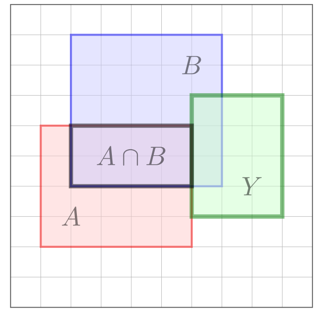


Figure 1: $\boldsymbol{P}(A) = 0.20$, $\boldsymbol{P}(B) = 0.25$, $\boldsymbol{P}(A \cap B) = 0.08$, $\gamma = 0.15$, so $\boldsymbol{P}(Y) = \gamma \boldsymbol{P}(\overline{A}) = (0.15)(0.80) = 0.12$. Then $\boldsymbol{P}(A \xrightarrow{.15} B) = \frac{0.08+0.12}{0.20+0.12} = 0.625$, $\boldsymbol{P}(A \xrightarrow{0} B) = \boldsymbol{P}(B \mid A) = 0.40$, and $\boldsymbol{P}(A \xrightarrow{1} B) = \boldsymbol{P}(A \to B) = 0.88$.

We can use the notation from Definition 1 to express the four basic regions in terms of a, b, m, c as $\boldsymbol{P}(A \cap B) = ac = m + a - 1$, $\boldsymbol{P}(A \cap \overline{B}) = a(1-c) = 1 - m$, $\boldsymbol{P}(\overline{A} \cap B) = b - ac = 1 - m - a + b$, $\boldsymbol{P}(\overline{A} \cap \overline{B}) = 1 - a - b + ac = m - b$. Note that if $a = 0$ then $x = \boldsymbol{P}(A \cap B) = 0$, even though c is undefined, and the expressions are still correct using this interpretation of ac (as the intersection). But typically, we work under the assumption that $a \neq 0$, so that c is defined. Furthermore, since a, b, x, m, c are probabilities, we obviously must have $0 \leq a, b, x, m, c, g \leq 1$.

3 THE GENERALIZED IMPLICATION

We start by investigating the relationships among the (probabilities of the) generalized implication $A \xrightarrow{\gamma} B$, the material implication $A \to B$, the Bayesian conditional $B \mid A$, the premise A and the intersection $A \cap B$.

3.1 FUNCTIONAL RELATIONSHIPS AMONG m, c, a, x, g

We first express all of our basic probabilities in terms of the classical probabilities m, c, a, x:

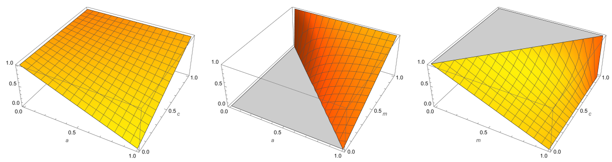
Lemma 2. *For m, c, a as in Definition 1, we have the following functional relationships:*

i) $m = 1 - a + ac = 1 - a(1-c)$,

ii) $c = \dfrac{m + a - 1}{a} = 1 - \dfrac{1-m}{a}$, *for* $a \neq 0$,

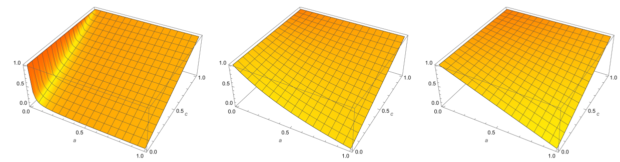
iii) $a = \dfrac{1-m}{1-c}$, *for* $c \neq 1$.

Furthermore, using these we can express $x = \boldsymbol{P}(A \cap B)$ in the following ways:



$m(a, c)$ $c(a, m)$ $a(m, c)$

Figure 2: Surfaces coming from m, c, a



$\gamma = 0.01$ $\gamma = 0.5$ $\gamma = 1.0$

Figure 3: $g(\gamma, a, c)$ for various γ

iv) $x = ac = a + m - 1 = \left(\dfrac{1-m}{1-c}\right) c.$

Proof. These are all just restatements of (1), or the definition of c. ☐

The following relationship between $m = \boldsymbol{P}(A \to B)$ and $c = \boldsymbol{P}(B \mid A)$ is an easy consequence of Lemma 2 and the definition of conditional probability.

Lemma 3. *$\boldsymbol{P}(A \to B) = m \geq c = \boldsymbol{P}(B \mid A)$, in fact,*

$$m \begin{cases} = 1 & \text{if } a = 0, \\ = c = b & \text{if } a = 1, \\ = c = 1 & \text{if } 0 < a < 1 \text{ and } a = x, \\ > c & \text{if } 0 < a < 1 \text{ and } a \neq x. \end{cases}$$

Proof. For the inequality $m > c$, first note that $a \neq x \Rightarrow c < 1$. Then compute $m - c = (1-a)(1-c) > 0$. ☐

Figure 2 shows the graphs of the functions in Lemma 2. These are, of course, the same piece of the quadratic surface of $1 - m - a + ac = 0$ with the axes rotated (but compare to the higher-dimensional case in Figures 3, 4). However, these different functional expressions are needed when doing iterated updates (see Figure 6) and when computing bounds.

The next lemma extends (and subsumes) Lemma 2 by expressing everything in terms of arbitrary γ, g rather than $\gamma = 0, g = c$ and $\gamma = 1, g = m$, as before.

Lemma 4. *Let m, c, a, x, γ, g be as in Definition 1 and Lemma 2. Then*

i) $$g = \frac{x + \gamma(1-a)}{a + \gamma(1-a)} = \frac{(1-\gamma)x + \gamma m}{(1-\gamma)a + \gamma \cdot 1} = \frac{ac + \gamma(1-a)}{a + \gamma(1-a)} = \frac{(a+m-1) + \gamma(1-a)}{(1-\gamma)a + \gamma \cdot 1}$$

ii) $x = (1-\gamma)ag + \gamma(a + g - 1),$

iii) $m = (1-\gamma)ag + \gamma(a + g - 1) + (1-a),$

iv) $c = \dfrac{(1-\gamma)ag + \gamma(a+g-1)}{a},$

v) $a = \dfrac{\gamma(1-g)}{\gamma(1-g) + g - c},$

vi) $\gamma = \dfrac{a(g-c)}{(1-a)(1-g)},$

and for arbitrary $\hat{\gamma}$ (not necessarily equal to γ), and letting $\hat{g} = g(\hat{\gamma}, x, a)$ from part (i) above, we have

vii) $\hat{g} = \dfrac{[(1-\gamma)ag + \gamma(a+g-1)] + \hat{\gamma}(1-a)}{a + \hat{\gamma}(1-a)},$

assuming the variables are in the domains of the relevant formulas.

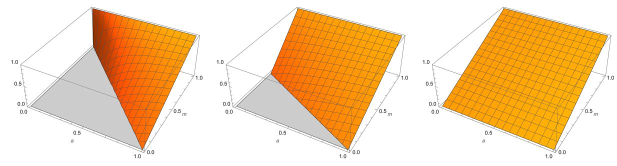
Proof. The first six come from (2) and straightforward arithmetic. To get Lemma 4.vii just substitute $x(\gamma, g, a)$ from Lemma 4.ii into $\hat{g} = \hat{g}(\hat{\gamma}, a, x)$ from Lemma 4.i. ☐

Note that all four of the expressions in Lemma 4.i have the form: an interpolation (some expression for) $x \mapsto$ (some expression for) m divided by an interpolation $a \mapsto 1$. It is in this sense that g interpolates from c to m as γ goes from 0 to 1. It is interesting to note that Lemma 4.ii, the expression for $x(\gamma, g, a)$ is an interpolation from one expression of x ($= ac$) to another ($= a + m - 1$), i.e., $x(a, c) \mapsto x(a, m)$, as γ goes from 0 to 1. The analogous claim is also true for Lemma 4.iii, 4.iv. The point of Lemma 4.vii is that the g defined in terms of one γ can also be defined in terms of another $\hat{\gamma}$ and its corresponding $\hat{g}$. This is just the general version of Lemma 4.i, 4.iii and 4.iv, and will be important later when we look at implication chains.

As above, all of these functions give a surface when graphed. An important point, however, is that all these expressions in Lemma 4.i give different functions when considered as functions of the same two variables x_1, x_2, say. For example, Figure 3 shows the graphs $g(\gamma, a, c)$, for various γ, and Figure 4 show the same series for $g(\gamma, a, m)$.

3.2 BOUNDS ON PROBABILITIES

Henceforth, all probabilities will be *imprecise*, i.e., we will typically not know the value p of a probability, but rather, will only be given $p_0, p_1 \in [0, 1]$ such that $p_0 \leq p \leq p_1$. The quantity p_0 is the *lower*, and p_1 is the *upper* probability for p. Of course, there is always the need to restrict to the



$\gamma = 0.01$ $\gamma = 0.5$ $\gamma = 1.0$

Figure 4: $g(\gamma, a, m)$ for various γ

interval $[0, 1]$ for the output of any formula giving probability bounds. In order to simplify notation, we will build this into our notation for probability bounds.

Definition 5 (The Lower- and Upper-bound operators)**.** Let p_0, p_1 be values for the lower and upper bounds on probability p, i.e., $p_0 \leq p \leq p_1$, but that p_0, p_1 are computed from functions that do not necessarily give values in $[0, 1]$. Then

$$p_{\boldsymbol{L}} = \max\{p_0, 0\}\ , \quad p_{\boldsymbol{U}} = \min\{p_1, 1\}.$$

For any probability p, we will write $p \in [p_{\boldsymbol{L}},\ p_{\boldsymbol{U}}]$, and we will denote the width of any probability interval $[p_{\boldsymbol{L}},\ p_{\boldsymbol{U}}]$ as $\Delta p = p_{\boldsymbol{U}} - p_{\boldsymbol{L}}$.

We next compute bounds on one of a, c, m, b given bounds on the other variables by observing that the relevant function is monotone increasing or decreasing (obvious from the graphs above), and then computing the relevant endpoints to get the extreme values:

Lemma 6. *Every function in Lemmas 2 and 4 is of the form $u(v^1, \ldots, v^k)$, for some $u, v^1, \ldots, v^k \in \{a, m, c, x, \gamma, \hat{\gamma}, g, \hat{g}\}$. Fix such $u(v^1, \ldots, v^k)$ and $v \in \{v^1, \ldots, v^k\}$ such that if $u(v^1, \ldots, v^k) = \hat{g}(\hat{\gamma}, \hat{g}, \gamma, a)$, then $v \neq a$. Then $\frac{\partial u}{\partial v}(v^1, \ldots, v^k)$ is either always positive, for all $v^1, \ldots, v^k \in (0, 1)$, or it is always negative, for all $v^1, \ldots, v^k \in (0, 1)$, assuming of course, that the partial is defined for these $v^1, \ldots, v^k$.*

We note that the one case not covered by Lemma 6, i.e., $u = \hat{g}(\hat{\gamma}, \hat{g}, \gamma, a)$, $v = a$ (see Lemma 4.vii), satisfies $\frac{\partial \hat{g}}{\partial a}(\hat{\gamma}, \hat{g}, \gamma, a) = \frac{(1-g)(\gamma - \hat{\gamma})}{a + \hat{\gamma}(1-a)} < 0$ if $\gamma < \hat{\gamma}$, $= 0$ if $\hat{\gamma} = \gamma$, and > 0 if $\gamma > \hat{\gamma}$, and hence, the function $\lambda a\, \hat{g}(\hat{\gamma}, \hat{g}, \gamma, a)$ is decreasing if $\gamma < \hat{\gamma}$ and increasing if $\gamma > \hat{\gamma}$.

Some specific examples of the computations in Lemma 6 are: $\partial m/\partial c = a > 0$, $\partial c/\partial a = (1-m)/a^2 > 0$, and $\partial a/\partial m = -1/(1-c) < 0$. We remark that the negative derivatives are to be expected, since they are a consequence of the cyclic chain rule. For example, $\frac{\partial m}{\partial c}\frac{\partial c}{\partial a}\frac{\partial a}{\partial m} = -1$.

An easy application of Lemma 6 is the observation that the probabilities of generalized rules are monotonic in γ:

Lemma 7. *If $a, c < 1$, then $\gamma_1 < \gamma_2$ implies $\boldsymbol{P}(A \xrightarrow{\gamma_1} B) < \boldsymbol{P}(A \xrightarrow{\gamma_2} B)$.*

Proof. From $\frac{\partial g(\gamma, a, c)}{\partial \gamma} = \frac{a(1-a)(1-c)}{(a + \gamma(1-a))^2} > 0$. □

3.3 BOUNDS ON m, c, a, g FROM PARTIAL DERIVATIVES

An immediate consequence of Lemma 6 is that all these functions are monotonic on $(0, 1)$ in each of their variables, with the exception of the one in Lemma 4.vii that has a min when $\hat{\gamma} = \gamma$. So to compute the bounds of any of our basic probabilities, in terms of the bounds of other variables, we just need to evaluate at the appropriate endpoints, i.e., the bounds of the independent variables.

Lemma 8. *Fix any function from Lemma 6, $u(v^1, \ldots, v^k)$, with $u, v^1, \ldots, v^k \in \{a, m, c, x, \gamma, g\}$, and $v^i \in [v^i_{\boldsymbol{L}},\ v^i_{\boldsymbol{U}}]$. Let $v^i_{\boldsymbol{E}} = \begin{cases} v^i_{\boldsymbol{L}} & \text{if } \partial u/\partial v^i > 0 \\ v^i_{\boldsymbol{U}} & \text{if } \partial u/\partial v^i < 0 \end{cases}$ and $v^i_{\boldsymbol{F}} = \begin{cases} v^i_{\boldsymbol{L}} & \text{if } \partial u/\partial v^i < 0 \\ v^i_{\boldsymbol{U}} & \text{if } \partial u/\partial v^i > 0, \end{cases}$ and note that $\{v^i_{\boldsymbol{E}}, v^i_{\boldsymbol{F}}\} = \{v^i_{\boldsymbol{L}}, v^i_{\boldsymbol{U}}\}$. Then we have the bounds $u_{\boldsymbol{L}} = u(v^1_{\boldsymbol{E}}, \ldots, v^k_{\boldsymbol{E}})$ and $u_{\boldsymbol{U}} = u(v^1_{\boldsymbol{F}}, \ldots, v^k_{\boldsymbol{F}})$.*

An easy example of Lemma 8 in action would be to use $\partial m(a, c)/\partial a < 0$ and $\partial m(a, c)/\partial c > 0$ to get $m_{\boldsymbol{L}} = m(a_{\boldsymbol{U}}, c_{\boldsymbol{L}}) = 1 - a_{\boldsymbol{U}}(1 - c_{\boldsymbol{L}})$. Similarly, from $\partial a/\partial m > 0$, $\partial a/\partial c < 0$, we get $a_{\boldsymbol{U}} = a(m_{\boldsymbol{L}}, c_{\boldsymbol{U}}) = (1 - m_{\boldsymbol{L}})/(1 - c_{\boldsymbol{U}})$, and from $\partial g/\partial \gamma > 0$, $\partial g/\partial a < 0$, and $\partial g/\partial c > 0$, we get $g_{\boldsymbol{L}}(\gamma, a, c) = g(\gamma_{\boldsymbol{L}}, a_{\boldsymbol{U}}, c_{\boldsymbol{L}}) = \frac{a_{\boldsymbol{U}} c_{\boldsymbol{L}} + \gamma_{\boldsymbol{L}}(1 - a_{\boldsymbol{U}})}{a_{\boldsymbol{U}} + \gamma_{\boldsymbol{L}}(1 - a_{\boldsymbol{U}})}$.

Note that when we are computing $m_{\boldsymbol{L}}$ we are treating m as a dependent variable, and a, c as independent, and when we compute the relevant partials we are holding either a or c constant. This is not the same as solving for $m_{\boldsymbol{L}}$ in the formula for $a_{\boldsymbol{U}}$ in order to compute $m_{\boldsymbol{L}}$. Also, if these formulas are being used to update a node in a network as new information comes in, we would consider $a_{\boldsymbol{U}}, c_{\boldsymbol{L}}$ to be fixed known quantities for which we want to compute the corresponding smallest consistent m-value, $m_{\boldsymbol{L}}$. This is clearly not the same as updating $a_{\boldsymbol{U}}$ given the newly measured (and fixed) quantities $m_{\boldsymbol{L}}, c_{\boldsymbol{U}}$.

We must also keep in mind that any updated probability is clipped to stay within its old bounds, as per Definition 5. Also, note that if there are multiple ways to compute a bound, e.g., $m'_{\boldsymbol{U}}(a_{\boldsymbol{L}}, c_{\boldsymbol{U}})$, $m'_{\boldsymbol{U}}(a'_{\boldsymbol{L}}, c_{\boldsymbol{U}})$, $m'_{\boldsymbol{U}}(a_{\boldsymbol{L}}, c'_{\boldsymbol{U}})$, $m'_{\boldsymbol{U}}(a'_{\boldsymbol{L}}, c'_{\boldsymbol{U}})$, they will not always yield the same result. We must always compute with the most recent data, since the order that the updates come to us matters to the final answer. The issues above are most salient in a network of nodes that each hold a probability and are being updated in real time

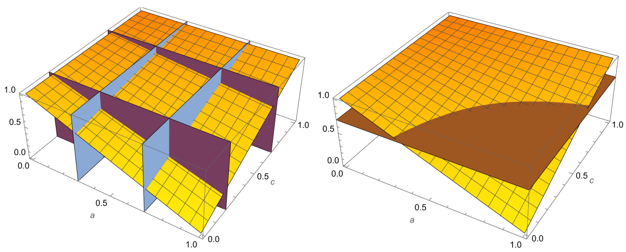


Figure 5: Geometric interpretation of bounds for $m(a,c)$ (a,c-bounds left, m-bounds right).

as new information becomes available. See Figure 6 for a example computation.

The bounds listed in Lemma 8 can be interpreted geometrically. The graph of $m(a,c)$ is given in Figure 5, and each point on this surface gives a consistent assignment of values to m,c,a. Bounds on a,c correspond to vertical planes, and bounds on m to horizontal planes. It is apparent that for even a single m-bound, m_L say, the set $\{a,c : m_L \le m(a,c) \le 1\}$ will be non-convex.

3.4 BOUNDS ON m,c,g FOR FIXED a,b

For fixed $[a_L,\ a_U]$, $[b_L,\ b_U]$, we want to find $[c_L,\ c_U]$, $[m_L,\ m_U]$, and $[g_L,\ g_U]$. As a first special case, assume $a = a_L = a_U$, $b = b_L = b_U$. Then note that $\boldsymbol{P}(A \cap B) = x = ac$, but we do not have c. Note also that x is *not* a function of a,b, but has the following bounds:

$$x_L = a + b - 1\ , \qquad x_U = \min\{a, b\}. \tag{3}$$

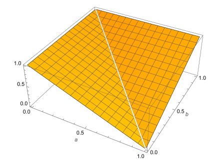
In general, if there are nontrivial probability intervals for a,b, then

$$x_L = a_L + b_L - 1\ , \qquad x_U = \min\{a_U,\ b_U\}. \tag{4}$$

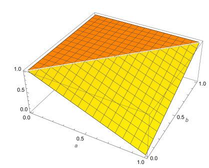
From this we want to compute bounds on c,m.

Of course, we do not have a function $c(a,b)$ to minimize/maximize, but $\frac{\partial}{\partial a}\left(\frac{a+b-1}{a}\right) \ge 0$, $\frac{\partial}{\partial b}\left(\frac{a+b-1}{a}\right) > 0$, and c_L, c_U will occur at the bounds of x, so $[c_L,\ c_U] = [\max\{(a_L + b_L - 1)/a_L, 0\},\ \min\{b_U/a_L, 1\}]$. From this c we can get bounds on $m(a,c(a,b))$ as before, but there might be clipping when recomputed using bounds gotten from $m(a,b)$ as below. Then $[m_L,\ m_U] = [1 - a_U(1 - c_L),\ 1 - a_L(1 - c_U)]$.

Using $m = 1 - a + x$, x depends on a,b (not a,c), so $[m_L,\ m_U] = [(1 - a + x)_L,\ (1 - a + x)_U] \neq [\underbrace{1 - a_U + (ac)_L}_{a,\ ac \text{ not indep}},\ 1 - a_L + (ac)_U]$. We need to minimize $m = 1 - a + x$, but expressed using the lower bound of x from (3). So we need to minimize $m = 1 - a + \max\{a + b - 1, 0\}$. From the graph



we see $\dfrac{\partial m}{\partial a} \le 0$ and $\dfrac{\partial m}{\partial b} \ge 0$, so we compute at the point (a_U, b_L), $m_L = 1 - a_U + \max\{a_U + b_L - 1, 0\} = \begin{cases} 1 - a_U & \text{if } b_L \le 1 - a_U \\ b_L & \text{if } b_L > 1 - a_U \end{cases}$. And to maximize m we need to use the upper bound for x from (3). So we maximize $m = 1 - a + x = 1 - a + \min\{a, b\}$. From the graph



we see that $\dfrac{\partial m}{\partial a} \le 0$ and $\dfrac{\partial m}{\partial b} \ge 0$, so we compute at the point (a_L, b_U), $m_U = 1 - a_L + \min\{a_L, b_U\} = \begin{cases} 1 - a_L + b_U & \text{if } b_U \le a_L \\ 1 & \text{if } b_U > a_L \end{cases}$.

3.5 BOUNDS ON b

We next investigate different ways to obtain bounds on b. Let $a \in [a_L,\ a_U]$. If we know both m,c (exactly), then we know a $(= \frac{1-m}{1-c})$ exactly, so using either material implication $(\gamma = 1)$ or conditional $(\gamma = 0)$ gives the same bounds $b \in [ac,\ m]$.

Now suppose we know m, but are not given any information about c. Hence we must compute $c \in [c_L,\ c_U] = [(m + a_L - 1)/a_L, (m + a_U - 1)/a_U]$. Then $b \in [a_L + m - 1,\ m]$, i.e., $b = [a_L c_L,\ m]$, and $\Delta b = m - a_L c_L = m - (a_L + m - 1) = 1 - a_L$. Also note: $\Delta c = \dfrac{a_U + m - 1}{a_U} - \dfrac{a_L + m - 1}{a_L} = \dfrac{\Delta a}{a_L a_U}(1 - m)$. But since $a + m \ge 1$, for all consistent a (in particular a_L), we have $\dfrac{\Delta a}{a_L a_U}(1 - m) \le \dfrac{\Delta a}{a_L a_U} a_L = \dfrac{\Delta a}{a_U} = 1 - \dfrac{a_L}{a_U}$.

Next suppose we know c, but are not given any information about m. Hence we must compute $m \in [m_L,\ m_U] = [1 - a_U(1 - c),\ 1 - a_L(1 - c)]$. Then $b \in [a_L c,\ m_U]$, and $\Delta b = m_U - a_L c = 1 - a_L(1 - c) - a_L c = 1 - a_L$. Also, $\Delta m = 1 - a_L(1 - c) - 1 + a_U(1 - c) = \Delta a(1 - c)$. So we see that knowing m (and not c) gives a better b_U, and knowing c (and not m) gives a better b_L. But the Δb is the same in both cases.

Finally, suppose we are given $m \in [m_L,\ m_U]$ and $c \in [c_L,\ c_U]$. Then inference with material implication gives

iteration	m_L $1-a_1(1-c_0)$	m_U $1-a_0(1-c_1)$	c_L $1-(1-m_0)/a_0$	c_U $1-(1-m_1)/a_1$	a_L $(1-m_1)/(1-c_0)$	a_U $(1-m_0)/(1-c_1)$
start	0	1	0	1	0	1
1	$\boxed{0.5}$	1	$\underline{0}$	1	0	$\overline{1}$
2	$\underline{0.5}$	$\boxed{0.8}$	$\underline{0}$	0.8	0.2	$\overline{1}$
3	$\underline{0.5}$	$\overline{0.8}$	$1/6$	0.8	$\boxed{0.6}$	$\overline{1}$
4	$\underline{0.5}$	$\overline{0.8}$	$1/6$	0.75	$\underline{0.6}$	$\boxed{0.8}$
start	0	1	0	1	0	1
1	0	1	0	1	$\boxed{0.6}$	1
2	0.2	1	$\underline{0}$	1	0.6	$\boxed{0.8}$
3	$\boxed{0.5}$	1	$1/6$	1	$\underline{0.6}$	$\overline{0.8}$
4	$\underline{0.5}$	$\boxed{0.8}$	$1/6$	0.75	$\underline{0.6}$	$\overline{0.8}$
start	$\boxed{0.2}$	$\boxed{0.8}$	0	1	$\boxed{0.1}$	$\boxed{0.9}$
1	$\underline{0.2}$	$\overline{0.8}$	$\underline{0}$	$7/9$	0.2	$\overline{0.9}$
start	$\boxed{0.3}$	$\boxed{0.8}$	0	1	0	1
1	0.3	0.8	0	0.8	0.2	1

Figure 6: Updating bounds with new data: $\boxed{q}$ means q was manually reset due to new information into to the system. If q was outside the old bounding box, $\underline{q}$ means q was clipped up to the old bound q_L, and $\overline{q}$ means clipped down to the old q_U.

$b_L = a_L c(a_L, m_L) = a_L + m_L - 1$, $b_U = m_U$ which gives $\Delta b = 1 - a_L + \Delta m$. Bayesian inference gives $b_L = a_L c_L$, $b_U = m(a_L, c_U) = 1 - a_L(1 - c_U)$, and $\Delta b = 1 - a_L(1 - \Delta c) = 1 - a_L + a_L \Delta c$. And if we kept the best bounds from both, we would have $b_L = \max\{a_L + m_L - 1,\ a_L c_L\}$, $b_U = \min\{m_U,\ 1 - a_L(1 - c_U)\}$.

A priori, any of these four bounds could be optimal. In practice, the values of c_L, m_U would already be reset to the computed values if the original values were less optimal than the computed ones in the optimization/clipping phase. In this case we would just have $b \in [a_L c_L, m_U]$. Then $\Delta b = m_U - a_L c_L$. We can't simplify algebraically, but $m_U \geq m(a_L, c_U)$ and $c_L \leq c(a_L, m_L) \leq c_U$ give $\Delta b \geq m(a_L, c_U) - a_L c_L = (1-a_L) + a_L \Delta c$, $\Delta b \geq m_U - a_L c(a_L, m_L) = (1-a_L) + \Delta m$.

Now we want to compute bounds on b when we have a generalized implication $A \xrightarrow{\gamma} B$. The equations in Lemma 2 are true for any B with $\boldsymbol{P}(B \mid A) = c$, but b is itself not functionally related to any of m, c, a. Consequently, we can only get bounds on b, even if we know all of m, c, a, x precisely. In this case, we use $\boldsymbol{P}(A \cap B) \leq \boldsymbol{P}(B) \leq \boldsymbol{P}(\overline{A} \cup B)$, to get the well-known (tight) bounds $x \leq b \leq m$, i.e., Using Lemma 4.ii, 4.iii, we can express these bounds in terms of g:

$$b_L = (1-\gamma)ag + \gamma(a + g - 1), \tag{5}$$

$$b_U = (1-\gamma)ag + \gamma(a + g - 1) + (1 - a). \tag{6}$$

Of course, these formulas apply when we have precise probabilities for a, g. If we only have bounds for a and g, we compute $\frac{\partial}{\partial a} b_L(a, g) > 0$, $\frac{\partial}{\partial g} b_L(a, g) > 0$, and $\frac{\partial}{\partial a} b_U(a, g) < 0$, $\frac{\partial}{\partial g} b_U(a, g) > 0$. Using Lemma 8 we can compute $(b_L)_L = x_L$, $(b_L)_U = x_U$, $(b_U)_L = m_L$, $(b_U)_U = m_U$. We will abuse notation and write $b_L = (b_L)_L$ and $b_U = (b_U)_U$, and discard the other two:

$$b_L = b_L(a_L, g_L) = (1-\gamma)a_L g_L + \gamma(a_L + g_L - 1) \tag{7}$$

$$\begin{aligned} b_U &= b_U(a_L, g_U) \\ &= (1-\gamma)a_L g_U + \gamma(a_L + g_U - 1) + (1 - a_L). \end{aligned} \tag{8}$$

Whether b_L refers to the function in (5) or the value in (7) will be clear from context. Furthermore, $\Delta b = (1 - a_L) + [(1-\gamma)a_L + \gamma \cdot 1]\Delta g$, and if g is precise we just have $\Delta b = 1 - a_L$, as noted above for the case of precise m, c.

4 IMPLICATION CHAINS WITH GENERALIZED IMPLICATION

Implications chains (i.e., *hypothetical syllogism*) are of particular interest for rule-based reasoning, hence we briefly summarize the properties of the generalized implication chains.

Suppose we have a sequence of generalized implications $B^i \xrightarrow{\gamma^i} B^{i+1}$, for $i = 0, 1, \ldots, N-1$. Let $A = B^0$, and suppose we are given bounds, $a \in [a_L,\ a_U]$, on the initial antecedent A. We want to compute the bounds for the final consequent b^N. We first do this for arbitrary γ. Then we easily derive the bounds for material implication (*modus ponens*, *hypothetical syllogism*) and Bayesian inference by setting $\gamma = 0, 1$.

Theorem 9. *Fix arbitrary γ and suppose we are given a sequence of generalized implications $B^i \xrightarrow{\gamma} B^{i+1}$, for $i = 0, 1, \ldots, N-1$. Suppose $A = B^0$ and that we are given bounds $a \in [a_{\boldsymbol{L}},\ a_{\boldsymbol{U}}]$, $g^i \in [g^i_{\boldsymbol{L}},\ g^i_{\boldsymbol{U}}]$, for $i < N$. Let $Y^i = \gamma(1-g^i) + g^i$ and $Z^i = \gamma(1-g^i)$. Then the bounds at the end of the generalized implication chain are*

$$b^N_{\boldsymbol{L}} = a_{\boldsymbol{L}} \prod_{i=0}^{N-1} Y^i_{\boldsymbol{L}} - \sum_{i=0}^{N-1} \left(Z^i_{\boldsymbol{U}} \prod_{j=i+1}^{N-1} Y^j_{\boldsymbol{L}} \right),$$

$$b^N_{\boldsymbol{U}} = 1 - Z^{N-1}_{\boldsymbol{L}} - (1 - Y^{N-1}_{\boldsymbol{U}}) \left[a_{\boldsymbol{L}} \prod_{i=0}^{N-2} Y^i_{\boldsymbol{U}} - \sum_{i=0}^{N-2} \left(Z^i_{\boldsymbol{L}} \prod_{j=i+1}^{N-2} Y^j_{\boldsymbol{U}} \right) \right].$$

Proof. We first note that $Y^i_{\boldsymbol{L}} = Y^i(g^i_{\boldsymbol{L}})$, $Y^i_{\boldsymbol{U}} = Y^i(g^i_{\boldsymbol{U}})$, $Z^i_{\boldsymbol{L}} = Z^i(g^i_{\boldsymbol{U}})$, $Z^i_{\boldsymbol{U}} = Z^i(g^i_{\boldsymbol{L}})$. To prove the claim we give a straightforward induction on n. For the base case $n = 1$, simple substitution and algebra gives $b^1_{\boldsymbol{L}} = a_{\boldsymbol{L}} Y^0_{\boldsymbol{L}} - Z^0_{\boldsymbol{U}} = (1-\gamma) a_{\boldsymbol{L}} g^0_{\boldsymbol{L}} + \gamma(a_{\boldsymbol{L}} + g^0_{\boldsymbol{L}} - 1)$, which is just (7). Similarly, $b^1_{\boldsymbol{U}} = 1 - a_{\boldsymbol{L}}(1 - Y^0_{\boldsymbol{U}}) - Z^0_{\boldsymbol{L}} = 1 - a_{\boldsymbol{L}} + (1-\gamma) a_{\boldsymbol{L}} g^0_{\boldsymbol{U}} + \gamma(a_{\boldsymbol{L}} + g^0_{\boldsymbol{U}} - 1)$, which is just (8).

For the inductive step, we get $b^{n+1}_{\boldsymbol{L}} = b^n_{\boldsymbol{L}} Y^n_{\boldsymbol{L}} - Z^n_{\boldsymbol{U}}$
$= \left[a_{\boldsymbol{L}} \prod_{i=0}^{n-1} Y^i_{\boldsymbol{L}} - \sum_{i=0}^{n-1} \left(Z^i_{\boldsymbol{U}} \prod_{j=i+1}^{n-1} Y^j_{\boldsymbol{L}} \right) \right] Y^n_{\boldsymbol{L}} - Z^n_{\boldsymbol{U}}$
$= a_{\boldsymbol{L}} \prod_{i=0}^{n} Y^i_{\boldsymbol{L}} - \sum_{i=0}^{n} \left(Z^i_{\boldsymbol{U}} \prod_{j=i+1}^{n-1} Y^j_{\boldsymbol{L}} \right) Y^n_{\boldsymbol{L}} - Z^n_{\boldsymbol{U}}$
$= a_{\boldsymbol{L}} \prod_{i=0}^{n} Y^i_{\boldsymbol{L}} - \sum_{i=0}^{n} \left(Z^i_{\boldsymbol{U}} \prod_{j=i+1}^{n} Y^j_{\boldsymbol{L}} \right) Y^n_{\boldsymbol{L}}$, as desired, and similarly for $b^{N+1}_{\boldsymbol{U}}$.

Recall that $b^i_{\boldsymbol{L}} = \cdots$, $b^i_{\boldsymbol{U}} = \cdots$ are automatically clipped to remain in $[0, 1]$, as per Definition 5, but that this clipping is lost if we combine and rearrange the RHS expressions for these quantities, rather than referring to the LHS quantities themselves. So the above formulas assume no clipping was necessary at intermediate stages of the recursion. □

Note that since $\gamma = 0 \Rightarrow Y = c,\ Z = 0$, and $\gamma = 1 \Rightarrow Y = 1,\ Z = 1 - m$, we can easily recover the classical cases below.

Corollary 10. If we are using Bayesian inference, then $\gamma = 0$. So $c^i \in [c^i_{\boldsymbol{L}},\ c^i_{\boldsymbol{U}}]$, for $i < N$ are given, and the bounds at the end of the implication chain are

$$b^N_{\boldsymbol{L}} = a_{\boldsymbol{L}} \prod_{i=0}^{N-1} c^i_{\boldsymbol{L}}\ , \quad b^N_{\boldsymbol{U}} = 1 - \left(a_{\boldsymbol{L}} \prod_{i=0}^{N-1} c^i_{\boldsymbol{L}} \right) \left(1 - c^{N-1}_{\boldsymbol{U}}\right)$$

We note that for typical data (not all $c^i_{\boldsymbol{L}} = 1$) we see $b^N_{\boldsymbol{L}} \to 0$ and $b^N_{\boldsymbol{U}} \to 1$.

Corollary 11. If we are using material implications in our chain, we just set $\gamma = 1$ in the formulas above. So $m^i \in [m^i_{\boldsymbol{L}},\ m^i_{\boldsymbol{U}}]$, for $i < N$, are given, and the bounds at the end of the implication chain are

$$b^N_{\boldsymbol{L}} = a_{\boldsymbol{L}} - \sum_{i=0}^{N-1} \left(1 - m^i_{\boldsymbol{L}}\right)\ , \qquad b^N_{\boldsymbol{U}} = m^{N-1}_{\boldsymbol{U}}.$$

We note that for typical data (not all $m^i_{\boldsymbol{L}} = 1$) we have $b^N_{\boldsymbol{L}} \to 0$. Also, $b^N_{\boldsymbol{U}}$ just depends on the last implication.

Lemma 12. *Suppose we are given a sequence of generalized implications $B^i \xrightarrow{\gamma^i} B^{i+1}$, for $i = 0, 1, \ldots, N-1$. Let $A = B^0$, and suppose we are given the bounds $a \in [a_{\boldsymbol{L}},\ a_{\boldsymbol{U}}]$, and the bounds $g^i \in [g^i_{\boldsymbol{L}},\ g^i_{\boldsymbol{U}}]$, for $i < N$. Then the bounds for the end of the generalized implication chain can be computed using Theorem 9.*

Proof. We just use Lemma 4.vii to write all of the γ^i, g^i pairs in terms of a single fixed $\hat{\gamma}$ (of our choice), and the newly computed $\hat{g}^i$. Then apply Theorem 9. However, these bounds might not be optimal for the reasons discussed in Section 3.5. □

Lemma 13. *The generalized implication chain, $B^i \xrightarrow{\gamma} B^{i+1}$, for $i = 0, 1, \ldots, N-1$, from Theorem 9, can be replaced with a single implication $A \xrightarrow{\gamma} B$, where $A = B^0$, $B = B^N$, and the bounds $[b^N_{\boldsymbol{L}},\ b^N_{\boldsymbol{U}}]$ are the same, whether computed from the implication chain or the single implication.*

Proof. Note that everything is specified except for the bounds on g. Since γ, $a_{\boldsymbol{L}}$, $b_{\boldsymbol{L}} = b^N_{\boldsymbol{L}}$, and $b_{\boldsymbol{U}} = b^N_{\boldsymbol{U}}$ are all fixed, equations (7), (8) give two linear equations each with one unknown. □

Several consequences of this result are noteworthy:

(a) The upper bound $a_{\boldsymbol{U}}$ of the initial antecedent A plays no role in the determination of the bounding interval of the final consequent B^N. This comes as no surprise because the same is true for a single rule (see (8)).

(b) The lower bound $b^N_{\boldsymbol{L}}$ of the consequent B^N is completely determined by the lower bounds of the antecedent A and the rules being cascaded together; their upper bounds play no role in its determination. In fact, if the lower bound of the antecedent A or any of the rules is zero, the lower bound of the consequent is zero.

(c) In the case $\gamma = 1$ (material implication) the upper bound $b^N_{\boldsymbol{U}}$ of the consequent B^N is completely determined by, in fact it is equal to, the upper bound of the last rule $B^{N-1} \to B^N$; neither its lower bound nor the antecedent A nor the other rules play a role in the determination of $b^N_{\boldsymbol{U}}$.

(d) In the case $\gamma = 1$ (material implication), we compute $\Delta b^N = b^N_{\boldsymbol{U}} - b^N_{\boldsymbol{L}} = \Delta b^N = \Delta m^{N-1} + (1 - a_{\boldsymbol{L}}) + \sum_{i=0}^{N-2} (1 - m^i_{\boldsymbol{L}})$. Interestingly, this implies that $\Delta b^N \geq \Delta m^{N-1}$, i.e., the uncertainty interval associated with the consequent B^N can never be narrower than the uncertainty interval associated with the rule at the end, viz., $B^{N-1} \to B^N$. In fact,

$$\Delta b^N \begin{cases} = \Delta m^{N-1}, & \text{if } m_{\boldsymbol{L}}^{N-1} = 0 \text{ or} \\ & \quad a_{\boldsymbol{L}} = m_{\boldsymbol{L}}^i = 1,\ 0 \le i \le n-2; \\ > \Delta m^{N-1}, & \text{otherwise.} \end{cases}$$

5 DISCUSSION

Our investigation of the properties of the generalized rule revealed several ways for expressing and computing bounds. In particular, it showed the close relationship between probabilistic material implication and Bayesian conditional, especially when used for inference with rules like modus ponens. Paired with the generalized rule, inference principles like modus ponens can generate bounds on the consequent that depend both on the γ value and the rule bounds, and the theorem about implications chains shows that it is possible to calculate inferences from any mixture of rule bounds and γ values (recursively in the worst case when clippings happens, see the discussion after Lemma 8). This then enables mixed rule-based systems with different γ values for rules, e.g., high γ values for rules that are based on little data and are more conceptual in nature, and low γ values for rules for which sufficient data exists. In general, one possible application of the γ parameter is to make it a function of the amount of data from which rules have been extracted: the more data is available, the closer the representation should go do the normatively correct Bayesian end point. The issue of the interpreting the generalized rule remains, and is not addressed here, but the starting point for that investigation would be the observation that if we consider the range of the conditional going down from $\top$ to A, i.e., going from $\boldsymbol{P}(A \to B \mid \top) = \boldsymbol{P}(A \to B)$ all the way to $\boldsymbol{P}(A \to B \mid A)$, we will get exactly the generalized rule.

6 CONCLUSION AND FUTURE WORK

The goal of this paper was to investigate the relationship of probabilistic material implications and Bayesian conditionals, and for this purpose we utilized a generalization that subsumed both as special cases. We provided methods for obtaining various bounds from different rules and also proved a generalization for implication chains that determines the bounds for mixed rule-based inference chains. An interesting extension of this would be to consider a system of nodes, each holding a probability that is updated in real-time as new information becomes available, and to then study how changes in the bounds propagate through the system. Furthermore, while the goal of this paper was not to speculate about interpretations of the γ parameter of the generalized rule, it would be interesting to see whether there is a natural interpretation of γ. We could only briefly allude to thinking about it in terms of conditional probabilities, or in terms of empirical support, i.e., the amount of data available for deriving probabilistic rules, but other interpretations are certainly possible and would be interesting to pursue.

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References

Ernest Adams. The logic of conditionals. *Inquiry: An Interdisciplinary Journal of Philosophy*, 8(1-4):166–197, 1965. doi: 10.1080/00201746508601430.

George Boole. *George Boole's Collected Logical Works*. Open Court, 1916.

Ivano Ciardelli and Adrian Ommundsen. Probabilities of conditionals: updating adams. *Noûs*, n/a (n/a), 11 2022. doi: https://doi.org/10.1111/nous.12437. URL `https://onlinelibrary.wiley.com/doi/abs/10.1111/nous.12437`.

Vincenzo Crupi and Andrea Iacona. Three ways of being non-material. *Studia Logica*, (1):1–47, 2021. doi: 10.1007/s11225-021-09949-y.

Didier Dubois and Henri Prade. The logical view of conditioning and its application to possibility and evidence theories. *International Journal of Approximate Reasoning*, 4(1):23–46, 1990. ISSN 0888-613X. doi: https://doi.org/10.1016/0888-613X(90)90007-O. URL `https://www.sciencedirect.com/science/article/pii/0888613X9090007O`.

Ronald Fagin and Joseph Y. Halpern. Uncertainty, belief, and probability. In *Proceedings of the 11th International Joint Conference on Artificial Intelligence - Volume 2*, IJCAI'89, page 1161–1167, San Francisco, CA, USA, 1989. Morgan Kaufmann Publishers Inc.

Nataniel Greene. An overview of conditionals and biconditionals in probability. In *MATH'08: Proceedings of the American Conference on Applied Mathematics*, page 171–177, 2008.

Theodore Hailperin. Best possible inequalities for the probability of a logical function of events. *The American Mathematical Monthly*, 72(4):343–359, 1965. ISSN 00029890, 19300972. URL `http://www.jstor.org/stable/2313491`.

Theodore Hailperin. Probability logic. *Notre Dame Journal of Formal Logic*, 25(3):198 – 212, 1984. doi:

10.1305/ndjfl/1093870625. URL https://doi.org/10.1305/ndjfl/1093870625.

Theodore Hailperin. *Sentential Probability Logic: Origins, Development, Current Status, and Technical Applications*. Lehigh University Press, 1996.

Justin Khoo and Matthew Mandelkern. Triviality Results and the Relationship between Logical and Natural Languages. *Mind*, 128(510):485–526, 05 2018. ISSN 0026-4423. doi: 10.1093/mind/fzy006. URL https://doi.org/10.1093/mind/fzy006.

Hung Nguyen, Masao Mukaidono, and Vladik Kreinovich. Probability of implication, logical version of bayes theorem, and fuzzy logic operations. In *2002 IEEE World Congress on Computational Intelligence. 2002 IEEE International Conference on Fuzzy Systems. FUZZ-IEEE'02. Proceedings (Cat. No.02CH37291)*, volume 1, pages 530 – 535, 02 2002. ISBN 0-7803-7280-8. doi: 10.1109/FUZZ.2002.1005046.

Nils J. Nilsson. Probabilistic logic. *Artificial Intelligence*, 28(1):71–87, 1986. ISSN 0004-3702. doi: https://doi.org/10.1016/0004-3702(86)90031-7. URL https://www.sciencedirect.com/science/article/pii/0004370286900317.

Nils J. Nilsson. Probabilistic logic revisited. *Artificial Intelligence*, 59(1):39–42, 1993. ISSN 0004-3702. doi: https://doi.org/10.1016/0004-3702(93)90167-A. URL https://www.sciencedirect.com/science/article/pii/000437029390167A.

J. Pearl. *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Morgan Kaufmann series in representation and reasoning. Elsevier Science, 1988. ISBN 9781558604797. URL https://books.google.com/books?id=AvNID7LyMusC.

Carl G. Wagner. Modus tollens probabilized. *The British Journal for the Philosophy of Science*, 55(4):747–753, 2004. ISSN 00070882, 14643537. URL http://www.jstor.org/stable/3541627.