

# Robust Cooperative Multi-Agent Reinforcement Learning: A Mean-Field Type Game Perspective

**Muhammad Aneeq uz Zaman**

*Coordinated Science Laboratory, University of Illinois at Urbana-Champaign*

MAZAMAN2@ILLINOIS.EDU

**Mathieu Laurière**

*Shanghai Frontiers Science Center of Artificial Intelligence and Deep Learning;  
NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai*

MATHIEU.LAURIERE@NYU.EDU

**Alec Koppel**

*Artificial Intelligence Research, JP Morgan Chase & Co.*

ALEC.KOPPEL@JPMCHASE.COM

**Tamer Başar**

*Coordinated Science Laboratory, University of Illinois at Urbana-Champaign*

BASAR1@ILLINOIS.EDU

**Editors:** A. Abate, M. Cannon, K. Margellos, A. Papachristodoulou

## Abstract

In this paper, we study the problem of robust cooperative multi-agent reinforcement learning (RL) where a large number of cooperative agents with distributed information aim to learn policies in the presence of *stochastic* and *non-stochastic* uncertainties whose distributions are respectively known and unknown. Focusing on policy optimization that accounts for both types of uncertainties, we formulate the problem in a worst-case (minimax) framework, which is intractable in general. Thus, we focus on the Linear Quadratic setting to derive benchmark solutions. First, since no standard theory exists for this problem due to the distributed information structure, we utilize the Mean-Field Type Game (MFTG) paradigm to establish guarantees on the solution quality in the sense of achieved Nash equilibrium of the MFTG. This in turn allows us to compare the performance against the corresponding original robust multi-agent control problem. Then, we propose a Receding-horizon Gradient Descent Ascent RL algorithm to find the MFTG Nash equilibrium and we prove a non-asymptotic rate of convergence. Finally, we provide numerical experiments to demonstrate the efficacy of our approach relative to a baseline algorithm.

## 1. Introduction

Reinforcement Learning (RL) has had many successes, such as autonomous driving (Sallab et al., 2017), robotics (Kober et al., 2013), and RL from human feedback (RLHF) (Ziegler et al., 2019), to name a few. These successes have been focused on single-agent scenarios, but many scenarios involving, e.g., financial markets, communication networks, distributed robotics involve multiple agents. Prevailing algorithms for Multi-Agent Reinforcement Learning (MARL) (Zhang et al., 2021b; Li et al., 2021), however, do not model the distinct effects of modeled and un-modeled uncertainties on the transition dynamics, which can result in practical instability in safety-critical applications (Riley et al., 2021).

In this paper we consider a large population multi-agent setting, with stochastic and non-stochastic (un-modeled, possibly adversarial) uncertainties. These types of formulations have been studied under the guise of robust control in the single-agent case (Başar and Bernhard, 2008). The uncertainties (modeled and un-modeled) affect the performance of the system and might even lead to instability. Robust control seeks the *robust* controller which guarantees a certain level of performance for the system in under a worst-case hypothesis on these uncertainties. We employ here the popular Linear-Quadratic (LQ) setting in order to rigorously characterize and synthesize the solution to the robust multi-agent problem in a data-driven manner. The LQ setting entails a class of models in which the dynamics are linear and the costs are quadratic in the state and the action of the agent. This setting has been used extensively in the literature due to its tractability: the optimal decisions can be computed analytically or almost analytically, up to solving Riccati equations, when one has access to all system matrices. Instances of applications include permanent income theory (Sargent and Ljungqvist, 2000), portfolio management (Cardaliaguet and Lehalle, 2018), and wireless power control (Huang et al., 2003), among many others. In the absence of knowledge

of system parameters, model-free RL methods have also been developed (Fazel et al., 2018; Malik et al., 2019) for single agent LQ settings. We refer to (Recht, 2019) for an overview. When one goes from single to multiple agents, the issue of communicating local state and control information among agents exhibit scalability problems, and in particular, practical algorithms require sharing state information that can scale exponential in the number of agents. Instead, here we consider a distributed information structure where each agent has access only to its own state and the average of states of the other agents. This distributed information structure causes the characterization of the solution to be very difficult, in that previous gradient dominance results from (Fazel et al., 2018) no longer hold. To overcome this difficulty, we utilize the mean-field game and control paradigm, first introduced in the purely non-cooperative agent setting in (Lasry and Lions, 2006; Huang et al., 2006), which replaces individual agents by a distribution over agent types, which enables characterization and computation of the solution. The approach has then been extended to the cooperative setting through the notion of mean field control (Bensoussan et al., 2013; Carmona and Delarue, 2018). Building on this paradigm, this work is the first to develop scalable algorithms for MARL that can handle model mis-specification or adversarial inputs in the sense of robust control in the very large or possibly infinite number of agents defined by the mean-field.

We start Section 2 by formulating a robust multi-agent control problem with stochastic and non-stochastic (un-modeled) noises. The agents have distributed information, such that they have access to their own states and the average behavior of all the agents. Solving this problem entails finding a *noise attenuation level* (noise-to-output gain) for the multi-agent system and the corresponding *robust controller*. As in the single-agent setting (Başar and Bernhard, 2008), the robust multi-agent control problem is reformulated into an equivalent zero-sum min-max game between the maximizing non-stochastic noise (which may be interpreted as an adversary) and the minimizing controller. Solving this problem is not possible in the finite agent case due to the limited information available to each agent. Thus, in Section 3 we consider the mean-field (infinite population) version of the problem, that we call the *Robust Mean-Field Control* (RMFC) problem. As in the finite-population setting, RMFC has an equivalent zero-sum min-max formulation, referred to as the 2-player *Zero-Sum Mean-Field Type Game* (ZS-MFTG) in (Carmona et al., 2020, 2021), where the controller is the minimizing player and the non-stochastic disturbance is the maximizing one.

In Section 4 we propose a bi-level RL algorithm to compute the Nash equilibrium for the ZS-MFTG (which equivalently yields the robust controller for the robust multi-agent problem) in the form of *Receding-horizon Gradient Descent Ascent* (RGDA) (Algorithm 1). The upper-level of RGDA, uses a receding-horizon approach, i.e., it finds the controller parameters starting from the last timestep  $T - 1$  and moving backwards-in-time (à la dynamic programming). The receding-horizon policy gradient approach was used in Kalman filtering (Zhang et al., 2023) and LQR problems (Zhang and Başar, 2023). The present work builds on this approach to multi-agent problems, which helps in simplifying the complex nature of the cost landscape (known to be non-coercive (Zhang et al., 2021a)) and renders it convex-concave. The lower-level employs gradient descent-ascent to find the saddle point (Nash equilibrium) for each timestep  $t$ . The convex-concave nature of the cost (due to the receding-horizon approach) proves to be a key component in proving linear convergence of the gradient descent-ascent to the saddle point (Theorem 4). Further analysis shows that the total accumulated error in the RGDA is small given that the lower level of RGDA has good convergence (Theorem 5). The gradient descent-ascent step requires computation of the stochastic gradient. We use a zero-order method (Fazel et al., 2018; Malik et al., 2019) which only requires access to the cost to compute stochastic gradients, and hence is *truly* model-free.

**Literature Review:** Robust control gained importance in the 1970s when control theorists realized the shortcomings of optimal control theory in dealing with model uncertainties (Athans et al., 1977; Harvey and Stein, 1978). The work of (Başar, 1989) was the first one to formulate the robust control problem as a zero-sum dynamic game between the controller and the uncertainty. Robust RL first introduced by (Morimoto and Doya, 2005) has recently had an increase in interest in for the single agent setting, where its ability to process trajectory data without explicit knowledge of system parameters can be used to learn robust controllers to address worst-case uncertainty (Zhang et al., 2020a; Kos and Song, 2017; Zhang et al., 2021c). Some recent works consider RL in scenarios with reward uncertainties (Zhang et al., 2020b), state uncertainty (He et al., 2023) or uncertainty in other agents’ policies (Sun et al., 2022). There have been some works on the intersection of RL for robust and multi-agent control (Li et al., 2019; He et al., 2023), yet there has not been any significant effort to provide (1) sufficient conditions for

solvability of the multi-agent robust control problem i.e. determining the noise attenuation level of a system and (2) provable Robust multi-agent RL (RMARL) algorithms in the large population setting, as proposed in this paper.

This is made possible due to the mean-field game and control paradigm, which considers the limiting case as the number of agents approaches infinity. This paradigm was first introduced in the context of non-cooperative game theory as Mean-Field Games (MFGs) concurrently by (Lasry and Lions, 2006; Huang et al., 2006). Since then, the question of learning equilibria in MFGs has gained momentum, see (Laurière et al., 2022b). In particular, there have been several works dealing with RL for MFGs (Guo et al., 2019; Elie et al., 2020; Perrin et al., 2020; Zaman et al., 2020; Xie et al., 2021; Anahtarci et al., 2023), deep RL for MFGs (Perrin et al., 2021; Cui and Koepl, 2021a; Laurière et al., 2022a), learning in multi-population MFGs (Pérolat et al., 2022; Zaman et al., 2021, 2023b), independent learning in MFGs (Yongacoglu et al., 2022; Yardim et al., 2023), oracle-free RL for MFGs (Angiuli et al., 2022; Zaman et al., 2023a) and RL for graphon games (Cui and Koepl, 2021b; Fabian et al., 2023). There have also been several works on RL for MFC, which is the cooperative counterpart, see e.g. (Carmona et al., 2019a,b; Gu et al., 2021; Mondal et al., 2022; Angiuli et al., 2022). But these works require ability to sample from the true transition model, and hence are inapplicable in the case of mis-specification or modeling errors. To address this setting, we introduce the Robust MFC problem. We will connect this problem to MFTGs Tembine (2017), which contain mixed cooperative-competitive elements. Zero-sum MFTG model a zero-sum competition between two infinitely large teams of agents. Prior work on the theoretical framework of zero-sum MFTG include (Choutri et al., 2019; Tembine, 2017; Cosso and Pham, 2019; Carmona et al., 2021; Guan et al., 2024). Related to RL, the works (Carmona et al., 2020, 2021) propose a data-driven RL algorithm based on Policy Gradient to compute the Nash equilibrium between the two coalitions in an LQ setting but do not provide a theoretical analysis of the algorithm.

## 2. Formulation

In this section we introduce the robust multi-agent control problem by first defining the dynamics of the multi-agent system along with its performance and noise indices. The performance and noise indices have been introduced in the literature (Başar and Bernhard, 2008) in order to quantify the affect of the accumulated noise (referred to as noise index) on the performance of the system (called the performance index). The noise attenuation level is then defined as an upper bound on the ratio between the performance and noise indices given that the agents employ a robust controller. Hence the robust multi-agent problem is that of finding the robust controller under which a certain noise attenuation is achieved. In order to solve this problem, we reformulate it as a min-max game problem as in the single-agent setting (Başar and Bernhard, 2008). Consider an  $N$  agent system. We let  $[N] = \{1, \dots, N\}$ . The  $i^{th}$  agent has dynamics which are linear in its state  $x_t^i \in \mathbb{R}^m$ , its action  $u_t^{1,i} \in \mathbb{R}^p$ , and the mean-field counterparts,  $\bar{x}_t$  and  $\bar{u}_t^i$ . The disturbance  $u_t^{2,i}$  is referred to as *non-stochastic noise*<sup>1</sup> since it is an un-modeled disturbance and can even be adversarial. This is similar in spirit to the works of (Simchowitz et al., 2020). Let  $T$  be a positive integer, interpreted as the horizon of the problem. The initial condition of agent  $i$ 's state,  $i \in [N]$ , is  $x_0^i = \omega^{0,i} + \bar{\omega}^0$ , where  $\omega^{0,i} \sim \mathcal{N}(0, \Sigma^0)$  and  $\bar{\omega}^0 \sim \mathcal{N}(0, \bar{\Sigma}^0)$  are i.i.d. noises. For  $t \in \{0, \dots, T-1\}$ ,

$$x_{t+1}^i = A_t x_t^i + \bar{A}_t \bar{x}_t + B_t u_t^{1,i} + \bar{B}_t \bar{u}_t^1 + u_t^{2,i} + \bar{u}_t^2 + \omega_t^i + \bar{\omega}_t, \forall i \in [N] \quad (1)$$

where  $u_t^{1,i}$  is the control action of the  $i^{th}$  agent,  $\bar{x}_t := \sum_{i=1}^N x_t^i / N$  is referred to as the state mean-field and  $\bar{u}_t^j := \sum_{i=1}^N u_t^{j,i} / N$  for  $j \in \{1, 2\}$  are the control and noise mean-fields respectively. Each agent's dynamics are perturbed by two types of noise:  $\omega_t^i$  and  $\bar{\omega}_t$  are referred to as stochastic noises since they are i.i.d. and their distributions are known ( $\omega_t^i \sim \mathcal{N}(0, \Sigma)$  and  $\bar{\omega}_t \sim \mathcal{N}(0, \bar{\Sigma})$ ). All of our results (excluding the finite-sample analysis of the RL Algorithm) can be readily generalized for zero-mean non-Gaussian disturbances with finite variance.

---

1. The non-stochastic noise is assumed to have identity coefficient in the dynamics (1) for simplicity of analysis but can be easily changed to some other matrix of appropriate size.

In order to define the robust control problem we define the *performance index* of the population which penalizes the deviation of the agents from their (state and control) mean-fields and also regulates the mean-fields:

$$J_N(u^1, u^2) = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \sum_{t=0}^{T-1} \left[ \|x_t^i - \bar{x}_t\|_{Q_t}^2 + \|\bar{x}_t\|_{\bar{Q}_t}^2 + \|u_t^{1,i} - \bar{u}_t^1\|^2 + \|\bar{u}_t^1\|^2 \right] + \|x_T^i - \bar{x}_T\|_{Q_T}^2 + \|\bar{x}_T\|_{\bar{Q}_T}^2 \quad (2)$$

where the matrices  $Q_t, \bar{Q}_t > 0$  are symmetric matrices,  $u^j = (u^{j,i})_{i \in [N]}$  where each  $u^{j,i}$  for  $j \in \{1, 2\}$  is adapted to the distribution information structure i.e.  $\sigma$ -algebra generated by  $x_t^i$  and  $\bar{x}_t$  and  $\mathcal{U}^1, \mathcal{U}^2$  represent the set of all possible  $u^1, u^2$ , respectively. We define the *noise index* of the population in a similar manner

$$\varpi_N(u^1, u^2) = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \sum_{t=0}^{T-1} \left[ \|u_t^{2,i} - \bar{u}_t^2\|^2 + \|\bar{u}_t^2\|^2 + \|\omega_t^i\|^2 + \|\bar{\omega}_t\|^2 \right]. \quad (3)$$

The robust control problem for this  $N$  agent system is that of finding the range of noise attenuation levels  $\gamma > 0$  such that:

$$\exists u^1 \in \mathcal{U}^1, \forall u^2 \in \mathcal{U}^2, \quad J_N(u^1, u^2) \leq \gamma^2 \varpi_N(u^1, u^2) \quad (4)$$

Any  $\gamma$  for which the above inequality is satisfied is referred to as a *viable attenuation level* and the least among them is called the *minimum attenuation level*. The controller  $u^1$  which ensures a particular level  $\gamma$  of noise attenuation is referred to as the *robust controller* corresponding to  $\gamma$  (or robust controller in short). Since the inequality (4) can also be reformulated as  $J_N(\cdot)/\varpi_N(\cdot) \leq \gamma^2$ , a viable attenuation parameter  $\gamma^2$  is also an upper bound on the noise-to-output gain of the system. As outlined in (Başar and Bernhard, 2008) for a single agent problem the condition (4) is equivalent to finding the range of value of  $\gamma > 0$  such that

$$\inf_{u^1} \sup_{u^2} (J_N(u^1, u^2) - \gamma^2 \varpi_N(u^1, u^2)) \leq 0, \quad (5)$$

where the infimizing controller  $u^1$  is the robust controller and the supremizing controller  $u^2$  is the worst-case non-stochastic noise. If we define the robust  $N$  agent cost  $J_N^\gamma$  as follows

$$J_N^\gamma(u^1, u^2) = J_N(u^1, u^2) - \gamma^2 \mathbb{E} \frac{1}{N} \sum_{i=1}^N \sum_{t=0}^{T-1} (\|u_t^{2,i} - \bar{u}_t^2\|^2 + \|\bar{u}_t^2\|^2),$$

then using (2) and (3), the robust  $N$  agent control problem (5) can be equivalently written as

$$\inf_{u^1} \sup_{u^2} J_N^\gamma(u^1, u^2) - \gamma^2 \mathbb{E} \frac{1}{N} \sum_{i=1}^N \sum_{t=0}^{T-1} (\|\omega_t^i\|^2 + \|\bar{\omega}_t\|^2) \leq 0. \quad (6)$$

Due to the distributed information structure of the agents the standard theory of single-agent robust control does not apply in this setting. Hence we are unable to provide sufficient conditions for a given  $\gamma > 0$  to be a viable attenuation level, and we resort to the mean-field limit as  $N \rightarrow \infty$ , which is of independent interest. The next section formulates the Robust Mean-Field Control (RMFC) problem and its equivalent 2-player zero-sum Mean-Field Type Game (ZS-MFTG) representation, and provides sufficient conditions for solvability of both.

### 3. Robust Mean-Field Control

Consider a system with infinitely many agents, where the generic agent has linear dynamics of its state  $x_t$  for a finite-horizon  $t \in \{0, \dots, T-1\}$ :

$$x_{t+1} = A_t x_t + \bar{A}_t \bar{x}_t + B_t u_t^1 + \bar{B}_t \bar{u}_t^1 + u_t^2 + \bar{u}_t^2 + \omega_t + \bar{\omega}_t, \quad (7)$$

where  $u_t^1$  is the control action of the generic agent,  $\bar{x}_t := \mathbb{E}[x_t | (\bar{\omega}_s)_{0 \leq s \leq t-1}]$  is referred to as the state mean-field and  $\bar{u}_t^j := \mathbb{E}[u_t^j | (\bar{\omega}_s)_{0 \leq s \leq t-1}]$  for  $j \in \{1, 2\}$  are the control and noise mean-fields respectively. The initial condition of the generic agent is  $x_0 = \omega^0 + \bar{\omega}^0$ , where  $\omega^0 \sim \mathcal{N}(0, \Sigma^0)$  and  $\bar{\omega}^0 \sim \mathcal{N}(0, \bar{\Sigma}^0)$  are i.i.d. noises. The stochastic noises  $\omega_t^i$  and  $\bar{\omega}_t$  are i.i.d. such that  $\omega_t^i \sim \mathcal{N}(0, \Sigma)$  and  $\bar{\omega}_t \sim \mathcal{N}(0, \bar{\Sigma})$ , whereas the non-stochastic noise  $u_t^2$  are un-modeled uncertainties. Similar to the  $N$  agent case, we define the robust mean-field cost  $J^\gamma$  as follows

$$J^\gamma(u^1, u^2) = \mathbb{E} \sum_{t=0}^{T-1} \left[ \|x_t - \bar{x}_t\|_{Q_t}^2 + \|\bar{x}_t\|_{\bar{Q}_t}^2 + \|u_t^1 - \bar{u}_t^1\|^2 + \|\bar{u}_t^1\|^2 - \gamma^2 (\|u_t^2 - \bar{u}_t^2\|^2 + \|\bar{u}_t^2\|^2) \right. \\ \left. + \|x_T - \bar{x}_T\|_{Q_T}^2 + \|\bar{x}_T\|_{\bar{Q}_T}^2 \right]. \quad (8)$$

Now the robust mean-field control problem which is the mean-field analog to (6) is defined as follows.

**Definition 1 (Robust Mean-Field Control problem)** *If for a given  $\gamma > 0$  the following inequality is satisfied, then  $\gamma$  is a viable noise attenuation level for the robust mean-field control problem.*

$$\inf_{u^1} \sup_{u^2} J^\gamma(u^1, u^2) - \gamma^2 \mathbb{E} \sum_{t=0}^{T-1} \|\omega_t\|^2 + \|\bar{\omega}_t\|^2 \leq 0. \quad (9)$$

Moreover, the infimizing controller  $u^1$  in (9) is a robust controller (corresponding to  $\gamma$ ).

Now, under the condition of interchangability of the inf and sup operations, the problem of finding  $\inf_{u^1} \sup_{u^2} J^\gamma(u^1, u^2)$  is that of finding the Nash equilibrium (equivalently, saddle point, in this case) of the *Zero-sum 2-player Mean-Field Type Game*; see (Carmona et al., 2020, 2021) for a very similar LQ setting without the theoretical analysis of the RL algorithm. In the following section we provide sufficient conditions for existence and uniqueness of a solution to this saddle point problem along with the value of  $\inf_{u^1} \sup_{u^2} J^\gamma(u^1, u^2)$ .

**2-player Zero-sum Mean-Field Type Games:** Let us define  $y_t = x_t - \bar{x}_t, z_t = \bar{x}_t$ . The dynamics of  $y_t$  and  $z_t$  can be written as

$$y_{t+1} = A_t y_t + B_t (u_t^1 - \bar{u}_t^1) + u_t^2 - \bar{u}_t^2 + \omega_t - \bar{\omega}_t, \quad z_{t+1} = \tilde{A}_t z_t + \tilde{B}_t \bar{u}_t^1 + 2\bar{u}_t^2 + 2\bar{\omega}_t,$$

where  $\tilde{A}_t = A_t + \bar{A}_t$  and  $\tilde{B}_t = B_t + \bar{B}_t$ . The optimal controls are known to be linear (Carmona et al., 2020), hence we restrict our attention the set of linear controls in  $y_t$  and  $z_t$ ,

$$u_t^1 = u_t^1(x_t, \bar{x}_t) = -K_t^1(x_t - \bar{x}_t) - L_t^1 \bar{x}_t, \quad u_t^2 = u_t^2(x_t, \bar{x}_t) = K_t^2(x_t - \bar{x}_t) + L_t^2 \bar{x}_t$$

which implies that  $\bar{u}_t^1 = -L_t^1 \bar{x}_t$  and  $\bar{u}_t^2 = L_t^2 \bar{x}_t$ . The dynamics of the processes  $y_t$  and  $z_t$  can be re-written as

$$y_{t+1} = (A_t - B_t K_t^1 + K_t^2) y_t + \omega_t - \bar{\omega}_t, \quad z_{t+1} = (\tilde{A}_t - \tilde{B}_t L_t^1 + L_t^2) z_t + 2\bar{\omega}_t. \quad (10)$$

Since the dynamics of  $y_t$  and  $z_t$  are decoupled, we can decompose the cost  $J^\gamma$  into the following two parts:

$$J^\gamma(K, L) = J_y^\gamma(K) + J_z^\gamma(L), \\ J_y^\gamma(K) = \mathbb{E} \left[ \sum_{t=0}^{T-1} y_t^\top (Q_t + (K_t^1)^\top K_t^1 - \gamma^2 (K_t^2)^\top K_t^2) y_t + y_T^\top Q_T y_T \right], \\ J_z^\gamma(L) = \mathbb{E} \left[ \sum_{t=0}^{T-1} z_t^\top (\bar{Q}_t + (L_t^1)^\top L_t^1 - \gamma^2 (L_t^2)^\top L_t^2) z_t + z_T^\top \bar{Q}_T z_T \right]. \quad (11)$$

The 2-player MFTG (7)-(8) has been decoupled into two 2-player LQ dynamic game problems as shown below:

$$\min_{K^1, L^1} \max_{K^2, L^2} J^\gamma((K^1, K^2), (L^1, L^2)) = \min_{K^1} \max_{K^2} J_y^\gamma(K) + \min_{L^1} \max_{L^2} J_z^\gamma(L)$$

where the dynamics of  $y_t$  and  $z_t$  are defined in (10). In the following section, using results in the literature, we specify the sufficient conditions for existence and uniqueness of Nash equilibrium of the 2-player MFTG and also present the *value* (Nash cost) of the game. Building on the techniques developed in (Başar and Olsder, 1998; Carmona et al., 2020), we can prove the following result.

**Theorem 2** Assume for a given  $\gamma > 0$ ,

$$\gamma^2 I - M_t^\gamma > 0 \text{ and } \gamma^2 I - \bar{M}_t^\gamma > 0, \quad (12)$$

where  $M_t^\gamma$  and  $\bar{M}_t^\gamma$  are positive semi-definite matrices which satisfy the Coupled Algebraic Riccati equations,

$$\begin{aligned} M_t^\gamma &= Q_t + A_t^\top M_{t+1}^\gamma \Lambda_t^{-1} A_t, \quad \Lambda_t = I + (B_t B_t^\top - \gamma^{-2} I) M_{t+1}^\gamma, \quad M_T^\gamma = Q_T, \\ \bar{M}_t^\gamma &= \bar{Q}_t + \bar{A}_t^\top \bar{M}_{t+1}^\gamma \bar{\Lambda}_t^{-1} \bar{A}_t, \quad \bar{\Lambda}_t = I + (\bar{B}_t \bar{B}_t^\top - \gamma^{-2} I) \bar{M}_{t+1}^\gamma, \quad \bar{M}_T^\gamma = \bar{Q}_T \\ N_t^\gamma &= N_{t+1}^\gamma + \text{Tr}(M_{t+1}^\gamma \Sigma), \quad N_T^\gamma = 0, \quad \bar{N}_t^\gamma = \bar{N}_{t+1}^\gamma + \text{Tr}(\bar{M}_{t+1}^\gamma \bar{\Sigma}), \quad \bar{N}_T^\gamma = 0. \end{aligned} \quad (13)$$

Then,  $u_t^{1*} = -K_t^{1*}(x_t - \bar{x}_t) - L_t^{1*} \bar{x}_t$  and  $u_t^{2*} = K_t^{2*}(x_t - \bar{x}_t) + L_t^{2*} \bar{x}_t$  (complete expressions provided in Supplementary Materials) are the unique Nash policies. Furthermore, the Nash equilibrium (equivalently, saddle point) value is

$$\inf_{u^1} \sup_{u^2} J^\gamma(u^1, u^2) = \text{Tr}(M_0^\gamma \Sigma^0) + \text{Tr}(\bar{M}_0^\gamma \bar{\Sigma}^0) + N_0^\gamma + \bar{N}_0^\gamma \quad (14)$$

This result can be proved using techniques in proofs of Theorem 3.2 in (Başar and Bernhard, 2008) or Proposition 36 in (Carmona et al., 2021). We now use the Nash value of the game (14) to come up with a condition for the attenuation level  $\gamma$  which solves the robust mean-field control problem (9). First we simplify expression in (9)  $\mathbb{E} \sum_{t=0}^{T-1} \|\omega_t\|^2 + \|\bar{\omega}_t\|^2 = T \text{Tr}(\Sigma + \bar{\Sigma})$  using the i.i.d. stochastic nature of the noise. Combining this fact with (14), we arrive at the conclusion that (9) will be satisfied if and only if

$$\sum_{t=1}^T \text{Tr}((M_t^\gamma - \gamma^2 I) \Sigma + (\bar{M}_t^\gamma - \gamma^2 I) \bar{\Sigma}) + \text{Tr}(M_0^\gamma \Sigma^0) + \text{Tr}(\bar{M}_0^\gamma \bar{\Sigma}^0) \leq 0 \quad (15)$$

Notice that the conditions (12) and (15) are different, as the first one requires positive definiteness of matrices and the second one requires a scalar inequality. Now we solve the robust  $N$  agent control problem by providing sufficient conditions for a given attenuation level  $\gamma$  satisfying (4).

**Theorem 3** Let  $\gamma > 0$ . Assume, in addition to (12), that we also have

$$\sum_{t=1}^T \text{Tr}((M_t^\gamma - \gamma^2 I) \Sigma + (\bar{M}_t^\gamma - \gamma^2 I) \bar{\Sigma}) + \text{Tr}(M_0^\gamma \Sigma^0) + \text{Tr}(\bar{M}_0^\gamma \bar{\Sigma}^0) \leq -\frac{CT}{N}, \quad (16)$$

where  $C$  is a constant which depends only on the model parameters and  $M_t^\gamma$  and  $\bar{M}_t^\gamma$  (13). Then  $\gamma$  is a viable attenuation level for the Robust  $N$  agent control problem (4). Moreover the robust controller for each agent  $i$  is given by  $u_t^{1,i*} = -K_t^{1*}(x_t^i - \bar{x}_t) - L_t^{1*} \bar{x}_t$ .

The proof of this result can be found in the full version of this paper (Zaman et al., 2024). The above theorem states that, if for a given  $\gamma$ , conditions (12) and (16) are satisfied (given that  $M_t^\gamma$  and  $\bar{M}_t^\gamma$  are defined by (13)), then not only is  $\gamma$  a viable attenuation level for the original Robust multi-agent control problem (1)-(4), but the Nash equilibrium for the ZS-MFTG also yields the robust controller  $u_t^{1,i*} = -K_t^{1*}(x_t^i - \bar{x}_t) - L_t^{1*} \bar{x}_t$  for the original finite-agent game. Condition (16) is strictly stronger than condition (15) but approaches (16) as  $N \rightarrow \infty$ .

## 4. Reinforcement Learning for Robust Mean-Field Control

In this section we present the Receding-horizon policy Gradient Descent Ascent (RGDA) algorithm to compute the Nash equilibrium (Theorem 2) of the 2-player MFTG (7)-(8), which will also generate the robust controller for a fixed noise attenuation level  $\gamma$ . For this section we assume access to only the finite-horizon costs of the agents under a set of control policies, and not the state trajectories. Under this setting the model of the agents cannot be constructed hence our approach is *truly* model free (Malik et al., 2019). Due to the non-convex non-concave (also non-coercive (Zhang et al., 2020b)) nature of the cost function  $J^\gamma$  in (11), instead we solve the receding-horizon

problem, for each  $t = \{T-1, \dots, 1, 0\}$  backwards-in-time. This entails solving  $2 \times T$  min-max problems, where each problem is convex-concave and aims at finding  $(K_t, L_t) = ((K_t^1, K_t^2), (L_t^1, L_t^2))$  at time step  $t$ , given the set of *future* controllers (controllers for times greater than  $t$ ),  $((\tilde{K}_{t+1}, \tilde{L}_{t+1}), \dots, (\tilde{K}_T, \tilde{L}_T))$  are held constant. But first we must approximate the mean-field term using finitely many agents.

**Approximation of mean-field terms using  $M$  agents:** Since simulating infinitely many agents is impractical, in this section we outline how to use a set of  $2 \leq M < \infty$  agents to approximately simulate the mean-field in a MFTG. Each of the  $M$  agents has state  $x_t^i$  at time  $t$  where  $i \in [M]$ . The agents follow controllers linear in their private state and empirical mean-field,  $x_t^i$  and  $\tilde{z}_t$ , respectively,  $u_t^1 = -K_t^1(x_t^i - \tilde{z}_t) - L_t^1 \tilde{z}_t$ ,  $u_t^2 = K_t^2(x_t^i - \tilde{z}_t) + L_t^2 \tilde{z}_t$ , where the empirical mean-field is  $\tilde{z}_t := \frac{1}{M} \sum_{i=1}^M x_t^i$ . Under these control laws, the dynamics of agent  $i \in [M]$  are

$$x_{t+1}^i = (A_t - B_t K_t^1 + K_t^2)(x_t^i - \tilde{z}_t) + (\tilde{A}_t - \tilde{B}_t L_t^1 + L_t^2) \tilde{z}_t + \omega_{t+1}^i + \bar{\omega}_t$$

and the dynamics of the empirical mean-field  $\tilde{z}_t$  is  $\tilde{z}_{t+1} = (\tilde{A}_t - \tilde{B}_t L_t^1 + L_t^2) \tilde{z}_t + \tilde{\omega}_{t+1}^0$ , where  $\tilde{\omega}_{t+1}^0 = \bar{\omega}_t + \frac{1}{M} \sum_{i=1}^M \omega_{t+1}^i$ . The cost of each agent is

$$\begin{aligned} \tilde{J}^{i,\gamma}(u_1, u_2) = & \mathbb{E} \left[ \sum_{t=0}^{T-1} (x_t^i - \tilde{z}_t)^\top [Q_t + (K_t^1)^\top K_t^1 - \gamma^2 (K_t^2)^\top K_t^2] (x_t^i - \tilde{z}_t) + (x_T^i - \tilde{z}_T)^\top Q_T (x_T^i - \tilde{z}_T) \right. \\ & \left. + \tilde{z}_t^\top [\bar{Q}_t + (L_t^1)^\top L_t^1 - \gamma^2 (L_t^2)^\top L_t^2] \tilde{z}_t + \tilde{z}_T^\top \bar{Q}_T \tilde{z}_T \right]. \end{aligned}$$

Now, similarly to the previous section, we define  $y_t^i = x_t^i - \tilde{z}_t$ . The dynamics of  $y_t^i$  are  $y_{t+1}^i = (A_t - B_t K_t^1 + K_t^2) y_t^i + \tilde{\omega}_{t+1}^i$ , where  $\tilde{\omega}_{t+1}^i = \frac{M-1}{M} \omega_{t+1}^i - \frac{1}{M} \sum_{j \neq i} \omega_{t+1}^j$ . The cost can then be decomposed in a manner similar to (11):

$$\begin{aligned} \tilde{J}^{i,\gamma}((K_t^1, K_t^2), (L_t^1, L_t^2)) &= \tilde{J}_y^{i,\gamma}(K_t^1, K_t^2) + \tilde{J}_z^{i,\gamma}(L_t^1, L_t^2), \\ \tilde{J}_y^{i,\gamma}(K_t^1, K_t^2) &= \mathbb{E} \left[ \sum_{t=0}^{T-1} (y_t^i)^\top [Q_t + (K_t^1)^\top K_t^1 - \gamma^2 (K_t^2)^\top K_t^2] y_t^i + (y_T^i)^\top Q_T y_T^i \right], \\ \tilde{J}_z^{i,\gamma}(L_t^1, L_t^2) &= \mathbb{E} \left[ \sum_{t=0}^{T-1} \tilde{z}_t^\top [\bar{Q}_t + (L_t^1)^\top L_t^1 - \gamma^2 (L_t^2)^\top L_t^2] \tilde{z}_t + \tilde{z}_T^\top \bar{Q}_T \tilde{z}_T \right]. \end{aligned} \quad (17)$$

**Receding-horizon approach:** Similar to the approach in Section 2, instead of finding the optimal,  $K^*$  and  $L^*$  which optimizes  $\tilde{J}$  in (17), we solve the receding-horizon problem for each  $t = \{T-1, \dots, 1, 0\}$  backwards-in-time. This forms two decoupled min-max convex-concave problems of finding  $(K_t, L_t) = ((K_t^1, K_t^2), (L_t^1, L_t^2))$  at each time step  $t$ , given the set of controllers for times greater than  $t$ ,  $((\tilde{K}_{t+1}, \tilde{L}_{t+1}), \dots, (\tilde{K}_T, \tilde{L}_T))$

$$\begin{aligned} \min_{(K_t^1, L_t^1)} \max_{(K_t^2, L_t^2)} \tilde{J}_t^{i,\gamma}(K_t, L_t) &= \\ \mathbb{E} \left[ \underbrace{y_t^\top (Q_t + (K_t^1)^\top K_t^1 - \gamma^2 (K_t^2)^\top K_t^2) y_t + \sum_{k=t+1}^T y_k^\top (Q_t + (\tilde{K}_k^1)^\top \tilde{K}_k^1 - \gamma^2 (\tilde{K}_k^2)^\top \tilde{K}_k^2) y_k}_{\tilde{J}_{y,t}^{i,\gamma}} \right. & (18) \\ \left. + \mathbb{E} \left[ \underbrace{z_t^\top (\bar{Q}_t + (L_t^1)^\top L_t^1 - \gamma^2 (L_t^2)^\top L_t^2) z_t + \sum_{k=t+1}^T z_k^\top (\bar{Q}_t + \tilde{L}_{1,k}^\top \tilde{L}_k^1 - \gamma^2 (\tilde{L}_k^2)^\top \tilde{L}_k^2) z_k}_{\tilde{J}_{z,t}^{i,\gamma}} \right] \right], \end{aligned}$$

for any  $i \in [M]$  and  $y_t \sim \mathcal{N}(0, \Sigma_y)$ ,  $z_t \sim \mathcal{N}(0, \Sigma_z)$ . This receding-horizon problem is solved using Receding-horizon policy Gradient Descent Ascent (RGDA) (Algorithm 1) where at each time instant  $t$  the Nash control is approached using gradient descent ascent. We anticipate a small approximation error between the optimal controller

and its computed approximation  $\tilde{K}_t$  (respectively  $\tilde{L}_t$ ). However, this error is shown to be well-behaved (Theorem 5), as we progress backwards-in-time, given that the hyper-parameters of RGDA satisfy certain bounds.

**Receding-horizon policy Gradient Descent Ascent (RGDA) Algorithm:** The RGDA Algorithm (Algorithm 1) is a bi-level optimization algorithm where the outer loop starts at time  $t = T - 1$  and moves backwards-in-time, and the inner loop is a gradient descent (for control parameters  $(K_t^1, L_t^1)$ ) ascent (for control policy  $(K_t^2, L_t^2)$ ) update with learning rate  $\eta_k$ . The gradient descent ascent step entails computing an approximation of the *exact* gradients of cost  $\tilde{J}_t^{i,\gamma}$  with respect to the controls variables  $(K_t^1, L_t^1), (K_t^2, L_t^2)$ . To obtain this approximation in a data driven manner we utilize a zero-order stochastic gradient  $\tilde{\nabla}_1 \tilde{J}_t^{i,\gamma}(K_t, L_t), \tilde{\nabla}_2 \tilde{J}_t^{i,\gamma}(K_t, L_t)$  (Fazel et al., 2018; Malik et al., 2019) which requires cost computation under a given set of controllers (18) as shown below.

$$\begin{aligned}\tilde{\nabla}_1 \tilde{J}_t^{i,\gamma}(K_t, L_t) &= \frac{n}{Mr^2} \sum_{j=1}^M \tilde{J}_t^{i,\gamma}((K_t^{j,1}, K_t^2), (L_t^{j,1}, L_t^2)) e_j, \quad \begin{pmatrix} K_t^{j,1} \\ L_t^{j,1} \end{pmatrix} = \begin{pmatrix} K_t^1 \\ L_t^1 \end{pmatrix} + e_j, \quad e_j \sim \mathbb{S}^{n-1}(r) \\ \tilde{\nabla}_2 \tilde{J}_t^{i,\gamma}(K_t, L_t) &= \frac{n}{Mr^2} \sum_{j=1}^M \tilde{J}_t^{i,\gamma}((K_t^1, K_t^{j,2}), (L_t^1, L_t^{j,2})) e_j, \quad \begin{pmatrix} K_t^{j,2} \\ L_t^{j,2} \end{pmatrix} = \begin{pmatrix} K_t^2 \\ L_t^2 \end{pmatrix} + e_j, \quad e_j \sim \mathbb{S}^{n-1}(r).\end{aligned}$$

Stochastic gradient computation entails computing the cost of  $N_b$  different *perturbed* controllers, with a perturbation magnitude if  $r$  also called the *smoothing radius*. This stochastic gradient provides us with a *biased* approximation of the exact gradient whose *bias* and *variance* can be controlled by tuning the values of  $N_b$  and  $r$ . Finally to ensure stability of the learning algorithm, we use projection  $\text{Proj}_D$  onto a  $D$ -ball such that the norm of the matrices is bounded by  $D$ ,  $\|(K_t, L_t)\|^2 \leq D$ . The radius of the ball  $D$  is chosen such that the Nash equilibrium controllers lie within this ball.

---

**Algorithm 1** RGDA Algorithm for 2-player MFTG
 

---

- 1: **for**  $t = T - 1, \dots, 1, 0$ , **do**
  - 2:   Initialize  $K_t = (K_t^1, K_t^2) = 0, L_t = (L_t^1, L_t^2) = 0$
  - 3:   **for**  $k = 0, \dots, K$  **do**
  - 4:     **Gradient Descent**  $\begin{pmatrix} K_t^1 \\ L_t^1 \end{pmatrix} \leftarrow \text{Proj}_D \left( \begin{pmatrix} K_t^1 \\ L_t^1 \end{pmatrix} - \eta_k \tilde{\nabla}_1 \tilde{J}_t^{i,\gamma}(K_t, L_t) \right)$ ,
  - 5:     **Gradient Ascent**  $\begin{pmatrix} K_t^2 \\ L_t^2 \end{pmatrix} \leftarrow \text{Proj}_D \left( \begin{pmatrix} K_t^2 \\ L_t^2 \end{pmatrix} + \eta_k \tilde{\nabla}_2 \tilde{J}_t^{i,\gamma}(K_t, L_t) \right)$ ,
  - 6:   **end for**
  - 7: **end for**
- 

**RGDA algorithm analysis:** In this section we start by showing linear convergence of the inner loop gradient descent ascent (Theorem 4), which is made possible by the convex-concave property of the cost function under the receding horizon approach (18). Then we show that if the error accumulated in each inner loop computation is small enough, the total accumulated error is well behaved (Theorem 5).

We first define some relevant notation. We define the *joint controllers* for each timestep  $t$  as  $\bar{K}_t = [(K_t^1)^\top, (K_t^2)^\top]^\top$  and  $\bar{L}_t = [(L_t^1)^\top, (L_t^2)^\top]^\top$ , for the sake of conciseness. For each timestep  $t \in \{T - 1, \dots, 1, 0\}$  let us also define the *target* joint controllers  $\tilde{K}_t^* = (\tilde{K}_t^{1*}, \tilde{K}_t^{2*}), \tilde{L}_t^* = (\tilde{L}_t^{1*}, \tilde{L}_t^{2*})$ , as the set of policies which exactly solve the receding-horizon min-max problem (18). Notice that the set of target controllers  $\tilde{K}_t^*, \tilde{L}_t^*$  are unique (due to convex-concave nature of (18)) but do depend on the set of future joint controllers  $(\bar{K}_s, \bar{L}_s)_{t < s < T}$ . On the other hand, the Nash joint controllers are denoted by  $\bar{K}_t^* = (K_t^{1*}, K_t^{2*})$  and  $\bar{L}_t^* = (L_t^{1*}, L_t^{2*})$ . Furthermore, the target joint controllers are equal to the Nash joint controllers  $(\tilde{K}_t^*, \tilde{L}_t^*) = (\bar{K}_t^*, \bar{L}_t^*)$  only if the future joint controllers are also Nash  $(\bar{K}_s, \bar{L}_s)_{t < s < T} = (\bar{K}_s^*, \bar{L}_s^*)_{t < s < T}$ .

**Theorem 4** *If the learning rate  $\eta_k$  is smaller than a certain function of model parameters, the number of inner loop iterations  $K = \Omega(\log(1/\epsilon))$ , the mini-batch size  $N_b = \Omega(1/\epsilon)$  and the smoothing radius  $r = \mathcal{O}(\epsilon)$ , then at each timestep  $t \in \{T - 1, \dots, 1, 0\}$  the optimality gaps are  $\|\bar{K}_t - \tilde{K}_t^*\|_2^2 \leq \epsilon$  and  $\|\bar{L}_t - \tilde{L}_t^*\|_2^2 \leq \epsilon$ .*



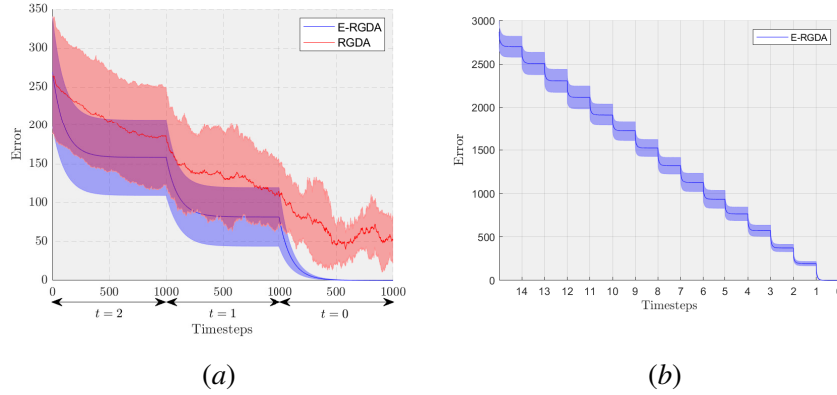


Figure 1: Performance of RGDA Algorithm

Closed form expressions of the bounds can be found in the proof given in the full version of the paper (Zaman et al., 2024). The linear rate of convergence is made possible by building upon the convergence analysis of descent ascent in (Fallah et al., 2020) due to the convex-concave nature of the cost function (18). The proof generalizes the techniques used in (Fallah et al., 2020) to stochastic unbiased gradients by utilizing the fact that the bias in stochastic gradients  $\tilde{\nabla}_j \tilde{J}_t^{i,\gamma}$  for  $j \in \{1, 2\}$  can be reduced by reducing the smoothing radius  $r$ . This in turn causes an increase in the variance of the stochastic gradient which is controlled by increasing the mini-batch size  $N_b$ .

Now we present the non-asymptotic convergence guarantee of the paper stating that even though each iteration of the outer loop (as timestep  $t$  moves backwards-in-time) accumulates error, if the error in each outer loop iteration is small enough, the total accumulated error will also be small enough. The proof can be found in the complete version of the paper (Zaman et al., 2024).

**Theorem 5** *If all conditions in Theorem 4 are satisfied, then  $\max_{j \in \{1,2\}} \|K_t^j - K_t^{j*}\| = \mathcal{O}(\epsilon)$  and  $\max_{j \in \{1,2\}} \|L_t^j - L_t^{j*}\| = \mathcal{O}(\epsilon)$  for a small  $\epsilon > 0$  and  $t \in \{T-1, \dots, 0\}$ .*

The Nash gaps at each time  $t$ ,  $\|K_t^j - K_t^{j*}\|$  and  $\|L_t^j - L_t^{j*}\|$  for  $j \in \{1, 2\}$  are due to a combination of the optimality gap in the inner loop  $\|\bar{K}_t - \tilde{K}_t^*\|_2^2$ ,  $\|\bar{L}_t - \tilde{L}_t^*\|_2^2$  and the accumulated Nash gap in the future joint controllers  $\|K_s^j - K_s^{j*}\|$  and  $\|L_s^j - L_s^{j*}\|$  for  $j \in \{1, 2\}$  and  $t < s < T$ . The proof of Theorem 5 characterizes these two quantities and then shows that if the optimality gap at each timestep  $t \in \{0, \dots, T-1\}$  never exceeds some small  $\epsilon$ , then the Nash gap at any time  $t$  never exceeds  $\epsilon$  scaled by a constant.

## 5. Numerical Analysis

First, we simulate the RGDA algorithm for time horizon  $T = 3$ , number of agents  $M = 1000$  and the dimension of the state and action spaces  $m = p = 2$ . For each timestep  $t \in \{2, 1, 0\}$ , the number of inner-loop iterations  $K = 1000$ , the mini-batch size  $N_b = 5 \times 10^4$  and the learning rate  $\eta_k = 0.001$ . In Figure 1(a) we compare the RGDA algorithm (Algorithm 1) with its exact version (E-RGDA) which has access to the exact policy gradients  $\nabla_1 \tilde{J}_t^{i,\gamma} = \delta \tilde{J}_t^{i,\gamma} / \delta(K_t^1, L_t^1)$  and  $\nabla_2 \tilde{J}_t^{i,\gamma} = \delta \tilde{J}_t^{i,\gamma} / \delta(K_t^2, L_t^2)$  at each iteration  $k \in [K]$ . The error plots in Figures 1(a) and 1(b) show the mean (solid lines) and standard deviation (shaded regions) of error, which is the norm of difference between iterates and Nash controllers. In Figure 1(a) the blue plot shows error convergence of the E-RGDA algorithm, which computes the Nash controllers for the last timestep  $t = 2$  (using gradient descent ascent with exact gradients) and moves backwards in time. Since at each timestep it has good convergence to Nash policies, the convexity-concavity of cost function at the next timestep is ensured, which results in linear convergence. The red plot in Figure 1(a) shows the error convergence in the RGDA algorithm which uses stochastic gradients, which results in a noisy but downward trend in error. Notice that RGDA imitates E-RGDA in a noisy fashion and at each timestep the iterates only approximate the Nash controllers. This approximation can be further sharpened by increasing the mini-batch size  $N_b$  and decreasing smoothing radius  $r$ . Figure 1(a) shows the error convergence of E-RGDA for a ZS-MFTG with  $T = 15$  and state and action space dimensions  $m = p = 2$ .

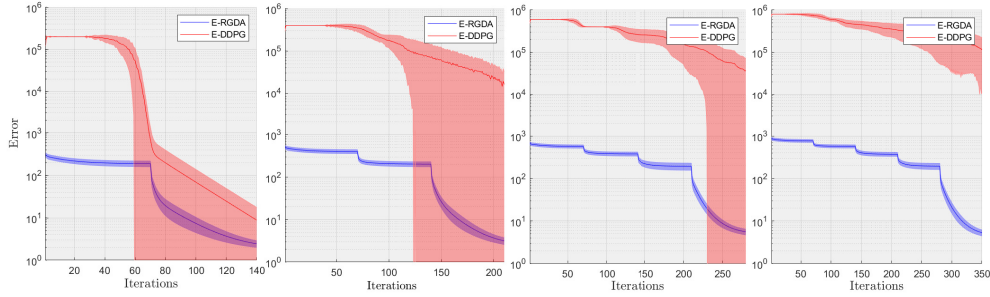


Figure 2: Comparison between E-RGDA and E-DDPG. The time-horizon is increasing from left to right with  $T = 2$  (left-most),  $T = 3$  (center left),  $T = 4$  (center right) and  $T = 5$  (right-most)

Figure 2 compares the E-RGDA algorithm with the exact 2-player zero-sum version of the MADPG algorithm (referred to as E-DDPG) (Lowe et al., 2017) which serves as a baseline as it does not use the receding-horizon approach. The number of inner-loop iterations for E-RGDA is  $K = 70$  and the learning rate for both algorithms is  $\eta = 0.025$ . The four figures represent the comparisons for  $T = \{2, 3, 4, 5\}$  and the y-axis is scaled in a logarithmic manner to best show the behavior of the algorithms. For all  $T > 1$  the E-DDPG first diverges until it reaches the projection threshold then eventually starts to converge. This is due to the fact that errors in later timesteps cause the convexity-concavity condition to fail resulting in divergence in earlier timesteps. Over time the error decreases in the later timesteps, which causes the error in earlier timesteps to gradually decrease as well. But as seen from Figure 2, the convergence for E-DDPG takes significantly longer as the time-horizon increases.

## 6. Conclusion

In this paper, we solve an MARL problem with the objective of designing robust controllers in the presence of modeled and un-modeled uncertainties. We introduce the concept of Robust Mean Field Control (RMFC) problem as the limiting problem when the number of agents grows to infinity. We then establish a connection with Zero-Sum Mean-Field Type Games (ZS-MFTG). We resort to the Linear-Quadratic (LQ) structure which, combined with the mean-field approximation, helps to have a more tractable model and to help resolve the analytical difficulty induced by the distributed information structure. This helps us obtain sufficient conditions for robustness of the problem as well as characterization of the robust control policy. We design and provide non-asymptotic analysis of a receding-horizon based RL algorithm which renders the non-coercive cost as convex-concave. Through numerical analysis the receding-horizon approach is shown to ameliorate the overshooting problem observed in the performance of the vanilla algorithm. In future work we would like to explore this type of robust mean-field problems beyond the LQ setting and to develop RL algorithms which go beyond the gradient descent-ascent updates used in this paper. Furthermore, our work is a first step in the direction of using mean-field approximations to study robust MARL problems which occur in many real-world scenarios, but the study of concrete examples is left for future work.<sup>2</sup>

## Acknowledgments

The authors would like to thank Xiangyuan Zhang (University of Illinois Urbana-Champaign) for useful discussions regarding the Receding-horizon Policy Gradient algorithm.

2. Disclaimer: This paper was prepared for informational purposes in part by the Artificial Intelligence Research group of JP Morgan Chase & Co and its affiliates (“JP Morgan”), and is not a product of the Research Department of JP Morgan. JP Morgan makes no representation and warranty whatsoever and disclaims all liability, for the completeness, accuracy or reliability of the information contained herein. This document is not intended as investment research or investment advice, or a recommendation, offer or solicitation for the purchase or sale of any security, financial instrument, financial product or service, or to be used in any way for evaluating the merits of participating in any transaction, and shall not constitute a solicitation under any jurisdiction or to any person, if such solicitation under such jurisdiction or to such person would be unlawful.

## References

- Berkay Anahtarci, Can Deha Kariksiz, and Naci Saldi. Q-learning in regularized mean-field games. *Dynamic Games and Applications*, 13(1):89–117, 2023.
- Andrea Angiuli, Jean-Pierre Fouque, and Mathieu Laurière. Unified reinforcement Q-learning for mean field game and control problems. *Mathematics of Control, Signals, and Systems*, pages 1–55, 2022.
- Michael Athans, David Castanon, K-P Dunn, C Greene, Wing Lee, N Sandell, and A Willsky. The stochastic control of the f-8c aircraft using a multiple model adaptive control (mmac) method–part i: Equilibrium flight. *IEEE Transactions on Automatic Control*, 22(5):768–780, 1977.
- Tamer Başar. A dynamic games approach to controller design: Disturbance rejection in discrete time. In *Proceedings of the 28th IEEE Conference on Decision and Control*,, pages 407–414. IEEE, 1989.
- Tamer Başar and Pierre Bernhard. *H-infinity optimal control and related minimax design problems: a dynamic game approach*. Springer Science & Business Media, 2008.
- Tamer Başar and Geert Jan Olsder. *Dynamic noncooperative game theory*. SIAM, 1998.
- Alain Bensoussan, Jens Frehse, Phillip Yam, et al. *Mean field games and mean field type control theory*, volume 101. Springer, 2013.
- Pierre Cardaliaguet and Charles-Albert Lehalle. Mean field game of controls and an application to trade crowding. *Mathematics and Financial Economics*, 12(3):335–363, 2018.
- Rene Carmona and François Delarue. *Probabilistic Theory of Mean Field Games with Applications I*. Springer, Cham, 2018.
- René Carmona, Mathieu Laurière, and Zongjun Tan. Linear-quadratic mean-field reinforcement learning: convergence of policy gradient methods. *arXiv preprint arXiv:1910.04295*, 2019a.
- René Carmona, Mathieu Laurière, and Zongjun Tan. Model-free mean-field reinforcement learning: mean-field MDP and mean-field Q-learning. *arXiv preprint arXiv:1910.12802*, 2019b.
- René Carmona, Kenza Hamidouche, Mathieu Laurière, and Zongjun Tan. Policy optimization for linear-quadratic zero-sum mean-field type games. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 1038–1043. IEEE, 2020.
- René Carmona, Kenza Hamidouche, Mathieu Laurière, and Zongjun Tan. Linear-quadratic zero-sum mean-field type games: Optimality conditions and policy optimization. *Journal of Dynamics & Games*, 8(4), 2021.
- Salah Eddine Choutri, Boualem Djehiche, and Hamidou Tembine. Optimal control and zero-sum games for markov chains of mean-field type. *Mathematical Control and Related Fields*, 9(3):571–605, 2019.
- Andrea Cosso and Huyên Pham. Zero-sum stochastic differential games of generalized mckean–vlasov type. *Journal de Mathématiques Pures et Appliquées*, 129:180–212, 2019.
- Kai Cui and Heinz Koepl. Approximately solving mean field games via entropy-regularized deep reinforcement learning. In *International Conference on Artificial Intelligence and Statistics*, pages 1909–1917. PMLR, 2021a.
- Kai Cui and Heinz Koepl. Learning graphon mean field games and approximate Nash equilibria. 2021b.
- Romuald Elie, Julien Perolat, Mathieu Laurière, Matthieu Geist, and Olivier Pietquin. On the convergence of model free learning in mean field games. 34(05):7143–7150, 2020.

- Christian Fabian, Kai Cui, and Heinz Koepl. Learning sparse graphon mean field games. In *International Conference on Artificial Intelligence and Statistics*, pages 4486–4514. PMLR, 2023.
- Alireza Fallah, Asuman Ozdaglar, and Sarath Pattathil. An optimal multistage stochastic gradient method for minimax problems. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 3573–3579. IEEE, 2020.
- Maryam Fazel, Rong Ge, Sham M Kakade, and Mehran Mesbahi. Global convergence of policy gradient methods for the linear quadratic regulator. In *International Conference on Machine Learning*, pages 1467–1476, 2018.
- Haotian Gu, Xin Guo, Xiaoli Wei, and Renyuan Xu. Mean-field controls with Q-learning for cooperative MARL: convergence and complexity analysis. *SIAM Journal on Mathematics of Data Science*, 3(4):1168–1196, 2021.
- Yue Guan, Mohammad Afshari, and Panagiotis Tsiotras. Zero-sum games between mean-field teams: Reachability-based analysis under mean-field sharing. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 38, pages 9731–9739, 2024.
- Xin Guo, Anran Hu, Renyuan Xu, and Junzi Zhang. Learning mean-field games. In *Advances in Neural Information Processing Systems*, 2019.
- Charles Harvey and Gunter Stein. Quadratic weights for asymptotic regulator properties. *IEEE Transactions on Automatic Control*, 23(3):378–387, 1978.
- Sihong He, Songyang Han, Sanbao Su, Shuo Han, Shaofeng Zou, and Fei Miao. Robust multi-agent reinforcement learning with state uncertainty. *Transactions on Machine Learning Research*, 2023.
- Minyi Huang, Peter E Caines, and Roland P Malhamé. Individual and mass behaviour in large population stochastic wireless power control problems: Centralized and Nash equilibrium solutions. In *IEEE International Conference on Decision and Control*, volume 1, pages 98–103. IEEE, 2003.
- Minyi Huang, Roland P Malhamé, and Peter E Caines. Large population stochastic dynamic games: Closed-loop Mckean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.
- Jens Kober, J Andrew Bagnell, and Jan Peters. Reinforcement learning in robotics: A survey. *The International Journal of Robotics Research*, 32(11):1238–1274, 2013.
- Jernej Kos and Dawn Song. Delving into adversarial attacks on deep policies. *arXiv preprint arXiv:1705.06452*, 2017.
- Jean-Michel Lasry and Pierre-Louis Lions. Jeux à champ moyen. i—le cas stationnaire. *Comptes Rendus Mathématique*, 343(9):619–625, 2006.
- Mathieu Laurière, Sarah Perrin, Sertan Girgin, Paul Muller, Ayush Jain, Theophile Cabannes, Georgios Piliouras, Julien Pérolat, Romuald Elie, Olivier Pietquin, et al. Scalable deep reinforcement learning algorithms for mean field games. In *International Conference on Machine Learning*, pages 12078–12095. PMLR, 2022a.
- Mathieu Laurière, Sarah Perrin, Julien Pérolat, Sertan Girgin, Paul Muller, Romuald Elie, Matthieu Geist, and Olivier Pietquin. Learning mean field games: A survey. *arXiv preprint arXiv:2205.12944*, 2022b.
- Shihui Li, Yi Wu, Xinyue Cui, Honghua Dong, Fei Fang, and Stuart Russell. Robust multi-agent reinforcement learning via minimax deep deterministic policy gradient. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 4213–4220, 2019.

- Yingying Li, Yujie Tang, Runyu Zhang, and Na Li. Distributed reinforcement learning for decentralized linear quadratic control: A derivative-free policy optimization approach. *IEEE Transactions on Automatic Control*, 67(12):6429–6444, 2021.
- Ryan Lowe, Yi I Wu, Aviv Tamar, Jean Harb, OpenAI Pieter Abbeel, and Igor Mordatch. Multi-agent actor-critic for mixed cooperative-competitive environments. *Advances in neural information processing systems*, 30, 2017.
- Dhruv Malik, Ashwin Pananjady, Kush Bhatia, Koulik Khamaru, Peter Bartlett, and Martin Wainwright. Derivative-free methods for policy optimization: Guarantees for linear quadratic systems. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 2916–2925. PMLR, 2019.
- Washim Uddin Mondal, Mridul Agarwal, Vaneet Aggarwal, and Satish V Ukkusuri. On the approximation of cooperative heterogeneous multi-agent reinforcement learning (marl) using mean field control (mfc). *The Journal of Machine Learning Research*, 23(1):5614–5659, 2022.
- Jun Morimoto and Kenji Doya. Robust reinforcement learning. *Neural computation*, 17(2):335–359, 2005.
- Julien Pérolat, Sarah Perrin, Romuald Elie, Mathieu Laurière, Georgios Piliouras, Matthieu Geist, Karl Tuyls, and Olivier Pietquin. Scaling mean field games by online mirror descent. In *Proceedings of the 21st International Conference on Autonomous Agents and Multiagent Systems*, pages 1028–1037, 2022.
- Sarah Perrin, Julien Pérolat, Mathieu Laurière, Matthieu Geist, Romuald Elie, and Olivier Pietquin. Fictitious play for mean field games: Continuous time analysis and applications. *Advances in Neural Information Processing Systems*, 33:13199–13213, 2020.
- Sarah Perrin, Mathieu Laurière, Julien Pérolat, Matthieu Geist, Romuald Elie, and Olivier Pietquin. Mean field games flock! The reinforcement learning way. In *proc. of IJCAI*, 2021.
- Benjamin Recht. A tour of reinforcement learning: The view from continuous control. *Annual Review of Control, Robotics, and Autonomous Systems*, 2:253–279, 2019.
- Joshua Riley, Radu Calinescu, Colin Paterson, Daniel Kudenko, and Alec Banks. Utilising assured multi-agent reinforcement learning within safety-critical scenarios. *Procedia Computer Science*, 192:1061–1070, 2021.
- Ahmad EL Sallab, Mohammed Abdou, Etienne Perot, and Senthil Yogamani. Deep reinforcement learning framework for autonomous driving. *arXiv preprint arXiv:1704.02532*, 2017.
- Thomas J Sargent and Lars Ljungqvist. Recursive macroeconomic theory. *Massachusetts Institute of Technology*, 2000.
- Max Simchowitz, Karan Singh, and Elad Hazan. Improper learning for non-stochastic control. In *Conference on Learning Theory*, pages 3320–3436. PMLR, 2020.
- Chuangchuang Sun, Dong-Ki Kim, and Jonathan P How. Romax: Certifiably robust deep multiagent reinforcement learning via convex relaxation. In *2022 International Conference on Robotics and Automation (ICRA)*, pages 5503–5510. IEEE, 2022.
- Hamidou Tembine. Mean-field-type games. *AIMS Math*, 2(4):706–735, 2017.
- Qiaomin Xie, Zhuoran Yang, Zhaoran Wang, and Andreea Minca. Learning while playing in mean-field games: Convergence and optimality. In *International Conference on Machine Learning*, pages 11436–11447. PMLR, 2021.
- Batuhan Yardim, Semih Cayci, Matthieu Geist, and Niao He. Policy mirror ascent for efficient and independent learning in mean field games. In *International Conference on Machine Learning*, pages 39722–39754. PMLR, 2023.

- Bora Yongacoglu, Gürdal Arslan, and Serdar Yüksel. Independent learning and subjectivity in mean-field games. In *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 2845–2850. IEEE, 2022.
- Muhammad Aneeq uz Zaman, Kaiqing Zhang, Erik Miehling, and Tamer Başar. Reinforcement learning in non-stationary discrete-time linear-quadratic mean-field games. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 2278–2284. IEEE, 2020.
- Muhammad Aneeq Uz Zaman, Sujay Bhatt, and Tamer Başar. Adversarial linear-quadratic mean-field games over multigraphs. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 209–214. IEEE, 2021.
- Muhammad Aneeq uz Zaman, Alec Koppel, Sujay Bhatt, and Tamer Başar. Oracle-free reinforcement learning in mean-field games along a single sample path. In *International Conference on Artificial Intelligence and Statistics*, pages 10178–10206. PMLR, 2023a.
- Muhammad Aneeq Uz Zaman, Erik Miehling, and Tamer Başar. Reinforcement learning for non-stationary discrete-time linear–quadratic mean-field games in multiple populations. *Dynamic Games and Applications*, 13(1):118–164, 2023b.
- Muhammad Aneeq uz Zaman, Mathieu Laurière, Alec Koppel, and Tamer Başar. Robust cooperative multi-agent reinforcement learning: A mean-field type game perspective. *arXiv preprint arXiv:2406.13992*, 2024.
- Huan Zhang, Hongge Chen, Chaowei Xiao, Bo Li, Mingyan Liu, Duane Boning, and Cho-Jui Hsieh. Robust deep reinforcement learning against adversarial perturbations on state observations. *Advances in Neural Information Processing Systems*, 33:21024–21037, 2020a.
- Kaiqing Zhang, Tao Sun, Yunzhe Tao, Sahika Genc, Sunil Mallya, and Tamer Başar. Robust multi-agent reinforcement learning with model uncertainty. *Advances in neural information processing systems*, 33:10571–10583, 2020b.
- Kaiqing Zhang, Bin Hu, and Tamer Başar. Policy optimization for  $H_2$  linear control with  $H_\infty$  robustness guarantee: Implicit regularization and global convergence. *SIAM Journal on Control and Optimization*, 59(6):4081–4109, 2021a.
- Kaiqing Zhang, Zhuoran Yang, and Tamer Başar. Multi-agent reinforcement learning: A selective overview of theories and algorithms. *Handbook of Reinforcement Learning and Control*, pages 321–384, 2021b.
- Kaiqing Zhang, Xiangyuan Zhang, Bin Hu, and Tamer Başar. Derivative-free policy optimization for linear risk-sensitive and robust control design: Implicit regularization and sample complexity. *Advances in Neural Information Processing Systems*, 34:2949–2964, 2021c.
- Xiangyuan Zhang and Tamer Başar. Revisiting LQR control from the perspective of receding-horizon policy gradient. *IEEE Control Systems Letters*, 2023.
- Xiangyuan Zhang, Bin Hu, and Tamer Başar. Learning the kalman filter with fine-grained sample complexity. *arXiv preprint arXiv:2301.12624*, 2023.
- Daniel M Ziegler, Nisan Stiennon, Jeffrey Wu, Tom B Brown, Alec Radford, Dario Amodei, Paul Christiano, and Geoffrey Irving. Fine-tuning language models from human preferences. *arXiv preprint arXiv:1909.08593*, 2019.