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Convex Approximations for a Bi-level Formulation of Data-Enabled Predictive Control

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Abstract

The Willems' fundamental lemma, which characterizes linear time-invariant (LTI) systems using input and output trajectories, has found many successful applications. Combining this with receding horizon control leads to a popular Data-EnablEd Predictive Control (DeePC) scheme. DeePC is first established for LTI systems and has been extended and applied for practical systems beyond LTI settings. However, the relationship between different DeePC variants, involving regularization and dimension reduction, remains unclear. In this paper, we first discuss a bi-level optimization formulation that combines a data pre-processing step as an inner problem (system identification) and predictive control as an outer problem (online control). We next introduce a series of convex approximations by relaxing some hard constraints in the bi-level optimization as suitable regularization terms, accounting for an implicit identification. These include some existing DeePC variants as well as two new variants, for which we establish their equivalence under appropriate settings. Notably, our analysis reveals a novel variant, called DeePC-SVD-Iter, which has remarkable empirical performance of direct methods on systems beyond deterministic LTI settings.

Keywords: Data-driven Control, Bi-level optimization, Convex approximation

1. Introduction

There has been a surging interest in utilizing data-driven techniques to control systems with unknown dynamics (Pillonetto et al., 2014; Markovsky and Dörfler, 2021). Existing methods can be generally categorized into indirect and direct data-driven control techniques: indirect data-driven control approaches typically include sequential system identification (system ID) (Ljung, 1998; Chiuso and Pillonetto, 2019) and model-based control (Kouvaritakis and Cannon, 2016), while direct data-driven control methods bypass system ID and directly design control strategies from input and output measured data (Markovsky and Dörfler, 2021).

In particular, Data-EnablEd Predictive Control (DeePC) (Coulson et al., 2019; Markovsky and Dörfler, 2021) that combines behavioral theory with receding horizon control has received increasing attention. It utilizes Willem's fundamental lemma (Willems, 2007) to construct a data-driven representation of a dynamic system and incorporates it with receding horizon control. DeePC is first established for deterministic linear time-invariant (LTI) systems, and its equivalence with subspace predictive control (SPC) has been discussed in (Fiedler and Lucia, 2021). Berberich et al. (2020) further investigate conditions for its closed-loop stability. The DeePC approach has shown promising results for the control of practical systems beyond LTI settings (Wang et al., 2023; Elokda et al., 2021; Shang et al., 2023; Lian et al., 2023). For non-deterministic or nonlinear systems, suitable regularizations are necessary for DeePC; see Breschi et al. (2023); Dörfler et al. (2022).

There are different regularization strategies for DeePC, ranging from some heuristics in Coulson et al. (2019) to principled analysis via bi-level formulations in Dörfler et al. (2022). Notably, indirect data-driven control is first formulated as a bi-level optimization problem involving both control and identification in Dörfler et al. (2022). Many regularized versions (such as l_1 or l_2 norms) of

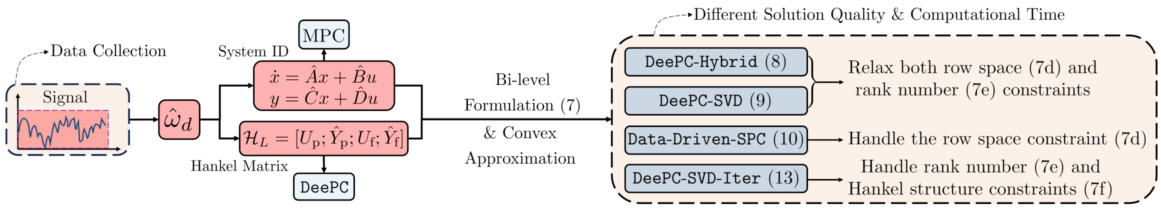


Figure 1: Schematic of data-driven control, which starts by collecting data (usually noisy) from the real system. Indirect methods identify a parametric model, while DeePC forms a Hankel matrix as the trajectory library for predictive control. Our bi-level formulation (7) integrates system ID techniques to DeePC. We introduce a series of convex approximations (8), (9), (10), and (13) that relax the bi-level formulation.

DeePC can be considered as convex relaxations of the bi-level optimization. Beyond regularization, some recent approaches aim to decrease the optimization dimensions and improve computational efficiency (Zhang et al., 2023; Alsalti et al., 2023). One simple strategy is to use a singular value decomposition (SVD) to pre-process the data-driven representation, which has shown promising performance (Zhang et al., 2023; Yin et al., 2021). However, the relationship between the recent variants of DeePC for non-deterministic and nonlinear systems, involving regularization and dimension reduction, remains unclear, and there is no analysis and comparison for their solution qualities.

In this paper, we introduce a new bi-level formulation incorporating both system ID techniques and predictive control, and discuss how existing and new variants of DeePC can be considered as convex approximations of this bi-level formulation; Figure 1 illustrates the overall process. Specifically, in our bi-level formulation for DeePC, the data pre-processing step is viewed as an inner optimization problem (identification), and the predictive control is viewed as an outer optimization problem (online control). Constraints for the inner optimization problem are derived from system ID methods (e.g., SPC Favoreel et al. (1999), and low-rank approximation Markovsky (2016)). We further discuss a series of convex approximations by relaxing some hard constraints as suitable regularization terms. In this process, we derive two new variants of DeePC by adapting existing methods: 1) Data-Driven-SPC is derived from classical SPC with the same structure as DeePC, and 2) DeePC-SVD-Iter refines the data-driven representation in DeePC-SVD from Zhang et al. (2023) and provides superior performance. We also investigate the equivalence of DeePC-Hybrid (Dörfler et al., 2022), DeePC-SVD (Zhang et al., 2023), and Data-Driven-SPC. Our analysis is more general than Zhang et al. (2023); Fiedler and Lucia (2021); Breschi et al. (2023). Numerical experiments confirm our analysis and show the superior performance of DeePC-SVD-Iter.

The rest of this paper is structured as follows. Section 2 introduces preliminaries and the problem statement of our bi-level formulation. A series of convex approximations are introduced in Section 3, their relationship is established in Section 4. Section 5 compares their control performance via numerical simulations. Finally, we conclude the paper in Section 6. Some technical proofs and extra discussions are provided in our technical report (Shang and Zheng, 2023).

2. Preliminaries and Problem Statement

2.1. Preliminaries

We consider a linear time-invariant (LTI) system in the discrete-time domain:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ \quad\;\; y(k) = Cx(k) + Du(k), \end{cases} \tag{1}$$

where the state, input, output at time k are $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $y(k) \in \mathbb{R}^p$, respectively. Given a desired reference trajectory $y_{\mathrm{r}} \in \mathbb{R}^{pN}$ with horizon $N > 0$, input constraint set $\mathcal{U} \subseteq \mathbb{R}^m$, output constraint set $\mathcal{Y} \subseteq \mathbb{R}^p$, we aim to design control inputs such that the system output tracks the reference trajectory. In particular, we consider the well-known receding horizon predictive control

$$
\begin{aligned}
\min_{x,u,y} \quad & \sum_{k=t}^{t+N-1} \left(\|y(k) - y_{\mathrm{r}}(k)\|_Q^2 + \|u(k)\|_R^2\right) \\
\text{subject to} \quad & x(k+1) = Ax(k) + Bu(k), \quad k \in [t, t+N-1] && \text{(2a)} \\
& y(k) = Cx(k) + Du(k), \quad k \in [t, t+N-1] && \text{(2b)} \\
& x(t) = x_{\mathrm{ini}}, && \text{(2c)} \\
& u(k) \in \mathcal{U},\ y(k) \in \mathcal{Y}, \quad k \in [t, t+N-1], && \text{(2d)}
\end{aligned}
$$

where $x_{\mathrm{ini}} \in \mathbb{R}^n$ is the initial state of system (1) and $\|u(k)\|_R^2$ denotes the quadratic norm $u(k)^{\mathsf{T}} R u(k)$ (similarly for $\|\cdot\|_Q$) with $R \in \mathbb{S}_+^m$ and $Q \in \mathbb{S}_+^p$. We assume $\mathcal{U}$ and $\mathcal{Y}$ are convex sets. Without loss of generality, we consider a regulation problem (i.e., $y_{\mathrm{r}}(k) = 0$) for the rest of the discussions.

It is clear that (2) is a convex optimization problem (it is indeed a quadratic program when $\mathcal{U}$ and $\mathcal{Y}$ are polytope), which admits an efficient solution when the model for the system (1) is known, i.e., matrices A, B, C and D are known. In this work, we focus on the case when the system model and the initial condition x_{ini} are unknown. Instead, we have access to 1) offline data, i.e., a length-T pre-collected input/output trajectory of (1), and 2) online data, i.e., the most recent past input/output sequence of length-T_{ini}. Then, (2) can be solved by either indirect system ID and model-based control (Åström and Eykhoff, 1971) or the recent emerging direct data-driven control, such as `DeePC` and its related approaches (Dörfler et al., 2022; Markovsky and Dörfler, 2021). As discussed in Dörfler et al. (2022), the indirect system ID approach is superior in the case of "variance" noise, while `DeePC` with suitable regularization terms has better performance in the case of "bias" errors.

2.2. Data-Enabled Predictive Control

We here review the basic setup of `DeePC`. First, let us introduce a notion of persistent excitation and we use $\mathrm{col}(A_1, A_2, \ldots, A_m) = [A_1^{\mathsf{T}}, A_2^{\mathsf{T}}, \ldots, A_m^{\mathsf{T}}]^{\mathsf{T}}$ in the rest of the paper.

Definition 1 (Persistently Exciting) *A sequence of signal $\omega = \mathrm{col}(\omega(1), \omega(2), \ldots, \omega(T))$ of the length T ($T \in \mathbb{N}$) is persistently exciting of order L ($L < T$) if its associated Hankel matrix with depth L, defined below, has full row rank,*

$$
\mathcal{H}_L(\omega) = \begin{bmatrix} \omega(1) & \omega(2) & \cdots & \omega(T-L+1) \\ \omega(2) & \omega(3) & \cdots & \omega(T-L+2) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(L) & \omega(L+1) & \cdots & \omega(T) \end{bmatrix}.
$$

Lemma 1 (Fundamental Lemma; Willems et al. (2005)) *Suppose that system (1) is controllable. Given a length T input/output trajectory: $u_{\mathrm{d}} = \mathrm{col}(u_{\mathrm{d}}(1), \ldots, u_{\mathrm{d}}(T)) \in \mathbb{R}^{mT}$, $y_{\mathrm{d}} = \mathrm{col}(y_{\mathrm{d}}(1), \ldots, y_{\mathrm{d}}(T)) \in \mathbb{R}^{pT}$ where u_{d} is persistently exciting of order $L+n$, then a length L input/output sequence $(u_{\mathrm{s}}, y_{\mathrm{s}})$ is a valid trajectory of (1) if and only if there exists a $g \in \mathbb{R}^{T-L+1}$ such that*

$$
\begin{bmatrix} \mathcal{H}_L(u_{\mathrm{d}}) \\ \mathcal{H}_L(y_{\mathrm{d}}) \end{bmatrix} g = \begin{bmatrix} u_{\mathrm{s}} \\ y_{\mathrm{s}} \end{bmatrix}. \tag{3}
$$

If length L is not smaller than the lag of the system, matrix $\mathrm{col}(\mathcal{H}_L(u_{\mathrm{d}}), \mathcal{H}_L(y_{\mathrm{d}}))$ has rank $mL+n$.

The DeePC approach in Coulson et al. (2019) employs (3) to build a predictor based on the pre-collected data. In particular, the Hankel matrix formed by the offline data is partitioned as

$$\begin{bmatrix} U_{\mathrm{P}} \\ U_{\mathrm{F}} \end{bmatrix} := \mathcal{H}_L(u_{\mathrm{d}}), \quad \begin{bmatrix} Y_{\mathrm{P}} \\ Y_{\mathrm{F}} \end{bmatrix} := \mathcal{H}_L(y_{\mathrm{d}}), \tag{4}$$

where U_{P} and U_{F} consist the first T_{ini} rows and the last N rows of $\mathcal{H}_L(u_{\mathrm{d}})$, respectively (similarly for Y_{P} and Y_{F}; so $L = T_{\mathrm{ini}} + N$). We denote the most recent past input trajectory of length T_{ini} and the future input trajectory of length N as $u_{\mathrm{ini}} = \mathrm{col}(u(t - T_{\mathrm{ini}}), u(t - T_{\mathrm{ini}} + 1), \ldots, u(t-1))$ and $u = \mathrm{col}(u(t), u(t+1), \ldots, u(t+N-1))$, respectively (similarly for y_{ini}, y).

Then, Lemma 1 ensures that the sequence $\mathrm{col}(u_{\mathrm{ini}}, y_{\mathrm{ini}}, u, y)$ is a valid trajectory of (1) if and only if there exists $g \in \mathbb{R}^{T-T_{\mathrm{ini}}-N+1}$ such that (5) holds. For notational simplicity, we further denote the matrix $\mathrm{col}(U_{\mathrm{P}}, Y_{\mathrm{P}}, U_{\mathrm{F}}, Y_{\mathrm{F}})$ associated with pre-collected data as H.

$$\begin{bmatrix} U_{\mathrm{P}} \\ Y_{\mathrm{P}} \\ U_{\mathrm{F}} \\ Y_{\mathrm{F}} \end{bmatrix} g = \begin{bmatrix} u_{\mathrm{ini}} \\ y_{\mathrm{ini}} \\ u \\ y \end{bmatrix}. \tag{5}$$

Note that H can be considered as a trajectory library since each of its columns is a valid trajectory of system (1). If T_{ini} is larger or equal to the lag of system (1), y is unique given any u_{ini}, y_{ini} and u in (5). The most basic version of DeePC (Coulson et al., 2019) utilizes the predictor (5) as the data-driven representation of (2a) to (2c) and reformulate the problem (2) as

$$\begin{aligned} \min_{g,u,y} \quad & \sum_{k=t}^{t+N-1} \left(\|y(k) - y_r(k)\|_Q^2 + \|u(k)\|_R^2 \right) \\ \text{subject to} \quad & (5), u \in \mathcal{U}, y \in \mathcal{Y} \end{aligned} \tag{6}$$

where we slightly abuse the notation and use $u \in \mathcal{U}, y \in \mathcal{Y}$ to denote input/output constraints (2d).

2.3. A bi-level formulation beyond deterministic LTI systems

It is not difficult to show that for LTI systems with noise-free data, problems (2) and (6) are fully equivalent (Coulson et al., 2019, Thm. 5.1). However, for the case beyond deterministic LTI systems, there exist different regularization terms or data pre-processing techniques that extend the basic DeePC (6). Indeed, an extensive discussion on bridging indirect and direct data-driven control was presented in Dörfler et al. (2022), where two different bi-level formulations were discussed.

Motivated by Dörfler et al. (2022), we introduce another bi-level formulation that incorporates data pre-processing techniques from system ID. In practice, the data predictor H in (5) may be corrupted by "variance" noises and/or "bias" errors. The key idea of our bi-level formulation is to pre-process the raw data H and construct a new trajectory library $\tilde{H} := \mathrm{col}(\tilde{U}_{\mathrm{P}}, \tilde{Y}_{\mathrm{P}}, \tilde{U}_{\mathrm{F}}, \tilde{Y}_{\mathrm{F}})$ satisfying specific structures in system ID:

$$\begin{aligned} \underset{g, \sigma_y, u \in \mathcal{U}, y \in \mathcal{Y}}{\text{minimize}} \quad & \sum_{k=t}^{t+N-1} \left(\|y(k) - y_r(k)\|_Q^2 + \|u(k)\|_R^2 \right) + \lambda_y \|\sigma_y\|_2^2 && \text{(7a)} \\ \text{subject to} \quad & \tilde{H}^* g = \mathrm{col}(u_{\mathrm{ini}}, y_{\mathrm{ini}} + \sigma_y, u, y), && \text{(7b)} \\ & \text{where} \quad \tilde{H}^* \in \arg\min_{\tilde{H}} \quad J(\tilde{H}, H), && \text{(7c)} \\ & \qquad \text{subject to} \quad \tilde{Y}_{\mathrm{F}} = Y_{\mathrm{F}} / \mathrm{col}(\tilde{U}_{\mathrm{P}}, \tilde{Y}_{\mathrm{P}}, \tilde{U}_{\mathrm{F}}) \ \text{(Row Space)}, && \text{(7d)} \\ & \qquad \qquad \mathrm{rank}(\tilde{H}) = mL + n \quad \text{(Rank Number)}, && \text{(7e)} \\ & \qquad \qquad \tilde{H} \in \mathcal{H} \quad \text{(Hankel Structure)}. && \text{(7f)} \end{aligned}$$

This bi-level problem structure in (7), which is consistent with those in Dörfler et al. (2022), reflects the sequential ID and control tasks, where we first fit a data-driven model $\tilde{H}$ from the raw data H (4) in the inner system ID before using the model for `DeePC` in the outer problem. The key difference between (7) and the bi-level formulation in Dörfler et al. (2022) is the inner problem in Dörfler et al. (2022) is to identify a *parametric* model while we directly identify a *data-driven* model, i.e., we process the data matrix H via (7d) to (7f).

In the outer problem (7a)-(7b), we have introduced a slack variable σ_y and its regularization term to handle the model mismatch and ensure feasibility, as discussed in Markovsky and Dörfler (2021). In the inner optimization problem (7c)-(7f), $J(\tilde{H}, H)$ in (7c) denotes system identification loss function with H being the raw Hankel matrix (4). In (7d), $Y_\mathrm{F}/\mathrm{col}(\tilde{U}_\mathrm{P}, \tilde{Y}_\mathrm{P}, \tilde{U}_\mathrm{F})$ denotes the orthogonal projection of Y_F onto the row space of $\mathrm{col}(\tilde{U}_\mathrm{P}, \tilde{Y}_\mathrm{P}, \tilde{U}_\mathrm{F})$. This row space constraint is derived from SPC (Favoreel et al., 1999) which will be discussed in detail in Section 3.2. The rank constraint (7e) and Hankel structure (7f) (where $\mathcal{H}$ is the set of all matrices with Hankel structure; cf. Definition 1) come from low-rank approximation in Markovsky (2016). We refer the interested readers to Fiedler and Lucia (2021) and Willems et al. (2005) for further details on row space and rank number respectively.

We will derive a series of convex approximations for this bi-level formulation (7) in Section 3 and discuss their equivalence (if possible) and relationship in Section 4.

3. Convex Approximations

While the bi-level formulation (7) is not solvable immediately, it provides useful guidance to derive new formulations/variants of `DeePC`. In this section, we present four convex approximations by adapting existing methods; see Figure 1 for an overview. These strategies relax the inner constraints (7d) to (7f) using suitable regularizers to the outer problem.

3.1. `DeePC` with regularization and dimension reduction

We first discuss two existing convex approximations of (7): `DeePC-Hybrid` from Dörfler et al. (2022) and `DeePC-SVD` from Zhang et al. (2023). Both of them use two different regularization terms to relax the row space constraint (7d) and the rank constraint (7e) while `DeePC-Hybrid` keeps the Hankel constraint (7f) and `DeePC-SVD` drops it.

Compared with the basic `DeePC` in (6), besides the regularizer $\|\sigma_y\|_2^2$, we introduce two extra regularizers $\|g\|_1$ and $\|(I - \Pi_1)g\|$ in `DeePC-Hybrid`, which reads as

$$\begin{aligned}
\min_{g,\sigma_y,u\in\mathcal{U},y\in\mathcal{Y}} \quad & \|u\|_R^2 + \|y\|_Q^2 + \lambda_1\|g\|_1 + \lambda_2\|(I-\Pi_1)g\|_2^2 + \lambda_y\|\sigma_y\|_2^2 \\
\text{subject to} \quad & \begin{bmatrix} U_\mathrm{P} \\ Y_\mathrm{P} \\ U_\mathrm{F} \\ Y_\mathrm{F} \end{bmatrix} g = \begin{bmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} + \sigma_y \\ u \\ y \end{bmatrix},
\end{aligned} \tag{8}$$

where $\Pi_1 = H_1^\dagger H_1$ with $H_1 = \mathrm{col}(U_\mathrm{P}, Y_\mathrm{P}, U_\mathrm{F})$. Throughout the rest of the discussion, we denote $\|u\|_R^2 = \sum_{k=t}^{t+N-1} \|u(k)\|_R^2$ (similarly for $\|y\|_Q^2$). We note that the l_1 regularization $\|g\|_1$ can be viewed as a convex relaxation of the rank constraint (7e) (Dörfler et al., 2022, Thm. IV.8), while the regularization $\|(I-\Pi_1)g\|_2^2$ relaxes row space constraint (7d) (Dörfler et al., 2022, Thm. IV.6).

Since the column number of H is usually larger than its row number in practice (i.e., H is typically a fat matrix), `DeePC-SVD` in Zhang et al. (2023) utilizes singular value decomposition (SVD) to pre-process H and reduce its column dimension. Denoting the SVD of H as $H = W\Sigma V^{\mathsf{T}}$, where Σ contains its non-zero singular values, we construct a new data matrix $\bar{H} = W\Sigma$, and partition its rows as $\bar{H} = \mathrm{col}(\bar{U}_{\mathrm{P}}, \bar{Y}_{\mathrm{P}}, \bar{U}_{\mathrm{F}}, \bar{Y}_{\mathrm{F}})$. Then, the formulation `DeePC-SVD` reads as

$$\begin{aligned} \min_{\bar{g},\sigma_y,u\in\mathcal{U},y\in\mathcal{Y}} \quad & \|u\|_R^2 + \|y\|_Q^2 + \lambda_1\|\bar{g}\|_1 + \lambda_2\|(I-\bar{\Pi}_1)\bar{g}\|_2^2 + \lambda_y\|\sigma_y\|_2^2 \\ \text{subject to} \quad & \begin{bmatrix} \bar{U}_{\mathrm{P}} \\ \bar{Y}_{\mathrm{P}} \\ \bar{U}_{\mathrm{F}} \\ \bar{Y}_{\mathrm{F}} \end{bmatrix} \bar{g} = \begin{bmatrix} u_{\mathrm{ini}} \\ y_{\mathrm{ini}} + \sigma_y \\ u \\ y \end{bmatrix} \end{aligned} \tag{9}$$

where $\bar{\Pi}_1 = \bar{H}_1^{\dagger}\bar{H}_1$ and $\bar{H}_1 = \mathrm{col}(\bar{U}_{\mathrm{P}}, \bar{Y}_{\mathrm{P}}, \bar{U}_{\mathrm{F}})$. The dimension of $\bar{g}$ in (9) can be much smaller than that in (8), and this simple fact can improve numerical efficiency.

3.2. Data-Driven Subspace Predictive Control (SPC)

We here introduce a new `Data-Driven-SPC` to approximate (7) and establish its equivalence with the classical SPC in Favoreel et al. (1999). Similar to `DeePC-Hybrid` (8), we drop the Hankel structure constraint (7f) and use a l_1 regularization to relax the rank constraint (7e). However, we will directly handle the row space constraint (7d) without using any relaxation. Let us consider

$$\begin{aligned} \min_{\tilde{H}} \quad & \|\mathrm{col}(\tilde{U}_{\mathrm{P}}, \tilde{Y}_{\mathrm{P}}, \tilde{U}_{\mathrm{F}}) - \mathrm{col}(U_{\mathrm{P}}, Y_{\mathrm{P}}, U_{\mathrm{F}})\|_2 \\ \text{subject to} \quad & \tilde{Y}_{\mathrm{F}} = Y_{\mathrm{F}}/\mathrm{col}(\tilde{U}_{\mathrm{P}}, \tilde{Y}_{\mathrm{P}}, \tilde{U}_{\mathrm{F}}). \end{aligned}$$

This inner problem has an analytical solution as $\tilde{H}^* = \mathrm{col}(U_{\mathrm{P}}, Y_{\mathrm{P}}, U_{\mathrm{F}}, M)$, with $M = Y_{\mathrm{F}}\Pi_1$ and Π_1 defined in Section 3.1. Then, we formulate `Data-Driven-SPC` as a problem in the form of (10). For self-completeness, we also present the classical SPC which is in the form of (11).

$$\begin{aligned} \min_{\sigma_y,g,u\in\mathcal{U},y\in\mathcal{Y}} \quad & \|u\|_R^2 + \|y\|_Q^2 + \lambda_1\|g\|_1 + \lambda_y\|\sigma_y\|_2^2 \\ \text{subject to} \quad & \begin{bmatrix} U_{\mathrm{P}} \\ Y_{\mathrm{P}} \\ U_{\mathrm{F}} \\ M \end{bmatrix} g = \begin{bmatrix} u_{\mathrm{ini}} \\ y_{\mathrm{ini}} + \sigma_y \\ u \\ y \end{bmatrix}. \end{aligned} \tag{10}$$

$$\begin{aligned} \min_{\sigma_y,u\in\mathcal{U},y\in\mathcal{Y}} \quad & \|u\|_R^2 + \|y\|_Q^2 + \lambda_y\|\sigma_y\|_2^2 \\ \text{subject to} \quad & y = Y_{\mathrm{F}} \begin{bmatrix} U_{\mathrm{P}} \\ Y_p \\ U_{\mathrm{F}} \end{bmatrix}^{\dagger} \begin{bmatrix} u_{\mathrm{ini}} \\ y_{\mathrm{ini}} + \sigma_y \\ u \end{bmatrix}. \end{aligned} \tag{11}$$

We show that (10) is indeed a direct data-driven version of (11) in the sense that they produce the same solution under a very mild condition. The proof is in our report (Shang and Zheng, 2023).

Theorem 1 *If $Q \succ 0, R \succ 0, \lambda_1 = 0$ and $H_1 = \mathrm{col}(U_{\mathrm{P}}, Y_{\mathrm{P}}, U_{\mathrm{F}})$ has full row rank, then (10) and (11) have the same optimal solution u^*, y^*, σ_y^*, $\forall \lambda_y > 0$.*

Note that our new `Data-Driven-SPC` (10) is more flexible than the classical SPC (11) thanks to the parameter λ_1, which was motivated from the relaxation of the rank constraint (7e).

3.3. `DeePC` with dominant range space and Hankel structure

All the convex approximations (8), (9) and (10) use different regularizations to relax the difficult constraints (7d) and (7e), but both (9) and (10) directly drop the Hankel constraint (7f). In this subsection, we derive another new convex approximation for (7) that also approximates the Hankel structure with a dominant range space from the SVD. We call it as `DeePC-SVD-Iter`.

$$\min_{\tilde{H}} \quad \|\tilde{H} - H\|_2 \tag{12a}$$
$$\text{subject to} \quad \text{rank}(\tilde{H}) = mL + n \tag{12b}$$
$$\tilde{H} \in \mathcal{H} \tag{12c}$$

In particular, we consider (12) as the inner problem in the bi-level formulation (7), where the row space constraint (7d) will be relaxed using regularization. Note that (12) is also difficult to solve due to the interplay between (12b) and (12c). There are extensive results in the field of structured low-rank approximation (SLRA) (Markovsky, 2008). One idea is to use an alternative optimization strategy by considering (12b) and (12c) sequentially. Specifically, we here adapt an iterative SLRA algorithm in Yin and Smith (2021) to get an approximation solution to (12). We first note that problem (12) without (12c) admits an analytical solution $\tilde{H}^* = W_\text{r}\Sigma_\text{r}V_\text{r}^\mathsf{T}$ where W_r, Σ_r and V_r represent the leading $mL + n$ singular vectors and singular values of H, i.e., the dominant range space.

The key idea of the iterative SLRA is to utilize SVD for low-rank approximation of noisy data and then project the low-rank matrix to the set of Hankel matrices. This process is summarized in Algorithm 1. For notational simplicity, We define $H_u = \mathcal{H}_L(u_\text{d})$, $H_y = \mathcal{H}_L(y_\text{d})$ to denote the Hankel matrices in (4). Thanks to the persistent excitation on u_d with no noise, we have $\text{rank}(H_u) = mL$. However, the measurement y_d usually contain "variance" noise and "bias" error, thus the data matrix satisfies $\text{rank}(\text{col}(H_u, H_y)) > mL + n$. We use an iterative procedure to denoise H_y while maintaining its Hankel structure (note that we do not change H_u since it is normally noisy-free).

Algorithm 1: Iterative SLRA

Input: H_y, Π_2, n, ϵ
$H_{y_1} \leftarrow H_y$
repeat
  $H_{y_2} \leftarrow \hat{H}(H_{y_1})$ `SVD step`
  $H_{y_1} \leftarrow \Pi_\text{H}(H_{y_2})$ `Hankel proj`
until $\|H_{y_1} - H_{y_2}\| \leq \epsilon \|H_{y_1}\|$;
Output: $H_y^* = H_{y_2}$

Let $\Pi_2 = H_u^\dagger H_u$ be the orthogonal projector onto the row space of H_u, and we first compute $H_y(I - \Pi_2)$ which is the component of H_y in the null space of H_u. In each iteration of Algorithm 1, we perform an SVD of $H_y(I - \Pi_2)$, estimate its rank-n approximation (since we have $\text{rank}(H_u) = mL$), and finally combine it with the component of H_y in the row space of H_u as follows

$$H_y(I - \Pi_2) = \sum_{i=1}^{pL} \sigma_i u_i v_i^\mathsf{T}, \quad \hat{H}(H_y) := H_y\Pi_2 + \sum_{i=1}^{n} \sigma_i u_i v_i^\mathsf{T}.$$

We then project $\hat{H}(H_y)$ onto the set of Hankel matrices by averaging skew-diagonal elements and denote Π_H as the corresponding operator. The resulting matrix H_y^* from Algorithm 1 is partitioned as $\text{col}(Y_\text{P}^*, Y_\text{F}^*)$, and we form a new Hankel matrix $\tilde{H}^* = \text{col}(U_\text{P}, Y_\text{P}^*, U_\text{F}, Y_\text{F}^*)$. Finally, we perform an SVD of $\tilde{H}^* = \tilde{W}_\text{r}\tilde{\Sigma}_\text{r}\tilde{V}_\text{r}^\mathsf{T}$ to reduce its column dimension and set $\tilde{W}_\text{r}\tilde{\Sigma}_\text{r} = \text{col}(\hat{U}_\text{P}, \hat{Y}_\text{P}, \hat{U}_\text{F}, \hat{Y}_\text{F})$ with rank $mL + n$ (in the final predictor, we only have the dominant $mL + n$ singular values). This new matrix $\text{col}(\hat{U}_\text{P}, \hat{Y}_\text{P}, \hat{U}_\text{F}, \hat{Y}_\text{F})$ is used as the predictor in `DeePC` as

$$\begin{aligned} \min_{\hat{g}, \sigma_y, u \in \mathcal{U}, y \in \mathcal{Y}} \quad & \|u\|_R^2 + \|y\|_Q^2 + \lambda_2 \|(I - \hat{\Pi}_1)\hat{g}\|_2^2 + \lambda_y \|\sigma_y\|_2^2 \\ \text{subject to} \quad & \begin{bmatrix} \hat{U}_\text{P} \\ \hat{Y}_\text{P} \\ \hat{U}_\text{F} \\ \hat{Y}_\text{F} \end{bmatrix} \hat{g} = \begin{bmatrix} u_\text{ini} \\ y_\text{ini} + \sigma_y \\ u \\ y \end{bmatrix} \end{aligned} \tag{13}$$

where $\hat{\Pi}_1 = \hat{H}_1^\dagger \hat{H}_1$, $\hat{H}_1 = \text{col}(\hat{U}_\text{P}, \hat{Y}_\text{P}, \hat{U}_\text{F})$ and $\|(I - \hat{\Pi}_1)\hat{g}\|_2^2$ is the relaxation term from the row space constraint. We call this formulation (13) as `DeePC-SVD-Iter`. In our experiments, Algorithm 1 terminated in a finite number of iterates (up to 2500) with ϵ chosen from 10^{-3} to 10^{-6}.

4. Relationship among Different Convex Approximations

As motivated above, (8) to (10) and (13) are all tractable convex approximations for the bi-level formulation (7). They all begin with the same data matrix H and apply different relaxation strategies to deal with the identification constraints (7d) to (7f). We here look into their relationships and establish certain equivalence. Proofs of Theorems 2 and 3 are presented in Shang and Zheng (2023).

First, it is not difficult to see that all of them are equivalent when the data matrix H comes from an LTI system with no noise. We summarize this simple fact below.

Proposition 1 *Suppose that the data matrix H in (4) comes from a controllable LTI system (1) with no noise, and the input u_{d} is persistently exciting of order $L+n$. Let $\lambda_1 = 0, \lambda_2 = 0$ and $\sigma_y = 0$. Then, all* `DeePC` *variants in (8), (9), (10), and (13) have the same unique optimal solution u^*, y^*.*

For a controllable system (1) with no noise, the data matrix H has already satisfied row space (7d), rank number (7e), and Hankel structure constraints (7f). Then the new matrix $\tilde{H}^*$ after pre-processing in `Data-Driven-SPC` (10) and `DeePC-SVD-Iter` (13) will remain the same as that in `DeePC-Hybrid` (8), and their range space are also equal to that of `DeePC-SVD` (9). Thus, all these formulations have the same feasible region and cost functions, and they are equivalent.

We next move away from noise-free LTI systems. The data matrix H may have 'variance" noise and/or "bias" errors; see Dörfler et al. (2022). In this case, we can still show that `DeePC-Hybrid` (8) and `DeePC-SVD` (9) produce the same optimal solution u^*, y^*, σ_y^*.

Theorem 2 *Fix any data matrix H, and suppose $\lambda_1 = 0$, $\mathcal{U}$ and $\mathcal{Y}$ are convex. Then,* `DeePC -Hybrid` (8) *and* `DeePC-SVD` (9) *have the same optimal solution u^*, y^*, σ_y^*, $\forall \lambda_2 > 0, \lambda_y > 0$.*

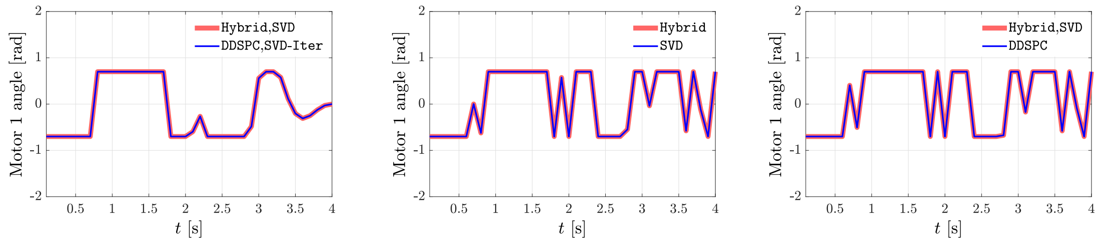
We establish Theorem 2 by expressing g and $\bar{g}$ in terms of u, y, σ_y. Then, using the SVD properties, we show that (8) and (9) become strictly convex optimization problems with the same objective function, decision variables, and feasible region. Note that Theorem 2 includes Theorem 1 of Zhang et al. (2023) as a special case, where Zhang et al. (2023) requires $\mathcal{U}$ and $\mathcal{Y}$ are convex polytopes that allow simple KKT conditions in their proof.

Finally, `DeePC-Hybrid`, `DeePC-SVD` and `Data-Driven-SPC` are also equivalent under certain conditions. This is summarized in Theorem 3 below.

Theorem 3 *Fix any data matrix H, and suppose $\lambda_1 = 0, \lambda_y > 0$ and $\mathcal{U}$ and $\mathcal{Y}$ are convex sets. If λ_2 is sufficiently large,* `DeePC-Hybrid` (8) *and* `DeePC-SVD` (9) *have the same unique optimal solution u^*, y^* and σ_y^* as* `Data-Driven-SPC` (10).

The key idea in the proof is to transform the regularizer $\lambda_2\|I - \Pi_1\|_2^2$ in (8) as the constraint $\|I - \Pi_1\|_2 = 0$ when λ_2 is sufficiently large via penalty arguments. Then, (8) and (10) have the same objective function and decision variables. The proof is completed by further establishing that they have the same feasible region. From Theorems 1 to 3, we conclude that `Data-Driven-SPC` (10), `DeePC-Hybrid` (8), `DeePC-SVD` (9) are equivalent to classical SPC (11) with noisy data when $\lambda_1 = 0, \lambda_y > 0$, λ_2 is sufficiently large and H_1 has full row rank. This is more general than Fiedler and Lucia (2021); Breschi et al. (2023): the equivalence for `DeePC-Hybrid` with regularizer $\|g\|_2^2$ and classical SPC is discussed in Fiedler and Lucia (2021), while `DeePC-Hybrid` and an approach similar to `Data-Driven-SPC` are proved to be equivalent in Breschi et al. (2023).

Note that the new variant `DeePC-SVD-Iter` involves an iterative algorithm to pre-process the noisy data (Algorithm 1), and thus it is non-trivial to formally establish its relationship with respect to other variants. Yet, our numerical experiments in Section 5 show that `DeePC-SVD-Iter` often has superior performance among all these convex approximations for noisy data.



(*a*) (*b*) (*c*)

Figure 2: Equivalent optimal solutions of different convex approximations in Theorems 2 and 3. (a) All methods with noise-free data. (b) `DeePC-Hybrid` and `DeePC-SVD` with $\lambda_1 = 0, \lambda_2 = 30$ and $\lambda_y = 100$. (c) `DeePC-Hybrid`, `DeePC-SVD` and `Data-Driven-SPC` with $\lambda_1 = 0, \lambda_2 = 10000$ and $\lambda_y = 100$.

5. Numerical Experiments

We perform numerical experiments to illustrate Theorems 2 and 3[1]. We also numerically investigate the effects of λ_1, λ_2, and confirm the superior performance of `DeePC-SDV-iter` (13). Additional numerical results on nonlinear systems are provided in our report (Shang and Zheng, 2023).

Experiment setup. We consider an LTI system from Fiedler and Lucia (2021). This is a triple-mass-spring system with $n = 8$ states, $m = 2$ inputs (two stepper motors), and $p = 3$ outputs (disc angles). In our experiments, the length of the pre-collected trajectory is $T = 200$, and the prediction horizon and the initial sequence are chosen as $N = 40$ and $T_{\text{ini}} = 4$, respectively. We choose $Q = I$, $R = 0.1I$ and $\mathcal{U} = [-0.7, 0.7]$.

Equivalence. We here numerically verify that the optimal solutions from different convex approximations are the same under appropriate settings. For noise-free pre-collected data (Proposition 1), all methods have the same optimal solution, and one solution instance is given in Figure 2(*a*). We next consider data collection with additive Gaussian measurement noises $\omega \sim \mathcal{N}(0, 0.01I)$. We choose $\lambda_1 = 0, \lambda_2 = 30$ and $\lambda_y = 100$ according to Theorem 2. One solution instance is shown in Figure 2(*b*), which shows that `DeePC-Hybrid` and `DeePC-SVD` provide the same optimal solution. Finally, for the equivalence of `Data-Driven-SPC`, `DeePC-Hybrid` and `DeePC-SVD`, we choose $\lambda_1 = 0$ and $\lambda_2 = 10000$ according to Theorem 3, and the results are shown in Figure 2(*c*).

Influence of λ_1 and λ_2. We then analyze the effect of hyperparameters λ_1 and λ_2. In particular, similar to Dörfler et al. (2022), we consider the realized control cost after applying the optimal control inputs from (8) to (10) and (13), which is computed as $\|u_{\text{opt}}\|_R^2 + \|y_{\text{true}}\|_Q^2$, where u_{opt} is the computed optimal control input and y_{true} is the realized trajectory after applying u_{opt}. We fixed $\lambda_y = 100$.

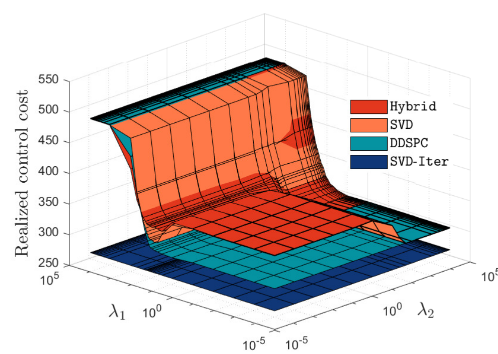


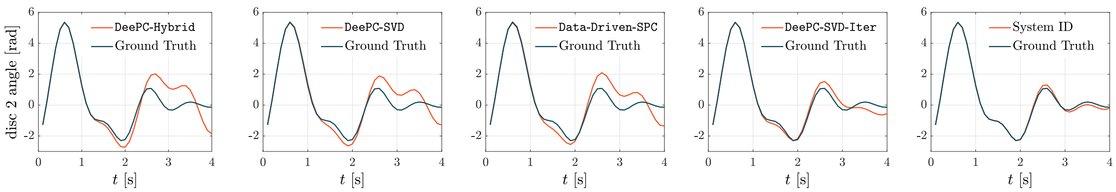
Figure 3: Realized control cost where sampling points are chosen from $\lambda_1, \lambda_2 \in [10^{-5}, 10^4]$ to capture the effect of hyperparameters.

Figure 3 shows the realized control performance over λ_1 and λ_2 for different convex approximations. The hyperparameters λ_1 and λ_2 indeed have a significant effect for `DeePC-Hybrid` (8), `DeePC-SVD` (9) and `Data-Driven-SPC` (10) (denoted as `DDSPC` in Figure 3). For these methods, λ_1 needs to be chosen more carefully, neither too large nor too small and λ_2 should not be chosen too small; similar phenomena also appeared

1. Our code is available at `https://github.com/soc-ucsd/Convex-Approximation-for-DeePC`.

Table 1: Realized Control Cost and Computational Time; GT denotes ground truth with noisy-free data.

	GT	`Hybrid`	`SVD`	`Data-Driven-SPC`	`SVD-Iter`	System ID
Realized Cost	277.25	388.42	370.20	365.08	288.17	279.86
Increasing Rate	N/A	40.1%	33.5%	31.7%	**3.9**%	0.9%
Compu. time [s]	N/A	0.133	0.131	0.097	0.104	N/A



(a) `Hybrid` (b) `SVD` (c) `DDSPC` (d) `SVD-Iter` (e) System ID

Figure 4: Open-loop trajectories (the angle of disc 2) of different methods. The blue trajectory and orange trajectories represent ground truth and different approximation methods, respectively.

in Dörfler et al. (2022). However, it is notable that `DeePC-SVD-Iter` (13) not only achieves the best performance but also is not very sensitive to λ_2 (note that $\lambda_1 = 0$ in (13)).

Comparison of `DeePC` variants. Finally, we compare the realized control cost and the computational time for different convex approximations. Motivated by Figure 3, we choose $\lambda_1 = \lambda_2 = 30$ and $\lambda_y = 100$. The performance of `DeePC` variants is related to the pre-collected trajectory. Thus, all presented realized control costs and computational time for different convex approximations are averaged over 100 pre-collected trajectories. The numerical results are listed in Table 1. The ground-truth cost is computed from (6) with noise-free data and we use the subspace approach N4SID proposed in Van Overschee and De Moor (1994) to compute the cost for System ID. From Table 1, we see that the realized control cost satisfies `DeePC-Hybrid` $>$ `DeePC-SVD` $>$ `Data-Driven-SPC` $>$ `DeePC-SVD-Iter` $>$ System ID. For the LTI system with noisy data, the inner problem in (7) forces the data-driven representation to be more structured, which enhances noise rejection performance for upper predictive control in (7). The increasing rate of realized cost for our new `DeePC-SVD-Iter` is 3.9%, which is much better than other `DeePC` variants.

Figure 4 shows one typical open-loop trajectory for all methods. In this case, the open-loop trajectories from (8) to (10) and (13) remain close to the ground truth up to 2 s. Then the trajectory is better aligned with the ground truth from Fig. 4(a) to Fig. 4(e) as the corresponding data-driven representation becomes more structured. Our numerical results also suggest that the indirect system ID approach is superior in the case of “variance” noise, consistent with Dörfler et al. (2022). In the report Shang and Zheng (2023), our numerical results on nonlinear systems further reveal that `DeePC-SVD-Iter` (13) also has enhanced performance in the case of “bias” errors.

6. Conclusion

In this paper, we have proposed a new bi-level formulation incorporating system ID techniques and predictive control. The existing `DeePC` (i.e., `DeePC-Hybrid` (8) and `DeePC-SVD` (9)) and also new variants (i.e., `Data-driven-SPC` (10) and `DeePC-SVD-Iter` (13)) can be considered as convex approximations of this bi-level formulation. We have further clarified their equivalence under appropriated settings (Theorems 1 to 3). Numerical simulations have validated our theoretical findings, and also revealed the superior performance of `DeePC-SVD-Iter` (13) with a more structured predictor. Interesting future directions include analyzing the effect of the length of pre-collected data, and investigating the closed-loop performance of different `DeePC` variants.

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References

Mohammad Alsalti, Ivan Markovsky, Victor G Lopez, and Matthias A Müller. Data-based system representations from irregularly measured data. *arXiv preprint arXiv:2307.11589*, 2023.

Karl Johan Åström and Peter Eykhoff. System identification—a survey. *Automatica*, 7(2):123–162, 1971.

Julian Berberich, Johannes Köhler, Matthias A Müller, and Frank Allgöwer. Data-driven model predictive control with stability and robustness guarantees. *IEEE Transactions on Automatic Control*, 66(4):1702–1717, 2020.

Valentina Breschi, Alessandro Chiuso, and Simone Formentin. Data-driven predictive control in a stochastic setting: A unified framework. *Automatica*, 152:110961, 2023.

Alessandro Chiuso and Gianluigi Pillonetto. System identification: A machine learning perspective. *Annual Review of Control, Robotics, and Autonomous Systems*, 2:281–304, 2019.

Jeremy Coulson, John Lygeros, and Florian Dörfler. Data-enabled predictive control: In the shallows of the deepc. In *2019 18th European Control Conference (ECC)*, pages 307–312. IEEE, 2019.

Florian Dörfler, Jeremy Coulson, and Ivan Markovsky. Bridging direct and indirect data-driven control formulations via regularizations and relaxations. *IEEE Transactions on Automatic Control*, 68(2):883–897, 2022.

Ezzat Elokda, Jeremy Coulson, Paul N Beuchat, John Lygeros, and Florian Dörfler. Data-enabled predictive control for quadcopters. *International Journal of Robust and Nonlinear Control*, 31 (18):8916–8936, 2021.

Wouter Favoreel, Bart De Moor, and Michel Gevers. Spc: Subspace predictive control. *IFAC Proceedings Volumes*, 32(2):4004–4009, 1999.

Felix Fiedler and Sergio Lucia. On the relationship between data-enabled predictive control and subspace predictive control. In *2021 European Control Conference (ECC)*, pages 222–229. IEEE, 2021.

Basil Kouvaritakis and Mark Cannon. Model predictive control. *Switzerland: Springer International Publishing*, 38, 2016.

Yingzhao Lian, Jicheng Shi, Manuel Koch, and Colin Neil Jones. Adaptive robust data-driven building control via bilevel reformulation: An experimental result. *IEEE Transactions on Control Systems Technology*, 2023.

Lennart Ljung. System identification. In *Signal analysis and prediction*, pages 163–173. Springer, 1998.

Ivan Markovsky. Structured low-rank approximation and its applications. *Automatica*, 44(4):891–909, 2008.

Ivan Markovsky. A missing data approach to data-driven filtering and control. *IEEE Transactions on Automatic Control*, 62(4):1972–1978, 2016.

Ivan Markovsky and Florian Dörfler. Behavioral systems theory in data-driven analysis, signal processing, and control. *Annual Reviews in Control*, 52:42–64, 2021.

Gianluigi Pillonetto, Francesco Dinuzzo, Tianshi Chen, Giuseppe De Nicolao, and Lennart Ljung. Kernel methods in system identification, machine learning and function estimation: A survey. *Automatica*, 50(3):657–682, 2014.

Xu Shang and Yang Zheng. Convex approximations for a bi-level formulation of data-enabled predictive control. *Technical report, available at arXiv preprint arXiv:2312.15431*, 2023.

Xu Shang, Jiawei Wang, and Yang Zheng. Smoothing mixed traffic with robust data-driven predictive control for connected and autonomous vehicles. *arXiv preprint arXiv:2310.00509*, 2023.

Peter Van Overschee and Bart De Moor. N4sid: Subspace algorithms for the identification of combined deterministic-stochastic systems. *Automatica*, 30(1):75–93, 1994.

Jiawei Wang, Yang Zheng, Keqiang Li, and Qing Xu. DeeP-LCC: Data-enabled predictive leading cruise control in mixed traffic flow. *IEEE Transactions on Control Systems Technology*, 31(6): 2760–2776, 2023.

Jan C. Willems. The behavioral approach to open and interconnected systems. *IEEE Control Systems Magazine*, 27(6):46–99, 2007. doi: 10.1109/MCS.2007.906923.

Jan C Willems, Paolo Rapisarda, Ivan Markovsky, and Bart LM De Moor. A note on persistency of excitation. *Systems & Control Letters*, 54(4):325–329, 2005.

Mingzhou Yin and Roy S Smith. On low-rank hankel matrix denoising. *IFAC-PapersOnLine*, 54 (7):198–203, 2021.

Mingzhou Yin, Andrea Iannelli, and Roy S Smith. Maximum likelihood estimation in data-driven modeling and control. *IEEE Transactions on Automatic Control*, 68(1):317–328, 2021.

Kaixiang Zhang, Yang Zheng, Chao Shang, and Zhaojian Li. Dimension reduction for efficient data-enabled predictive control. *IEEE Control Systems Letters*, 7:3277–3282, 2023.