Regularized Online DR-Submodular Optimization (Supplementary Material)

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A PROOF OF LEMMA 2

Proof. The proof of Lemma 2 can be derived from [Zhang et al., 2022]. For the readers' convenience, we also give a proof here. First, we have a inequality about $\langle \mathbf{x}, \nabla F(\mathbf{x}) \rangle$, that is,

$$\begin{aligned} \langle \mathbf{x}, \nabla F(\mathbf{x}) \rangle &= \int_0^1 e^{z-1} \langle \mathbf{x}, \nabla f(z * \mathbf{x}) \rangle \mathrm{d}z \\ &= \int_0^1 e^{z-1} \mathrm{d}f(z * \mathbf{x}) \\ &= e^{z-1} f(z * \mathbf{x}) \big|_{z=0}^{z=1} - \int_0^1 f(z * \mathbf{x}) e^{z-1} \mathrm{d}z \\ &\leq f(\mathbf{x}) - \int_0^1 f(z * \mathbf{x}) e^{z-1} \mathrm{d}z. \end{aligned} \tag{1}$$

Second, we also have an inequality about $\langle \mathbf{y}, \nabla F(\mathbf{x}) \rangle$, that is,

$$\begin{aligned} \langle \mathbf{y}, \nabla F(\mathbf{x}) \rangle &= \int_0^1 e^{z-1} \langle \mathbf{y}, \nabla f(z * \mathbf{x}) \rangle \mathrm{d}z \\ &\geq \int_0^1 e^{z-1} \langle \mathbf{y} \vee (z * \mathbf{x}) - z * \mathbf{x}, \nabla f(z * \mathbf{x}) \rangle \mathrm{d}z \\ &\geq \int_0^1 e^{z-1} (f(\mathbf{y} \vee (z * \mathbf{x})) - f(z * \mathbf{x})) \mathrm{d}z \\ &\geq (1 - \frac{1}{e}) f(\mathbf{y}) - \int_0^1 f(z * \mathbf{x}) e^{z-1} \mathrm{d}z \end{aligned} \tag{2}$$

where the first inequality holds because $\mathbf{y} \geq \mathbf{y} \vee (z * \mathbf{x}) - z * \mathbf{x} \geq \mathbf{0}$ and $\nabla f(z * \mathbf{x}) \geq \mathbf{0}$; the second one comes from the property that DR-submodular function is concave along any non-negative and non-positive direction Bian et al. [2017]; the final one comes from $f(\mathbf{y} \vee (z * \mathbf{x})) \geq f(\mathbf{y})$.

Finally, putting the inequality (1) and inequality (2) together, we have

$$\begin{aligned} \langle \mathbf{y} - \mathbf{x}, \nabla F(\mathbf{x}) \rangle &\geq (1 - \frac{1}{e}) f(\mathbf{y}) - \int_0^1 f(z * \mathbf{x}) e^{z-1} \mathrm{d}z - \left(f(\mathbf{x}) - \int_0^1 f(z * \mathbf{x}) e^{z-1} \mathrm{d}z \right) \\ &\geq (1 - \frac{1}{e}) f(\mathbf{y}) - f(\mathbf{x}). \end{aligned}$$

□

Accepted for the 39th Conference on Uncertainty in Artificial Intelligence (UAI 2023).

B PROOF OF THEOREM 1

Proof. Let τ be the stopping time of Algorithm 1, i.e. when $B_\tau < 1$. We will complete the proof in three steps.

Step 1: We will bound the regret of $\mathcal{L}_t^P$ up to τ.

Let $\mathbf{x}_*^P = \arg\sup_{\mathbf{x}\in\mathcal{P}} \sum_{t=1}^{\tau} \mathcal{L}_t^P(\mathbf{x})$. We define $\nabla_t = \nabla\mathcal{L}_t^P(\mathbf{x}_t)$, and $\tilde{\nabla}_t = \nabla\tilde{\mathcal{L}_t^P}(\mathbf{x}_t) = \nabla\left(F(\mathbf{x}_t) + r(\mathbf{x}) - \langle\lambda_t, c_t(\mathbf{x})\rangle\right)$. By the definition of $\mathbf{x}_{t+1}$ and properties of the projection operator for a convex set, we have

$$\begin{aligned}\left\|\mathbf{x}_{t+1} - \mathbf{x}_*^P\right\|^2 &= \left\|\Pi_{\mathcal{P}}\left(\mathbf{x}_t + \eta_t\tilde{\nabla}_t\right) - \mathbf{x}_*^P\right\|^2 \le \left\|\mathbf{x}_t + \eta_t\tilde{\nabla}_t - \mathbf{x}_*^P\right\|^2 \\ &\le \left\|\mathbf{x}_t - \mathbf{x}_*^P\right\|^2 + \eta_t^2\left\|\tilde{\nabla}_t\right\|^2 - 2\eta_t\tilde{\nabla}_t^\top\left(\mathbf{x}_*^P - \mathbf{x}_t\right).\end{aligned}$$

Therefore we further have

$$\begin{aligned}\tilde{\nabla}_t^\top\left(\mathbf{x}_*^P - \mathbf{x}_t\right) &\le \frac{\left\|\mathbf{x}_t - \mathbf{x}_*^P\right\|^2 - \left\|\mathbf{x}_{t+1} - \mathbf{x}_*^P\right\|^2 + \eta_t^2\left\|\tilde{\nabla}_t\right\|^2}{2\eta_t} \\ &\le \frac{\left\|\mathbf{x}_t - \mathbf{x}_*^P\right\|^2 - \left\|\mathbf{x}_{t+1} - \mathbf{x}_*^P\right\|^2}{2\eta_t} + \frac{\eta_t G^2}{2},\end{aligned}$$

where $G = \sup_t\left\|\tilde{\nabla}_t\right\|$.

If we define $\frac{1}{\eta_0} \triangleq 0$ and in light of Lemma 2, it can be deduced that

$$\begin{aligned}&\sum_{t=1}^{\tau}(1-\frac{1}{e})f_t(\mathbf{x}_*^P) + r\left(\mathbf{x}_*^P\right) - \lambda_t c_t(\mathbf{x}_*^P) - f_t(\mathbf{x}_t) - r\left(\mathbf{x}_t\right) + \lambda_t c_t(\mathbf{x^t}) \\ &\le \sum_{t=1}^{\tau}\langle\nabla F(\mathbf{x}_t), \mathbf{x}_*^P - \mathbf{x}_t\rangle + \langle\nabla(r(\mathbf{x}_t) - \lambda_t c_t(\mathbf{x})), \mathbf{x}_*^P - \mathbf{x}_t\rangle \\ &= \sum_{t=1}^{\tau}\langle\tilde{\nabla}_t, \mathbf{x}_*^P - \mathbf{x}_t\rangle \\ &\le \frac{1}{2\eta_t}\sum_{t=1}^{\tau}\left\|\mathbf{x}_t - \mathbf{x}_*^P\right\|^2 - \left\|\mathbf{x}_{t+1} - \mathbf{x}_*^P\right\|^2 + \frac{G^2}{2}\sum_{t=1}^{\tau}\eta_t \\ &\le \frac{1}{2}(\sum_{t=1}^{\tau}\left\|\mathbf{x}_t - \mathbf{x}_*^P\right\|^2(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}})) + \frac{G^2}{2}\sum_{t=1}^{\tau}\eta_t \\ &\le \frac{D^2}{2\eta_\tau} + \frac{G^2}{2}\sum_{t=1}^{\tau}\eta_t \\ &\le O(\sqrt{\tau}),\end{aligned}$$

where $D = \sup_{\mathbf{x},\mathbf{y}\in\mathcal{P}}\|\mathbf{x} - \mathbf{y}\|$.

Step 2: We will bound the regret of $\mathcal{L}_t^D$ up to τ.

Because $\mathcal{L}_t^D$ is a linear function, using the online gradient descent, we have $\sup_{\lambda\in\mathcal{D}}\sum_{t=1}^{\tau}(\mathcal{L}_t^D(\lambda) - \mathcal{L}_t^D(\lambda_t)) \le O(\sqrt{\tau})$ for any λ.

Step 3: Using the results of Steps 1 and 2, we can complete the proof.

From Step 1, we have

$$\sup_{x\in\mathcal{P}}\sum_{t=1}^{\tau}((1-\frac{1}{e})f_t(\mathbf{x}) + r(\mathbf{x}) - \langle\lambda_t, c_t(\mathbf{x})\rangle - f_t(\mathbf{x}_t) - r(\mathbf{x}_t) + \langle\lambda_t, c_t(\mathbf{x}_t)\rangle) \le O(\sqrt{\tau}).$$

Then, by rearranging,

$$\sum_{t=1}^{\tau} f_t(\mathbf{x}_t) + r(\mathbf{x}_t) \geq \sup_{x\in\mathcal{P}} \sum_{t=1}^{\tau}((1-\frac{1}{e})f_t(\mathbf{x}) + r(\mathbf{x}) - T\langle\lambda_t, c_t(\mathbf{x})\rangle + \langle\lambda_t, c_t(\mathbf{x}_t)\rangle) - O(\sqrt{\tau}).$$

From Step 2, $\forall\lambda$ we have $\sum_{t=1}^{\tau}(\mathcal{L}_t^D(\lambda) - \mathcal{L}_t^P(\lambda_t)) \leq O(\sqrt{\tau})$. Then, by the definition of $\mathcal{L}_t^D$,

$$\sum_{t=1}^{\tau}\langle\lambda_t, c_t(\mathbf{x}_t)\rangle \geq \sum_{t=1}^{\tau}(\langle\lambda_t, \rho\rangle - \langle\lambda, \rho\rangle + \langle\lambda, c_t(\mathbf{x}_t)\rangle) - O(\sqrt{\tau}).$$

Therefore,

$$\sum_{t=1}^{\tau} f_t(\mathbf{x}_t) + r(\mathbf{x}_t) \geq -O(\sqrt{\tau}) + \sup_{x\in\mathcal{P}} \sum_{t=1}^{\tau}((1-\frac{1}{e})f_t(\mathbf{x}) + r(\mathbf{x}) + \langle\lambda_t, \rho - c_t(\mathbf{x})\rangle - \langle\lambda, \rho - c_t(\mathbf{x}_t)\rangle). \tag{3}$$

Next, we provide a lower bound on the following term.

$$\text{Ⓐ} = \sup_{x\in\mathcal{P}} \sum_{t=1}^{\tau}((1-\frac{1}{e})f_t(\mathbf{x}) + r(\mathbf{x}) + \langle\lambda_t, \rho - c_t(\mathbf{x})\rangle).$$

Let $\text{APO}_\tau^* = \sup_{x\in\mathcal{P}}\sum_{t=1}^{\tau}\big((1-\frac{1}{e})f_t(\mathbf{x}) + r(\mathbf{x})\big)$ and $\mathbf{x}^* = \arg\sup_{x\in\mathcal{P}}\sum_{t=1}^{\tau}\big((1-\frac{1}{e})f_t(\mathbf{x}) + r(\mathbf{x})\big)$. APO_τ^* represents the $(1-\frac{1}{e}, 1)$ approximate optimal value without constraints. We shall show that

$$\text{Ⓐ} \geq \rho\text{APO}_\tau^*. \tag{4}$$

To do so, we consider two cases. First, if $\sum_{t=1}^{\tau}(1-\frac{1}{e})f_t(\mathbf{x}^*) \geq \sum_{t=1}^{\tau}\langle\lambda_t, c_t(\mathbf{x}^*)\rangle$, then the value of the function for $\mathbf{x}^*$ is at least

$$\begin{aligned}
\text{Ⓐ} &= \sup_{x\in\mathcal{P}} \sum_{t=1}^{\tau}((1-\frac{1}{e})f_t(\mathbf{x}) + r(\mathbf{x}) + \langle\lambda_t, \rho - c_t(\mathbf{x})\rangle) \\
&\geq \sum_{t=1}^{\tau}((1-\frac{1}{e})f_t(\mathbf{x}^*) + r(\mathbf{x}^*) + \langle\lambda_t, \rho - c_t(\mathbf{x}^*)\rangle) \\
&\geq \sum_{t=1}^{\tau}((1-\frac{1}{e})f_t(\mathbf{x}^*) + r(\mathbf{x}^*) + \langle\lambda_t, \rho\cdot c_t(\mathbf{x}^*) - c_t(\mathbf{x}^*)\rangle) \\
&\geq \sum_{t=1}^{\tau}(1-\frac{1}{e})f_t(\mathbf{x}^*) + r(\mathbf{x}^*) + (1-\rho)\sum_{t=1}^{\tau}\langle\lambda_t, c_t(\mathbf{x}^*)\rangle \\
&\geq \rho\sum_{t=1}^{\tau}(1-\frac{1}{e})f_t(\mathbf{x}^*) + r(\mathbf{x}^*) = \rho\text{APO}_\tau^*,
\end{aligned}$$

where the second inequality holds since $c_t(\cdot) \in [0,1]$, for each $t \in [T]$. Otherwise, if $\sum_{t=1}^{\tau}(1-\frac{1}{e})f_t(\mathbf{x}^*) < \sum_{t=1}^{\tau}\langle\lambda_t, c_t(\mathbf{x}^*)\rangle$, we have that

$$\begin{aligned}
\text{Ⓐ} &\geq \sum_{t=1}^{\tau}\langle\lambda_t, \rho\rangle \geq \sum_{t=1}^{\tau}\langle\lambda_t, \rho\cdot c_t(\mathbf{x}^*)\rangle \\
&\geq \rho\sum_{t=1}^{\tau}(1-\frac{1}{e})f_t(\mathbf{x}^*) + r(\mathbf{x}^*) = \rho\text{APO}_\tau^*.
\end{aligned}$$

Combining inequality (3) and inequality (4), we get

$$\sum_{t=1}^{\tau}(f_t(\mathbf{x}_t) + r(\mathbf{x}_t) - \langle\lambda, c_t(\mathbf{x}_t)\rangle) \geq -O(\sqrt{\tau}) + \rho\text{APO}_\tau^* - \tau\langle\lambda, \rho\rangle.$$

In particular, we have

$$\rho\text{APO}_\tau^* \geq \rho\text{APO}_\tau \geq \rho(\text{APO}_\tau - T + \tau),$$

where, APO_τ is the $(1-\frac{1}{e},1)$ approximate optimal reward with constraints. By definition, $\text{REW} = \sum_{t=1}^{\tau} f_t(\mathbf{x}_t) + r(\mathbf{x}_t)$. Then,

$$\text{REW} = \sum_{t=1}^{\tau} f_t(\mathbf{x}_t) + r(\mathbf{x}_t) \geq -O(\sqrt{\tau}) + \rho\text{APO}^*_\tau - \sum_{t=1}^{\tau}\langle\lambda, \rho - c_t(\mathbf{x}_t)\rangle$$
$$\geq -O(\sqrt{\tau}) + \rho(\text{APO}_\tau - T + \tau) - \sum_{t=1}^{\tau}\langle\lambda, \rho - c_t(\mathbf{x}_t)\rangle.$$

If $\tau = T$, in order to get the result, it is enough to set $\lambda = 0$, and to substitute the above expression in the definition of regret. Otherwise, if $\tau < T$, which means that

$$\sum_{t=1}^{\tau} c_t(\mathbf{x}_t) + 1 \geq \rho T,$$

where, in our setting, the largest possible cost is 1. Then, we set $\lambda = 1/\rho$ and thus,

$$\sum_{t=1}^{\tau}\langle\lambda, \rho - c_t(\mathbf{x}_t)\rangle = 1/\rho\sum_{t=1}^{\tau}(\rho - c_t(\mathbf{x}_t)) \leq \tau - T + 1/\rho.$$

Then, by substituting the above expression

$$\text{REW} \geq -O(\sqrt{\tau}) + \rho(\text{APO}_\tau - T + \tau) - (\tau - T) - 1/\rho.$$

Finally, we have

$$\rho(1-\frac{1}{e},1)\text{OPT} - \text{REW} \leq \rho\text{APO}_\tau - \text{REW} \leq O(\sqrt{\tau}) + (T-\tau)(\rho-1) + 1/\rho$$
$$\leq O(\sqrt{\tau}) + 1/\rho = O(\sqrt{T}).$$

□

C PROOF OF PROPOSITION 1

Proof. We use induction to prove this proposition. For the case when $n = 2$, let $\mathbf{x_1} \leq \mathbf{x_2}, \mathbf{z} = \lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2}$, we have $\mathbf{x_1} - \mathbf{z} \leq \mathbf{0}$, and $\mathbf{x_2} - \mathbf{z} \geq \mathbf{0}$. Using the property that DR-submodular function is concave along any non-negative and non-positive direction Bian et al. [2017], we get

$$f(\mathbf{x_1}) \leq f(\mathbf{z}) + \nabla f(\mathbf{z})(\mathbf{x_1} - \mathbf{z}),$$
$$f(\mathbf{x_2}) \leq f(\mathbf{z}) + \nabla f(\mathbf{z})(\mathbf{x_2} - \mathbf{z}).$$

Multiplying the first inequality by λ, the second equation by $1-\lambda$, and then adding the two inequalities together, we get the result for $n = 2$.

To show that this is true for all natural numbers, we proceed by induction. Assume the proposition is true for some n and then,

$$f(\sum_{i=1}^{n+1}\lambda_i\mathbf{x_i}) = f(\lambda_1\mathbf{x_1} + \sum_{i=2}^{n+1}\lambda_i\mathbf{x_i}) = f(\lambda_1\mathbf{x_1} + (1-\lambda_1)\frac{1}{1-\lambda_1}\sum_{i=2}^{n+1}\lambda_i\mathbf{x_i})$$

because $\mathbf{x_1} \leq \mathbf{x_i}, \forall i = 2, \ldots, n+1$, and $\sum_{i=2}^{n+1}\frac{\lambda_i}{1-\lambda_1} = 1$, we get $\mathbf{x_1} \leq \frac{1}{1-\lambda_1}\sum_{i=2}^{n+1}\lambda_i\mathbf{x_i}$, finally we have

$$f(\sum_{i=1}^{n+1}\lambda_i\mathbf{x_i}) \geq \lambda_1 f(\mathbf{x_1}) + (1-\lambda_1)f(\frac{1}{1-\lambda_1}\sum_{i=2}^{n+1}\lambda_i\mathbf{x_i})$$
$$= \lambda_1 f(\mathbf{x_1}) + (1-\lambda_1)f(\sum_{i=2}^{n+1}\frac{\lambda_i}{1-\lambda_1}\mathbf{x_i})$$
$$\geq \lambda_1 f(\mathbf{x_1}) + (1-\lambda_1)\sum_{i=2}^{n+1}\frac{\lambda_i}{1-\lambda_1}f(\mathbf{x_i})$$
$$= \sum_{i}^{n}\lambda_i f(\mathbf{x_i}).$$

□

D PROOF OF THEOREM 2

Proof. Let τ be the stopping time of Algorithm 2, i.e. when $B_\tau < 1$. First, we will bound the regret up to τ. Let x^* be the best fixed action for Problem (2) defined in the main paper. Because $c_t(\mathbf{x}_t) \le 1$, we have $\tau \ge \rho T$. Using the L-smoothness of f and r and the update rule of Algorithm 2, we have

$$
\begin{aligned}
f(\mathbf{x}_{t+1}) + r(\mathbf{x}_{t+1}) &\overset{(a)}{\ge} f(\mathbf{x}_t) + r(\mathbf{x}_t) + \frac{1}{T}\langle \mathbf{v}_t, \nabla f(\mathbf{x}_t) + \nabla r(\mathbf{x}_t)\rangle - \frac{L}{2T^2}\|\mathbf{v}_t\|^2 \\
&\ge f(\mathbf{x}_t) + r(\mathbf{x}_t) + \frac{1}{T}\langle \mathbf{v}_t, \mathbf{d}_t + \nabla r(\mathbf{x}_t)\rangle + \frac{1}{T}\langle \mathbf{v}_t, \nabla f(\mathbf{x}_t) - \mathbf{d}_t\rangle - \frac{LD^2}{2T^2} \\
&\overset{(b)}{\ge} f(\mathbf{x}_t) + r(\mathbf{x}_t) + \frac{1}{T}\langle \mathbf{x}^*, \mathbf{d}_t + \nabla r(\mathbf{x}_t)\rangle + \frac{1}{T}\langle \mathbf{v}_t, \nabla f(\mathbf{x}_t) - \mathbf{d}_t\rangle - \frac{LD^2}{2T^2} \\
&= f(\mathbf{x}_t) + r(\mathbf{x}_t) + \frac{1}{T}\langle \mathbf{x}^*, \nabla f(\mathbf{x}_t) + \nabla r(\mathbf{x}_t)\rangle + \frac{1}{T}\langle \mathbf{v}_t - \mathbf{x}^*, \nabla f(\mathbf{x}_t) - \mathbf{d}_t\rangle \\
&- \frac{LD^2}{2T^2} \\
&\overset{(c)}{\ge} f(\mathbf{x}_t) + r(\mathbf{x}_t) + \frac{1}{T}\langle (\mathbf{x}^* - \mathbf{x}_t) \vee 0, \nabla f(\mathbf{x}_t) + \nabla r(\mathbf{x}_t)\rangle \\
&+ \frac{1}{T}\langle \mathbf{v}_t - \mathbf{x}^*, \nabla f(\mathbf{x}_t) - \mathbf{d}_t\rangle - \frac{LD^2}{2T^2} \\
&\overset{(d)}{\ge} f(\mathbf{x}_t) + r(\mathbf{x}_t) + \frac{1}{T}(f(\mathbf{x}^* \vee \mathbf{x}_t) + r(\mathbf{x}^* \vee \mathbf{x}_t)) - \frac{1}{T}(f(\mathbf{x}_t) + r(\mathbf{x}_t)) + \\
&\frac{1}{T}\langle \mathbf{v}_t - \mathbf{x}^*, \nabla f(\mathbf{x}_t) - \mathbf{d}_t\rangle - \frac{LD^2}{2T^2} \\
&\overset{(e)}{\ge} f(\mathbf{x}_t) + r(\mathbf{x}_t) + \frac{1}{T}(f(\mathbf{x}^*) + r(\mathbf{x}^*)) - \frac{1}{T}(f(\mathbf{x}_t) + r(\mathbf{x}_t)) + \\
&\frac{1}{T}\langle \mathbf{v}_t - \mathbf{x}^*, \nabla f(\mathbf{x}_t) - \mathbf{d}_t\rangle - \frac{LD^2}{2T^2}
\end{aligned}
$$

where $D = \sup_{\mathbf{x},\mathbf{y}\in\mathcal{P}}\|\mathbf{x} - \mathbf{y}\|$. The inequality (a) comes from the L-smoothness of f and r, inequality (b) holds because the update rule of Algorithm 2, (c) and (e) are due to the monotonocity of f, and (d) comes from the property that DR-submodular function is concave along any non-negative and non-positive direction. Defining $\epsilon_t := \mathbf{d}_t - \nabla f(\mathbf{x}_t)$ and rearranging the term in the above inequality, we have

$$
f(\mathbf{x}^*) + r(\mathbf{x}^*) - f(\mathbf{x}_{t+1}) - r(\mathbf{x}_{t+1}) \le (1 - \frac{1}{T})(f(\mathbf{x}^*) + r(\mathbf{x}^*) - f(\mathbf{x}_t) - r(\mathbf{x}_t)) + \frac{D}{T}\|\epsilon_t\| + \frac{LD^2}{2T^2}.
$$

Applying the above inequality recursively, we further have

$$
f(\mathbf{x}^*) + r(\mathbf{x}^*) - f(\mathbf{x}_{t+1}) - r(\mathbf{x}_{t+1}) \le (1 - \frac{1}{T})^t(f(\mathbf{x}^*) + r(\mathbf{x}^*) - f(\mathbf{x}_1) - r(\mathbf{x}_1))) + \frac{D}{T}\sum_{s=1}^{t}\|\epsilon_s\| + \frac{LD^2}{2T}.
$$

Using the above inequality and the fact that $\sum_{t=1}^{\tau-1}(1 - \frac{1}{T})^t \le \sum_{t=1}^{\tau-1} e^{-t/T} \le \tau(e^{-1/T} - 1/e^\rho) \le \tau(1 - 1/e^\rho)$, we get the $(\frac{1}{e^\rho}, \frac{1}{e^\rho})$-$\mathcal{SR}_\tau$ is bounded by $\frac{\rho LD^2}{2} + \frac{D}{T}\sum_{t=1}^{\tau-1}\sum_{s=1}^{t}\epsilon_s$. Thus, we get

$$
\begin{aligned}
\epsilon_t &= \mathbf{d}_t - \nabla f(\mathbf{x}_t) \\
&= (1 - \eta_t)\epsilon_{t-1} + \eta_t(\nabla f_t(\mathbf{x}_t) - \nabla f(\mathbf{x}_t)) \\
&+ (1 - \eta_t)(\nabla f_t(\mathbf{x}_t) - \nabla f_t(\mathbf{x}_{t-1}) - (\nabla f(\mathbf{x}_t) - \nabla f(\mathbf{x}_{t-1}))).
\end{aligned}
$$

Applying the above equality recursively, we obtain

$$
\begin{aligned}
\epsilon_t = &\prod_{s=2}^{\tau}(1 - \eta_t)\epsilon_1 + \sum_{m=1}^{t}\prod_{s=m}^{t}(1 - \eta_t)(\nabla f_m(\mathbf{x}_m) - \nabla f_m(\mathbf{x}_{m-1}) - (\nabla f(\mathbf{x}_m) - \nabla f(\mathbf{x}_{m-1}))) \\
&+ \sum_{m=2}^{t}\eta_t\prod_{m+1}^{t}(1 - \eta_t)(\nabla f_m(\mathbf{x}_m) - \nabla f(\mathbf{x}_m)).
\end{aligned}
$$

Let $\epsilon_1 = \sum_{m=1}^t \xi_{t,m}$, where $\xi_{t,1} = \prod_{s=2}^{\tau}(1-\eta_t)\epsilon_1$ and $\xi_{t,m} = \prod_{s=m}^{t}(1-\eta_t)(\nabla f_m(\mathbf{x}_m) - \nabla f_m(\mathbf{x}_{m-1}) - (\nabla f(\mathbf{x}_m) - \nabla f(\mathbf{x}_{m-1}))) + \eta_t \prod_{m+1}^{t}(1-\eta_t)(\nabla f_m(\mathbf{x}_m) - \nabla f(\mathbf{x}_m))$ for $m > 1$. Let $\mathcal{F}_t$ be the σ-field generated by $\{f_s\}_{s=1}^{m-1}$. Clearly, $\mathbb{E}[\xi_{t,1}] = 0$. Also, for $m > 1$, we have

$$\begin{aligned}\mathbb{E}[\xi_{t,m}|\mathcal{F}_m] = &\prod_{s=m}^{t}(1-\eta_t)(\nabla f(\mathbf{x}_m) - \nabla f(\mathbf{x}_{m-1}) - (\nabla f(\mathbf{x}_m) - \nabla f(\mathbf{x}_{m-1})))\\ &+ \eta_t \prod_{m+1}^{t}(1-\eta_t)(\nabla f(\mathbf{x}_m) - \nabla f(\mathbf{x}_m))\\ &= 0.\end{aligned}$$

Therefore, for all $t \in \tau$, $\{\xi_{t,m}\}_{m=1}^t$ is a martingale difference sequence. For any $m \in [t]$, we can write

$$\prod_{s=m}^{t}(1-\eta_t) = \prod_{s=m}^{t}(1-\frac{1}{s+1}) = \prod_{s=m}^{t}(\frac{s}{s+1}) = \frac{m}{t+1}.$$

Thus we have $\|\xi_{t,1}\| = \frac{2}{t+1}\|\nabla f_1(\mathbf{x}_1) - \nabla f(\mathbf{x}_1)\| \leq \frac{2\sigma}{t+1}$. For $m > 1$, we have

$$\begin{aligned}\|\xi_{t,m}\| \leq &\prod_{s=m}^{t}(1-\eta_t)(\|\nabla f(\mathbf{x}_m) - \nabla f(\mathbf{x}_{m-1})\| + \|(\nabla f(\mathbf{x}_m) - \nabla f(\mathbf{x}_{m-1}))\|)\\ &+ \eta_t \prod_{m+1}^{t}(1-\eta_t)\|\nabla f(\mathbf{x}_m) - \nabla f(\mathbf{x}_m)\|\\ \leq &\frac{2Lm}{t+1}\|\mathbf{x}_m - \mathbf{x}_{m-1}\| + \frac{\sigma}{t+1}\\ \leq &\frac{2LDm/\tau + m}{t+1}\\ \leq &\frac{2LD + m}{t+1}.\end{aligned}$$

Using the concentration inequality for vector-valued martingales, we have

$$\mathbb{P}(\|\epsilon_t\| \geq \lambda_t) \leq 2\exp\left(-\frac{\lambda_t^2}{(\frac{2\sigma}{t+1})^2 + (t-1)(\frac{2LD+m}{t+1})^2}\right) \leq 2\exp\left(-\frac{\lambda_t^2(t+1)}{(2LD+2\sigma)^2}\right).$$

Therefore, we can get the expected regret bound of the algorithm:

$$\begin{aligned}\mathbb{E}\|\epsilon_t\| &= \int_{\lambda=0}^{\infty}\mathbb{P}(\|\epsilon_t\| \geq \lambda)d\lambda\\ &\leq \int_{\lambda=0}^{\infty} 2\exp\left(-\frac{\lambda_t^2(t+1)}{(2LD+2\sigma)^2}\right)d\lambda\\ &= \int_{\lambda=0}^{\infty} 2\exp(-x^2)\frac{2LD+2\sigma}{\sqrt{t=1}}dx\\ &= \frac{2\sqrt{\pi}(LD+\sigma)}{\sqrt{t+1}}.\end{aligned}$$

As a consequence, the expected regret bound is $O(\sqrt{\tau})$.

Now, we get the regret up to τ. Let REW_τ be the reward that we get, and OPT_τ be the optimal reward till τ. So we have:

$$\mathbb{E}(\frac{1}{e^\rho}\mathrm{OPT}_\tau - \mathrm{REW}_\tau) \leq O(\sqrt{\tau}) = O(\sqrt{T}).$$

Because the f_t, c_t are sampled i.i.d from $\mathcal{D}$ and $\tau \geq \rho T$, we get $\mathbb{E}(\mathrm{OPT}_\tau) \geq \rho\mathbb{E}(\mathrm{OPT}_T)$. Therefore,

$$\mathbb{E}(\frac{\rho}{e^\rho}\mathrm{OPT}_T - \mathrm{REW}_\tau) \leq \mathbb{E}(\frac{1}{e^\rho}\mathrm{OPT}_\tau - \mathrm{REW}_\tau) \leq O(\sqrt{T}).$$

□

Deferences

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