Random Features Approximation for Control-Affine Systems

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Editors: A. Abate, M. Cannon, K. Margellos, A. Papachristodoulou

Abstract

Modern data-driven control applications call for flexible nonlinear models that are amenable to principled controller synthesis and realtime feedback. Many nonlinear dynamical systems of interest are *control affine*. We propose two novel classes of nonlinear feature representations which capture control affine structure while allowing for arbitrary complexity in the state dependence. Our methods make use of random features (RF) approximations, inheriting the expressiveness of kernel methods at a lower computational cost. We formalize the representational capabilities of our methods by showing their relationship to the Affine Dot Product (ADP) kernel proposed by Castañeda et al. (2021) and a novel Affine Dense (AD) kernel that we introduce. We further illustrate the utility by presenting a case study of data-driven optimization-based control using control certificate functions (CCF). Simulation experiments on a double pendulum empirically demonstrate the advantages of our methods.

Keywords: Random Features, Control-Affine Systems, Control Certificate Functions.

1. Introduction

Modern control applications require modelling systems with complex and nonlinear dynamics. Modern machine learning techniques offer a data-driven solution. From deep learning to kernel methods, learning-based approaches fit models to data. Highly expressive models can approximate arbitrary functions, and therefore model arbitrarily complex phenomena. However, this comes at a cost—they can be computationally expensive to train and difficult to use for the purpose of synthesizing a controller. This poses a challenge in real-time feedback systems.

Linear regression is a straightforward approach for learning dynamical models from data, so long as a suitable nonlinear feature representation, i.e., set of basis functions, is known (Mania et al., 2020). However, selecting proper basis functions is often challenging and requires modelling detailed properties of the unknown dynamics. One solution is to choose a set of random basis functions to generate feature vectors of fixed dimension. This approach, called *random features (RF)*, can achieve high expressiveness as long as the dimension of the feature vectors is large enough (Rahimi and Recht, 2008). Random features have proven useful for dynamical systems forecasting (Giannakis et al., 2023), receding horizon control (Lale et al., 2021), and policy learning (Lale et al., 2022).

We propose two novel classes of random feature representations suitable for principled data-driven control (Section 3). Our key insight is to leverage the control-affine structure of many nonlinear dynamics of interest, which enables principled optimization-based approaches for controller synthesis. We propose two distinct methods for incorporating this structure into random basis func-

tions and formalize their representation guarantees by showing that they approximate functions in a Reproducing Kernel Hilbert Space (RKHS). One of our methods approximates the *Affine Dot Product (ADP) kernel* proposed by Castañeda et al. (2021), while the other corresponds to a novel *Affine Dense (AD) kernel* that we propose. The RF methods significantly reduce the computational time and memory complexity compared to their kernel counterparts.

To showcase the utility of an explicit control-affine structure, we present a case study for nonlinear control in Section 4. Our data-driven approach is based on *Control Certificate Functions* (CCFs), which are utilized to synthesize controllers that provably achieve properties such as safety and stability (Taylor et al., 2021). CCFs have been used in a range of applications from robotics to multi-agent systems (Artstein, 1983; Ames et al., 2014; Nguyen et al., 2016; Pickem et al., 2017), including in a data-driven manner (Castañeda et al., 2021; Castañeda et al., 2021; Taylor et al., 2021; Choi et al., 2023). Simulations on a double inverted pendulum illustrate the benefits of our models when used in a *certainty-equivalent* manner. In the our extended paper (Kazemian et al., 2023), we additionally derive uncertainty estimates analogous to those of Gaussian process (GP) regression, and use them to propose a robust data-driven controller. We highlight that the approximation methods that we propose may be broadly of interest for any control application which makes use of GPs (Koller et al., 2018; Caldwell and Marshall, 2021; Bradford et al., 2019; Hewing et al., 2020; Li et al., 2021).

2. Problem Setting and Preliminaries

In this work, we consider an affine modelling and prediction problem inspired by applications in data-driven control. We first define the general problem of interest, and then give several examples that arise in the context of learning for dynamics and control.

Definition 1 (Control-affine modelling problem) *For data of the form* $\{(\boldsymbol{x}_i, \boldsymbol{u}_i, z_i)\}_{i=1}^N$*, find a function* $\hat{h} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ *which i) is affine in its second argument and ii) accurately models the relationship between* $(\boldsymbol{x}, \boldsymbol{u})$ *and* z*, i.e.* $\hat{h}(\boldsymbol{x}_i, \boldsymbol{u}_i)$ *is not far from* z_i*.*

Such a modelling problem naturally arises in applications involving nonlinear control-affine systems. The dynamics are described in either continuous or discrete time:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} \quad \text{or} \quad \boldsymbol{x}_{t+1} = \boldsymbol{f}(\boldsymbol{x}_t) + \boldsymbol{g}(\boldsymbol{x}_t)\boldsymbol{u}_t. \tag{1}$$

Here, $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ is the system state, and $\boldsymbol{u} \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input. The nonlinear function $\boldsymbol{f}$ determines the evolution of the state in the absence of control inputs, while $\boldsymbol{g}$ models state-dependent actuation. Control-affine dynamics arise naturally from manipulator equations (Murray and Hauser, 1991; Tedrake, 2023), and are thus prevalent in applications like robotics. While many systems are known to follow dynamics of the form (1), the precise form of $\boldsymbol{f}$ and/or $\boldsymbol{g}$ may be unknown. Data-driven approaches enable the control of systems with entirely or partially unknown dynamics. There are many examples of modelling tasks that arise in such data-driven control settings.

Example 1 *Consider a model predictive control setting in which the evolution of the state itself must be predicted (Lale et al., 2021). For a discrete-time control-affine system* (1) *with unknown dynamics and direct state observation, a sequence of states and inputs* $\{(\boldsymbol{x}_i, \boldsymbol{u}_i)\}_{i=0}^N$ *defines* n *modelling problems of the form presented in Definition 1: one for each state dimension.*

Example 2 *Consider again model predictive control, now for continuous time control-affine dynamics (1). A sequence of sampled states $\{\boldsymbol{x}_i\}_{i=0}^N$ can be used to approximate $\{\dot{\boldsymbol{x}}_i\}_{i=1}^N$ with forward finite differencing and define n modelling problems of the form presented in Definition 1.*

Example 3 *Consider Certificate Function Control (Taylor et al., 2021), which enforces safety or stability using a known certificate function $C : \mathcal{X} \to \mathbb{R}$. For continuous time control-affine dynamics (1), such controllers require computing $\dot{C} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, which cannot be done directly when the dynamics are unknown. However, a sequence $\{C(\boldsymbol{x}_i)\}_{i=0}^N$ can be computed from a sequence of sampled states and the known function C. Then, finite differencing approximates the time derivative, resulting in a problem of the form presented in Definition 1.*

Example 4 *For any of the previous examples, suppose that an approximate model of the dynamics $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{g}}$ is known. Then learning residual error dynamics also results in a problem of the form presented in Definition 1 (see e.g., Taylor et al. (2021); Castañeda et al. (2021)).*

The examples above serve to motivate the relevance of the modelling problem in Definition 1. We now turn to background and preliminaries on solving it. Our focus is on nonparametric techniques which can model phenomena of arbitrary complexity. We review kernel regression, which is both nonparametric and amenable to uncertainty quantification, and random features approximation, which allows for computational efficiency.

2.1. From Linear to Kernel Regression

We begin by reviewing regression approaches for general data containing input vectors $\{\boldsymbol{s}_i\}_{i=1}^N \subset \mathbb{R}^d$ and a target output variable $\{z_i\}_{i=1}^N \subset \mathbb{R}$. Our starting point is parametric linear regression, in which predictions depend linearly on a known nonlinear feature function of the inputs. Let $\boldsymbol{\phi} : \mathbb{R}^d \to \mathbb{R}^D$ map an input vector $\boldsymbol{s} \in \mathbb{R}^d$ to a feature vector $\boldsymbol{\phi}(\boldsymbol{s}) \in \mathbb{R}^D$. The feature function, also known as a basis function, maps the input vectors to a higher-dimensional feature space, where a linear relationship can be established more easily.

Linear least-squares regression (Watson, 1967) models the relationship as $\hat{h}(\boldsymbol{s}) = \hat{\boldsymbol{w}}^\top \boldsymbol{\phi}(\boldsymbol{s})$, where the parameter $\hat{\boldsymbol{w}} \in \mathbb{R}^D$ is learned from data by solving

$$\min_{\boldsymbol{w} \in \mathbb{R}^D} \sum_{i=1}^N (\boldsymbol{\phi}(\boldsymbol{s}_i)^\top \boldsymbol{w} - z_i)^2 + \lambda \|\boldsymbol{w}\|_2^2, \tag{2}$$

where $\lambda \geq 0$ is a regularization parameter. Let the matrix $\Phi \in \mathbb{R}^{N \times D}$ and the vector $\boldsymbol{z} \in \mathbb{R}^N$ be the aggregation of rows $\{\boldsymbol{\phi}(\boldsymbol{s}_i)^\top\}_{i=1}^N$ and $\{z_i\}_{i=1}^N$, respectively. Then the prediction is

$$\hat{h}(\boldsymbol{s}) = \boldsymbol{\phi}(\boldsymbol{s})^\top (\Phi^\top \Phi + \lambda \boldsymbol{I}_D)^{-1} \Phi^\top \boldsymbol{z} = \boldsymbol{\phi}(\boldsymbol{s})^\top \Phi^\top (\Phi \Phi^\top + \lambda \boldsymbol{I}_N)^{-1} \boldsymbol{z}. \tag{3}$$

The first equality is the closed-form solution to the least squares objective, and the second leverages the *kernel trick* (Scholkopf and Smola, 2018; Müller et al., 2018). The significance of this reformulation is that the vector $\Phi \boldsymbol{\phi}(\boldsymbol{s}) =: \boldsymbol{k}(\boldsymbol{s})$ and matrix $\Phi \Phi^\top =: K$ can be computed using only inner products of basis functions evaluated on training data. This is attractive because the class of basis functions determine the complexity and richness of the modelled relationship between inputs $\boldsymbol{s}$ and outputs z. While low-dimensional bases may suffice for highly structured processes, generally, a suitable compact basis may not be known a priori.

Kernel methods allow for expressive basis functions of arbitrarily high or infinite dimension. A kernel function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ generalizes inner products between basis functions, and is used as a nonparametric approach for representing complex functions. Appropriately defined, kernel ridge regression corresponds to regression in Reproducing Kernel Hilbert Spaces (RKHS) (Wendland, 2004). Many RKHS are dense in the set of continuous functions, enabling arbitrarily accurate representation of continuous functions via kernel regression. The following lemma presents a sufficient condition for checking that a kernel function defines a RKHS. It is a direct implication of the Moore–Aronszajn theorem and Lemma 1 in Berlinet and Thomas-Agnan (2011).

Lemma 2 *Let $\mathcal{H}$ be some Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a reproducing kernel if there exists a mapping $\varphi : \mathbb{R}^d \to \mathcal{H}$ such that $k(\boldsymbol{s}, \boldsymbol{s}') = \langle \varphi(\boldsymbol{s}), \varphi(\boldsymbol{s}') \rangle$.*

In addition to their expressivity, kernel methods are amenable to theoretical guarantees and uncertainty characterization. Popularized from the Bayesian perspective as Gaussian Process (GP) regression (Williams and Rasmussen, 2006), confidence intervals on kernel predictions can be derived even in frequentist settings (Srinivas et al., 2009). We discuss this perspective further in our extended paper (Kazemian et al., 2023) as it is useful for robust control. The drawback of kernel methods is computation. Algorithms have superlinear complexity in the number of data points. In particular, computing the kernel weights can be prohibitively expensive for large datasets. Solving (3) generally requires $O(N^3)$ time and $O(N^2)$ memory.

2.2. Random Feature Approximation

Rather than using kernel methods directly, we propose basis functions which are expressive, general purpose, and yet finite-dimensional. Consider a parametric family of basis functions $b : \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}$. Then for parameters $\{\boldsymbol{\vartheta}_j\}_{j=1}^D \subset \mathbb{R}^p$ sampled i.i.d. from a fixed probability distribution $p(\boldsymbol{\vartheta})$, the random basis is defined as $\boldsymbol{\phi}(\boldsymbol{s}) = \begin{bmatrix} b(\boldsymbol{s}; \boldsymbol{\vartheta}_1) & b(\boldsymbol{s}; \boldsymbol{\vartheta}_2) & \dots & b(\boldsymbol{s}; \boldsymbol{\vartheta}_D) \end{bmatrix}^\top$. Random basis functions of this form approximate rich class of functions in the sense that $\boldsymbol{\phi}(\boldsymbol{s})^\top \boldsymbol{\phi}(\boldsymbol{s}')$ is a Monte-Carlo estimator which converges uniformly to a kernel $k(\boldsymbol{s}, \boldsymbol{s}')$ (Rahimi and Recht, 2008). The rate of convergence is controlled by the feature dimension D and the particular kernel depends on the definition of $b(\cdot; \boldsymbol{\vartheta})$ and $p(\boldsymbol{\vartheta})$.

The underlying observation behind random features is a simple consequence of Bochner's Theorem (Avron et al., 2017): For every normalized shift-invariant kernel (i.e., $k(0) = 1$), there is a probability density function $p(\cdot)$ on $\mathbb{R}^d$ such that

$$k(\boldsymbol{s}, \boldsymbol{s}') = \int_{\boldsymbol{\vartheta} \in \mathbb{R}^d} e^{-i2\pi \boldsymbol{\vartheta}^\top (\boldsymbol{s} - \boldsymbol{s}')} p(\boldsymbol{\vartheta}) d\boldsymbol{\vartheta} =: \mathcal{F}(p(\boldsymbol{\vartheta})). \tag{4}$$

In other words, the inverse Fourier transform $\mathcal{F}^{-1}$ of the kernel $k(\cdot)$ is the probability density function $p(\cdot)$. This implies a one to one correspondence between any shift invariant kernel and a random features basis.

Example 5 (Random Fourier basis) *The random Fourier basis consists of sinusoidal nonlinearities of the form $b(\boldsymbol{s}; \boldsymbol{\vartheta}) = \begin{bmatrix} \cos(\boldsymbol{\vartheta}^\top \boldsymbol{s}) & \sin(\boldsymbol{\vartheta}^\top \boldsymbol{s}) \end{bmatrix}$. When $\boldsymbol{\vartheta}$ is sampled from a Gaussian distribution, i.e., $p(\boldsymbol{\vartheta}) \sim \mathcal{N}(0, 2\gamma \boldsymbol{I}_d)$, then the random Fourier basis approximates the radial basis function (RBF) kernel $k(\boldsymbol{s}, \boldsymbol{s}') = e^{-\frac{1}{\gamma} \|\boldsymbol{s} - \boldsymbol{s}'\|_2^2}$.*

The randomized nonlinear expansions provide a compact and computationally efficient alternative to the RKHS representations. This is particularly attractive when the number of data points is large. Recall from the prior section the feature matrix $\Phi \in \mathbb{R}^{N\times D}$ appearing in the prediction (3). Since $\boldsymbol{\phi}(\boldsymbol{s}) \in \mathbb{R}^D$ and $\Phi^\top\Phi$ is a $D\times D$ matrix, the computation only depends on the dimension of our feature space. Hence, we can compute a random feature approximation in $O(ND^2)$ time and $O(ND)$ memory, which is computationally attractive when $D<N$.

3. Random Features for Control-Affine Modelling

In this section, we use ideas from kernel regression and random feature approximations to propose representations which capture the control-affine structure from Definition 1. We first present two general approaches for defining basis functions that are affine in the control variable. Then, we present RKHS representation guarantees by showing that the random basis approximates particular kernels. Finally, we present experiments which illustrate the predictive modelling capabilities of the proposed methods.

3.1. Control-Affine Basis Functions

The control-affine modelling problem (Definition 1) allows for complex dependence on the state variable, but imposes a restriction on the control variable. Given any arbitrary state-dependent bases $\boldsymbol{\psi}_i : \mathcal{X} \to \mathbb{R}^D$ for $i = 1, ..., m+1$, we propose the following two basis functions that are affine in the control variable $\boldsymbol{u}$.

Definition 3 (Affine dot product (ADP) bases) *The basis $\boldsymbol{\phi}_c : \mathcal{X}\times\mathcal{U} \to \mathbb{R}^{D(m+1)}$, given by*

$$\boldsymbol{\phi}_c(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} u_1\boldsymbol{\psi}_1(\boldsymbol{x})^\top & \dots & u_m\boldsymbol{\psi}_m(\boldsymbol{x})^\top & \boldsymbol{\psi}_{m+1}(\boldsymbol{x})^\top \end{bmatrix}^\top,$$

is the ADP basis of $m+1$ individual basis functions $\boldsymbol{\psi}_i : \mathcal{X} \to \mathbb{R}^D$, $i = 1, \dots, m+1$.

As we show in the following section (see Theorem 6), the ADP bases approximate the affine dot product (ADP) kernel, which was first proposed by Castañeda et al. (2021). Note that the ADP bases can also be written as the product of $\mathrm{blkdiag}(\boldsymbol{\psi}_1(\boldsymbol{x}), ..., \boldsymbol{\psi}_{m+1}(\boldsymbol{x}))$ with the vector $[\boldsymbol{u}^\top \ 1]^\top$. This basis expands the feature dimension for every dimension of the control input, resulting in dimension which scales by $m+1$. This observation motivates a second proposed representation.

Definition 4 (Affine dense (AD) bases) *The basis $\boldsymbol{\phi}_d : \mathcal{X}\times\mathcal{U} \to \mathbb{R}^D$, given by*

$$\boldsymbol{\phi}_d(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} \boldsymbol{\psi}_1(\boldsymbol{x}) \dots \boldsymbol{\psi}_{m+1}(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ 1 \end{bmatrix}$$

is the AD basis of $m+1$ individual basis functions $\boldsymbol{\psi}_i : \mathcal{X} \to \mathbb{R}^D$, $i = 1, \dots m+1$.

Compared with the ADP basis, the AD basis is more compact. For individual basis functions of dimension D, the AD basis will be of dimension D, whereas the ADP basis will be of dimension $D(m+1)$. Considering the linear regression use case, this means that AD has $O(ND^2)$ time and $O(ND)$ memory complexity, whereas for ADP it is $O(N(m+1)^2D^2)$ and $O(N(m+1)D)$.

Leveraging ideas from random Fourier features, we propose control-affine basis functions constructed with state-dependent random Fourier basis functions ψ_i for $i = 1, \ldots, m+1$:

$$\boldsymbol{\psi}_i(\boldsymbol{x}) := \sqrt{2/D} \begin{bmatrix} \sin(\boldsymbol{\vartheta}_{i,1}^\top \boldsymbol{x}) & \cos(\boldsymbol{\vartheta}_{i,1}^\top \boldsymbol{x}) & \ldots & \sin(\boldsymbol{\vartheta}_{i,D/2}^\top \boldsymbol{x}) & \cos(\boldsymbol{\vartheta}_{i,D/2}^\top \boldsymbol{x}) \end{bmatrix}^\top, \tag{5}$$

where weights $\{\boldsymbol{\vartheta}_{i,j}\}_{j=1}^{D/2}$ are drawn i.i.d. from the distribution $p_i(\boldsymbol{\vartheta})$ for $i = 1, \ldots, m+1$. As described in Example 5, each of these individual basis functions approximates a shift invariant kernel corresponding to the Fourier transform of the density $p_i(\boldsymbol{\vartheta})$ (Rahimi and Recht, 2008, 2007). In other words, $\mathbb{E}_{\boldsymbol{\vartheta} \sim p_i(\cdot)}[\boldsymbol{\psi}_i(\boldsymbol{x})^\top \boldsymbol{\psi}_i(\boldsymbol{x}')] = k_i(\boldsymbol{x} - \boldsymbol{x}')$ where $k_i(\boldsymbol{v}) = \mathcal{F}(p_i(\boldsymbol{\vartheta}))[\boldsymbol{v}]$.

3.2. Representation Guarantees

We now develop representation guarantees for the compound bases by showing which kernels they approximate. The affine dot product (ADP) kernel was first proposed by Castañeda et al. (2021) for systems with control-affine dynamics.

Definition 5 (Affine dot product (ADP) kernel) *Define $k_c : \mathcal{X} \times U \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, given by*

$$k_c((\boldsymbol{x}, \boldsymbol{u}), (\boldsymbol{x}', \boldsymbol{u}')) := \begin{bmatrix} \boldsymbol{u}^\top & 1 \end{bmatrix} \operatorname{diag}(k_1(\boldsymbol{x}, \boldsymbol{x}'), \cdots, k_{m+1}(\boldsymbol{x}, \boldsymbol{x}')) \begin{bmatrix} \boldsymbol{u}'^\top & 1 \end{bmatrix}^\top,$$

as the Affine Dot Product (ADP) compound kernel of $m+1$ individual kernels $k_i : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$.

The following theorem shows that the ADP basis approximates the ADP kernel.

Theorem 6 (ADP Approximation) *For $i = 1, ..., m+1$, suppose the basis functions ψ_i are defined according to (5) with p_i the inverse Fourier transform of a shift invariant kernel k_i. Let ϕ_c be the ADP compound basis of ψ_i and let k_c the compound ADP kernel of k_i. Then*

$$\mathbb{E}[\boldsymbol{\phi}_c(\boldsymbol{x}, \boldsymbol{u})^\top \boldsymbol{\phi}_c(\boldsymbol{x}', \boldsymbol{u}')] = k_c((\boldsymbol{x}, \boldsymbol{u}), (\boldsymbol{x}', \boldsymbol{u}')).$$

The result follows by relating the dot product of features to the diagonal matrix in the ADP kernel. We defer all formal proofs to our extended paper (Kazemian et al., 2023).

An alternative way to understand the ADP random feature approximation is to interpret the $(m+1) \times (m+1)$ diagonal matrix of kernels as an operator valued kernel. This operator valued kernel is the sum of $m+1$ *decomposable kernels*, as defined by Brault et al. (2016) (Definition 3). The ADP block diagonal matrix of basis functions can be interpreted through their framework as a random feature approximation for this operator valued kernel.

We now turn to the affine dense basis. First, we define a novel Affine Dense compound kernel.

Definition 7 (Affine dense (AD) kernel) *For $m+1$ individual kernels $k_i : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, let $\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{x}')$ be the diagonal matrix with i^{th} entry as $k_i(\boldsymbol{x} - \boldsymbol{x}')$, and $\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{x}')$ a matrix with zero on the diagonal and $[\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{x}')]_{ij} = k_i(\boldsymbol{x}) k_j(\boldsymbol{x}')$ for $i \neq j \in [m+1]$. Then, define the Affine Dense (AD) compound kernel as $k_d : \mathcal{X} \times \mathcal{U} \times \mathcal{X} \times U \to \mathbb{R}$, given by*

$$k_d((\boldsymbol{x}, \boldsymbol{u}), (\boldsymbol{x}', \boldsymbol{u}')) := \begin{bmatrix} \boldsymbol{u}^\top & 1 \end{bmatrix} \left(\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{x}') + \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{x}') \right) \begin{bmatrix} \boldsymbol{u}'^\top & 1 \end{bmatrix}^\top.$$

Notice that the diagonal matrix $\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{x}')$ in the AD kernel is similar to the ADP kernel. However, the AD kernel additionally includes the dense matrix $\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{x}')$. Due to this second dense term, the AD compound kernel is not shift invariant in $\boldsymbol{x}$. As a result, it is not possible to view $\boldsymbol{A}+\boldsymbol{D}$ as a shift invariant operator-valued kernel, and thus the results of Brault et al. (2016) cannot be used to derive a random features approximation. Furthermore, it is not immediately clear whether the AD kernel is indeed a valid reproducing kernel. We therefore begin by showing that it is.

Theorem 8 (AD kernel) *Let $k_d((\boldsymbol{x}, \boldsymbol{u}), (\boldsymbol{x}', \boldsymbol{u}'))$ be as in Definition 7. Suppose each $k_i(\boldsymbol{x}, \boldsymbol{x}')$ is a normalized shift invariant reproducing kernel. Then, $k_d((\boldsymbol{x}, \boldsymbol{u}), (\boldsymbol{x}', \boldsymbol{u}'))$ is a reproducing kernel.*

To prove this result, we use the crucial (but non-obvious) claim that if $k(\boldsymbol{x}, \boldsymbol{x}')$ is a normalized shift invariant reproducing kernel, then $k(\boldsymbol{x}, \boldsymbol{x}') - k(\boldsymbol{x})k(\boldsymbol{x}')$ is also a reproducing kernel. To prove that the claim is true, we construct an explicit feature mapping of the form required in Lemma 2. With the claim in hand, the proof follows by algebraic manipulations and the fact that the set of reproducing kernels is closed under addition. Therefore, the AD kernel k_d is a reproducing kernel and thus defines a RKHS. We next show that the AD basis functions approximate this RKHS.

Theorem 9 (Affine-dense kernel approximation) *Suppose that for $i = 1, ..., m+1$ the basis functions ψ_i are defined according to (5) with p_i the inverse Fourier transform of a shift invariant kernel k_i. Let $\boldsymbol{\phi}_d$ be the AD compound basis of $\boldsymbol{\psi}_i$ and let k_d be the compound AD kernel of k_i. Then $\mathbb{E}[\boldsymbol{\phi}_d(\boldsymbol{x}, \boldsymbol{u})^\top \boldsymbol{\phi}_d(\boldsymbol{x}', \boldsymbol{u}')] = k_d((\boldsymbol{x}, \boldsymbol{u}), (\boldsymbol{x}', \boldsymbol{u}'))$.*

So far our results show that the basis functions we propose approximate kernel regression in expectation. When the dimension D is large enough, the approximation error can be bounded with high probability (Rahimi and Recht, 2007; Sutherland and Schneider, 2015). In particular, Sutherland and Schneider (2015) show conditions under which the pointwise approximation error is no more than ϵ with probability depending on D and ϵ. We therefore conclude with a result which shows that when the individual kernels have bounded approximation errors, so do the compound kernels. In our extended paper (Kazemian et al., 2023), we further derive bounds on the prediction errors and confidence intervals for use in robust control.

Proposition 10 (Kernel Approximation Errors) *Consider the ADP kernel $k_c(\boldsymbol{s}, \boldsymbol{s}')$ and the AD kernel $k_d(\boldsymbol{s}, \boldsymbol{s}')$ from Definitions 5 and 7, respectively. Consider $\{k_i(\boldsymbol{x})\}_{i=1}^{m+1}$ which are the individual kernels used to construct the ADP and the AD kernels. Recall $\boldsymbol{\psi}$ from 5. Suppose $|k_i(\boldsymbol{x})| \leq 1$ and $|k_i(\boldsymbol{x}) - \boldsymbol{\psi}_i^\top(\boldsymbol{x})\boldsymbol{\psi}_i(\boldsymbol{x})| \leq \epsilon$ for all $\boldsymbol{x} \in \mathcal{X}$ and $i \in [m+1]$. Then, we have*

$$\max\{|k_c(\boldsymbol{s}, \boldsymbol{s}') - \boldsymbol{\phi}_c(\boldsymbol{s})^\top \boldsymbol{\phi}_c(\boldsymbol{s}')|,\ |k_d(\boldsymbol{s}, \boldsymbol{s}') - \boldsymbol{\phi}_d(\boldsymbol{s})^\top \boldsymbol{\phi}_d(\boldsymbol{s}')|\} \leq \epsilon(\boldsymbol{u}^\top \boldsymbol{u}' + 1) \tag{6}$$

where $\boldsymbol{\phi}_c$ and $\boldsymbol{\phi}_d$ are defined in Theorems 6 and 9 respectively.

3.3. Numerical demonstration

In this section, we empirically[1] study the performance of the the two random features methods (ADP-RF and AD-RF) as well as the corresponding kernel methods (ADP-K and AD-K). We focus on performance in terms of prediction accuracy. In the next section, we also demonstrate the utility of these models for data-driven control.

1. Code is available at https:github.com/kimzemian/swift_affine_mastery

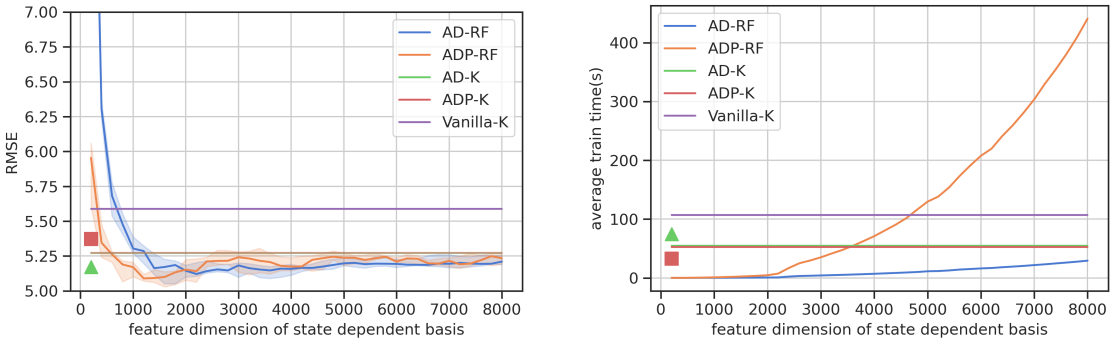

Figure 1: Evaluation of models, comparing prediction accuracy on test data (left) and training time on 8859 points (right) for a prediction problem on a double pendulum system. Horizontal lines and markers correspond to kernel methods. Random features are sampled 10 times at varying dimensions; the left panel displays median and quartiles over the trials while the right panel shows the mean. Increasing the feature dimension of each state-dependent basis $\boldsymbol{\psi}_i(x)$ results in lower RMSE but longer training time, especially for ADP-RF.

We consider a prediction task relating to a double pendulum with actuation at both joints. The state of this system $\boldsymbol{x} \in \mathbb{R}^4$ consists of two angle variables and two angular velocities, while the control input $\boldsymbol{u} \in \mathbb{R}^2$ consists of the two joint actuation torques. In our extended paper (Kazemian et al., 2023) we present a full derivation of the dynamics equation, which is affine in the control inputs. We simulate the system under closed-loop control and sample at 10 Hz. The controller is imperfectly designed to bring the system to an upright and balanced configuration; Further controller details are deferred to the following section. We collect a dataset containing 226 trajectories, each starting at a different initial point and lasting 5 seconds. The dataset if of the form $\{\{(\boldsymbol{x}_i^e, \boldsymbol{u}_i^e), z_i^e\}_{i=1}^L\}_{e=1}^E$ where z_i is the time derivative of a scalar function of the state (Example 3); details are described in the following section. We split the data into train and evaluation subsets with an $80/20$ split, so the train size is 8859 and test size is 2215, formulating a prediction task of the form in Definition 1.

We compare the performance of five models: three kernel methods (Vanilla-K, ADP-K, AD-K) and two random features methods (ADP-RF, AD-RF). Vanilla-K is an RBF kernel (Example 5) that operates on the concatenated state and input without any affine structure. ADP-K and AD-K (Definitions 5 and 7) use RBF kernel on the state variable, and ADP-RF and AD-RF defined in Theorems 6 and 9 with the corresponding random Fourier bases (5) as in Example 5. For all models, $\gamma = 1$ and $\lambda = 1$. Figure 1 plots the performance in terms of test accuracy and training time. The left panel shows median and quartile RMSE on the evaluation split and the right panel shows the training time. The kernels are represented by horizontal lines and markers. Vanilla-K performs worse than the affine kernels since it does not capture the affine structure, while AD-K and ADP-K have similar performance. For the RF models, we examine the effect of the random features approximation of state-dependent samples $\boldsymbol{\psi}_i(x)$ of dimension D. ADP-RF has lower error for smaller feature dimension, but both RF methods quickly approach the performance of the kernel methods. For small D, the RF methods are both much faster. Train time increases in D more quickly for ADP-RF than AD-RF, as training ADP-RF scales quadratically with $m+1$. Comparing the RF models, Figure 1 suggests that, although training with AD-RF is faster as compared to ADP-RF, the later has smaller RMSE. We attribute this to the higher dimensionality of the ADP compound basis,

which allows for greater expressivity. In our extended paper (Kazemian et al., 2023), we present extensive experiments with synthetic data demonstrating the relationship between training time and RMSE. These show that for fixed training time, AD-RF outperforms ADP-RF on accuracy when D is sufficiently large, and this performance advantage grows as input dimension m increases.

4. Case Study: Certificate Function Control

A key motivation for our work is that the affine structure of our data-driven models is amenable for use in control tasks. We therefore describe how to incorporate these models into a particular approach to nonlinear control. We then evaluate closed loop performance of our models.

Background As a case study, we demonstrate a nonlinear control technique based on *control certificate functions* as proposed by Taylor et al. (2021), which we refer to for a more rigorous and precise introduction. This approach generalizes and unifies the use of control Lyapunov functions (CLFs) to guarantee stability (Galloway et al., 2015) and control barrier functions (CBFs) to guarantee safety (Ames et al., 2016). Certificate function control requires a continuously differentiable *certificate function* $C : \mathbb{R}^n \to \mathbb{R}$ satisfying certain properties (Taylor et al., 2021), along with a *comparison function* $\alpha : \mathbb{R} \to \mathbb{R}_+$. Then, an optimization-based state feedback controller can be defined which will guarantee desired properties such as stability or safety by construction (Ames et al., 2019). Given some "desired" control input $\boldsymbol{u}_d(\boldsymbol{x})$, the CCF quadratic program (QP) is:

$$\begin{aligned} \boldsymbol{u}^*(\boldsymbol{x}) = \ & \underset{\boldsymbol{u}\in\mathbb{R}^m}{\arg\min} \ \|\boldsymbol{u}\|_2^2 + c_1 \|\boldsymbol{u} - \boldsymbol{u}_d(\boldsymbol{x})\|_2^2 \qquad \textbf{(CCF-QP)} \\ & \text{s.t.} \ \underbrace{\nabla C(\boldsymbol{x})^\top (\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u})}_{\dot{C}(\boldsymbol{x},\boldsymbol{u})} + \alpha(C(\boldsymbol{x})) \leq 0. \end{aligned}$$

Data-driven Control We now suppose that the dynamics $\boldsymbol{f}$ and $\boldsymbol{g}$ are unknown, so the CCF-QP controller cannot be directly implemented. We assume that a valid CCF C and comparison function α for the unknown true system is given[2]. Given a sampled trajectory $\{(\boldsymbol{x}_i, \boldsymbol{u}_i)\}_{i=1}^N$, we construct a control affine modelling problem for $\dot{C} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ as described in Example 3. We therefore use methods discussed in Section 3 to create a control-affine model $\hat{h}(\boldsymbol{x}, \boldsymbol{u})$ which can be used in place of the unknown $\dot{C}(\boldsymbol{x}, \boldsymbol{u})$ function in (**CCF-QP**) in a *certainty equivalent* (CE) manner. Because the model $\hat{h}$ is affine in $\boldsymbol{u}$, the resulting optimization problem is still a QP. In our extended paper (Kazemian et al., 2023), we provide additional details on constructing this QP for both kernel and RF methods. We also discuss methods for *robust*, rather than CE, data-driven control. The robust approach requires estimates of uncertainty (e.g. as in Gaussian process regression) as well as the pointwise RF approximation errors, and results in a second order cone program (SOCP).

Simulation experiments We simulate data-driven CCF control of the double pendulum introduced in Section 3.3, where the goal is to swing up and balance in the upright position $\boldsymbol{x} = 0$ with only an incorrect model of the dynamics $\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}$. Knowing only its degree of actuation, we may conclude that the dynamics are *feedback linearizable* and therefore we can define a Control Lyapunov Function (CLF) without the exact dynamics model (Taylor et al., 2019). Specifically, we define $C(\boldsymbol{x}) = \boldsymbol{x}^\top P \boldsymbol{x}$, $\alpha(\boldsymbol{x}) = .725\boldsymbol{x}$, $\boldsymbol{u}_d(\boldsymbol{x})$ a feedback linearizing controller for the incorrect $\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}$, and $c_1 = 25$. Full details are provided in our extended paper (Kazemian et al., 2023).

2. This assumption is met for feedback linearizable systems as long as the the degree of actuation of the true dynamics model is known (Taylor et al., 2019). For example, many robotic systems satisfy this assumption.

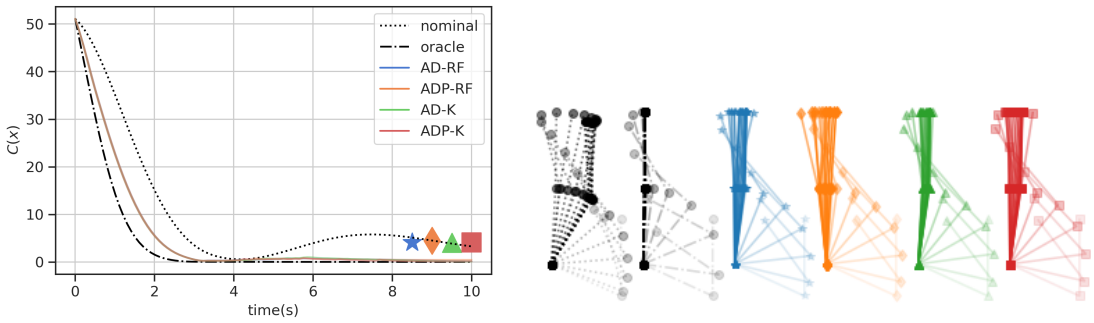

Figure 2: Left: The value of the Lyapunov function $C(\boldsymbol{x})$ over time for nominal, oracle, and data-driven controllers with initial state $[2, 0, 0, 0]$. Right: Illustration of the pendulum configurations over time. Nominal fails to balance the pendulum; data-driven methods succeed.

We first define a "nominal" QP controller which selects inputs according to (**CCF-QP**) with the nominal model $\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}$. We use this controller to gather trajectories and define a dataset as described in Section 3.3. We subsample the data by $1/5$ and derive data-driven models of $\dot{C}(\boldsymbol{x}, \boldsymbol{u})$ as outlined in the paragraph above. We consider four data-driven QP controllers using the four affine models: AD-K, ADP-K, AD-RF, and ADP-RF. For the data-driven controllers, we augment the initial dataset with episodic data collection: we run the controller for 10 seconds at 10 Hz, retrain, and repeat for ten episodes. The RF dimension is $D = N/5$ for N the size of the training data. Finally, we compare the performance of the the nominal and data-driven methods with an "oracle" controller that solves (**CCF-QP**) with the true dynamics. Figure 2 plots the system trajectory in terms of the Lyapunov function $C(\boldsymbol{x})$ and the pendulum configuration. While the nominal controller fails to balance the pendulum, the data-driven controllers succeed and are similar to each other.

5. Conclusion

This work considers a control affine modelling problem and proposes two classes of random basis functions as a solution: ADP and AD. The representation guarantees of these methods are made formal by connection to kernel regression in corresponding RKHSs. A case study in nonlinear control with CCF illustrates the utility of control affine models. Numerical experiments demonstrate the performance of the RF and kernel methods in terms of accuracy, computation time, and closed loop control performance. The extended version additionally presents uncertainty estimates analogous to Gaussian process (GP) regression, as well as a corresponding robust data-driven control. We highlight that the approximation methods that we propose may be broadly of interest for any control application which makes use of GPs.

Our work opens the door to many future questions of interest. It would be interesting to develop kernels and random features tailored to particular control applications. One could explore the application of our methods to additional control techniques, like feedback linearization or model predictive control. It would be interesting to develop principled techniques for acquiring data, expanding from the simple warm start episodic approach that we used. Furthermore, additional methods for approximating kernels would provide alternatives to speeding up kernel and GP regression for data-driven control.

Acknowledgments

We thank Jaosn Choi for helping us understand their code base for ADP kernel. This work was partly funded by NSF CCF 2312774 and NSF OAC-2311521, a LinkedIn Research Award, and a gift from Wayfair.

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