Follow-the-Perturbed-Leader with Fréchet-type Tail Distributions: Optimality in Adversarial Bandits and Best-of-Both-Worlds

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Abstract

This paper studies the optimality of the Follow-the-Perturbed-Leader (FTPL) policy in both adversarial and stochastic K-armed bandits. Despite the widespread use of the Follow-the-Regularized-Leader (FTRL) framework with various choices of regularization, the FTPL framework, which relies on random perturbations, has not received much attention, despite its inherent simplicity. In adversarial bandits, there has been conjecture that FTPL could potentially achieve $\mathcal{O}(\sqrt{KT})$ regrets if perturbations follow a distribution with a Fréchet-type tail. Recent work by Honda et al. (2023) showed that FTPL with Fréchet distribution with shape $\alpha = 2$ indeed attains this bound and, notably logarithmic regret in stochastic bandits, meaning the Best-of-Both-Worlds (BOBW) capability of FTPL. However, this result only partly resolves the above conjecture because their analysis heavily relies on the specific form of the Fréchet distribution with this shape. In this paper, we establish a sufficient condition for perturbations to achieve $\mathcal{O}(\sqrt{KT})$ regrets in the adversarial setting, which covers, e.g., Fréchet, Pareto, and Student-t distributions. We also demonstrate the BOBW achievability of FTPL with certain Fréchet-type tail distributions. Our results contribute not only to resolving existing conjectures through the lens of extreme value theory but also potentially offer insights into the effect of the regularization functions in FTRL through the mapping from FTPL to FTRL.
Keywords: Multi-armed bandits, Best-of-both-worlds, Extreme value theory

1. Introduction

In the multi-armed bandit (MAB) problem, an agent plays an arm I_t from a set of K arms at each round $t \in [T] := \{1, \ldots, T\}$ over a time horizon T. The agent only observes the loss ℓ_{t,I_t} generated from the played arm, where the loss vectors $\ell_t = (\ell_{t,1}, \ldots, \ell_{t,K})^\top \in [0, 1]^K$ are determined by the environment. Given the constraints of partial feedback, the agent must handle the tradeoff between gathering information about the arms and playing arms strategically to minimize total loss. The performance of the policy is measured by pseudo-regret, defined as $\mathbb{E}[\sum_t \ell_{t,I_t}] - \min_i \mathbb{E}[\sum_t \ell_{t,i}]$.

There are two primary formulations of the environment to determine loss vectors: the stochastic setting (Lai and Robbins, 1985; Katehakis and Robbins, 1995), and the adversarial setting (Auer et al., 2002b; Audibert and Bubeck, 2009). In the stochastic setting, the loss vector ℓ_t is independent

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and identically distributed (i.i.d.) from an unknown but fixed distribution $\mathcal{D}$ over $[0,1]^K$. Therefore, one can define the expected losses of arms $\mu_i := \mathbb{E}_{\ell\sim\mathcal{D}}[\ell_i]$ and the optimal arm $i^* \in \arg\min_{i\in[K]} \mu_i$. The suboptimality gap of each arm is denoted by $\Delta_i = \mu_i - \mu_i^*$ and the optimal problem-dependent regret bound is known to be $\sum_{i:\Delta_i>0} \mathcal{O}\left(\frac{\log T}{\Delta_i}\right)$ (Lai and Robbins, 1985), which can be achieved by several policies such as UCB (Auer et al., 2002a) and Thompson sampling (Agrawal and Goyal, 2017; Riou and Honda, 2020).

On the other hand, in the adversarial setting, an (adaptive) adversary determines the loss vector based on the history of the decisions, and thus specific assumptions about the loss distribution are not made. In this particular environment, the optimal regret bound stands at $\mathcal{O}(\sqrt{KT})$ (Auer et al., 2002b) and some Follow-The-Regularized-Leader (FTRL) policies have demonstrated their capability to attain this bound (Audibert and Bubeck, 2009; Zimmert and Lattimore, 2019).

In practical scenarios, a priori knowledge regarding the nature of the environment is often unavailable. Therefore, there arises a need for an algorithm that can adeptly address both stochastic and adversarial settings at the same time. While several policies have been proposed to tackle this problem (Bubeck and Slivkins, 2012; Seldin and Lugosi, 2017), the Tsallis-INF policy, based on FTRL framework, has demonstrated its effectiveness in achieving optimality in *both* setting (Zimmert and Seldin, 2021), a status referred to as the Best-of-Both-Worlds (BOBW) (Bubeck and Slivkins, 2012). Moreover, FTRL framework has been successfully adapted to achieve BOBW in various domains such as combinatorial semi-bandits (Ito, 2021; Tsuchiya et al., 2023a), linear bandits (Lee et al., 2021; Dann et al., 2023), dueling bandits (Saha and Gaillard, 2022) and partial monitoring (Tsuchiya et al., 2023b).

However, FTRL policies require the explicit computation of the probability of arm selections per step, by solving an optimization problem in general. In light of this limitation, the Follow-the-Perturbed-Leader (FTPL) framework, which simply selects the arm with the minimum cumulative estimated loss along with a random perturbation, has gained attention for its computational efficiency in adversarial bandits (Abernethy et al., 2015), combinatorial semi-bandits (Neu, 2015), and linear bandits (McMahan and Blum, 2004). It has been established that FTPL, when coupled with perturbations satisfying several conditions, can achieve nearly optimal $\mathcal{O}\left(\sqrt{KT\log K}\right)$ regret in adversarial bandits (Poland, 2005; Abernethy et al., 2015; Kim and Tewari, 2019). Subsequently, Kim and Tewari (2019) conjectured that if FTPL achieves minimax optimality, then the corresponding perturbations should be of Fréchet-type tail distribution.

Recently, Honda et al. (2023) showed that FTPL with Fréchet perturbations with shape $\alpha = 2$ indeed achieves $\mathcal{O}(\sqrt{KT})$ regret in adversarial bandits and $\mathcal{O}\left(\sum_i \frac{\log T}{\Delta_i}\right)$ regret in stochastic bandits, highlighting the effectiveness of FTPL. However, their analysis heavily relies on the specific form of Fréchet distribution, providing only a partial solution to the above conjecture. It is noteworthy that any FTPL policy can be expressed as FTRL policy (Abernethy et al., 2016). Therefore, investigating the properties of more general perturbations not only extends our understanding of FTPL but also can clarify the impact of regularization functions used in FTRL, where several regularization functions in FTRL beyond Tsallis entropy have been used to achieve BOBW in various settings (Jin et al., 2023).

Contribution This paper proves that FTPL with Fréchet-type tail distributions satisfying some mild conditions can achieve $\mathcal{O}(\sqrt{KT})$ regret in adversarial bandits, which resolves an open question raised by Kim and Tewari (2019) comprehensively. Moreover, we provide a problem-dependent regret bound in stochastic bandits, demonstrating that some of them can achieve BOBW, which generalizes the results of Honda et al. (2023). Given that our analysis is grounded in the language of extreme value theory, we expect that our analysis can provide insights for constructing an FTPL counterpart of FTRL in settings beyond the standard MAB. In particular, one main obstacle in constructing FTPL counterparts to FTRL would be the use of hybrid regularization, such as combining Tsallis entropy, Shannon entropy and log-barrier (Zimmert et al., 2019; Ito, 2021; Tsuchiya et al., 2023a), which might correspond to the combination of Fréchet perturbation with other perturbation in FTPL. Our results would be beneficial for such construction, as our results cover a broad class of perturbation distributions.

2. Preliminaries

In this section, we formulate the problem and provide a brief overview of extreme value theory and the framework of regular variation, based on which Fréchet-type tail is formulated. For a thorough understanding of extreme value theory and related discussions, we refer the reader to Appendix A and the references therein.

2.1. Problem formulation

At every round $t \in [T]$, the environment determines the loss vector $\ell_t = (\ell_{t,1}, \dots, \ell_{t,K}) \in [0,1]^K$ through either a stochastic or adversarial process. Then the agent plays an arm I_t according to their policy and observes the corresponding loss ℓ_{t,I_t} of the played arm. Then, the pseudo-regret, a measure to evaluate the performance of a policy, is defined as

$$\mathcal{R}(T) = \mathbb{E}\left[\sum_{t=1}^{T}(\ell_{t,I_t} - \ell_{t,i^*})\right], \quad i^* \in \operatorname*{arg\,min}_{i\in[K]} \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,i}\right],$$

where i^* denotes the optimal arm. Since only partial feedback is available, FTRL and FTPL policies use an estimator $\hat{\ell}_t$ of the loss vector ℓ_t specified in Section 2.2. We denote the cumulative loss at round t by $L_t = \sum_{s=1}^{t-1} \ell_t$ and its estimation by $\hat{L}_t = \sum_{s=1}^{t-1} \hat{\ell}_s$.

2.2. Follow-the-Perturbed-Leader policy

In the MAB problems, FTPL is a policy that plays an arm

$$I_t \in \operatorname*{arg\,min}_{i\in[K]} \left\{ \hat{L}_{t,i} - \frac{r_{t,i}}{\eta_t} \right\},$$

where η_t denotes the learning rate specified later and $r_t = (r_{t,1}, \dots, r_{t,K})$ denotes the random perturbation i.i.d. from a common distribution $\mathcal{D}$ with a distribution function F. Then, the probability of playing an arm $i \in [K]$ given $\hat{L}_t$ is written as $w_{t,i} = \phi_i(\eta_t \hat{L}_t; \mathcal{D})$, where for $\lambda \in [0,\infty)^K$

$$\begin{aligned}
\phi_i(\lambda; \mathcal{D}) &:= \Pr_{r_1,\dots,r_K \sim \mathcal{D}}\left[i = \operatorname*{arg\,min}_{j\in[K]} \{\lambda_j - r_j\} \right] \\
&= \int_{\nu - \min_{j\in[K]} \lambda_j}^{\infty} \prod_{j\neq i} F(z+\lambda_j)\, \mathrm{d}F(z+\lambda_i) \\
&= \int_{\nu}^{\infty} \prod_{j\neq i} F(z+\underline{\lambda}_j)\, \mathrm{d}F(z+\underline{\lambda}_i), \qquad (1)
\end{aligned}$$

Algorithm 1 FTPL with geometric resampling

Initialization : $\hat{L}_1 = 0$ and set distribution $\mathcal{D}$
for $t = 1$ *to* T **do**
  Sample $r_t = (r_{t,1}, \ldots, r_{t,K})$ i.i.d. from $\mathcal{D}$.
  Play $I_t \in \arg\min_{i\in[K]} \left\{ \hat{L}_{t,i} - \frac{r_{t,i}}{\eta_t} \right\}$.
  Observe ℓ_{t,I_t} and set $m = 0$.
  repeat
    $m := m + 1$. // Geometric resampling
    Sample $r' = (r'_1, \ldots, r'_K)$ i.i.d. from $\mathcal{D}$.
  until $I_t = \arg\min_{i\in[K]} \left\{ \hat{L}_{t,i} - \frac{r'_i}{\eta_t} \right\}$
  Set $\widehat{w_{t,I_t}^{-1}} := m$ and $\hat{L}_{t+1} := \hat{L}_t + \ell_{t,I_t} \widehat{w_{t,I_t}^{-1}} e_{I_t}$.
end

where ν denotes the left endpoint of the support of F. Here, underlines denote the gap of a vector from its minimum, i.e., $\underline{\lambda} = \lambda - \mathbf{1} \min_{i\in[K]} \lambda_i$ for all-one vector $\mathbf{1}$.

For the unbiased loss estimator, FTRL policies often employ an importance-weighted estimator, $\hat{\ell}_t = (\ell_{t,I_t}/w_{t,I_t}) e_{I_t}$, where w_{t,I_t} is explicitly computed. On the other hand in FTPL, we use an unbiased estimator $\widehat{w_{t,i}^{-1}}$ of $w_{t,i}^{-1}$ by geometric resampling (Neu and Bartók, 2016), whose pseudo-code is given in Lines 6–10 of Algorithm 1. Simply speaking, the process involves repeated samplings of perturbations r' until $\arg\min_i \left\{ \hat{L}_{t,i} - r'_{t,i}/\eta_t \right\}$ coincides with I_t and $\widehat{w_{t,i}^{-1}}$ is then set as the number of resampling. For more details, refer to Neu and Bartók (2016) and Honda et al. (2023).

2.3. Fréchet maximum domain of attraction

In the adversarial setting, it has been conjectured that FTPL might achieve $\mathcal{O}(\sqrt{KT})$ regrets if perturbations follow a distribution with a Fréchet-type tail (Kim and Tewari, 2019). In the following, we explain the terminology and basic concepts related to this description.

Extreme value theory is a branch of statistics to study the distributions of maxima of random variables. One of the most important results in this theory is that the distribution of the maxima of i.i.d. random variables can *only* converge in distribution to three types of extreme value distributions: Fréchet, Gumbel, and Weibull, after appropriate normalization (Fisher and Tippett, 1928; Gnedenko, 1943). Among these, a distribution is called Fréchet-type if its limiting distribution is Fréchet distribution. The family of Fréchet-type distributions is called Fréchet maximum domain of attraction (FMDA), and its representation is known to be associated with the notion of regular variation (Embrechts et al., 1997; Haan and Ferreira, 2006; Resnick, 2007) defined as follows.

Definition 1 (Regular variation (Haan and Ferreira, 2006)) *An eventually positive function g, which becomes positive after a certain point, is called* regularly varying *at infinity with index α, $g \in \mathrm{RV}_\alpha$ if*

$$\lim_{x\to\infty} \frac{g(tx)}{g(x)} = t^\alpha, \quad \forall t > 0.$$

If $g(x)$ is regularly varying with index 0, then g is called slowly varying.

Table 1: Some well-known Fréchet-type tail distributions with parameters $\alpha, \beta, m, n > 0$. $S_F(x)$ denotes the corresponding slowly varying function that characterizes the tail distribution. More examples such as LogGamma can be found in Beirlant et al. (2006, Table 2.1). Here, $B(a, b)$ and $B(x; a, b)$ denote the Beta function and incomplete Beta function, respectively.

Distribution ($\mathcal{D}$)	$1 - F(x)$	$f(x)$	$S_F(x)$	Support	Index
Fréchet ($\mathcal{F}_\alpha$)	$1 - e^{-x^{-\alpha}}$	$\alpha \frac{e^{-x^{-\alpha}}}{x^{\alpha+1}}$	$x^\alpha(1 - e^{-x^{-\alpha}})$	$x > 0$	α
Pareto ($\mathcal{P}_\alpha$)	$x^{-\alpha}$	$\frac{\alpha}{x^{\alpha+1}}$	1	$x \geq 1$	α
Generalized Pareto ($\mathcal{GP}_{\alpha,\beta}$)	$\left(1 + \frac{x}{\alpha\beta}\right)^{-\alpha}$	$\frac{1}{\beta}\left(1 + \frac{x}{\alpha\beta}\right)^{-(\alpha+1)}$	$(\alpha\beta)^\alpha \left(1 + \frac{\alpha\beta}{x}\right)^{-\alpha}$	$x \geq 0$	α
Student-t ($\mathcal{T}_n$)	$\int_{-\infty}^{x} \frac{(1+t^2/n)^{-\frac{n+1}{2}}}{\sqrt{n}B(n/2,1/2)} \mathrm{d}t$	$\frac{1}{\sqrt{n}B(n/2,1/2)}\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$	$\frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\Gamma(n/2)} n^{\frac{n-1}{2}} \left(1 - \frac{n^2(n+1)}{2(n+2)} x^{-2} + o(x^{-2})\right)$	$\mathbb{R}$	n
Snedecor's F ($\mathcal{S}_{m,n}$)	$1 - \frac{B\left(\frac{mx}{mx+n}; \frac{m}{2}, \frac{n}{2}\right)}{B\left(\frac{m}{2}, \frac{n}{2}\right)}$	$\frac{(m/n)^{\frac{m}{2}}}{B\left(\frac{m}{2}, \frac{n}{2}\right)} x^{\frac{m}{2}-1}\left(1 + \frac{m}{n}x\right)^{-\frac{m+n}{2}}$	$\frac{(m/n)^{\frac{m}{2}}}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \left(\frac{m}{n} + \frac{1}{x}\right)^{-\frac{m+n}{2}} (1 + o(1))$	$x > 0$	$\frac{n}{2}$

From the definition, one can see that any regularly varying function with index α can be written with a product of a slowly varying function and x^α, i.e., if $g \in \mathrm{RV}_\alpha$, then $g = x^\alpha S(x)$ for some $S \in \mathrm{RV}_0$ and all $x > 0$. A necessary and sufficient condition for a distribution to belong to FMDA is known to be expressed in terms of regular variation as shown below.

Proposition 2 (Gnedenko (1943); Resnick (2008)) *A distribution $\mathcal{D}_\alpha$ belongs to FMDA with index $\alpha > 0$ if and only if its right endpoint is infinite and the tail function, $1 - F$, is regularly varying at infinity with index $-\alpha$, i.e., $1 - F \in \mathrm{RV}_{-\alpha}$. In this case,*

$$F^n(a_n x) \to \begin{cases} \exp(-x^{-\alpha}), & x \geq 0, \\ 0, & x < 0, \end{cases} \quad n \to \infty, \tag{2}$$

where $a_n = \inf\left\{x : F(x) \geq 1 - \frac{1}{n}\right\}$.

Let $\mathfrak{D}_\alpha^{\text{all}}$ denote the class of FMDA with index $\alpha > 0$. From its definition, if $\mathcal{D} \in \mathfrak{D}_\alpha^{\text{all}}$, we can express the tail distribution with $S_F \in \mathrm{RV}_0$ as

$$1 - F(x) = x^{-\alpha} S_F(x), \quad \forall x > 0. \tag{3}$$

In other words, a Fréchet-type tail distribution can be characterized by a slowly varying function S_F and an index α, where Table 1 provides examples of well-known distributions and their associated slowly varying functions.

Notably, $\mathfrak{D}_\alpha^{\text{all}}$ encompasses exceptionally diverse distributions since its definition generally allows for any slowly varying functions, even those that are discontinuous. In this paper, we consider a set of Fréchet-type distributions denoted by $\mathfrak{D}_\alpha \subset \mathfrak{D}_\alpha^{\text{all}}$, which is defined as follows.

Definition 3 *$\mathfrak{D}_\alpha$ is a set of distributions that belong to FMDA with index $\alpha > 0$ satisfying the following assumptions.*

Assumption 1 *$F(x)$ has a density function $f(x)$ that is decreasing in $x \geq z_0$ for some $z_0 > \nu$.*

Assumption 2 *$\mathfrak{D}_\alpha$ is supported over $[\nu, \infty)$ for some $\nu \geq 0$ and the hazard function $\frac{f(x)}{1-F(x)}$ is bounded.*

Assumption 3 *There exist positive constants $M = M(\mathcal{D}_\alpha)$ and $m = m(\mathcal{D}_\alpha)$ satisfying*

$$\mathbb{E}_{X_1,\dots,X_k\sim\mathcal{D}_\alpha}\left[\max_{i\in[k]} X_i/a_k\right] \le M \tag{4}$$

$$\mathbb{E}_{X_1,\dots,X_k\sim\mathcal{D}_\alpha}\left[\frac{1}{\max_{i\in[k]} X_i/a_k}\right] \le m \tag{5}$$

for $a_k = \inf\{x : F(x) \ge 1-1/k\}$ and and it satisfies $A_l k^{\frac{1}{\alpha}} \le a_k \le A_u k^{\frac{1}{\alpha}}$ for some positive constants A_l, A_u.

Assumption 4 $\lim_{x\to\infty} \frac{-xf'(x)}{f(x)} = \alpha + 1$ *and* $\frac{-f'(x)}{f(x)}$ *is bounded almost everywhere on* $[\nu, \infty)$.

Assumption 5 $\frac{f(x)}{F(x)}$ *is monotonically decreasing in $x \ge \nu$.*

These assumptions offer easy-to-check *sufficient* conditions for perturbations to achieve the optimal order and verifying necessary conditions would be interesting for future work. In the following, we explain the implication of the assumptions in Definition 3.

Assumption 1 states that the density eventually monotonically decreases and does not have a fluctuated tail. This is known as a sufficient condition that $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha^{\text{all}}$ satisfies von Mises condition (von Mises, 1936, see also Resnick, 2008, Proposition 1.15), which is given by

$$\lim_{x\to\infty} \frac{xf(x)}{1-F(x)} = \alpha. \tag{6}$$

The von Mises condition is known to play an important role in the analysis of the FMDA. For example, it is known that any $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha^{\text{all}}$ (possibly without a density) is tail-equivalent to some distribution in $\mathfrak{D}_\alpha^{\text{all}}$ satisfying von Mises condition (Embrechts et al., 1997, Corollary 3.3.8). Here, a distribution $F(x)$ is called to be tail-equivalent to $F^*(x)$ if they have the same right endpoint x_r and $\lim_{x\to x_r}(1-F(x))/(1-F^*(x)) = c$ for some constant $c > 0$.

In Assumption 2, the bounded hazard function is also assumed in the existing analysis of near-optimality in adversarial bandits (Abernethy et al., 2015; Kim and Tewari, 2019). The assumption of the nonnegative left-endpoint $\nu \ge 0$ is mainly for notational simplicity. This is because $S_F(x)$ in (3) is not well-defined for $x \le 0$. Although the requirements in Assumption 2 are not satisfied for some distributions such as t-distribution, we can easily construct a tail-equivalent distribution satisfying the assumption by considering the truncated version F^* of F given by

$$F^*(x) = \Pr[X \ge 1 + x | X > 1] = \frac{F(x+1) - F(1)}{1 - F(1)}, \quad x > 0, \tag{7}$$

which is also considered in Abernethy et al. (2015, Appendix B.2).

Eq. (5) in Assumption 3 is the term that directly appears in the regret bound. As described in Proposition 2, $\max_{i\in[k]} X_i/a_k$ converges weakly to Fréchet distribution with shape α, which satisfies $\mathbb{E}_{X\sim\mathcal{F}_\alpha}[X] = \Gamma\big(1-\frac{1}{\alpha}\big)$, $\mathbb{E}_{X\sim\mathcal{F}_\alpha}[1/X] = \Gamma\big(1+\frac{1}{\alpha}\big)$ and $a_k \approx k^{\frac{1}{\alpha}}$. Therefore, (4) and (5) roughly require that it also converges in the sense of expectation and expectation of the inverse. The assumption of $a_k = \Theta(k^{\frac{1}{\alpha}})$ does not hold in general, but it holds if we ignore the sub-polynomial

Table 2: Verification of distributions in (7) whether satisfying the assumptions. ✓ and × denote whether the distribution satisfies the assumption or not, respectively, regardless of the parameters. (∗) denotes that the truncated distribution in (7) satisfies the assumption.

Distribution ($\mathcal{D}$)	$\mathcal{F}_\alpha$	$\mathcal{P}_\alpha$	$\mathcal{GP}_{\alpha,\beta}$	$\mathcal{T}_n$	$\mathcal{S}_{m,n}$
Assumption 1	✓	✓	✓	✓	✓
Assumption 2	✓	✓	✓	× (*)	× (*)
Assumption 3	✓	✓	✓	× (*)	✓
Assumption 4	✓	✓	✓	✓	✓
Assumption 5	✓	✓	✓	× (*)	✓

factor. As a result, if we remove this assumption the bound becomes sub-polynomially worse in terms of K. An easy-to-verify sufficient condition for Assumption 3 is

$$\begin{aligned} \limsup_{x\to\infty} S_F(x) &= \limsup_{x\to\infty} x^\alpha(1-F(x)) < \infty \\ \liminf_{x\to\infty} S_F(x) &= \liminf_{x\to\infty} x^\alpha(1-F(x)) > 0, \end{aligned} \tag{8}$$

while (8) becomes the necessary condition for $a_k^{-1} = \mathcal{O}(k^{-\frac{1}{\alpha}})$ if we replace $\liminf$ with $\limsup$. Note that both F and F^* in (7) for all distributions in Table 1 satisfy (8) with explicit forms of m and A_l as shown in Appendix A.2 and Lemma 10.

Assumptions 4 and 5 may appear somewhat restrictive, but many Fréchet-type distributions, including several well-known examples such as $\mathcal{F}_\alpha$ and $\mathcal{P}_\alpha$, satisfy this condition, as shown in Table 2. Assumption 4 is a condition slightly stronger than von Mises condition, because $\frac{-xf'(x)}{f(x)} \to \alpha+1$ implies (6) by L'hôpital's rule. We expect that Assumption 5 can be relaxed to the monotonicity of $f(x)/F(x)$ in $x > z_1$ for some $z_1 \geq \nu$ as in Assumption 1, which is satisfied in all examples in Table 2. Still, this relaxation makes the case-analysis somewhat too long and is left as a future work.

In the rest of this paper, we always assume that the distribution satisfies $\nu \geq 1$ rather than $\nu \geq 0$ for notational simplicity except for the specific analysis for Fréchet and Pareto distributions, where the density functions are written in simple forms. This is without loss of generality because the shifted distribution $G(x) = F(x-1)$ has the left-endpoint $\nu + 1 \geq 1$ and clearly satisfies Assumptions 1–5, while the arm-selection probability is the same between $F(x)$ and $G(x)$.

3. Main result

In this section, we present our main theoretical results that show the optimality of FTPL with perturbation distribution $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$ in adversarial bandits. Furthermore, we provide regret upper bounds of FTPL with perturbations under a mild additional condition on $\mathcal{D}_\alpha$ in stochastic bandits.

Theorem 4 *In the adversarial bandits, there exist some constants $C_1(\mathcal{D}_\alpha, c)$, $C_2(\mathcal{D}_\alpha)$ and $C_3(\mathcal{D}_\alpha, c, K)$ such that FTPL with $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$ and learning rates $\eta_t = \frac{c}{\sqrt{t}} K^{\frac{1}{\alpha}-\frac{1}{2}}$ for $c > 0$ and $\alpha > 1$ satisfies*

$$\mathcal{R}(T) \leq C_1(\mathcal{D}_\alpha, c)\sqrt{KT} + C_2(\mathcal{D}_\alpha)\log(T+1) + \frac{MA_u\sqrt{K}}{c}.$$

This result shows the minimax optimality of FTPL with the Fréchet-type distributions including Fréchet distributions and generalized Pareto distributions, which not only generalizes the results of

Honda et al. (2023) but also resolves the open question in Kim and Tewari (2019) in the sense that we provide conditions for a very large class of Fréchet-type perturbations.

Here, our result requires that $\alpha > 1$ holds. This is because (4) in Assumption 3 does not hold for $\alpha \leq 1$ since the extreme distribution of $\mathcal{D}_\alpha$ (that is, $\mathcal{F}_\alpha$) has infinite mean. This corresponds to the assumption of the finite expected block maxima $\mathbb{E}_{X_1,\dots,X_k\sim\mathcal{D}}[\max_i X_i] < \infty$ considered in Abernethy et al. (2015) and Kim and Tewari (2019).

The following result shows that FTPL with $\mathfrak{D}_2$ can achieve the logarithmic regret in the stochastic bandits. Note that all Fréchet-type tail distributions in Table 1 belong to $\mathfrak{D}_\alpha$.

Theorem 5 *Assume that $i^* = \arg\min_{i\in[K]} \mu_i$ is unique and let $\Delta_i = \mu_i - \mu_i^*$. Then, FTPL with learning rate $\eta_t = \frac{c}{\sqrt{t}}$ for $c > 0$ and $\mathcal{D} \in \mathfrak{D}_2$ satisfies*

$$\mathcal{R}(T) \leq \mathcal{O}\left(\sum_{i\neq i^*} \frac{\log T}{\Delta_i}\right).$$

This result shows that FTPL achieves BOBW if the limiting distribution of the perturbation under mild conditions is Fréchet distribution with shape $\alpha = 2$. It can be interpreted as a counterpart of FTRL with Tsallis entropy regularization, where the logarithmic regret is known only for $1/2$-Tsallis entropy without any knowledge of the gaps (see Zimmert and Seldin, 2021, Remarks 5 and 6), while Tsallis entropy with any parameter achieves the optimal adversarial regret.

Although there is no stochastic perturbation that yields the same arm-selection probability as Tsallis entropy regularizer for $K \geq 4$, in two-armed setting, it has been shown that β-Tsallis entropy regularizer can be reduced to a Fréchet-type perturbation with index $\alpha = \frac{1}{1-\beta}$ satisfying von Mises condition (Kim and Tewari, 2019, Appendix C.2). Therefore, the success of $\alpha = 2$ perturbation seems intuitive since it roughly corresponds to $1/2$-Tsallis entropy regularizer. In addition, β-Tsallis entropy becomes the log-barrier for $\beta \to 0$ (Zimmert and Seldin, 2021), which corresponds to $\alpha \to 1$. The BOBW achievability of log-barrier regularization without adaptive learning rate has not been known, which seems to correspond to our requirement of $\alpha > 1$.

Beyond the case $\alpha = 2$, we obtain the following results.

Theorem 6 *Assume that $i^* = \arg\min_{i\in[K]} \mu_i$ is unique and let $\Delta_i = \mu_i - \mu_i^*$. Then, FTPL with learning rate $\eta_t = \frac{c}{\sqrt{t}} K^{\frac{1}{\alpha}-\frac{1}{2}}$ for $c > 0$ and $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$ for $\alpha > 2$ satisfies*

$$\mathcal{R}(T) \leq \mathcal{O}\left(\sum_{i\neq i^*} \frac{1}{\alpha-2} \frac{T^{\frac{\alpha-2}{2(\alpha-1)}} - 1}{\Delta_i^{\frac{1}{\alpha-1}} K^{\frac{\alpha-2}{2(\alpha-1)}}}\right).$$

If $\alpha \in (1, 2)$, then

$$\mathcal{R}(T) \leq \mathcal{O}\left(\sum_{i\neq i^*} \frac{1}{2-\alpha} \frac{T^{1-\frac{\alpha}{2}} - 1}{\Delta_i^{\alpha-1} K^{1-\frac{\alpha}{2}}}\right).$$

Although our regret upper bound for FTPL with index $\alpha \neq 2$ does not match the regret lower bound for the stochastic case, this result shows that the regret of FTPL has better dependence on T in the stochastic case than $\mathcal{O}(\sqrt{T})$ in the adversarial case because $\frac{\alpha-2}{2(\alpha-1)} < \frac{1}{2}$ for $\alpha > 2$ and $1 - \frac{\alpha}{2} < \frac{1}{2}$ for $\alpha \in (1, 2)$.

We expect that FTPL with $\alpha \neq 2$ can attain (poly-)logarithmic regret in the stochastic setting by using arm-dependent learning rate as Jin et al. (2023) showed the BOBW results for FTRL with β-Tsallis entropy regularization for $\beta \in (0, 1)$. However, the results of Jin et al. (2023) in the adversarial setting are $\mathcal{O}(\sqrt{KT \log T})$ when $\beta \neq 1/2$, which does not achieve the adversarial optimality in the strict sense. It is highly nontrivial whether FTPL with $\alpha \neq 2$ can achieve both logarithmic regret in the stochastic case and $\mathcal{O}(\sqrt{KT})$ regret in the adversarial case.

4. Proof Outline

In this section, we first provide a proof outline of Theorem 4 and then sketch the proof of Theorems 5 and 6, whose detailed proofs are given in Appendices C, D and E.

While our analysis draws inspiration from the structure in Honda et al. (2023), a naive application of their analysis does not yield a bound for the general case. This is mainly because, while the use of Fréchet distribution in Kim and Tewari (2019) and Honda et al. (2023) is inspired by the extreme value theory, their actual analysis is not based on this theory. Instead, it is highly specific to the Fréchet distribution with shape $\alpha = 2$. Consequently, the representations of Fréchet-type distributions in extreme value theory are not directly associated with their analysis. To address this challenge, we demonstrate that the general representation in (3) under von Mises condition can be specifically tailored for the regret analysis.

4.1. Regret decomposition

To evaluate the regret of FTPL, we first decompose regret into three terms, which generalizes Lemma 3 of Honda et al. (2023). The proofs of lemmas in this section are given in Appendix B.

Lemma 7 *For any $\alpha > 1$ and $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$,*

$$\mathcal{R}(T) \leq \sum_{t=1}^{T} \mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1} \right\rangle\right] + \sum_{t=1}^{T} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) \mathbb{E}\left[r_{t+1,I_{t+1}} - r_{t+1,i^*}\right] + \frac{M A_u \sqrt{K}}{c}. \quad (9)$$

The proof of this lemma is essentially the same as that of Honda et al. (2023), except that we need to evaluate the block maxima $\mathbb{E}_{X_i \sim \mathcal{D}_\alpha}[\max_{i \in [K]} X_i]$ for general $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$. Following the convention in the analysis of BOBW policies (Zimmert and Seldin, 2021; Ito et al., 2022; Honda et al., 2023), we refer to the first and second terms of (9) as *stability term* and *penalty term*, respectively.

Here, we can further decompose the stability term into two terms as follows.

Lemma 8 *For any $\alpha > 1$ and $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$,*

$$\sum_{t=1} \mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1} \right\rangle\right] \leq 2C_2(\mathcal{D}_\alpha) \log\left(\frac{\eta_1}{\eta_{T+1}}\right) + \sum_{t=1}^{T} \mathbb{E}\left[\left\langle \hat{\ell}_t, \phi(\eta_t \hat{L}_t) - \phi(\eta_t(\hat{L}_t + \hat{\ell}_t)) \right\rangle\right], \quad (10)$$

where $\phi = (\phi_1, \ldots, \phi_K)$ for ϕ_i defined in (1),

$$C_2(\mathcal{F}_\alpha) = \frac{\alpha}{2}, \quad \text{and} \quad C_2(\mathcal{D}_\alpha) \leq \frac{\rho_1(e^2+1)}{2}, \quad \mathcal{D}_\alpha \in \mathfrak{D}_\alpha.$$

Here, $\rho_1 = \rho_1(\mathcal{D}_\alpha)$ is a positive distribution-dependent constant satisfying

$$\frac{x f(x)}{1 - F(x)} \leq \rho_1. \quad (11)$$

The proof of this lemma is based on the representation of the distribution function using the slowly varying function in (3). Note that Assumption 2 under von Mises condition implies the existence of ρ_1 in (11). From this result, it remains to derive upper bounds of the second term of (10) and the penalty term to conclude the proof of Theorem 4.

4.2. Stability term

The analysis of the arm-selection probability ϕ has been recognized as the central and most challenging aspect of the regret analysis for FTPL (Abernethy et al., 2015; Honda et al., 2023). The key to the analysis of the stability for general Fréchet-type distribution is another representation called Karamata's representation, which is an essential tool to express the slowly varying functions (Bingham et al., 1989). In the analysis, we interchangeably use this representation along with the representation in (3) and von Mises condition in (6), which utilizes a coherent connection between general representations and those under von Mises conditions. See Appendices A.1 and C for details of Karamata's representation and the proofs, respectively.

For the arm selection probability function $\phi_i(\lambda)$ in (1), define for any $\alpha > 0$, $\phi_i'(\lambda; \mathcal{D}_\alpha) = \frac{\partial \phi_i}{\partial \lambda_i}(\lambda; \mathcal{D}_\alpha)$ and

$$I_{i,n}(\lambda; \alpha) = \int_0^\infty \frac{1}{(z+\lambda_i)^n} \exp\left(-\sum_{j\in[K]} \frac{1}{(z+\lambda_j)^\alpha}\right) \mathrm{d}z, \tag{12}$$

$$J_i(\lambda; \mathcal{D}_\alpha) = \int_1^\infty \frac{f(z+\lambda_i)}{(z+\lambda_i)} \prod_{j\neq i} F(z+\lambda_j) \mathrm{d}z. \tag{13}$$

We will employ $I_{i,n}$ and J_i to analyze the stability term for $\mathcal{F}_\alpha$ and $\mathfrak{D}_\alpha \setminus \{\mathcal{F}_\alpha\}$, respectively. Although the analysis for J_i can cover $\mathcal{F}_\alpha$, we consider the specific form of $\mathcal{F}_\alpha$ in I_i without any truncation or shift to derive a tighter upper bound.

Note that $\phi_i'(\lambda) \leq 0$ holds since it denotes the probability of $\lambda_i - r_i < \min_{i\neq j} \{\lambda_i - r_j\}$ when each r_i is generated from $\mathcal{D}_\alpha$. By the same reason, $\phi_i(\lambda)$ is non-decreasing with respect to λ_j for $i \neq j$. To derive an upper bound of the stability term, we provide lemmas that are related to the relation between the arm-selection probability and its derivatives, which plays a central role in the regret analysis of FTPL.

Lemma 9 *For any $\alpha > 0$ and $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$, $\frac{I_{i,\alpha+2}(\lambda;\alpha)}{I_{i,\alpha+1}(\lambda;\alpha)}$ and $\frac{J_i(\lambda;\mathcal{D}_\alpha)}{\phi_i(\lambda;\mathcal{D}_\alpha)}$ are monotonically increasing with respect to λ_j for any $j \neq i$.*

Assumption 5 plays a key role in simplifying the proof of this lemma. Still, we conjecture that it can be weakened to the monotonicity of $\frac{f(x)}{F(x)}$ in $x \geq z_2$ for some $z_2 > 0$ rather than the current assumption requiring $z_2 = \nu$. This is because the role of Lemma 9 is to control the behavior of the algorithm when the perturbation becomes large.

Based on this result, the following lemma holds. To prove this lemma, we first examine the Pareto distribution as the proof for the Pareto case offers insights into the proofs for general Fréchet-type distributions. Specifically, we utilized the relationship between corresponding slowly varying functions for *tail functions* and those for *tail quantile functions*.

Lemma 10 *If λ_i is the σ_i-th smallest among $\lambda_1, \dots, \lambda_K$ (ties are broken arbitrarily), then*

$$\frac{I_{i,\alpha+2}(\underline{\lambda};\alpha)}{I_{i,\alpha+1}(\underline{\lambda};\alpha)} \le \frac{\alpha}{(\alpha+1)\underline{\lambda}_i} \wedge \frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{\sqrt[\alpha]{\sigma_i}} \quad \textit{and} \quad \frac{J_i(\underline{\lambda};\mathcal{D}_\alpha)}{\phi_i(\underline{\lambda};\mathcal{D}_\alpha)} \le \frac{m}{A_l}\sigma_i^{-\frac{1}{\alpha}} \wedge \frac{\alpha}{\alpha+1}\frac{eA_u}{A_l\underline{\lambda}_i},$$

where m, A_l, and A_u are given in Assumption 3. Moreover, if $\mathcal{D}_\alpha$ satisfies

$$\frac{xf(x)}{1-F(x)} \le \alpha, \tag{14}$$

then, $m \le 2\Gamma\left(1+\frac{1}{\alpha}\right)$, $A_l = 1$, and $A_u = \lim_{x\to\infty} S_F^{1/\alpha}(x)$ holds

Note that all distributions in Table 1 satisfy (14) as shown in Appendix A.2. Similarly to (11), from Assumption 4, there exists some constants $\rho_2 > 0$ satisfying

$$\frac{-xf'(x)}{f(x)} \le \rho_2. \tag{15}$$

Then, by following the same steps in Lemma 7 of Honda et al. (2023) based on our result in Lemma 10, we obtain the following lemma.

Lemma 11 *For any $i \in [K]$, if $\hat{L}_{t,i}$ is the $\sigma_{t,i}$-th smallest among $\{\hat{L}_{t,j}\}_j$, then for $\alpha > 1$ and $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$*

$$\mathbb{E}\left[\hat{\ell}_{t,i}\left(\phi_i\left(\eta_t\hat{L}_t;\mathcal{D}_\alpha\right) - \phi_i\left(\eta_t\left(\hat{L}_t+\hat{\ell}_t\right);\mathcal{D}_\alpha\right)\right)\middle|\hat{L}_t\right] \le \psi_s(\underline{\hat{L}}_{t,i};\mathcal{D}_\alpha) \wedge 2\eta_t\frac{\rho_2 m A_u}{A_l\sqrt[\alpha]{\sigma_i}}, \tag{16}$$

where $\rho_2 = \alpha+1$ holds for $\mathcal{F}_\alpha$ and $\mathcal{P}_\alpha$, $m(\mathcal{F}_\alpha) = \Gamma\left(1+\frac{1}{\alpha}\right)$, and

$$\psi_s(\underline{\hat{L}}_{t,i};\mathcal{D}_\alpha) = \begin{cases} \frac{2\alpha}{\underline{\hat{L}}_{t,i}} & \textit{if } \mathcal{D}_\alpha = \mathcal{F}_\alpha, \\ \frac{2\rho_2\alpha}{\alpha+1}\frac{eA_u}{A_l\underline{\hat{L}}_{t,i}} & \textit{if } \mathcal{D}_\alpha \in \mathfrak{D}_\alpha \setminus \{\mathcal{F}_\alpha\}. \end{cases}$$

The second term of RHS of (16) finally leads to the bound on the stability term, which is used for both the adversarial and stochastic bandits. For the stochastic bandits, we use the tighter bound with ψ_s to apply the self-bounding technique.

Lemma 12 *For any $\hat{L}_t$ and $\alpha > 1$ and $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$,*

$$\mathbb{E}\left[\left\langle\hat{\ell}_t, \phi\left(\eta_t\hat{L}_t;\mathcal{D}_\alpha\right) - \phi\left(\eta_t\left(\hat{L}_t+\hat{\ell}_t\right);\mathcal{D}_\alpha\right)\right\rangle\middle|\hat{L}_t\right] \le 2\frac{\alpha\rho_2}{\alpha-1}\frac{mA_u}{A_l}K^{1-\frac{1}{\alpha}}\eta_t.$$

4.3. Penalty term

Next, we establish an upper bound for the penalty term.

Lemma 13 *For any $\alpha > 1$ and $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$,*

$$\mathbb{E}\left[r_{t,I_t} - r_{t,i^*}\middle|\hat{L}_t\right] \le \psi_p(\underline{\hat{L}}_{t,i},\mathcal{D}_\alpha) \wedge C_{1,1}(\mathcal{D}_\alpha)\sqrt[\alpha]{K}, \tag{17}$$

where $C_{1,1}(\mathcal{D}_\alpha)$ is a distribution-dependent constant, which satisfies $C_{1,1}(\mathcal{F}_\alpha) = C_{1,1}(\mathcal{P}_\alpha)/e$ for

$$C_{1,1}(\mathcal{P}_\alpha) = \frac{2\alpha^3 + (e-2)\alpha^2}{(\alpha-1)(2\alpha-1)},$$

and

$$\psi_p(\hat{\underline{L}}_{t,i}; \mathcal{D}_\alpha) = \begin{cases} \sum_{i\neq i^*} \frac{1}{(\eta_t \hat{\underline{L}}_{t,i})^{\alpha-1}} & \text{if } \mathcal{D}_\alpha = \mathcal{F}_\alpha, \\ \frac{e\rho_1 A_u^\alpha}{\alpha-1} \sum_{i\neq i^*} \frac{1}{(\eta_t \hat{\underline{L}}_{t,i})^{\alpha-1}} & \text{if } \mathcal{D}_\alpha \in \mathfrak{D}_\alpha \setminus \{\mathcal{F}_\alpha\}. \end{cases}$$

The expression of $C_{1,1}(\mathcal{D}_\alpha)$ for general $\mathcal{D}_\alpha$ is given in the proof of this lemma in Appendix D.3, which is expressed in terms of A_u. For the adversarial bandits, we only utilize the bound with $K^{1/\alpha}$ in (17), which induces $\mathcal{O}(\sqrt{KT})$ regret by using learning rate $\eta_T = \mathcal{O}(K^{\frac{1}{\alpha}-\frac{1}{2}} T^{-\frac{1}{2}})$. Similarly to the stability term, we use ψ_p to apply the self-bounding technique for the stochastic bandits.

Remark 14 *Specifically, there are many characterizations and theories associated with general Frechet-type distributions, such as (i) von Mises conditions, (ii) slowly varying functions and (iii) Karamata's representation, from which we carefully chose an adequate one depending on the desired result. We also need to appropriately choose the representations based on (a) density function f, (b) tail distribution $1-F$, and (c) tail quantile function. For example, we choose (ii) with (b) and (c) for Lemma 10 and choose (i) and (iii) with (c) for Lemma 13.*

4.4. Proof of Theorem 4

By combining Lemmas 7, 8, 12 and 13 with $\eta_t = \frac{c}{\sqrt{t}} K^{\frac{1}{\alpha}-\frac{1}{2}}$, we have

$$\begin{aligned}\mathcal{R}(T) &\leq \frac{2\alpha\rho_2 m A_u c\sqrt{K}}{A_l(\alpha-1)} \sum_{t=1}^{T} \frac{1}{\sqrt{t}} + \frac{C_{1,1}(\mathcal{D}_\alpha)\sqrt{K}}{c} \sum_{t=1}^{T} \left(\sqrt{t+1} - \sqrt{t}\right) \\ &\qquad + 2C_2(\mathcal{D}_\alpha) \log\left(\sqrt{T+1}\right) + \frac{MA_u\sqrt{K}}{c} \\ &\leq \left(\frac{4\alpha\rho_2 m A_u c}{A_l(\alpha-1)} + \frac{C_{1,1}(\mathcal{D}_\alpha)}{c}\right)\sqrt{KT} + C_2(\mathcal{D}_\alpha)\log(T+1) + \frac{MA_u\sqrt{K}}{c},\end{aligned}$$

where letting $C_1(\mathcal{D}_\alpha, c) = \frac{4\alpha\rho_2 m A_u c}{A_l(\alpha-1)} + \frac{C_{1,1}(\mathcal{D}_\alpha)}{c}$ concludes the proof.

4.5. Proof sketch of Theorems 5 and 6

Since the overall proof for $\alpha \geq 2$ and $\alpha \in (1,2)$ are very similar, we provide a sketch for the case $\alpha \geq 2$. Let us begin by restating the regret in stochastic bandits, which is

$$\mathcal{R}(T) = \mathbb{E}\left[\sum_{t=1} \sum_{i\neq i^*} \Delta_i w_{t,i}\right].$$

To apply the proof techniques in Honda et al. (2023), we define an event D_t based on the tail quantile function where $\hat{L}_{t,i}$ is sufficiently large compared to that of the optimal arm so that $\hat{\underline{L}}_{t,i^*} = 0$.

In Appendix E.2, we show that the stability term corresponding to the optimal arm is bounded by $\mathcal{O}\Big(\sum_{i\neq i^*} 1/\hat{\underline{L}}_{t,i}\Big)$ on D_t, which provides for $\alpha \geq 2$

$$\mathcal{R}(T) \leq \mathbb{E}\left[\sum_{t=1}^{T} \mathcal{O}\left(\mathbb{1}[D_t]\sum_{i\neq i^*}\frac{1}{\hat{\underline{L}}_{t,i}} + \mathbb{1}[D_t^c]\sqrt{K/t}\right)\right].$$

To apply the self-bounding technique, we obtain

$$\mathcal{R}(T) \geq \mathbb{E}\left[\sum_{t=1}^{T} \Omega\left(\mathbb{1}[D_t]\sum_{i\neq i^*}\frac{t^{\frac{\alpha}{2}}\Delta_i}{K^{1-\frac{\alpha}{2}}\hat{\underline{L}}_{t,i}^{\alpha}} + \mathbb{1}[D_t^c]\Delta\right)\right],$$

where $\Delta = \min_{i\neq i^*}\Delta_i$, Ω denotes the big-Omega notation and the proof is given in Appendix E.1. By combining these results, we have

$$\frac{\mathcal{R}(T)}{2} \leq \mathbb{E}\left[\sum_{t=1}^{T} \mathcal{O}\left(\mathbb{1}[D_t]\sum_{i\neq i^*}\left(\frac{1}{\hat{\underline{L}}_{t,i}} - \frac{t^{\frac{\alpha}{2}}\Delta_i}{2K^{1-\frac{\alpha}{2}}\hat{\underline{L}}_{t,i}^{\alpha}}\right)\right)\right] + \mathbb{E}\left[\sum_{t=1}^{T} \mathcal{O}\Big(\mathbb{1}[D_t^c](\sqrt{K/t} - \Delta/2)\Big)\right].$$

Since $Ax - Bx^{\alpha} \leq A\frac{\alpha-1}{\alpha}\left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha-1}}$ holds for $A, B > 0$ and $\alpha > 1$, we obtain

$$\mathcal{R}(T) \leq \sum_{t=1}^{T} \mathcal{O}\left(\sum_{i\neq i^*}\frac{K^{\frac{2-\alpha}{2(\alpha-1)}}}{\Delta_i^{\frac{1}{\alpha-1}}t^{\frac{\alpha}{2(\alpha-1)}}}\right) + \mathcal{O}(K),$$

which concludes the proof. Note that the dependency on K in the leading term stems from the choice of learning rate.

5. Conclusion

In this paper, we considered FTPL policy with perturbations belonging to FMDA in the adversarial and stochastic settings. We provided a sufficient condition for perturbation distributions to achieve optimality, which solves the open problem by Kim and Tewari (2019) in a comprehensive direction. Furthermore, we provide the stochastic regret bound for FTPL, where Fréchet-type distributions with mild assumptions can achieve BOBW. While our analysis for FTPL with index $\alpha \neq 2$ does not attain logarithmic stochastic regrets, these findings align with observations in FTRL policies, offering insights that might help understand the effect of regularization of FTRL through the lens of FTPL.

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Appendix A. Details on extreme value theory

When contemplating the asymptotic properties of sample statistics, the sample means and central limit theorem often comes to mind, elucidating the behavior of partial sums of samples. Conversely, interest might shift towards extremes, focusing on maxima or minima of samples, particularly when singular rare events pose challenges, such as substantial insurance claims arising from catastrophic events like earthquakes and tsunamis. Extreme value theory is the field of studying the behavior of maxima of random variables, especially the behavior of the distribution function in the tail. One of the fundamental results in extreme value theory is the Fisher–Tippett–Gnedenko theorem, which provides a general result regarding the asymptotic distribution of normalized extreme order statistics of i.i.d. sequence of random variables.

Proposition 15 (Fisher–Tippett–Gnedenko theorem (Fisher and Tippett, 1928; Gnedenko, 1943)) *Let $M_n = \vee_{i=1}^n X_i$ where $\{X_i\}_{i=1}^n$ be an i.i.d. sequence of random variables with common distribution function $F(x)$. Suppose there exist $a_n > 0$, $b_n \in \mathbb{R}$, $n \geq 1$ such that*

$$\Pr[(M_n - b_n)/a_n \leq x] = F^n(a_n x + b_n) \to G(x),$$

weakly as $n \to \infty$ where G is assumed nondegenerate. Then, G is of the type of one of the following three classes:

(i) (Fréchet-type) $\Phi_\alpha(x) = \begin{cases} 0, & x < 0, \\ \exp(-x^{-\alpha}), & x \geq 0, \end{cases}$ *for some* $\alpha > 0$.

(ii) (Weibull-type) $\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^{\alpha}), & x < 0, \\ 1, & x \geq 0, \end{cases}$ *for some* $\alpha > 0$.

(iii) (Gumbel-type) $\Lambda(x) = \exp(-e^{-x})$ *for* $x \in \mathbb{R}$.

Among these three types of extreme value distributions, we are interested in Fréchet-type distributions, where the equivalence was established in Proposition 2, which states

$$F^n(a_n x) \to \Phi_\alpha(x)$$

with $a_n = \inf\left\{x : F(x) \geq 1 - \frac{1}{n}\right\}$.

However, verifying whether a distribution belongs to a domain of attraction can often be challenging. Therefore, a convenient sufficient condition, known as the von Mises condition, is often considered (von Mises, 1936; Beirlant et al., 2006), which is

$$\lim_{x \to \infty} \frac{x f(x)}{1 - F(x)} = \alpha.$$

It is worth noting that $\mathfrak{D}_\alpha^{\text{all}}$ consists of distributions satisfying von Mises condition and their tail-equivalent distributions (Embrechts et al., 1997).

Existence of the density Here, it is known that if $g \in \mathrm{RV}_\alpha$, for $\alpha \neq 0$, then there exists g^* that is absolutely continuous, strictly monotone, and $g(x) \sim g^*(x)$ as $x \to \infty$, i.e., tail-equivalent (Resnick, 2008, Propostition 0.8.). Therefore, Assumption 1 implies that we fix our interest solely on distribution with a continuous density among their tail-equivalent distributions.

Tail quantile function When $1 - F \in \mathrm{RV}_{-\alpha}$, its tail quantile function U is regularly varying with index $\frac{1}{\alpha}$, i.e., $U \in \mathrm{RV}_{1/\alpha}$, where $U(t) = \inf\{x : F(x) \geq 1 - 1/t\}$ on $[1, \infty)$ (Beirlant et al., 2006). Therefore, one can directly obtain that $a_n = n^{\frac{1}{\alpha}} S_U(n)$, where S_U denotes the corresponding slowly varying function. Here, it is known that S_U is the de Bruijn conjugate (or de Bruyn in some literature) of $S_F^{-1/\alpha}$, which satisfies $S_U(x) S_F^{-1/\alpha}(x S_U(x)) \sim S_F^{-1/\alpha}(x) S_U(x S_F^{-1/\alpha}(x)) \to 1$. This implies that if S_F is upper-bounded by some constants, then S_U is also upper-bounded regardless of K. For more details, we refer readers to Charras-Garrido and Lezaud (2013), which provides a concise introduction to the extreme value theory.

Karamata's theorem Since all tail distributions in FMDA are regularly varying, the following results are useful to represent the regularly varying functions.

Proposition 16 (Karamata's theorem (Haan and Ferreira, 2006, Theorem B.1.5)) *Suppose $f \in \mathrm{RV}_\alpha$. There exists $t_0 > 0$ such that $g(t)$ is positive and locally bounded for $t \geq t_0$. If $\alpha \geq -1$, then*

$$\lim_{t\to\infty} \frac{t g(t)}{\int_{t_0}^{t} g(s)\mathrm{d}s} = \alpha + 1.$$

If $\alpha < -1$ and $\int_0^\infty g(s)\mathrm{d}s < \infty$, then

$$\lim_{t\to\infty} \frac{t g(t)}{\int_{t}^{\infty} g(s)\mathrm{d}s} = -\alpha - 1. \tag{18}$$

Conversely, if (18) holds with $\alpha \in (-\infty, -1)$, then $g \in \mathrm{RV}_\alpha$.

Therefore, one can see that von Mises condition and the existence of density imply $f \in \mathrm{RV}_{-\alpha-1}$. Furthermore, from (18), Assumption 4 is equivalent to $-f' \in \mathrm{RV}_{-\alpha-2}$ and boundedness of $-f'(x)/f(x)$.

A.1. Karamata's representation

From (3), one can specify a distribution in FMDA with index α and the slowly varying function $S_F(x)$. Here, several representations of slowly varying functions can be considered (Galambos and Seneta, 1973), and we follow Karamata's representation described in Resnick (2008), which is

$$S_F(x) = c(x) \exp\left(\int_1^x \frac{\varepsilon_F(t)}{t} \mathrm{d}t \right), \quad x \geq 1 \tag{19}$$

where $c(x)$ and $\varepsilon(x)$ are bounded functions such that $\lim_{x\to\infty} c(x) = c > 0$ and $\lim_{x\to\infty} \varepsilon_F(x) = 0$. Here, the representation is not unique and it depends on the choice of $c(x)$, $\varepsilon_F(x)$, and the interval of the integral. For example, $c(x)$ and $\varepsilon_F(x)$ can be written as (Resnick, 2008, Corollary of Theorem 0.6.)

$$c(x) = \frac{x S_F(x)}{\int_0^x S_F(t)\mathrm{d}t} \int_0^1 S_F(t)\mathrm{d}t,$$
$$\varepsilon_F(x) = \frac{x S_F(x)}{\int_0^x S_F(t)\mathrm{d}t} - 1.$$

One can check that $\lim_{x\to\infty} \varepsilon_F(x) \to 0$ from Proposition 16 with $\alpha = 0$.

On the other hand, when F is absolutely continuous, we can rewrite the tail distribution as for $x \geq 1$

$$1 - F(x) = \exp(\log(1 - F(x))) = \exp\left(\int_1^x \frac{-f(t)}{1 - F(t)} \mathrm{d}t\right). \tag{20}$$

Since $1 - F(x) = x^{-\alpha} S_F(x)$ holds for $x \geq 1$, it holds that

$$\begin{aligned} S_F(x) = x^\alpha(1 - F(x)) = x^\alpha \exp\left(\int_1^x \frac{-f(t)}{1 - F(t)} \mathrm{d}t\right) & \quad \text{by (20)} \\ = \exp\left(\alpha \log x - \int_1^x \frac{f(t)}{1 - F(t)} \mathrm{d}t\right) & \\ = \exp\left(\int_1^x \frac{\alpha}{t} \mathrm{d}t - \int_1^x \frac{f(t)}{1 - F(t)} \mathrm{d}t\right). & \end{aligned}$$

By letting $\varrho(t) = \frac{tf(t)}{1-F(t)}$, we obtain

$$\begin{aligned} S_F(x) &= \exp\left(\int_1^x \frac{\alpha}{t} \mathrm{d}t - \int_1^x \frac{\varrho(t)}{t} \mathrm{d}t\right) \\ &= \exp\left(\int_1^x \frac{\alpha - \varrho(t)}{t} \mathrm{d}t\right). \end{aligned} \tag{21}$$

Here, from the definition of ϱ, von Mises condition can be written as $\varrho(t) \to \alpha$, as $t \to \infty$, which satisfies $\lim_{t\to\infty} \alpha - \varrho(t) = 0$ and thus indicates the existence of the upper bound of S_F. In this paper, we use the representation of S_F in (21), where $c(x)$ is given as the ultimate constant. Therefore, when F satisfies (14), one can see that S_F is monotonically increasing for $x \geq 1$. Note that $\varepsilon_F(t)$ in (19) are not necessarily the same as $\varrho(t) - \alpha$ unless $c(x) = 1 - F(1)$.

The von Mises condition (6) with Assumption 1 implies $f \in \mathrm{RV}_{-1-\alpha}$ from Proposition 16 (see Embrechts et al., 1997, Proposition A3.8), i.e., $f = x^{-\alpha+1} S_f(x)$. Therefore, from $1 - F(x) = x^{-\alpha} S_F(x)$ with (21), we have

$$f(x) = \frac{S_F(x)\alpha}{x^{\alpha+1}} - \frac{S_F'(x)}{x^\alpha} = \frac{S_F(x)}{x^{\alpha+1}} \varrho(x),$$

which implies

$$S_f(x) = S_F(x)\varrho(x). \tag{22}$$

One can check that $S_f \in \mathrm{RV}_0$ since $\lim_{t\to\infty} \varrho(x) = \alpha$ holds by von Mises condition and $S_F \in \mathrm{RV}_0$.

A.2. Proofs for Table 2

It is straightforward to check whether the Fréchet, Pareto, and Generalized Pareto satisfy the assumptions. Therefore, we showed that the Student-t distribution satisfies Assumption 3 but not Assumption 5 and the Snedecor's F distribution satisfies all Assumptions. In this section, we prove Assumption 3 by showing $\frac{xf(x)}{1-F(x)} \leq \alpha$, which implies $S_F(x)$ is increasing so that satisfying the sufficient condition (8).

A.2.1. STUDENT-t

Since it is easy to verify Assumptions 1, 2 and 4, we focus on Assumptions 3 and 5

Assumption 3 Here, we show that (14) holds. Since $\frac{xf(x)}{1-F(x)} \leq 0$ is obvious for $x \leq 0$, let us consider the case $x > 0$. In this case, $1 - F(x) = \frac{1}{2}\frac{B\left(\frac{n}{x^2+n}; \frac{n}{2}, \frac{1}{2}\right)}{B\left(\frac{n}{2}, \frac{1}{2}\right)}$ holds. Therefore,

$$
\begin{aligned}
\frac{xf(x)}{1-F(x)} &= \frac{x\left(\frac{n+x^2}{n}\right)^{-\frac{n+1}{2}}}{\sqrt{n}B\left(\frac{n}{2}, \frac{1}{2}\right)} \left(\frac{1}{2}\frac{B\left(\frac{n}{x^2+n}; \frac{n}{2}, \frac{1}{2}\right)}{B\left(\frac{n}{2}, \frac{1}{2}\right)}\right)^{-1} \\
&= \frac{2}{\sqrt{n}} \frac{x\left(\frac{x^2+n}{n}\right)^{-\frac{n+1}{2}}}{B\left(\frac{n}{x^2+n}; \frac{n}{2}, \frac{1}{2}\right)} \\
&= \frac{2}{\sqrt{n}} \frac{x\left(\frac{x^2+n}{n}\right)^{-\frac{n+1}{2}}}{\frac{2}{n}\left(\frac{n}{x^2+n}\right)^{\frac{n}{2}}\left(\frac{x^2}{x^2+n}\right)^{\frac{1}{2}} {}_2F_1\left(\frac{n+1}{2}, 1; \frac{n+2}{2}; \frac{n}{x^2+n}\right)} \qquad (23) \\
&= \frac{n}{{}_2F_1\left(\frac{n+1}{2}, 1; \frac{n+2}{2}; \frac{n}{x^2+n}\right)},
\end{aligned}
$$

In (23), we used the results in (Olver et al., 2010, 8.17.8) that provide the relationship between the incomplete Beta function and the (Gaussian) hypergeometric function ${}_2F_1$, which is

$$
B(x; a, b) = \frac{x^a(1-x)^b}{a} {}_2F_1(a+b, 1; a+1; x). \qquad (24)
$$

Here, the hypergeometric function is defined by the Gauss series, which is defined for $|x| < 1$ and $c > 0$ by

$$
{}_2F_1(a, b; c; x) = \sum_{s=0}^{\infty} \frac{(a)_s(b)_s}{(c)_s s!} x^s = 1 + \frac{ab}{c}z + \cdots,
$$

where $(a)_n$ denotes the rising factorial, i.e., $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$. Therefore, we have for $x \geq 0$

$$
\frac{xf(x)}{1-F(x)} \leq n,
$$

which verifies that $\mathcal{T}_n$ satisfies Assumption 3 by (14). Here, one can see that the hazard function $\frac{f(x)}{1-F(x)}$ diverges as $x \to 0$, while $\frac{f^*(x)}{1-F^*(x)} \leq n$ holds.

Assumption 5 Since the density of $\mathcal{T}_n$ is symmetric, it holds for any $t \geq 0$ that

$$
f(t) = f(-t), \qquad F(t) = 1 - F(-t).
$$

Then, we have

$$
\frac{f(-t)}{F(-t)} = \frac{f(t)}{1-F(t)} \geq \frac{f(t)}{F(t)},
$$

where the inequality follows from $F(t) \geq \frac{1}{2}$ for $t \geq 0$. Therefore, $\mathcal{T}_n$ does not satisfy Assumption 5. However, when one considers only for $t \geq 0$, $f(t)$ is decreasing while $F(t)$ is increasing, which implies that f/F is decreasing for $x \geq 0$. This implies that both the half-t distribution, $|\mathcal{T}_n|$ and truncated one in (7) satisfy Assumption 5.

A.2.2. F DISTRIBUTION

Since it is easy to verify Assumptions 1, 2 and 4, we focus on Assumptions 3 and 5

Assumption 3 Here, we show that (14) holds. Let $I(x;a,b) = \frac{B(x;a,b)}{B(a,b)}$ denote the regularized incomplete beta function. From the definition of the incomplete beta function, one can see that $I(x;a,b) = 1 - I(1-x;b,a)$ holds. Then, it holds that

$$\frac{xf(x)}{1-F(x)} = \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} x^{\frac{m}{2}} \left(\frac{mx+n}{n}\right)^{-\frac{m+n}{2}}}{B\left(\frac{m}{2}, \frac{n}{2}\right) I\left(\frac{n}{mx+n}; \frac{n}{2}, \frac{m}{2}\right)}.$$

Since $B(a,b) = B(b,a)$, we obtain

$$\begin{aligned}
\frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} x^{\frac{m}{2}} \left(\frac{mx+n}{n}\right)^{-\frac{m+n}{2}}}{B\left(\frac{m}{2}, \frac{n}{2}\right) I\left(\frac{n}{mx+n}; \frac{n}{2}, \frac{m}{2}\right)} &= \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} x^{\frac{m}{2}} \left(\frac{mx+n}{n}\right)^{-\frac{m+n}{2}}}{B\left(\frac{n}{mx+n}; \frac{n}{2}, \frac{m}{2}\right)} \\
&= \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} x^{\frac{m}{2}} \left(\frac{mx+n}{n}\right)^{-\frac{m+n}{2}}}{\frac{2}{n}\left(\frac{n}{mx+n}\right)^{\frac{n}{2}} \left(\frac{mx}{mx+n}\right)^{\frac{m}{2}} {}_2F_1\left(\frac{m+n}{2}, 1; \frac{n}{2}+1; \frac{n}{mx+n}\right)} && \text{by (24)} \\
&= \frac{n}{2} \frac{1}{{}_2F_1\left(\frac{m+n}{2}, 1; \frac{n}{2}+1; \frac{n}{mx+n}\right)} \leq \frac{n}{2},
\end{aligned}$$

which verifies Assumption 3 by (14). Here, one can observe that the hazard function $\frac{f(x)}{1-F(x)}$ diverges as $x \to 0$, while $\frac{f^*(x)}{1-F^*(x)} \leq n$ holds.

Assumption 5 If f/F is monotonically decreasing, it should hold that for any $x \geq y > 0$

$$\frac{F(y)}{F(x)} \leq \frac{f(y)}{f(x)}.$$

Here, it holds that

$$\begin{aligned}
\frac{F(y)}{F(x)} = \frac{B\left(\frac{my}{my+n}; \frac{m}{2}, \frac{n}{2}\right)}{B\left(\frac{mx}{my+n}; \frac{m}{2}, \frac{n}{2}\right)} &= \frac{\left(\frac{my}{my+n}\right)^{\frac{m}{2}} \left(\frac{n}{my+n}\right)^{\frac{n}{2}} {}_2F_1\left(\frac{m+n}{2}, 1; 1+\frac{m}{2}; \frac{my}{my+n}\right)}{\left(\frac{mx}{mx+n}\right)^{\frac{m}{2}} \left(\frac{n}{mx+n}\right)^{\frac{n}{2}} {}_2F_1\left(\frac{m+n}{2}, 1; 1+\frac{m}{2}; \frac{mx}{mx+n}\right)} && \text{by (24)} \\
&= \left(\frac{y}{x}\right)^{\frac{m}{2}} \left(\frac{mx+n}{my+n}\right)^{\frac{m+n}{2}} \frac{{}_2F_1\left(\frac{m+n}{2}, 1; 1+\frac{m}{2}; \frac{my}{my+n}\right)}{{}_2F_1\left(\frac{m+n}{2}, 1; 1+\frac{m}{2}; \frac{mx}{mx+n}\right)},
\end{aligned}$$

Therefore, we have for any $x \geq y > 0$

$$\frac{F(y)}{F(x)} \leq \left(\frac{y}{x}\right)^{\frac{m}{2}} \left(\frac{mx+n}{my+n}\right)^{\frac{m+n}{2}}.$$

since $\frac{mx}{mx+n}$ is increasing with respect to $x > 0$. We have for $x \geq y > 0$

$$\frac{f(y)}{f(x)} = \left(\frac{y}{x}\right)^{\frac{m}{2}-1}\left(\frac{mx+n}{my+n}\right)^{\frac{m+n}{2}} = \left(\frac{y}{x}\right)^{\frac{m}{2}}\left(\frac{mx+n}{my+n}\right)^{\frac{m+n}{2}}\frac{x}{y}$$
$$\geq \left(\frac{y}{x}\right)^{\frac{m}{2}}\left(\frac{mx+n}{my+n}\right)^{\frac{m+n}{2}} \geq \frac{F(y)}{F(x)},$$

which verifies Assumption 5.

Appendix B. Proofs for regret decomposition

Here, we provide the proofs for Lemmas 7 and 8.

B.1. Proof of Lemma 7

Firstly, we present the regret decomposition that can be applied to general distributions.

Lemma 17 (Lemma 3 of Honda et al. (2023))

$$\mathcal{R}(T) \leq \sum_{t=1}^{T}\mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1}\right\rangle\right] + \sum_{t=1}^{T}\left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right)\mathbb{E}_{r_{t+1}\sim\mathcal{D}}\left[r_{t+1,I_{t+1}} - r_{t+1,i^*}\right] + \frac{1}{\eta_1}\mathbb{E}_{r_1\sim\mathcal{D}}[r_{1,I_1}],$$

where $r_{1,I_1} = \max_{i\in[K]} r_{1,i}$.

Here, notice that $\mathbb{E}_{r_1\sim\mathcal{D}}[r_{1,I_1}]$ is the expected block maxima when K samples are given. For the Fréchet distributions and Pareto distributions, we can explicitly compute the upper bound $\mathbb{E}[M_K]$ as follows.

Lemma 18 *For* $\alpha > 1$,

$$\mathbb{E}_{r_{1,1},\dots,r_{1,K}\sim\mathcal{D}_\alpha}[r_{1,I_1}] \leq \begin{cases} MA_u K^{\frac{1}{\alpha}} & \text{if } \mathcal{D}_\alpha \in \mathfrak{D}_\alpha \\ K^{\frac{1}{\alpha}}\Gamma\left(1-\frac{1}{\alpha}\right) & \text{if } \mathcal{D}_\alpha = \mathcal{F}_\alpha, \\ K^{\frac{1}{\alpha}}\Gamma\left(1-\frac{1}{\alpha}\right)\frac{\alpha}{\alpha-1} & \text{if } \mathcal{D}_\alpha = \mathcal{P}_\alpha. \end{cases}$$

Proof The proof for the general $\mathfrak{D}_\alpha$ can be directly obtained by (4) in Assumption 3. As explained in Appendix A, the tail quantile function U is regularly varying with index $\frac{1}{\alpha}$, which implies

$$a_K = K^{\frac{1}{\alpha}} S_U(K)$$

for some $S_U \in \mathrm{RV}_0$. Thus, Assumption 3 implies the boundedness of S_U. Here, from the definition of a_K, it holds that

$$1 - F(a_K) = \frac{1}{K} = \frac{S_F(a_K)}{(a_K)^\alpha},$$

which implies

$$a_K = K^{\frac{1}{\alpha}} S_F^{\frac{1}{\alpha}}(a_K). \tag{25}$$

Therefore, $S_U(K) = S_F^{\frac{1}{\alpha}}(a_K)$ holds. The upper-bounded assumption (the existence of A_u) is not restrictive from Karamata's representation with von Mises condition in (21), where $\frac{\alpha-\varrho(t)}{t} \to 0$ as $t \to \infty$ and $c(x)$ is given as ultimate constants.

Case 1. Fréchet distribution It is well-known that when $X_i \sim \mathcal{F}(\alpha, s, m)$ where $(\alpha, s, m) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ denotes the shape, scale, and location of the Fréchet distribution, then $Y = \max(X_1, \ldots, X_n)$ follows $\mathcal{F}(\alpha, n^{1/\alpha}, m)$. One can easily check by observing its CDF is given by e^{-K/x^α} or the max-stability of Fréchet distributions. The fact that the expected value of $\mathcal{F}(\alpha, s, m) = m + s\Gamma(1 - \frac{1}{\alpha})$ for $\alpha > 1$ and $\mathcal{F}_\alpha = \mathcal{F}(\alpha, 1, 0)$ completes the proof.

Case 2. Pareto distribution Since $r_{1,I_1} = \max_{i\in[K]} r_{1,i}$, its CDF is $(1 - z^{-\alpha})^K$ with density $\frac{\alpha K}{z^{\alpha+1}}(1 - z^{-\alpha})^{K-1}$. By letting $w = z^{-\alpha}$,

$$\begin{aligned}
\mathbb{E}_{r\sim\mathcal{F}_\alpha}[r_{1,I_1}] &= \int_1^\infty \frac{\alpha K}{z^\alpha}(1 - z^{-\alpha})^{K-1}\mathrm{d}z \\
&= K\int_0^1 w^{-\frac{1}{\alpha}}(1 - w)^{K-1}\mathrm{d}w \\
&= KB\left(1 - \frac{1}{\alpha}, K\right) = K\frac{\Gamma\big(1 - \frac{1}{\alpha}\big)\Gamma(K)}{\Gamma\big(K + 1 - \frac{1}{\alpha}\big)},
\end{aligned} \tag{26}$$

where $B(z_1, z_2) := \int_0^1 w^{z_1-1}(1 - w)^{z_2-1}\mathrm{d}w$ denotes the Beta function. Then, by applying Lemma 27, Gautschi's inequality, we obtain for $\alpha > 1$

$$\begin{aligned}
\frac{\Gamma\big(1 - \frac{1}{\alpha}\big)\Gamma(K + 1)}{\Gamma\big(K + 1 - \frac{1}{\alpha}\big)} &= \frac{K}{K - \frac{1}{\alpha}}\frac{\Gamma(K)}{\Gamma\big(K - \frac{1}{\alpha}\big)} \\
&\leq \frac{K}{K - \frac{1}{\alpha}}\Gamma\left(1 - \frac{1}{\alpha}\right)K^{\frac{1}{\alpha}} \\
&\leq \frac{\alpha}{\alpha - 1}\Gamma\left(1 - \frac{1}{\alpha}\right)K^{\frac{1}{\alpha}},
\end{aligned}$$

where the last inequality follows from $K \geq 1$. Here, one can directly apply Gautschi's inequality in (26), which results in $\Gamma\big(1 - \frac{1}{\alpha}\big)(K + 1)^{\frac{1}{\alpha}}$. ■

B.2. Proof of Lemma 8

From the definition of $w_t = \phi(\eta_t \hat{L}_t; \mathcal{D}_\alpha)$, we have

$$\begin{aligned}
w_t - w_{t+1} &= \phi(\eta_t\hat{L}_t) - \phi(\eta_{t+1}\hat{L}_{t+1}) \\
&= \phi(\eta_t\hat{L}_t) - \phi(\eta_{t+1}(\hat{L}_t + \hat{\ell}_t)) \\
&= \phi(\eta_t\hat{L}_t) - \phi(\eta_t(\hat{L}_t + \hat{\ell}_t)) + \phi(\eta_t(\hat{L}_t + \hat{\ell}_t)) - \phi(\eta_{t+1}(\hat{L}_t + \hat{\ell}_t)),
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_{t=1} \mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1}\right\rangle\right] \leq &\sum_{t=1}^T \mathbb{E}\left[\left\langle \hat{\ell}_t, \phi(\eta_t\hat{L}_t) - \phi(\eta_t(\hat{L}_t) + \hat{\ell}_t)\right\rangle\right] \\
&+ \sum_{t=1}^T \mathbb{E}\left[\left\langle \hat{\ell}_t, \phi(\eta_t(\hat{L}_t + \hat{\ell}_t)) - \phi(\eta_{t+1}(\hat{L}_t + \hat{\ell}_t))\right\rangle\right].
\end{aligned} \tag{27}$$

Therefore, it remains to bound the second term of (27).

Case 1. Fréchet distribution By explicitly substituting the density function and CDF of $\mathcal{F}_\alpha$, $\phi_i(\lambda; \mathcal{F}_\alpha)$ is expressed by

$$\phi_i(\lambda;\mathcal{F}_\alpha) := \Pr_{r\sim\mathcal{F}_\alpha}\left[i = \arg\min_{j\in[K]}\{\lambda_j - r_j\}\right] = \int_{-\min_{j\in[K]}\lambda_j}^{\infty} \frac{\alpha}{(z+\lambda_i)^{\alpha+1}} \exp\left(-\sum_{l\in[K]}\frac{1}{(z+\lambda_l)^\alpha}\right)\mathrm{d}z$$

$$= \int_0^\infty \frac{\alpha}{(z+\underline{\lambda}_i)^{\alpha+1}} \exp\left(-\sum_{l\in[K]}\frac{1}{(z+\underline{\lambda}_l)^\alpha}\right)\mathrm{d}z,$$

Then, for generic $L \in \mathbb{R}^K$, $\underline{L} = L - \mathbf{1}\min_i L_i$, and any $i \in [K]$

$$\begin{aligned}
&\frac{\partial}{\partial\eta}\phi_i(\eta L;\mathcal{F}_\alpha)\\
&= \alpha\int_0^\infty \left[\frac{1}{(z+\eta\underline{L}_i)^{\alpha+1}}\sum_{j\in[K]}\frac{\alpha\underline{L}_j}{(z+\eta\underline{L}_j)^{\alpha+1}} - \frac{(\alpha+1)\underline{L}_i}{(z+\eta\underline{L}_i)^{\alpha+2}}\right]\exp\left(-\sum_{j\in[K]}\frac{1}{(z+\eta\underline{L}_j)^\alpha}\right)\mathrm{d}z\\
&\le \alpha\int_0^\infty \frac{1}{(z+\eta\underline{L}_i)^{\alpha+1}}\exp\left(-\sum_{j\in[K]}\frac{1}{(z+\eta\underline{L}_j)^\alpha}\right)\sum_{j\in[K]}\frac{\alpha\underline{L}_j}{(z+\eta\underline{L}_j)^{\alpha+1}}\mathrm{d}z\\
&\le \alpha\int_0^\infty \frac{1}{(z+\eta\underline{L}_i)^{\alpha+1}}\exp\left(-\sum_{j\in[K]}\frac{1}{(z+\eta\underline{L}_j)^\alpha}\right)\max_{l\in[K]}\frac{\alpha\underline{L}_l}{(z+\eta\underline{L}_l)}\sum_{j\in[K]}\frac{1}{(z+\eta\underline{L}_j)^{\alpha}}\mathrm{d}z\\
&\le \alpha\int_0^\infty \frac{1}{(z+\eta\underline{L}_i)^{\alpha+1}}\exp\left(-\sum_{j\in[K]}\frac{1}{(z+\eta\underline{L}_j)^\alpha}\right)\frac{\alpha}{\eta}\sum_{j\in[K]}\frac{1}{(z+\eta\underline{L}_j)^{\alpha}}\mathrm{d}z.
\end{aligned}$$

Let $L = \hat{L}_t + \hat{\ell}_t$. Since $\hat{\ell}_t = l_t\widehat{w_t^{-1}}e_{I_t}$ and $l_{t,i} \in [0,1]$,

$$\begin{aligned}
&\sum_{t=1}^T \mathbb{E}\left[\left\langle \hat{\ell}_t, \phi(\eta_t(\hat{L}_t+\hat{\ell}_t);\mathcal{F}_\alpha) - \phi(\eta_{t+1}(\hat{L}_t+\hat{\ell}_t));\mathcal{F}_\alpha\right\rangle\right] \qquad (28)\\
&\quad= \sum_{t=1}^T\sum_{i\in[K]}\mathbb{E}\left[\mathbb{1}[I_t=i]l_{t,i}\widehat{w_{t,i}^{-1}}(\phi_i(\eta_t L;\mathcal{F}_\alpha) - \phi_i(\eta_{t+1}L;\mathcal{F}_\alpha))\right]\\
&\quad= \sum_{t=1}^T\mathbb{E}\left[\int_{\eta_{t+1}}^{\eta_t}\sum_{i\in[K]} l_{t,i}\frac{\partial}{\partial\eta}\phi_i(\eta L;\mathcal{F}_\alpha)\mathrm{d}\eta\right]\\
&\quad\le \alpha\sum_{t=1}^T\mathbb{E}\left[\int_{\eta_{t+1}}^{\eta_t}\frac{1}{\eta}\int_0^\infty\sum_{i\in[K]} l_{t,i}\frac{\alpha}{(z+\eta\underline{L}_i)^{\alpha+1}}\exp\left(-\sum_{j\in[K]}\frac{1}{(z+\eta\underline{L}_j)^\alpha}\right)\sum_{j\in[K]}\frac{1}{(z+\eta\underline{L}_j)^\alpha}\mathrm{d}z\mathrm{d}\eta\right]\\
&\quad\le \alpha\sum_{t=1}^T\mathbb{E}\left[\int_{\eta_{t+1}}^{\eta_t}\frac{1}{\eta}\int_0^\infty we^{-w}\mathrm{d}w\mathrm{d}\eta\right] \qquad \left(\because \ell_{t,i}\le 1 \text{ and } w = \sum_{j\in[K]}\frac{1}{(z+\eta\underline{L}_j)^\alpha}\right)\\
&\quad= \alpha\sum_{t=1}^T\log\left(\frac{\eta_t}{\eta_{t+1}}\right) = \alpha\log\left(\frac{\eta_1}{\eta_{T+1}}\right).
\end{aligned}$$

Case 2. Distributions in $\mathfrak{D}_\alpha$ From the definition of ϕ in (1), for generic $L \in \mathbb{R}^K$, $\underline{L} = L - \mathbf{1}\min_i L_i$, and any $i \in [K]$

$$\frac{\partial}{\partial \eta}\phi_i(\eta L) = \int_1^\infty \underline{L}_i f'(z+\eta\underline{L}_i)\prod_{j\neq i} F(z+\eta\underline{L}_j)\mathrm{d}z + \int_1^\infty \sum_{j\neq i}\left(\underline{L}_j f(z+\eta\underline{L}_i) f(z+\eta\underline{L}_j)\prod_{l\neq i,j} F(z+\eta\underline{L}_l)\right)\mathrm{d}z. \quad (29)$$

Recall the definition of $\varrho(x) = \frac{xf(x)}{1-F(x)}$, which implies

$$f(x) = \frac{\varrho(x)}{x}(1-F(x)). \quad (30)$$

Then, the first term of (29) can be bounded by

$$\begin{aligned}
&\int_1^\infty \underline{L}_i f'_\alpha(z+\eta\underline{L}_i)\prod_{j\neq i} F(z+\eta\underline{L}_j)\mathrm{d}z \\
&\quad\leq \int_1^{z_0} \underline{L}_i f'_\alpha(z+\eta\underline{L}_i)\prod_{j\neq i} F(z+\eta\underline{L}_j)\mathrm{d}z && \text{(by Assumption 1)} \\
&\quad= \underline{L}_i f(z+\eta\underline{L}_i)\prod_{j\neq i} F(z+\eta\underline{L}_j)\Big|_{z=1}^{z=z_0} - \int_1^{z_0} \underline{L}_i f(z+\eta\underline{L}_i)\sum_{j\neq i} f(z+\eta\underline{L}_j)\prod_{l\neq i,j} F(z+\eta\underline{L}_l)\mathrm{d}z \\
&\quad\leq \underline{L}_i f(z_0+\eta\underline{L}_i)\prod_{j\neq i} F(z_0+\eta\underline{L}_j) \\
&\quad\leq \frac{\underline{L}_i\varrho(z_0+\eta\underline{L}_i)}{z_0+\eta\underline{L}_i}(1-F(z_0+\eta\underline{L}_i)) \leq \frac{\rho_1}{\eta}, && \text{(by (11) and (30))}
\end{aligned}$$

Next, for the second term of (29), by representation in (21), we obtain

$$\begin{aligned}
\prod_{l\neq i,j} F(z+\eta\underline{L}_l)) &= \prod_{l\neq i,j}\left(1 - \frac{S_F(z+\eta\underline{L}_l)}{(z+\eta\underline{L}_l)^\alpha}\right) \\
&\leq \exp\left(-\sum_{l\neq i,j}\frac{S_F(z+\eta\underline{L}_l)}{(z+\eta\underline{L}_l)^\alpha}\right) && (\because 1-x\leq e^{-x}, \forall x\geq 0) \\
&\leq e^2\exp\left(-\sum_{j\in[K]}(1-F(z+\eta\underline{L}_j))\right),
\end{aligned}$$

where the last inequality follows from $F(x)\in[0,1]$ for all $x\in[1,\infty)$, i.e., $e^{1-F(x)}\leq e$ for any $x\in[1,\infty)$.

Then, we have

$$\begin{aligned}
&\int_1^\infty \sum_{j\neq i}\left(\underline{L}_j f(z+\eta\underline{L}_i) f(z+\eta\underline{L}_j)\prod_{l\neq i,j} F(z+\eta\underline{L}_l)\right)\mathrm{d}z \\
&\quad\le e^2\int_1^\infty f(z+\eta\underline{L}_i)\sum_{j\in[K]}\big(\underline{L}_j f(z+\eta\underline{L}_j)\big)\exp\left(-\sum_{l\in[K]}(1-F(z+\eta\underline{L}_j))\right)\mathrm{d}z \\
&\quad= e^2\int_1^\infty f(z+\eta\underline{L}_i)\left(\sum_{j\in[K]}\underline{L}_j f(z+\eta\underline{L}_j)\right)\exp\left(-\sum_{j\in[K]}(1-F(z+\eta\underline{L}_j))\right)\mathrm{d}z.
\end{aligned}$$

Here, by (30) again, for generic $L\in\mathbb{R}^K$, we obtain for $z\in[1,\infty)$

$$\begin{aligned}
\sum_{j\in[K]}\underline{L}_j f(z+\eta\underline{L}_j)) &= \sum_{j\in[K]}\frac{\underline{L}_j\varrho(z+\eta\underline{L}_j)}{z+\eta\underline{L}_j}(1-F(z+\eta\underline{L}_j)) \\
&\le \sum_{j\in[K]}\frac{\underline{L}_j\rho_1}{z+\eta\underline{L}_j}(1-F(z+\eta\underline{L}_j)) \le \sum_{j\in[K]}\frac{\rho_1}{\eta}(1-F(z+\eta\underline{L}_j)),
\end{aligned}$$

which implies

$$\frac{\partial}{\partial\eta}\phi_i(\eta L)\le\frac{\rho_1 e^2}{\eta}\int_1^\infty f(z+\eta\underline{L}_i)\left(\sum_{j\in[K]}(1-F(z+\eta\underline{L}_j))\right)\exp\left(-\sum_{j\in[K]}(1-F(z+\eta\underline{L}_j))\right)\mathrm{d}z+\frac{\rho_1}{\eta}.$$

By noticing that

$$\sum_{i\in[K]}-f(z+\eta\underline{L}_i)=\frac{\mathrm{d}}{\mathrm{d}z}\sum_{j\in[K]}(1-F(z+\eta\underline{L}_j)),$$

one can reproduce the proof in Case 1 from (28), which implies

$$\begin{aligned}
&\sum_{t=1}^T\mathbb{E}\Big[\Big\langle\hat{\ell}_t,\phi(\eta_t(\hat{L}_t+\hat{\ell}_t);\mathcal{D}_\alpha)-\phi(\eta_{t+1}(\hat{L}_t+\hat{\ell}_t);\mathcal{D}_\alpha)\Big\rangle\Big] \\
&\quad\le\rho_1\sum_{t=1}^T\mathbb{E}\Bigg[l_{t,i}\int_{\eta_{t+1}}^{\eta_t}\frac{e^2}{\eta}\Bigg\{\int_1^\infty\sum_{i\in[k]}f(z+\eta\underline{L}_i) \\
&\qquad\qquad\cdot\left(\sum_{j\in[K]}(1-F(z+\eta\underline{L}_j))\right)\exp\left(-\sum_{j\in[K]}(1-F(z+\eta\underline{L}_j))\right)\mathrm{d}z\Bigg\}+\frac{1}{\eta}\mathrm{d}\eta\Bigg] \\
&\quad\le\rho_1\sum_{t=1}^T\mathbb{E}\left[\int_{\eta_{t+1}}^{\eta_t}\frac{1}{\eta}\left\{e^2\int_0^K we^{-w}\mathrm{d}w+1\right\}\mathrm{d}\eta\right] \\
&\quad\le\rho_1\big(e^2+1\big)\sum_{t=1}^T\log\left(\frac{\eta_t}{\eta_{t+1}}\right)=\rho_1\big(e^2+1\big)\log\left(\frac{\eta_1}{\eta_{T+1}}\right).
\end{aligned}$$

Appendix C. Regret bound for adversarial bandits: Stability

Here, we provide the proofs for Lemmas 9–12.

C.1. Proof of Lemma 9: monotonicity

Let us consider the Fréchet distributions first.

C.1.1. FRÉCHET DISTRIBUTION

From the definitions of ϕ and I,

$$\phi_i(\lambda; \mathcal{F}_\alpha) = \alpha I_{i,\alpha+1}(\underline{\lambda}; \alpha), \qquad \phi_i'(\lambda; \mathcal{F}_\alpha) = -\alpha(\alpha+1) I_{i,\alpha+2}(\underline{\lambda}; \alpha) + \alpha^2 I_{i,2(\alpha+1)}(\underline{\lambda}; \alpha). \quad (31)$$

Here, $\phi_i'(\lambda; \mathcal{D}_\alpha) \leq 0$ holds for any $\alpha > 0$ as it denotes the probability of $\{\lambda_i - r_i < \min_{i \neq j} \{\lambda_j - r_j\}\}$ when each r_i follows $\mathcal{D}_\alpha$.

Define

$$I_{i,j,n}(\lambda; \alpha) = \int_0^\infty \frac{1}{(z+\lambda_i)^n} \frac{1}{(z+\lambda_j)^{\alpha+1}} \exp\left(-\sum_j \frac{1}{(z+\lambda_j)^\alpha}\right) \mathrm{d}z.$$

For simplicity, we write $I_{i,j,n}(\lambda; \alpha) = I_{i,j,n}(\lambda)$ and $I_{i,n}(\lambda; \alpha) = I_{i,n}(\lambda)$ when n is written with α. Then,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_j} \frac{I_{i,\alpha+2}(\lambda)}{I_{i,\alpha+1}(\lambda)} = \alpha \frac{I_{i,j,\alpha+2}(\lambda) I_{i,\alpha+1}(\lambda) - I_{i,j,\alpha+1}(\lambda) I_{i,\alpha+2}(\lambda)}{I_{i,\alpha+1}^2(\lambda)}. \quad (32)$$

By letting $k(z) = \frac{1}{(z+\lambda_i)^{\alpha+1}} \exp\left(-\sum_j \frac{1}{(z+\lambda_j)^\alpha}\right)$, each term of the numerator of (32) is written as

$$\begin{aligned}
I_{i,j,\alpha+2}(\lambda) I_{i,\alpha+1}(\lambda) &= \iint_{z,w \geq 0} \frac{k(z)k(w)}{(z+\lambda_i)(z+\lambda_j)^{\alpha+1}} \mathrm{d}z\mathrm{d}w \\
&= \frac{1}{2} \iint_{z,w \geq 0} k(z)k(w) \left(\frac{1}{(z+\lambda_i)(z+\lambda_j)^{\alpha+1}} + \frac{1}{(w+\lambda_i)(w+\lambda_j)^{\alpha+1}} \right) \mathrm{d}z\mathrm{d}w, \\
I_{i,j,\alpha+1}(\lambda) I_{i,\alpha+2}(\lambda) &= \iint_{z,w \geq 0} \frac{k(z)k(w)}{(z+\lambda_i)(w+\lambda_j)^{\alpha+1}} \mathrm{d}z\mathrm{d}w \\
&= \frac{1}{2} \iint_{z,w \geq 0} k(z)k(w) \left(\frac{1}{(z+\lambda_i)(w+\lambda_j)^{\alpha+1}} + \frac{1}{(w+\lambda_i)(z+\lambda_j)^{\alpha+1}} \right) \mathrm{d}z\mathrm{d}w.
\end{aligned}$$

Then, the integrand for $I_{i,j,\alpha+2}(\lambda) I_{i,\alpha+1}(\lambda) - I_{i,j,\alpha+1}(\lambda) I_{i,\alpha+2}(\lambda)$ is expressed as

$$\begin{aligned}
&\frac{1}{(z+\lambda_i)(z+\lambda_j)^{\alpha+1}} + \frac{1}{(w+\lambda_i)(w+\lambda_j)^{\alpha+1}} - \frac{1}{(z+\lambda_i)(w+\lambda_j)^{\alpha+1}} - \frac{1}{(w+\lambda_i)(z+\lambda_j)^{\alpha+1}} \\
&\quad = \frac{(w+\lambda_i)(w+\lambda_j)^{\alpha+1} + (z+\lambda_i)(z+\lambda_j)^{\alpha+1} - (w+\lambda_i)(z+\lambda_j)^{\alpha+1} - (z+\lambda_i)(w+\lambda_j)^{\alpha+1}}{(z+\lambda_i)(z+\lambda_j)^{\alpha+1}(w+\lambda_i)(w+\lambda_j)^{\alpha+1}} \\
&\quad = (w-z) \frac{(w+\lambda_j)^{\alpha+1} - (z+\lambda_j)^{\alpha+1}}{(z+\lambda_i)(z+\lambda_j)^{\alpha+1}(w+\lambda_i)(w+\lambda_j)^{\alpha+1}}.
\end{aligned}$$

Here, one can see that when $w \geq z$, the integrand is non-negative since $\lambda_j > 0$ and $\alpha > 0$. On the other hand, if $w < z$, then both $(w-z)$ and $(w+\lambda_j)^{\alpha+1} - (z+\lambda_j)^{\alpha+1}$ becomes negative, i.e., integrand is again positive. Therefore, $I_{i,j,\alpha+2}(\lambda) I_{i,\alpha+1}(\lambda) - I_{i,j,\alpha+1}(\lambda) I_{i,\alpha+2}(\lambda)$ is an integral of a positive function, which concludes the proof.

C.1.2. FRÉCHET-TYPE DISTRIBUTIONS

As discussed in Appendix A.1, when F is absolute continuous and satisfies von Mises condition, $f \in \mathrm{RV}_{-\alpha-1}$, which implies $f(x) = x^{-\alpha-1}S_f(x)$ for some $S_f \in \mathrm{RV}_0$. Let $g_i(z) = \frac{S_f(z+\lambda_i)}{(z+\lambda_i)^{\alpha+2}}$. Then, for $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$, we can rewrite J_i as

$$J_i(\lambda; \mathcal{D}_\alpha) = \int_1^\infty \frac{S_f(z+\lambda_i)}{(z+\lambda_i)^{\alpha+2}} \prod_{j \neq i} F(z+\lambda_j)\mathrm{d}z.$$

For simplicity, let $f_i(z) = f(z+\lambda_i)$ and $F_i(z) = F(z+\lambda_i)$ for any $i \in [K]$, which denotes the density function and CDF of $\mathcal{D}_\alpha$, respectively. From the definition of ϕ in (1) and J_i in (13), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda_j} J_i(\lambda; \mathcal{D}_\alpha) &= \int_1^\infty \frac{S_f(z+\lambda_i)}{(z+\lambda_i)^{\alpha+2}} \frac{\mathrm{d}}{\mathrm{d}\lambda_j} \prod_{j \neq i} F(z+\lambda_j)\mathrm{d}z \\ &= \int_1^\infty \frac{S_f(z+\lambda_i)}{(z+\lambda_i)^{\alpha+2}} f(z+\lambda_j) \prod_{l \neq i,j} F(z+\lambda_l)\mathrm{d}z\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda_j} \phi_i(\lambda; \mathcal{D}_\alpha) &= \int_1^\infty f(z+\lambda_i) \frac{\mathrm{d}}{\mathrm{d}\lambda_j} \prod_{j \neq i} F(z+\lambda_j)\mathrm{d}z \\ &= \int_1^\infty f(z+\lambda_i) f(z+\lambda_j) \prod_{l \neq i,j} F(z+\lambda_l)\mathrm{d}z.\end{aligned}$$

Then, we have for $k(z) = \prod_{l \neq i,j} F_l(z)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda_j} \frac{J_i(\lambda; \mathcal{D}_\alpha)}{\phi_i(\lambda; \mathcal{D}_\alpha)} &= \frac{1}{\phi_i^2(\lambda; \mathcal{D}_\alpha)} \Bigg(\iint_{w,z \geq 1} g_i(z) f_j(z) \left(\prod_{l \neq i,j} F_l(z)\right) f_i(w) \left(\prod_{l \neq i} F_l(w)\right) \mathrm{d}w\mathrm{d}z \\ &\qquad - \iint_{w,z \geq 1} g_i(z) \left(\prod_{l \neq i} F_l(z)\right) f_i(w) f_j(w) \left(\prod_{l \neq i,j} F_l(w)\right) \mathrm{d}w\mathrm{d}z \Bigg) \\ &= \frac{1}{\phi_i^2(\lambda; \mathcal{D}_\alpha)} \Bigg(\iint_{w,z \geq 1} g_i(z) f_j(z) k(z) f_i(w) k(w) F_j(w) \mathrm{d}w\mathrm{d}z \\ &\qquad - \iint_{w,z \geq 1} g_i(z) k(z) F_j(z) f_i(w) f_j(w) k(w) \mathrm{d}w\mathrm{d}z \Bigg).\end{aligned}$$

Here, one can see that

$$\begin{aligned}&\iint_{w,z \geq 1} g_i(z) f_j(z) k(z) f_i(w) k(w) F_j(w) \mathrm{d}w\mathrm{d}z \\ &\qquad = \iint_{w,z \geq 1} \frac{k(z)k(w)}{2} (g_i(z) f_j(z) f_i(w) F_j(w) + g_i(w) f_j(w) f_i(z) F_j(z)) \mathrm{d}w\mathrm{d}z, \\ &\iint_{w,z \geq 1} g_i(z) k(z) F_j(z) f_i(w) f_j(w) k(w) \mathrm{d}w\mathrm{d}z \\ &\qquad = \iint_{w,z \geq 1} \frac{k(z)k(w)}{2} (g_i(z) F_j(z) f_i(w) f_j(w) + g_i(w) F_j(w) f_i(z) f_j(z)) \mathrm{d}w\mathrm{d}z.\end{aligned}$$

Then, by elementary calculation, we obtain

$$\begin{aligned}
&g_i(z)f_j(z)f_i(w)F_j(w) + g_i(w)f_j(w)f_i(z)F_j(z) - (g_i(z)F_j(z)f_i(w)f_j(w) + g_i(w)F_j(w)f_i(z)f_j(z)) \\
&\qquad = F_j(z)f_j(w)(g_i(w)f_i(z) - g_i(z)f_i(w)) + F_j(w)f_j(z)(g_i(z)f_i(w) - g_i(w)f_i(z)) \\
&\qquad = (g_i(w)f_i(z) - g_i(z)f_i(w)) \cdot (F_j(z)f_j(w) - F_j(w)f_j(z)). \qquad (33)
\end{aligned}$$

Obviously, (33) becomes 0 when $z = w$.

Firstly, let us consider the case $z \geq w$, where Assumption 5 implies

$$\frac{f(z+\underline{L}_j)}{F(z+\underline{L}_j)} \leq \frac{f(w+\underline{L}_j)}{F(w+\underline{L}_j)} \implies F_j(w)f_j(z) \leq F_j(z)f_j(w).$$

On the other hand, we have

$$g_i(z)f_i(w) = \frac{S_f(z)}{z^{\alpha+2}}\frac{S_f(w)}{w^{\alpha+1}},$$

which implies

$$\begin{aligned}
g_i(w)f_i(z) - g_i(z)f_i(w) &= \frac{S_f(z)}{z^{\alpha+2}}\frac{S_f(w)}{w^{\alpha+1}} - \frac{S_f(w)}{w^{\alpha+2}}\frac{S_f(z)}{z^{\alpha+1}} \\
&= \frac{S_f(w)S_f(z)}{w^{\alpha+1}z^{\alpha+1}}\left(\frac{1}{w} - \frac{1}{z}\right) \geq 0, \qquad z \geq w.
\end{aligned}$$

Therefore, when $z \geq w$, the integrand becomes positive. For the case $z \leq w$, one can easily reverse the inequalities above, which results in the positive integrand again. Therefore, $\frac{J_i(\lambda;\mathcal{D}_\alpha)}{\phi_i(\lambda;\mathcal{D}_\alpha)}$ is monotonically increasing.

C.2. Proof of Lemma 10

Here, we assume $\lambda_1 \leq \ldots \leq \lambda_K$ without loss of generality, where $\sigma_i = i$ holds.

C.2.1. FRÉCHET DISTRIBUTION

By the monotonicity of $I_{i,\alpha+2}(\lambda)/I_{i,\alpha+1}(\lambda)$ in Lemma 9, we have

$$\frac{I_{i,\alpha+2}(\underline{\lambda})}{I_{i,\alpha+1}(\underline{\lambda})} \leq \frac{I_{i,\alpha+2}(\lambda^*)}{I_{i,\alpha+1}(\lambda^*)}, \qquad \text{where} \quad \lambda_j^* = \begin{cases} \underline{\lambda}_i, & j \leq i, \\ \infty, & j > i. \end{cases}$$

From the definition of $I_{i,n}(\lambda;\alpha)$ in (12), we have

$$\begin{aligned}
I_{i,n}(\lambda^*;\alpha) &= \int_0^\infty \frac{1}{(z+\underline{\lambda}_i)^n}\exp\left(-\frac{i}{(z+\underline{\lambda}_i)^\alpha}\right)\mathrm{d}z \\
&= \frac{i^{-\frac{n-1}{\alpha}}}{\alpha}\int_0^{\frac{i}{\underline{\lambda}_i^\alpha}} u^{\frac{n-1}{\alpha}-1}e^{-u}\mathrm{d}u \\
&= \frac{i^{-\frac{n-1}{\alpha}}}{\alpha}\gamma\left(\frac{n-1}{\alpha}, \frac{i}{\underline{\lambda}_i^\alpha}\right),
\end{aligned}$$

where $\gamma(n,x) = \int_0^x t^{n-1}e^{-t}\mathrm{d}t$ denotes the lower incomplete gamma function.

By substituting this result, we obtain

$$\frac{I_{i,\alpha+2}(\underline{\lambda};\alpha)}{I_{i,\alpha+1}(\underline{\lambda};\alpha)} \le \frac{1}{\sqrt[\alpha]{i}} \frac{\gamma\Big(1+\frac{1}{\alpha}, \frac{i}{\underline{\lambda}_i^\alpha}\Big)}{\gamma\Big(1, \frac{i}{\underline{\lambda}_i^\alpha}\Big)}.$$

Note that $\gamma(1,x) = 1-e^{-x}$ holds for any $x>0$, and for any $\alpha>0$

$$\gamma\bigg(1+\frac{1}{\alpha}, x\bigg) \le \frac{\sqrt[\alpha]{x}}{1+1/\alpha}(1-e^{-x}) = \frac{\sqrt[\alpha]{x}}{1+1/\alpha}\gamma(1,x)$$

by Lemma 26, which proves the first inequality of Lemma 10.

Then, let us assume there exists a constant $C<\infty$ satisfying for any $x>0$

$$\gamma(1+1/\alpha, x) - (1-e^{-x})C \le 0. \tag{34}$$

The derivative of the LHS of (34) is given as

$$\sqrt[\alpha]{x}e^{-x} - Ce^{-x},$$

which achieves the minimum at $x = C^\alpha$, i.e., its maximum is achieved at $x=0$ or $x=\infty$. Applying this finding in (34) gives $C \ge \Gamma\big(1+\frac{1}{\alpha}\big)$, which concludes the proof.

C.2.2. FRÉCHET-TYPE DISTRIBUTIONS

By the monotonicity of $\frac{J_i(\lambda;\mathcal{D}_\alpha)}{\phi_i(\lambda;\mathcal{D}_\alpha)}$ in Lemma 9, we have

$$\frac{J_i(\underline{\lambda};\mathcal{D}_\alpha)}{\phi_i(\underline{\lambda};\mathcal{D}_\alpha)} \le \frac{J_i(\lambda^*;\mathcal{D}_\alpha)}{\phi_i(\lambda^*;\mathcal{D}_\alpha)}, \qquad \text{where} \quad \lambda_j^* = \begin{cases} \underline{\lambda}_i, & j\le i, \\ \infty, & j>i. \end{cases}$$

From the definition of $J_i(\lambda;\mathcal{D}_\alpha)$ in (13), we have

$$J_i(\lambda^*;\mathcal{D}_\alpha) = \int_1^\infty \frac{S_f(z+\underline{\lambda}_i)}{(z+\underline{\lambda}_i)^{\alpha+2}} F^{i-1}(z+\underline{\lambda}_i)\mathrm{d}z$$

and

$$\phi_i(\lambda^*;\mathcal{D}_\alpha) = \int_1^\infty f(z+\underline{\lambda}_i)F^{i-1}(z+\underline{\lambda}_i)\mathrm{d}z.$$

Here, we begin by examining the Pareto distribution, as the proof for this case offers insights into the generalization of our results.

Pareto distribution Let us consider the $\mathcal{D}_\alpha = \mathcal{P}_\alpha$, where

$$\begin{aligned} J_i(\lambda^*;\mathcal{P}_\alpha) &= \int_1^\infty \frac{\alpha}{(z+\underline{\lambda}_i)^{\alpha+2}}\bigg(1-\frac{1}{(z+\underline{\lambda}_i)^\alpha}\bigg)^{i-1}\mathrm{d}z \\ &= \int_0^{\frac{1}{(1+\underline{\lambda}_i)^\alpha}} w^{\frac{1}{\alpha}}(1-w)^{i-1}\mathrm{d}w \\ &= B\bigg(\frac{1}{(1+\underline{\lambda}_i)^\alpha}; 1+\frac{1}{\alpha}, i\bigg), \end{aligned}$$

where $B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t$ denotes the incomplete Beta function. Similarly,

$$\begin{aligned}\phi_i(\lambda^*;\mathcal{P}_\alpha) &= \int_1^\infty \frac{\alpha}{(z+\underline{\lambda}_i)^{\alpha+1}}\left(1-\frac{1}{(z+\underline{\lambda}_i)^\alpha}\right)^{i-1}\mathrm{d}z \\ &= \int_0^{\frac{1}{(1+\underline{\lambda}_i)^\alpha}} w^0(1-w)^{i-1}\mathrm{d}w \\ &= B\left(\frac{1}{(1+\underline{\lambda}_i)^\alpha};1,i\right).\end{aligned}$$

Therefore, by Lemma 28

$$\frac{J_i(\lambda^*;\mathcal{P}_\alpha)}{\phi_i(\lambda^*;\mathcal{P}_\alpha)} = \frac{B\left(\frac{1}{(1+\underline{\lambda}_i)^\alpha};1+\frac{1}{\alpha},i\right)}{B\left(\frac{1}{(1+\underline{\lambda}_i)^\alpha};1,i\right)} \leq \frac{B\left(1+\frac{1}{\alpha},i\right)}{B(1,i)}.$$

Since $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $i \geq 1$, and $\alpha > 1$, applying Gautschi's inequality provides

$$\begin{aligned}\frac{B\left(1+\frac{1}{\alpha},i\right)}{B(1,i)} = \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma(i+1)}{\Gamma\left(1+\frac{1}{\alpha}+i\right)} &= \frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{i+\frac{1}{\alpha}}\frac{\Gamma(i+1)}{\Gamma\left(i+\frac{1}{\alpha}\right)} \\ &\leq \frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{i+\frac{1}{\alpha}}(i+1)^{1-\frac{1}{\alpha}} \\ &\leq \frac{2\alpha}{\alpha+1}\Gamma\left(1+\frac{1}{\alpha}\right)\frac{1}{(i+1)^{\frac{1}{\alpha}}} \\ &\leq 2\Gamma\left(1+\frac{1}{\alpha}\right)\frac{1}{\sqrt[\alpha]{i}}.\end{aligned} \tag{35}$$

On the other hand, for $x \in [0,1]$ it holds that $B(x;1,i) = (1-(1-x)^i)$ and

$$\begin{aligned}B\left(x;1+\frac{1}{\alpha},i\right) = \int_0^x t^{\frac{1}{\alpha}}(1-t)^{i-1}\mathrm{d}t &\leq \int_0^x t^{\frac{1}{\alpha}}e^{-t(i-1)}\mathrm{d}t \\ &\leq e\int_0^x t^{\frac{1}{\alpha}}e^{-ti}\mathrm{d}t \\ &= \frac{e}{i^{1+\frac{1}{\alpha}}}\int_0^{xi} w^{\frac{1}{\alpha}}e^{-w}\mathrm{d}w = \frac{e}{i^{1+\frac{1}{\alpha}}}\gamma\left(1+\frac{1}{\alpha},xi\right).\end{aligned}$$

Then, by Lemma 26, we have

$$\frac{B\left(x;1+\frac{1}{\alpha},i\right)}{B(x;1,i)} \leq \frac{e}{i^{1+\frac{1}{\alpha}}}\frac{(xi)^{\frac{1}{\alpha}}}{1+1/\alpha}\frac{1-e^{-xi}}{(1-(1-x)^i)} \leq \frac{e}{i}\frac{(x)^{\frac{1}{\alpha}}}{1+1/\alpha}, \tag{36}$$

where the last inequality follows from $\lim_{x\to 0}\frac{1-e^{-xi}}{(1-(1-x)^i)} = 1$ and $\lim_{x\to 1}\frac{1-e^{-xi}}{(1-(1-x)^i)} < 1$. Therefore, by substituting $x = \frac{1}{(1+\underline{\lambda}_i)^\alpha}$, we have

$$\frac{B\left(\frac{1}{(1+\underline{\lambda}_i)^\alpha};1+\frac{1}{\alpha},i\right)}{B\left(\frac{1}{(1+\underline{\lambda}_i)^\alpha};1,i\right)} \leq \frac{e\alpha}{\alpha+1}\frac{1}{1+\underline{\lambda}_i} \leq \frac{e\alpha}{\alpha+1}\frac{1}{\underline{\lambda}_i}.$$

Generalization to $\mathfrak{D}_\alpha$ Let us define a function for $x \geq 1$

$$k(x) = k(x; \mathcal{D}_\alpha) := \frac{\int_x^\infty \frac{S_f(z)}{z^{\alpha+2}} F^{i-1}(z)\mathrm{d}z}{\int_x^\infty f(z)F^{i-1}(z)\mathrm{d}z}.$$

Then, it holds that

$$\begin{aligned}
\frac{\mathrm{d}k(x)}{\mathrm{d}x} &= \frac{1}{\left(\int_x^\infty f(z)F^{i-1}(z)\mathrm{d}z\right)^2}\Big(f(x)F^{i-1}(x)\int_x^\infty \frac{S_f(z)}{z^{\alpha+2}}F^{i-1}(z)\mathrm{d}z \\
&\qquad\qquad - \frac{S_f(x)}{x^{\alpha+2}}F^{i-1}(x)\int_x^\infty f(z)F^{i-1}(z)\mathrm{d}z\Big) \\
&= \frac{F^{i-1}(x)}{\left(\int_x^\infty f(z)F^{i-1}(z)\mathrm{d}z\right)^2}\left(f(x)\int_x^\infty \frac{S_f(z)}{z^{\alpha+2}}F^{i-1}(z)\mathrm{d}z - \frac{S_f(x)}{x^{\alpha+2}}\int_x^\infty f(z)F^{i-1}(z)\mathrm{d}z\right) \\
&\leq \frac{F^{i-1}(x)}{\left(\int_x^\infty f(z)F^{i-1}(z)\mathrm{d}z\right)^2}\left(\frac{S_f(x)}{x^{\alpha+1}}\int_x^\infty \frac{S_f(z)}{z^{\alpha+2}}F^{i-1}(z)\mathrm{d}z - \frac{S_f(x)}{x^{\alpha+2}}\int_x^\infty \frac{S_f(z)}{z^{\alpha+1}}F^{i-1}(z)\mathrm{d}z\right) \\
&= \frac{F^{i-1}(x)}{\left(\int_x^\infty f(z)F^{i-1}(z)\mathrm{d}z\right)^2}\frac{S_f(x)}{x^{\alpha+2}}\left(\int_x^\infty \frac{xS_f(z)}{z^{\alpha+2}}F^{i-1}(z)\mathrm{d}z - \int_x^\infty \frac{S_f(z)}{z^{\alpha+1}}F^{i-1}(z)\mathrm{d}z\right) \\
&= \frac{F^{i-1}(x)}{\left(\int_x^\infty f(z)F^{i-1}(z)\mathrm{d}z\right)^2}\frac{S_f(x)}{x^{\alpha+2}}\left(\int_x^\infty \left(\frac{x}{z}-1\right)\left(\frac{S_f(z)}{z^{\alpha+1}}F^{i-1}(z)\right)\mathrm{d}z\right) \leq 0,
\end{aligned}$$

which implies $k(x)$ is decreasing with respect to $x \geq 1$. Therefore,

$$\begin{aligned}
\frac{J_i(\lambda^*; \mathcal{D}_\alpha)}{\phi_i(\lambda^*; \mathcal{D}_\alpha)} \leq \frac{\int_1^\infty \frac{S_f(z)}{z^{\alpha+2}}F^{i-1}(z)\mathrm{d}z}{\int_1^\infty f(z)F^{i-1}(z)\mathrm{d}z} &= i\int_1^\infty \frac{f(z)}{z}F^{i-1}(z)\mathrm{d}z \\
&= \mathbb{E}\left[\frac{1}{M_i}\right] \leq \frac{m}{A_l\sqrt[\alpha]{i}}, \qquad (37)
\end{aligned}$$

where (37) follows from Assumption 3.

Next, let us consider the case (14) holds, where $S_F(x)$ is increasing. Let $1 - F(z) = t$, which implies $z = U(1/t)$ for $t \in [1, \infty)$. Then, we have

$$\begin{aligned}
\int_1^\infty \frac{f(z)}{z}F^{i-1}(z)\mathrm{d}z &= \int_0^1 \frac{1}{U(1/t)}(1-t)^{(i-1)}\mathrm{d}t \\
&= \int_0^1 \frac{t^{\frac{1}{\alpha}}}{S_U(1/t)}(1-t)^{(i-1)}\mathrm{d}t, \qquad (38) \\
&\leq B\left(1+\frac{1}{\alpha}; i\right).
\end{aligned}$$

where (38) follows from $S_U \in \mathrm{RV}_{1/\alpha}$. Here, $S_U(1/t) = S_F^{\frac{1}{\alpha}}(U(1/t))$ holds from (25), which implies that $\frac{1}{S_U(1/t)}$ is increasing when S_F is increasing function since $U(1/t)$ is decreasing. From the definition of $U(1) = 1 = 1^{\frac{1}{\alpha}}S_U(1)$, we obtain $S_U(1) = 1$, i.e., $A_l = 1$. Therefore, the analysis of the Pareto distributions from (35) implies that $m \leq 2\Gamma\left(1 + \frac{1}{\alpha}\right)$.

Next, we obtain

$$\int_{1+\underline{\lambda}_i}^{\infty} f(z)F^{i-1}(z)\mathrm{d}z = \frac{1}{i}\big(1-F^i(1+\underline{\lambda}_i)\big) = B(1-F(1+\underline{\lambda}_i);1,i).$$

By Assumption 3 and (38), we have

$$\int_{1+\underline{\lambda}_i}^{\infty} \frac{S_f(z)}{z^{\alpha+2}}F^{i-1}(z)\mathrm{d}z \leq \frac{1}{A_l}B\left(1-F(1+\underline{\lambda}_i);1+\frac{1}{\alpha},i\right).$$

Therefore, following the same steps from (36), we have

$$\begin{aligned}
\frac{\int_{1+\underline{\lambda}_i}^{\infty} \frac{S_f(z)}{z^{\alpha+2}}F^{i-1}(z)\mathrm{d}z}{\int_{1+\underline{\lambda}_i}^{\infty} f(z)F^{i-1}(z)\mathrm{d}z} &\leq \frac{1}{A_l}\frac{B\big(1-F(1+\underline{\lambda}_i);1+\frac{1}{\alpha},i\big)}{B(1-F(1+\underline{\lambda}_i);1,i)} \\
&\leq \frac{1}{A_l}\frac{e}{i}\frac{\alpha}{\alpha+1}\left(\frac{S_F(1+\underline{\lambda}_i)}{(1+\underline{\lambda}_i)^{\alpha}}\right)^{\frac{1}{\alpha}} \qquad (39) \\
&\leq \frac{\alpha e}{A_l(\alpha+1)}\frac{A_u}{1+\underline{\lambda}_i} \\
&\leq \frac{\alpha e}{A_l(\alpha+1)}\frac{A_u}{\underline{\lambda}_i}.
\end{aligned}$$

where (39) follows from Assumption 3. Here, when S_F is an increasing function, then $A_u = \lim_{x\to\infty} S_F^{\frac{1}{\alpha}}(x)$ from (25).

Remark 19 *When one considers the shifted distribution, (14) does not necessarily hold even when its original distribution satisfies it. In such cases, it suffices to consider the shifted distribution function after the conditioning trick, where we have*

$$G(x) = F^*(x-1) = \frac{F(x)-F(1)}{1-F(1)}, \quad x \geq 1,$$

which implies

$$1-G(x) = \frac{1-F(x)}{1-F(1)} = x^{-\alpha}S_G(x), \quad x \geq 1.$$

Therefore, $S_G(x) = \frac{1}{1-F(1)}S_F(x)$ holds for $x \geq 1$ and thus they are tail-equivalent. Furthermore, if F satisfies (14), then

$$\frac{xg(x)}{1-G(x)} = \frac{xf(x)}{1-F(x)} \leq \alpha, \quad x \geq 1$$

holds. Therefore, $S_G(x)$ is monotonically increasing for $x \geq 1$ with $S_G(1) = 1$, which implies that $A = 1$ and $m \leq 2\Gamma\big(1+\frac{1}{\alpha}\big)$.

C.3. Proof of Lemma 11

Although the overall proof is almost the same and follows the proofs of Honda et al. (2023), we provide the proofs for completeness.

C.3.1. FRÉCHET DISTRIBUTION

From the definition of $\hat{\ell}_t = \left(\ell_{t,I_t}\widehat{w_{t,I_t}^{-1}}\right)e_{I_t}$, when $I_t = i$, we have

$$\begin{aligned}
\phi_i\left(\eta_t\hat{L}_t;\mathcal{F}_\alpha\right) - \phi_i\left(\eta_t\left(\hat{L}_t + \left(\ell_{t,i}\widehat{w_{t,i}^{-1}}\right)e_i\right);\mathcal{F}_\alpha\right)&\\
&= \int_0^{\eta_t\ell_{t,i}\widehat{w_{t,i}^{-1}}} -\phi_i'(\eta_t\hat{L}_t + xe_i;\mathcal{F}_\alpha)\mathrm{d}x\\
&\leq \alpha(\alpha+1)\int_0^{\eta_t\ell_{t,i}\widehat{w_{t,i}^{-1}}} I_{i,\alpha+2}(\underline{\eta_t\hat{L}_t + xe_i};\alpha)\mathrm{d}x \qquad \text{(by (31))}\\
&\leq \alpha(\alpha+1)\int_0^{\eta_t\ell_{t,i}\widehat{w_{t,i}^{-1}}} I_{i,\alpha+2}(\underline{\eta_t\hat{L}_t};\alpha)\mathrm{d}x \qquad (40)\\
&= \alpha(\alpha+1)\eta_t\ell_{t,i}\widehat{w_{t,i}^{-1}} I_{i,\alpha+2}(\underline{\eta_t\hat{L}_t};\alpha),
\end{aligned}$$

where (40) follows from the monotonicity of $I_{i,\alpha}$. Since $\widehat{w_{t,i}^{-1}}$ follows the geometric distribution with mean $w_{t,i}^{-1}$ given $\hat{L}_t$ and I_t, it holds that

$$\mathbb{E}\left[\widehat{w_{t,I_t}^{-1}}^2 \middle| \hat{L}_t, I_t\right] = \frac{2}{w_{t,I_t}^2} - \frac{1}{w_{t,I_t}} \leq \frac{2}{w_{t,I_t}^2}.$$

Since $I_t \neq i$ implies $\hat{\ell}_{t,i} = 0$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\hat{\ell}_{t,i}\left(\phi_i\left(\eta_t\hat{L}_t\right) - \phi_i\left(\eta_t\left(\hat{L}_t + \hat{\ell}_t\right)\right)\right)\middle|\hat{L}_t\right]&\\
&= \mathbb{E}\left[\mathbb{1}[I_t = i]\hat{\ell}_{t,i}\left(\phi_i\left(\eta_t\hat{L}_t\right) - \phi_i\left(\eta_t\left(\hat{L}_t + \hat{\ell}_t\right)\right)\right)\middle|\hat{L}_t\right]\\
&= \mathbb{E}\left[\mathbb{1}[I_t = i]\ell_{t,i}\widehat{w_{t,i}^{-1}}\left(\phi_i\left(\eta_t\hat{L}_t\right) - \phi_i\left(\eta_t\left(\hat{L}_t + \hat{\ell}_t\right)\right)\right)\middle|\hat{L}_t\right]\\
&\leq \mathbb{E}\left[w_{t,i}\ell_{t,i}\widehat{w_{t,i}^{-1}}\cdot\alpha(\alpha+1)\eta_t\ell_{t,i}\widehat{w_{t,i}^{-1}} I_{i,\alpha+2}(\eta_t\underline{\hat{L}}_t)\middle|\hat{L}_t\right]\\
&\leq 2\alpha(\alpha+1)\eta_t\mathbb{E}\left[w_{t,i}\frac{\ell_{t,i}^2 I_{i,\alpha+2}(\eta_t\underline{\hat{L}}_t)}{w_{t,i}^2}\middle|\hat{L}_t\right]\\
&\leq 2(\alpha+1)\eta_t\mathbb{E}\left[\frac{I_{i,\alpha+2}(\eta_t\underline{\hat{L}}_t)}{I_{i,\alpha+1}(\eta_t\underline{\hat{L}}_t)}\middle|\hat{L}_t\right] \qquad \left(\text{by } w_{t,i} = \alpha I_{i,\alpha+1}(\eta_t\underline{\hat{L}}_t),\ \ell_{t,i}\leq 1\right)\\
&\leq \frac{2\alpha}{\eta\underline{\hat{L}}_{t,i}} \wedge 2(\alpha+1)\eta_t\frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{\sqrt[\alpha]{\sigma_i}},
\end{aligned}$$

where the last inequality follows from Lemma 10 for $\mathcal{F}_\alpha$.

C.3.2. FRÉCHET-TYPE DISTRIBUTIONS

From the definition of ϕ in (1) and (15) from Assumption 4, we have

$$-\phi_i'(\lambda;\mathcal{D}_\alpha) = \int_1^\infty -f'(z+\lambda_i)\prod_{j\neq i}F(z+\lambda_j)\mathrm{d}z$$

$$\leq \int_1^\infty \rho_2 \frac{f(z+\lambda_i)}{z+\lambda_i} F^{i-1}(z+\lambda_i)\mathrm{d}z = \rho_2 J_i(\lambda; \mathcal{D}_\alpha).$$

Therefore, we can replace $\alpha(\alpha+1)I_{i,\alpha+2}$ with $\rho_2 J_i$, which gives

$$\begin{aligned}
\mathbb{E}\left[\hat{\ell}_{t,i}\Big(\phi_i\Big(\eta_t\hat{L}_t\Big) - \phi_i\Big(\eta_t\Big(\hat{L}_t + \hat{\ell}_t\Big)\Big)\Big)\middle|\hat{L}_t\right] &\leq \mathbb{E}\left[w_{t,i}\ell_{t,i}\widehat{w_{t,i}^{-1}} \cdot \eta_t \ell_{t,i}\widehat{w_{t,i}^{-1}}\rho_2 J_i(\eta_t \underline{\hat{L}}_t; \mathcal{D}_\alpha)\middle|\hat{L}_t\right] \\
&\leq 2\eta_t \mathbb{E}\left[w_{t,i}\frac{\ell_{t,i}^2 \rho_2 J_i(\eta_t\underline{\hat{L}}_t; \mathcal{D}_\alpha)}{w_{t,i}^2}\middle|\hat{L}_t\right] \\
&\leq 2\rho_2\eta_t \mathbb{E}\left[\frac{J_i(\eta_t\underline{\hat{L}}_t; \mathcal{D}_\alpha)}{\phi_i(\eta_t\underline{\hat{L}}_t; \mathcal{D}_\alpha)}\middle|\hat{L}_t\right],
\end{aligned}$$

where Lemma 10 concludes the proof.

C.4. Proof of Lemma 12

By Lemmas 10 and 11, for $\mathcal{F}_\alpha$, we have

$$\begin{aligned}
\mathbb{E}\left[\hat{\ell}_t\Big(\phi_i\Big(\eta_t\hat{L}_t\Big) - \phi_i\Big(\eta_t\Big(\hat{L}_t + \hat{\ell}_t\Big)\Big)\Big)\middle|\hat{L}_t\right] &\leq \sum_{i\in[K]} 2(\alpha+1)\eta_t \frac{\Gamma\big(1+\frac{1}{\alpha}\big)}{\sqrt[\alpha]{\sigma_i}} \\
&\leq 2(\alpha+1)\eta_t\Gamma\left(1+\frac{1}{\alpha}\right)\left(1+\int_1^K x^{-1/\alpha}\mathrm{d}x\right) \\
&= 2(\alpha+1)\eta_t\Gamma\left(1+\frac{1}{\alpha}\right)\frac{\alpha K^{1-1/\alpha}-1}{\alpha-1} \\
&\leq \frac{2\alpha(\alpha+1)}{\alpha-1}\eta_t\Gamma\left(1+\frac{1}{\alpha}\right)K^{1-1/\alpha}.
\end{aligned}$$

Similarly, for $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$, we have

$$\mathbb{E}\left[\hat{\ell}_t\Big(\phi_i\Big(\eta_t\hat{L}_t\Big) - \phi_i\Big(\eta_t\Big(\hat{L}_t + \hat{\ell}_t\Big)\Big)\Big)\middle|\hat{L}_t\right] \leq \frac{2\alpha\rho_2}{\alpha-1}\eta_t\frac{m}{A_l}K^{1-1/\alpha}.$$

Appendix D. Regret bound for adversarial bandits: Penalty

This section provides the proofs on Lemma 13.

D.1. Penalty term analysis for the Fréchet distributions

By letting $k_\alpha(z) = \sum_i \frac{1}{(z+\eta_t\underline{\hat{L}}_{t,i})^\alpha} \in \big(0, \frac{K}{z^\alpha}\big]$, we have

$$\begin{aligned}
\mathbb{E}\left[r_{t,I_t} - r_{t,i^*}\middle|\hat{L}_t\right] &\leq \sum_{i\neq i^*}\mathbb{E}\left[\mathbb{1}[I_t = I]r_{t,i}\middle|\hat{L}_t\right] \\
&= \alpha\int_0^\infty \sum_{i\neq i^*}\frac{1}{(z+\eta_t\underline{\hat{L}}_{t,i})^\alpha}e^{-k_\alpha(z)}\mathrm{d}z
\end{aligned}$$

$$\leq \alpha \int_0^\infty \sum_{i \neq i^*} \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \mathrm{d}z = \frac{\alpha}{\alpha - 1} \sum_{i \neq i^*} \frac{1}{(\eta_t \underline{\hat{L}}_{t,i})^{\alpha - 1}}.$$

On the other hand,

$$\begin{aligned}
\alpha \int_0^\infty \sum_{i \neq i^*} \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} e^{-k_\alpha(z)} \mathrm{d}z &\leq \alpha \int_0^\infty k_\alpha(z) e^{-k_\alpha(z)} \mathrm{d}z \\
&= \alpha \int_0^{\sqrt[\alpha]{K}} k_\alpha(z) e^{-k_\alpha(z)} \mathrm{d}z + \alpha \int_{\sqrt[\alpha]{K}}^\infty k_\alpha(z) e^{-k_\alpha(z)} \mathrm{d}z \quad (41) \\
&\leq \alpha \int_0^{\sqrt[\alpha]{K}} e^{-1} \mathrm{d}z + \alpha \int_{\sqrt[\alpha]{K}}^\infty \frac{K}{z^\alpha} e^{-\frac{K}{z^\alpha}} \mathrm{d}z \\
&= \alpha e^{-1} \sqrt[\alpha]{K} + \sqrt[\alpha]{K} \int_0^1 w^{-\frac{1}{\alpha}} e^{-w} \mathrm{d}w \\
&= \left(\alpha e^{-1} + \gamma\left(1 - \frac{1}{\alpha}, 1\right) \right) \sqrt[\alpha]{K},
\end{aligned}$$

where the first term of (41) follows from the fact that $xe^{-x} \leq e^{-1}$ and the second term follows from the fact that xe^{-x} is increasing for $x \leq 1$ and $k_\alpha(z) \leq 1$ holds for $z \geq \sqrt[\alpha]{K}$. From the definition of the lower incomplete gamma function, one can obtain

$$\gamma(s+1, x) = s\gamma(s, x) - x^s e^{-x} \implies \gamma\left(2 - \frac{1}{\alpha}, 1\right) = \left(1 - \frac{1}{\alpha}\right) \gamma\left(1 - \frac{1}{\alpha}, 1\right) - e^{-1},$$

which implies

$$\begin{aligned}
\gamma\left(1 - \frac{1}{\alpha}, 1\right) &= \frac{\alpha}{\alpha - 1} \gamma\left(2 - \frac{1}{\alpha}, 1\right) + \frac{\alpha e^{-1}}{\alpha - 1} \\
&\leq \frac{\alpha}{\alpha - 1} \frac{\alpha}{2\alpha - 1} (1 - e^{-1}) + \frac{\alpha e^{-1}}{\alpha - 1}
\end{aligned}$$

by Lemma 26 again. Therefore, by doing elementary calculations, we obtain that

$$\alpha \int_0^\infty \sum_{i \neq i^*} \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} e^{-k_\alpha(z)} \mathrm{d}z \leq \left(\frac{0.74\alpha^3 + 0.27\alpha^2}{(\alpha - 1)(2\alpha - 1)} \right) \sqrt[\alpha]{K}.$$

D.2. Penalty term analysis for the Pareto distributions

By letting $k_\alpha(z) = \sum_i \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \in \left(0, \frac{K}{z^\alpha}\right]$, we have

$$\begin{aligned}
\mathbb{E}\left[r_{t,I_t} - r_{t,i^*} \middle| \hat{L}_t \right] &\leq \sum_{i \neq i^*} \mathbb{E}\left[\mathbb{1}[I_t = I] r_{t,i} \middle| \hat{L}_t \right] \\
&= \alpha \int_1^\infty \sum_{i \neq i^*} \left(\frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \prod_{j \neq i} \left(1 - \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,j})^\alpha} \right) \right) \mathrm{d}z \\
&\leq e\alpha \int_1^\infty \sum_{i \neq i^*} \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} e^{-k_\alpha(z)} \mathrm{d}z.
\end{aligned}$$

Therefore, the proof in Section D.1 immediately concludes the Pareto case.

D.3. Penalty term for the Fréchet-type distributions

Here, let us consider the inverse of the tail function, which is the tail quantile function defined as

$$U(t) := \inf\left\{x : F(x) \geq \frac{1}{t}\right\}. \tag{42}$$

Note that when F and U are continuous, $1 - F(U(t)) = \frac{1}{t}$ holds. Then, as in the other cases, we have

$$\begin{aligned}
\mathbb{E}\left[r_{t,I_t} - r_{t,i^*} \middle| \hat{L}_t\right] &\leq \sum_{i\neq i^*} \mathbb{E}\left[\mathbb{1}[I_t = I] r_{t,i} \middle| \hat{L}_t\right] \\
&= \int_1^\infty \sum_{i\neq i^*} \left((z+\eta_t\underline{\hat{L}}_{t,i}) f(z+\eta_t\underline{\hat{L}}_{t,i}) \prod_{j\neq i} F(z+\eta_t\underline{\hat{L}}_{t,j})\right) \mathrm{d}z \\
&= \int_1^{U(K)} \sum_{i\neq i^*} \left(\frac{S_f(z+\eta_t\underline{\hat{L}}_{t,i})}{(z+\eta_t\underline{\hat{L}}_{t,i})^\alpha} \prod_{j\neq i} F(z+\eta_t\underline{\hat{L}}_{t,j})\right) \mathrm{d}z \\
&\quad + \int_{U(K)}^\infty \sum_{i\neq i^*} \left(\frac{S_f(z+\eta_t\underline{\hat{L}}_{t,i})}{(z+\eta_t\underline{\hat{L}}_{t,i})^\alpha} \prod_{j\neq i} F(z+\eta_t\underline{\hat{L}}_{t,j})\right) \mathrm{d}z.
\end{aligned} \tag{43}$$

The first term of (43) can be bounded as

$$\begin{aligned}
&\int_1^{U(K)} \sum_{i\neq i^*} \left(\frac{S_f(z+\eta_t\underline{\hat{L}}_{t,i})}{(z+\eta_t\underline{\hat{L}}_{t,i})^\alpha} \prod_{j\neq i} F(z+\eta_t\underline{\hat{L}}_{t,j})\right) \mathrm{d}z \\
&\quad = \int_1^{U(K)} \sum_{i\neq i^*} \left(\frac{\varrho(z+\eta_t\underline{\hat{L}}_{t,i}) S_F(z+\eta_t\underline{\hat{L}}_{t,i})}{(z+\eta_t\underline{\hat{L}}_{t,i})^\alpha} \prod_{j\neq i} F(z+\eta_t\underline{\hat{L}}_{t,j})\right) \mathrm{d}z \qquad \text{by (22)} \\
&\quad \leq \int_1^{U(K)} \sum_{i\neq i^*} \left(\frac{\rho_1 S_F(z+\eta_t\underline{\hat{L}}_{t,i})}{(z+\eta_t\underline{\hat{L}}_{t,i})^\alpha} \prod_{j\neq i} F(z+\eta_t\underline{\hat{L}}_{t,j})\right) \mathrm{d}z \\
&\quad \leq e\rho_1 \int_1^{U(K)} \left(\sum_{i\in[K]} \frac{S_F(z+\eta_t\underline{\hat{L}}_{t,i})}{(z+\eta_t\underline{\hat{L}}_{t,i})^\alpha}\right) \exp\left(-\sum_{i\in[K]} \frac{S_F(z+\eta_t\underline{\hat{L}}_{t,i})}{(z+\eta_t\underline{\hat{L}}_{t,i})^\alpha}\right) \mathrm{d}z \\
&\quad \leq \rho_1 e \int_1^{U(K)} e^{-1} \mathrm{d}z \leq \rho_1 U(K) \leq \rho_1 A_u K^{\frac{1}{\alpha}}
\end{aligned}$$

where the second last inequality follows from $xe^{-x} \leq e^{-1}$.

For the second term of (43), from $S_f(x) = \varrho(x)S_F(x)$ in (22), we have

$$\begin{aligned}
&\int_{U(K)}^{\infty} \sum_{i\neq i^*} \left(\frac{S_f(z+\eta_t \underline{\hat{L}}_{t,i})}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha} \prod_{j\neq i} F(z+\eta_t \underline{\hat{L}}_{t,j}) \right) \mathrm{d}z \\
&\qquad = \int_{U(K)}^{\infty} \sum_{i\neq i^*} \left(\frac{\varrho(z+\eta_t \underline{\hat{L}}_{t,i}) S_F(z+\eta_t \underline{\hat{L}}_{t,i})}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha} \prod_{j\neq i} F(z+\eta_t \underline{\hat{L}}_{t,j}) \right) \mathrm{d}z \\
&\qquad \leq \rho_1 \int_{U(K)}^{\infty} \sum_{i\neq i^*} \left(\frac{S_F(z+\eta_t \underline{\hat{L}}_{t,i})}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha} \prod_{j\neq i} F(z+\eta_t \underline{\hat{L}}_{t,j}) \right) \mathrm{d}z \\
&\qquad = \rho_1 \int_{U(K)}^{\infty} \left(\sum_{i\in[K]} (1-F(z+\eta_t \underline{\hat{L}}_{t,i})) \prod_{j\neq i} F(z+\eta_t \underline{\hat{L}}_{t,j}) \right) \mathrm{d}z \\
&\qquad \leq \rho_1 \int_{U(K)}^{\infty} \left(\sum_{i\in[K]} (1-F(z+\eta_t \underline{\hat{L}}_{t,i})) \exp\left(-\sum_{j\neq i} (1-F(z+\eta_t \underline{\hat{L}}_{t,j})) \right) \right) \mathrm{d}z \\
&\qquad \leq e\rho_1 \int_{U(K)}^{\infty} \left(\sum_{i\in[K]} (1-F(z)) \right) \exp\left(-\sum_{j\in[K]} (1-F(z)) \right) \mathrm{d}z \qquad (44) \\
&\qquad = e\rho_1 \int_{U(K)}^{\infty} K(1-F(z)) \exp(-K(1-F(z))) \mathrm{d}z \\
&\qquad = e\rho_1 \int_{U(K)}^{\infty} K \frac{S_F(z)}{z^\alpha} \exp\left(-K \frac{S_F(z)}{z^\alpha} \right) \mathrm{d}z,
\end{aligned}$$

where (44) holds since xe^{-x} is increasing with respect to $x \in [0,1]$ and $\sum_{i\in[K]}(1-F(z+\eta_t \underline{\hat{L}}_{t,i}))) \leq \sum_{i\in[K]}(1-F(z)) \leq 1$ for $z \geq U(K)$. Here, $S_F(z)$ is increasing function with respect to $z \geq \nu$, which implies

$$\begin{aligned}
&e\rho_1 \int_{U(K)}^{\infty} K \frac{S_F(z)}{z^\alpha} \exp\left(-K \frac{S_F(z)}{z^\alpha} \right) \mathrm{d}z \\
&\qquad \leq e\rho_1 \int_{U(K)}^{\infty} K \frac{S_F(z)}{z^\alpha} \exp\left(-K \frac{S_F(U(K))}{z^\alpha} \right) \mathrm{d}z \\
&\qquad = \frac{e\rho_1}{\alpha} \int_{U(K)}^{\infty} \frac{S_F(z) z}{S_F(U(K))} \frac{K\alpha S_F(U(K))}{z^{\alpha+1}} \exp\left(-K \frac{S_F(U(K))}{z^\alpha} \right) \mathrm{d}z \\
&\qquad = \frac{e\rho_1}{\alpha} \int_{U(K)}^{\infty} \frac{S_F(z) z}{S_F(U(K))} \left(-\frac{\mathrm{d}}{\mathrm{d}x} \frac{K S_F(U(K))}{z^\alpha} \right) \exp\left(-K \frac{S_F(U(K))}{z^\alpha} \right) \mathrm{d}z.
\end{aligned}$$

By Potter's bound (Lemma 29) with arbitrary chosen $\delta > 0$, there exists some constants b_δ such that for any $z \geq U(K)$

$$\frac{S_F(z)}{S_F(U(K))} \leq b_\delta \left(\frac{z}{U(K)} \right)^\delta.$$

Therefore, for $\delta > 0$

$$\begin{aligned}
&e\rho_1 \int_{U(K)}^{\infty} K \frac{S_F(z)}{z^\alpha} \exp\left(-K \frac{S_F(z)}{z^\alpha}\right) \mathrm{d}z \\
&\quad \le \frac{e\rho_1}{\alpha} \int_{U(K)}^{\infty} b_\delta \frac{z^{1+\delta}}{U^\delta(K)} \left(-\frac{\mathrm{d}}{\mathrm{d}x} \frac{K S_F(U(K))}{z^\alpha}\right) \exp\left(-K \frac{S_F(U(K))}{z^\alpha}\right) \mathrm{d}z \\
&\quad = \frac{e\rho_1}{\alpha} \int_{U(K)}^{\infty} b_\delta \frac{K^{\frac{1+\delta}{\alpha}}}{U^\delta(K)} S_F(U(K))^{\frac{1+\delta}{\alpha}} \left(K \frac{S_F(U(K))}{z^\alpha}\right)^{-\frac{1+\delta}{\alpha}} \\
&\qquad\qquad \cdot \left(-\frac{\mathrm{d}}{\mathrm{d}x} \frac{K S_F(U(K))}{z^\alpha}\right) \exp\left(-K \frac{S_F(U(K))}{z^\alpha}\right) \mathrm{d}z \\
&\quad = \frac{e\rho_1}{\alpha} \int_{U(K)}^{\infty} b_\delta K^{\frac{1}{\alpha}} S_F(U(K))^{\frac{1}{\alpha}} \left(K \frac{S_F(U(K))}{z^\alpha}\right)^{-\frac{1+\delta}{\alpha}} \left(-\frac{\mathrm{d}}{\mathrm{d}x} \frac{K S_F(U(K))}{z^\alpha}\right) \\
&\qquad\qquad \cdot \exp\left(-K \frac{S_F(U(K))}{z^\alpha}\right) \mathrm{d}z,
\end{aligned}$$

where the last equality follows from the definition of the tail quantile function,

$$1 - F(U(K)) = \frac{1}{K} = \frac{S_F(U(K))}{U^\alpha(K)} \iff \frac{S_F^{\frac{\delta}{\alpha}}(U^\delta(K))}{U(K)} = K^{-\frac{\delta}{\alpha}}.$$

By letting $w = K \frac{S_F(U(K))}{z^\alpha}$, we have for any $\delta \in (0, \alpha - 1)$ and $K \ge 2$

$$\begin{aligned}
e\rho_1 \int_{U(K)}^{\infty} K \frac{S_F(z)}{z^\alpha} \exp\left(-K \frac{S_F(z)}{z^\alpha}\right) \mathrm{d}z &\le \frac{e\rho_1}{\alpha} \int_0^1 b_\delta K^{\frac{1}{\alpha}} S_F^{\frac{1}{\alpha}}(U(K)) w^{-\frac{1+\delta}{\alpha}} e^{-w} \mathrm{d}w \\
&= \frac{e\rho_1}{\alpha} b_\delta S_F^{\frac{1}{\alpha}}(U(K)) K^{\frac{1}{\alpha}} \gamma\left(1 - \frac{1+\delta}{\alpha}, 1\right) \\
&\le \frac{e\rho_1}{\alpha} b_\delta A_u \gamma\left(1 - \frac{1+\delta}{\alpha}, 1\right) K^{\frac{1}{\alpha}}.
\end{aligned}$$

Letting $C_{1,1}(\mathcal{D}_\alpha) = \min_{\delta \in (0, \alpha-1)} \frac{e\rho_1}{\alpha} b_\delta A_u \gamma\big(1 - \frac{1+\delta}{\alpha}, 1\big) + \rho_1 A_u$ concludes the proof.

D.4. Penalty term analysis dependent on the loss estimation

Similarly to Section D.3, we have

$$\begin{aligned}
\mathbb{E}\left[r_{t,I_t} - r_{t,i^*} \middle| \hat{L}_t\right] &\leq \sum_{i \neq i^*} \mathbb{E}\left[\mathbb{1}[I_t = i] r_{t,i} \middle| \hat{L}_t\right] \\
&= \int_1^\infty \sum_{i \neq i^*} \left(\frac{S_f(z + \eta_t \underline{\hat{L}}_{t,i})}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \prod_{j \neq i} F(z + \eta_t \underline{\hat{L}}_{t,j}) \right) \mathrm{d}z \\
&\leq \int_1^\infty \sum_{i \neq i^*} \frac{S_f(z + \eta_t \underline{\hat{L}}_{t,i})}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \mathrm{d}z \\
&\leq \int_1^\infty \sum_{i \neq i^*} \left(\frac{\rho_1 A_u^\alpha}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \right) \mathrm{d}z \qquad (45) \\
&\leq \frac{\rho_1 A_u^\alpha}{\alpha - 1} \sum_{i \neq i^*} \frac{1}{(\eta_t \underline{\hat{L}}_{t,i})^{\alpha - 1}}, `
\end{aligned}$$

where (45) follows from (22), $S_f(x) = S_F(x)\varrho(x)$, and the boundedness of $S_F(x)$ and $\varrho(x)$.

Remark 20 *When $\nu < 0$, the perturbation $r_{t,i}$ can be negative. In such cases, we have*

$$\begin{aligned}
\sum_{i \neq i^*} \mathbb{E}\left[\mathbb{1}[I_t = I] r_{t,i} - r_{t,i^*} \middle| \hat{L}_t\right] &\leq \sum_{i \neq i^*} \mathbb{E}\left[\mathbb{1}[I_t = i] r_{t,i} \middle| \hat{L}_t\right] - \mathbb{E}\left[r_{t,i^*} \middle| \hat{L}_t\right] \\
&= \sum_{i \neq i^*} \mathbb{E}\left[\mathbb{1}[I_t = i] r_{t,i} \middle| \hat{L}_t\right] - \mathbb{E}[r_{t,i^*}] \\
&\leq \int_0^\infty \sum_{i \neq i^*} \left(\frac{S_f(z + \eta_t \underline{\hat{L}}_{t,i})}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \prod_{j \neq i} F(z + \eta_t \underline{\hat{L}}_{t,j}) \right) \mathrm{d}z - \mathbb{E}[r_{t,i^*}].
\end{aligned}$$

Therefore, when $\nu < 0$, adding a constant is enough (at most) to provide the upper bound.

Appendix E. Regret bound for stochastic bandits

In this section, we provide the proof of Theorem 5 based on the self-bounding technique, which requires a regret lower bound of the policy (Zimmert and Seldin, 2021). We first generalize the results of Honda et al. (2023) to Fréchet distributions with index $\alpha > 1$ and then generalize it to $\mathfrak{D}_\alpha$. Here, we consider two events F_t and D_t, which are defined by

$$\begin{aligned}
F_t &:= \left\{ \sum_{i \neq i^*} \frac{1}{(\eta_t \underline{\hat{L}}_{t,i})^\alpha} \leq 1 \right\}, \\
D_t &:= \left\{ \sum_{i \neq i^*} 1 - F(U(2) + \eta_t \underline{\hat{L}}_{t,i}) \leq 1 - F(U(2) + 1) \right\},
\end{aligned}$$

where $U(2)$ denotes the median of $\mathcal{D}_\alpha$. Note that $F(U(2)+1) < 1$ holds since $F(x) < 1$ holds for any finite x if $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha^{\text{all}}$. The key property on these events are

$$\underline{\hat{L}}_{t,i^*} = 0, \quad \text{and} \quad \eta_t \underline{\hat{L}}_{t,j} \geq 1, \, \forall j \neq i^*. \tag{46}$$

Note that the choice of RHS, 1 and $1 - F(U(2)+1)$ is not mandatory, and thus one can choose any real values for F_t and $1 - F(U(b)+1)$ with $b > 1$ for D_t.

E.1. Regret lower bounds

Here, we provide the regret lower bounds for $\mathcal{F}_\alpha$ and $\mathfrak{D}_\alpha$, respectively.

Lemma 21 *Let $\Delta := \min_{i \neq i^*} \Delta_i$. Then, for any $\alpha > 1$, there exists some constants $c_{s,1}(\mathcal{F}_\alpha) \in (0,1)$ that only depend on α such that*

(i) On F_t, $\sum_{i \neq i^*} \Delta_i w_{t,i} \geq c_{s,1}(\mathcal{F}_\alpha) \sum_{i \neq i^*} \frac{\Delta_i}{(\eta_t \underline{\hat{L}}_{t,i})^\alpha}$ and $w_{t,i^*} \geq 1/e$.

(ii) On F_t^c, $\sum_{i \neq i^*} \Delta_i w_{t,i} \geq \frac{\Delta}{2^{\alpha+1}+1}$.

Proof Let $\underline{\hat{L}}' = \min_{i \neq i^*} \underline{\hat{L}}_{t,i}$. Then, for any $b > 0$ we have

$$\begin{aligned}
\sum_{i \neq i^*} \Delta_i w_{t,i} &= \alpha \int_0^\infty \left(\sum_{i \neq i^*} \frac{\Delta_i}{(z + \eta_t \underline{\hat{L}}_{t,i})^{\alpha+1}} \right) \exp\left(- \sum_{i \in [K]} \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \right) \mathrm{d}z \\
&\geq \alpha \int_{b\eta_t \underline{\hat{L}}'}^\infty \left(\sum_{i \neq i^*} \frac{\Delta_i}{(z + \eta_t \underline{\hat{L}}_{t,i})^{\alpha+1}} \right) \exp\left(- \sum_{i \in [K]} \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \right) \mathrm{d}z.
\end{aligned}$$

(i) Consider the case $\sum_{i \neq i^*} \frac{1}{(\eta_t \underline{\hat{L}}_{t,i})^\alpha} \leq 1$, we have

$$\begin{aligned}
\sum_{i \neq i^*} \Delta_i w_{t,i} &\geq \alpha \int_{b\eta_t \underline{\hat{L}}'}^\infty \left(\sum_{i \neq i^*} \frac{\Delta_i}{(z + \eta_t \underline{\hat{L}}_{t,i})^{\alpha+1}} \right) \exp\left(- \sum_{i \in [K]} \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \right) \mathrm{d}z \\
&\geq \alpha \int_{b\eta_t \underline{\hat{L}}'}^\infty \left(\sum_{i \neq i^*} \frac{\Delta_i}{(z + \eta_t \underline{\hat{L}}_{t,i})^{\alpha+1}} \right) \exp\left(- \frac{1}{(b\eta_t \underline{\hat{L}}')^\alpha} - \sum_{i \neq i^*} \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \right) \mathrm{d}z \\
&\geq \alpha \int_{b\eta_t \underline{\hat{L}}'}^\infty \left(\sum_{i \neq i^*} \frac{\Delta_i}{(z + \eta_t \underline{\hat{L}}_{t,i})^{\alpha+1}} \right) \exp\left(- \left(1 + \frac{1}{b^\alpha}\right) \sum_{i \neq i^*} \frac{1}{(z + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \right) \mathrm{d}z \\
&\geq \alpha \int_{b\eta_t \underline{\hat{L}}'}^\infty \left(\sum_{i \neq i^*} \frac{\Delta_i}{(z + \eta_t \underline{\hat{L}}_{t,i})^{\alpha+1}} \right) \exp\left(- \left(1 + \frac{1}{b^\alpha}\right) \right) \mathrm{d}z \\
&= \exp\left(- \left(1 + \frac{1}{b^\alpha}\right) \right) \left(\sum_{i \neq i^*} \frac{\Delta_i}{(b\eta_t \underline{\hat{L}}' + \eta_t \underline{\hat{L}}_{t,i})^\alpha} \right)
\end{aligned}$$

$$\geq \left(\sum_{i\neq i^*} \frac{\Delta}{((1+b)\eta_t \underline{\hat{L}}_{t,i})^\alpha}\right) \exp\Big(-\Big(1+\frac{1}{b^\alpha}\Big)\Big)$$
$$= \frac{\exp\big(-\big(1+\frac{1}{b^\alpha}\big)\big)}{(1+b)^\alpha} \left(\sum_{i\neq i^*} \frac{\Delta}{(\eta_t \underline{\hat{L}}_{t,i})^\alpha}\right).$$

Since $b > 0$ is arbitrary chose, we can set $c_{s,1}(\mathcal{F}_\alpha) = \max_{b>0} \frac{\exp\left(-\left(1+\frac{1}{b^\alpha}\right)\right)}{(1+b)^\alpha} \in (0,1)$.

Since $\underline{\hat{L}}_{t,i^*} = 0$ holds on F_t, we have

$$w_{t,i^*} = \int_0^\infty \frac{\alpha}{z^{\alpha+1}} \exp\left(-\sum_{i\in[K]} \frac{1}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha}\right) \mathrm{d}z$$
$$\geq \int_0^\infty \frac{\alpha}{z^{\alpha+1}} \exp\left(-\sum_{i\neq i^*} \frac{1}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha} - \frac{1}{z^\alpha}\right) \mathrm{d}z$$
$$\geq e^{-1} \int_0^\infty \frac{\alpha}{z^{\alpha+1}} \exp\Big(-\frac{1}{z^\alpha}\Big) \mathrm{d}z = \frac{1}{e},$$

which concludes the proof of the case (i).

(ii) When $\sum_{i\neq i^*} \frac{1}{(\eta_t \underline{\hat{L}}_{t,i})^\alpha} \geq 1$, we have for any $z \geq b\eta_t \underline{\hat{L}}'$

$$\sum_{i\in[K]} \frac{1}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha} \leq \sum_{i\neq i^*} \frac{1}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha} + \frac{1}{z^\alpha}$$
$$\leq \sum_{i\neq i^*} \frac{1}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha} + \frac{1}{(\frac{z+b\eta_t \underline{\hat{L}}'}{2})^\alpha}$$
$$\leq \sum_{i\neq i^*} \frac{1}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha} + \sum_{i\neq i^*} \frac{2^\alpha}{(z+b\eta_t \underline{\hat{L}}_{t,i})^\alpha}.$$

Therefore, by letting $b = 1$, we obtain that

$$\sum_{i\neq i^*} \Delta_i w_{t,i} \geq \alpha\Delta \int_{\eta_t \underline{\hat{L}}'}^\infty \left(\sum_{i\neq i^*} \frac{1}{(z+\eta_t \underline{\hat{L}}_{t,i})^{\alpha+1}}\right) \exp\left(-\sum_{i\neq i^*} \frac{2^\alpha+1}{(z+\eta_t \underline{\hat{L}}_{t,i})^\alpha}\right) \mathrm{d}z$$
$$= \frac{\Delta}{2^\alpha+1}\left(1-\exp\left(-\sum_{i\neq i^*} \frac{2^\alpha+1}{(\eta_t \underline{\hat{L}}'+\eta_t \underline{\hat{L}}_{t,i})^\alpha}\right) \mathrm{d}z\right)$$
$$\geq \frac{\Delta}{2^\alpha+1}\left(1-\exp\left(-\sum_{i\neq i^*} \frac{2^\alpha+1}{2^\alpha(\eta_t \underline{\hat{L}}_{t,i})^\alpha}\right) \mathrm{d}z\right)$$
$$\geq \frac{\Delta}{2^\alpha+1}\Big(1-e^{-\frac{2^\alpha+1}{2^\alpha}}\Big)$$
$$\geq \frac{\Delta}{2^\alpha+1} \frac{2^\alpha+1}{2^{\alpha+1}+1} = \frac{\Delta}{2^{\alpha+1}+1},$$

where the last inequality follows from $\frac{x}{1+x} < 1 - e^{-x}$ for $x > -1$. ■

Lemma 22 *Let $\Delta := \min_{i \neq i^*} \Delta_i$. Then, for any $\alpha > 1$ and $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$, there exists some distribution-dependent constants $c_{s,1}(\mathcal{D}_\alpha), c_{s,2}(\mathcal{D}_\alpha) \in (0,1)$ such that*

(i) On D_t, $\sum_{i \neq i^*} \Delta_i w_{t,i} \geq c_{s,1}(\mathcal{D}_\alpha) \sum_{i \neq i^*} \frac{\Delta_i}{(\eta_t \underline{\hat{L}}_{t,i})^\alpha}$ and $w_{t,i^*} \geq 0.14$.

(ii) On D_t^c, $\sum_{i \neq i^*} \Delta_i w_{t,i} \geq c_{s,2}(\mathcal{D}_\alpha)\Delta$.

Proof Here, for any $\hat{L}_t$, we have

$$\begin{aligned}
\sum_{i \neq i^*} \Delta_i w_{t,i} &= \int_1^\infty \left(\sum_{i \neq i^*} \Delta_i f(z + \eta_t \underline{\hat{L}}_{t,i}) \prod_{j \neq i} F(z + \eta_t \underline{\hat{L}}_{t,j}) \right) \mathrm{d}z \\
&\geq \int_1^\infty \left(\sum_{i \neq i^*} \Delta_i f(z + \eta_t \underline{\hat{L}}_{t,i}) \right) \prod_{j \in [K]} F(z + \eta_t \underline{\hat{L}}_{t,j}) \mathrm{d}z \\
&\geq \int_1^\infty \left(\sum_{i \neq i^*} \Delta_i f(z + \eta_t \underline{\hat{L}}_{t,i}) \right) \exp\left(- \sum_{j \in [K]} \frac{1 - F(z + \eta_t \underline{\hat{L}}_{t,i})}{F(z + \eta_t \underline{\hat{L}}_{t,i})} \right) \mathrm{d}z \qquad (47) \\
&\geq \int_1^\infty \left(\sum_{i \neq i^*} \Delta_i f(z + \eta_t \underline{\hat{L}}_{t,i}) \right) \exp\left(- \sum_{j \neq i^*} \frac{1 - F(z + \eta_t \underline{\hat{L}}_{t,i})}{F(z + \eta_t \underline{\hat{L}}_{t,i})} \right) \exp\left(- \frac{1 - F(z)}{F(z)} \right) \mathrm{d}z
\end{aligned}$$

where (47) holds since $e^{-\frac{x}{1-x}} < 1 - x$ holds for $x < 1$.

(i) When D_t holds, we obtain

$$\begin{aligned}
&\int_1^\infty \left(\sum_{i \neq i^*} \Delta_i f(z + \eta_t \underline{\hat{L}}_{t,i}) \right) \exp\left(- \sum_{j \neq i^*} \frac{1 - F(z + \eta_t \underline{\hat{L}}_{t,i})}{F(z + \eta_t \underline{\hat{L}}_{t,i})} \right) \exp\left(- \frac{1 - F(z)}{F(z)} \right) \mathrm{d}z \\
&\qquad \geq e^{-1} \int_{U(2)}^\infty \left(\sum_{i \neq i^*} \Delta_i f(z + \eta_t \underline{\hat{L}}_{t,i}) \right) \exp\left(-2 \sum_{j \neq i^*} (1 - F(z + \eta_t \underline{\hat{L}}_{t,i})) \right) \mathrm{d}z \\
&\qquad \geq e^{-1} \int_{U(2)}^\infty \left(\sum_{i \neq i^*} \Delta_i f(z + \eta_t \underline{\hat{L}}_{t,i}) \right) \mathrm{d}z \\
&\qquad = e^{-1} \sum_{i \neq i^*} \Delta_i \left(1 - F\left(U(2) + \eta_t \underline{\hat{L}}_{t,i} \right) \right) \\
&\qquad = e^{-1} \sum_{i \neq i^*} \Delta_i \frac{S_F\left(U(2) + \eta_t \underline{\hat{L}}_{t,i} \right)}{\left(U(2) + \eta_t \underline{\hat{L}}_{t,i} \right)^\alpha} \\
&\qquad \geq e^{-1} \sum_{i \neq i^*} \Delta_i \frac{A_l^\alpha}{\left(U(2) + \eta_t \underline{\hat{L}}_{t,i} \right)^\alpha} \\
&\qquad \geq e^{-1} \frac{A_l^\alpha}{(U(2) + 1)^\alpha} \sum_{i \neq i^*} \frac{\Delta_i}{(\eta_t \underline{\hat{L}}_{t,i})^\alpha} = c_{s,1}(\mathcal{D}_\alpha) \frac{\Delta_i}{(\eta_t \underline{\hat{L}}_{t,i})^\alpha},
\end{aligned}$$

where the last inequality holds for $\eta_t \hat{\underline{L}}_{t,j} \geq 1$ holds for $j \neq i^*$ on D_t. When S_F is increasing, one can replace A_l^α with $S_F(U(2))$, where $c_{s,1}(\mathcal{D}_\alpha) \approx \frac{e^{-1}}{2}$ holds. Note that one can replace $U(2)$ with $U(b)$ for any $b > 1$ and choose

$$c_{s,1}(\mathcal{D}_\alpha) = \min_{b>1} e^{1-b} \frac{S_F(U(b))}{(U(b)+1)^\alpha} \in (0,1),$$

which will provide a tighter lower bound.

Since $\hat{\underline{L}}_{t,i^*} = 0$ holds on D_t, we have

$$\begin{aligned}
w_{t,i^*} &\geq e^{-1} \int_{U(2)}^{\infty} f(z) \exp\left(-\sum_{i \neq i^*} 1 - F(z + \eta_t \hat{\underline{L}}_{t,i}) \right) \mathrm{d}z \\
&\geq e^{-1} \int_{U(2)}^{\infty} f(z) \exp\left(-\sum_{i \neq i^*} 1 - F(z + \eta_t \hat{\underline{L}}_{t,i}) - (1 - F(z)) \right) \mathrm{d}z \\
&\geq e^{-1} \int_{1}^{\infty} f(z) \exp(F(z) - 1) \exp(F(U(2)+1) - 1) \mathrm{d}z \\
&\geq e^{-1} \int_{1}^{\infty} f(z) \exp(F(z) - 1) \exp(F(U(2)) - 1) \mathrm{d}z \\
&= e^{-\frac{3}{2}} (1 - e^{-1}) \geq 0.14
\end{aligned}$$

which concludes the proof of the case (i).

(ii) Recall the definition of the tail quantile function $U(x)$ defined in (42). Then, we have

$$\begin{aligned}
&\int_{1}^{\infty} \left(\sum_{i \neq i^*} \Delta_i f(z + \eta_t \hat{\underline{L}}_{t,i}) \right) \exp\left(-\sum_{j \neq i^*} \frac{1 - F(z + \eta_t \hat{\underline{L}}_{t,i})}{F(z + \eta_t \hat{\underline{L}}_{t,i})} \right) \exp\left(-\frac{1 - F(z)}{F(z)} \right) \mathrm{d}z \\
&\geq \Delta \int_{U(2)}^{\infty} \left(\sum_{i \neq i^*} f(z + \eta_t \hat{\underline{L}}_{t,i}) \right) \exp\left(-\sum_{j \neq i^*} \frac{1 - F(z + \eta_t \hat{\underline{L}}_{t,i})}{F(z + \eta_t \hat{\underline{L}}_{t,i})} \right) \exp\left(-\frac{1 - F(z)}{F(z)} \right) \mathrm{d}z \\
&\geq \Delta e^{-1} \int_{U(2)}^{\infty} \left(\sum_{i \neq i^*} f(z + \eta_t \hat{\underline{L}}_{t,i}) \right) \exp\left(-2 \sum_{j \neq i^*} (1 - F(z + \eta_t \hat{\underline{L}}_{t,i})) \right) \mathrm{d}z \qquad (48) \\
&= \Delta \frac{e^{-1}}{2} \left(1 - \exp\left(-2 \sum_{j \neq i^*} (1 - F(U(2) + \eta_t \hat{\underline{L}}_{t,j})) \right) \right) \\
&\geq \Delta \frac{e^{-1}}{2} (1 - \exp(-2(1 - F(U(2)+1)))) = c_s(\mathcal{D}_\alpha) \Delta
\end{aligned}$$

where (48) holds since $e^{-\frac{1-x}{x}}$ is increasing with respect to $x \in [0,1]$, and $z \geq U(b)$ and $F(z) \geq \frac{1}{b}$ for $z \geq B$. Note that $c_s(\mathcal{D}_\alpha) \in (0,1)$ is a distribution-dependent constant and can be approximated as $\frac{e^{-1}}{2}(1 - e^{-1})$. ∎

E.2. Regret for the optimal arm

To apply the self-bounding technique to FTPL, it is necessary to represent the regret associated with the optimal arm in terms of statistics of the other arms. We begin by extending the findings of Honda et al. (2023) to Fréchet distributions with an index $\alpha > 1$ and subsequently generalize it to $\mathfrak{D}_\alpha$. Before diving into the proofs, we first introduce the lemma by Honda et al. (2023).

Lemma 23 (Partial result of Lemma 11 in Honda et al. (2023)) *For any $\hat{L}_t$ and $\zeta \in (0,1)$, it holds that*

$$\mathbb{E}\left[\mathbb{1}\left[\hat{\ell}_{t,i^*} > \frac{\zeta}{\eta_t}\right]\hat{\ell}_{t,i^*}\middle|\hat{L}_t\right] \leq \frac{1}{1-e^{-1}}(1-e^{-1})^{\frac{\zeta}{\eta_t}}\left(\frac{\zeta}{\eta_t}+e\right)$$

and when $\eta_t = \frac{cK^{\frac{1}{\alpha}-\frac{1}{2}}}{\sqrt{t}}$

$$\sum_{t=1}^{\infty}\frac{1}{1-e^{-1}}(1-e^{-1})^{\frac{\zeta}{\eta_t}}\left(\frac{\zeta}{\eta_t}+e\right) \leq \mathcal{O}\left(c^2K^{\frac{2}{\alpha}-1}\right).$$

Lemma 24 *On F_t, for any $\zeta \in (0,1)$ and $\alpha > 1$, we have*

$$\mathbb{E}\left[\hat{\ell}_{t,i^*}\left(\phi_{i^*}(\eta_t\hat{L}_t;\mathcal{F}_\alpha) - \phi_{i^*}(\eta_t(\hat{L}_t+\hat{\ell}_t);\mathcal{F}_\alpha)\right)\middle|\hat{L}_t\right]$$
$$\leq \frac{2\alpha e}{(1-\zeta)^{\alpha+1}}\sum_{j\neq i^*}\frac{1}{\underline{\hat{L}}_{t,j}} + \frac{1}{1-e^{-1}}(1-e^{-1})^{\frac{\zeta}{\eta_t}}\left(\frac{\zeta}{\eta_t}+e\right).$$

Proof Recall (46), which shows that any $j \neq i^*$ satisfies $\underline{\hat{L}}_{t,j} \geq \frac{1}{\eta_t}$ and $\arg\min_{i\in[K]}\hat{L}_t = \hat{L}_{t,i^*}$ holds on F_t. Following Honda et al. (2023), we consider the cases (a) $\widehat{w_{t,i^*}^{-1}} \leq \frac{\zeta}{\eta_t}$ and (b) $\widehat{w_{t,i^*}^{-1}} > \frac{\zeta}{\eta_t}$, separately.

(a) Let us consider the first case, where $\arg\min_{i\in[K]}\hat{L}_t + xe_{i^*} = i^*$ holds since $\underline{\hat{L}}_t \geq \frac{1}{\eta_t}$ and

$$\hat{\ell}_{t,i^*} = \ell_{t,i^*}\widehat{w_{t,i^*}^{-1}} \leq \frac{\zeta}{\eta_t} < \frac{1}{\eta_t} \leq \min_{i\neq i^*}\underline{\hat{L}}_{t,i}.$$

Therefore, we have for $x \leq \frac{\zeta}{\eta_t}$

$$\phi_{i^*}(\eta_t(\hat{L}_t + xe_{i^*})) = \int_0^\infty \frac{\alpha}{z^{\alpha+1}}\exp\left(-\sum_{i\in[K]}\frac{1}{(z+\eta_t(\underline{\hat{L}}_{t,j}-x))^\alpha}\right)\mathrm{d}z,$$

which implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\phi_{i^*}(\eta_t(\underline{\hat{L}}_t + xe_{i^*})) &= \int_0^\infty -\frac{\alpha}{z^{\alpha+1}}\sum_{j\neq i^*}\frac{\alpha\eta_t}{(z+\eta_t(\underline{\hat{L}}_{t,j}-x))^{\alpha+1}}\exp\left(-\sum_{j\neq i^*}\frac{1}{(z+\eta_t(\underline{\hat{L}}_{t,j}-x))^\alpha}-\frac{1}{z^\alpha}\right)\mathrm{d}z \\ &\geq \int_0^\infty -\frac{\alpha}{z^{\alpha+1}}\sum_{j\neq i^*}\frac{\alpha\eta_t}{(z+\eta_t(\underline{\hat{L}}_{t,j}-x))^{\alpha+1}}\exp\left(-\sum_{j\neq i^*}\frac{1}{(z+\eta_t(\underline{\hat{L}}_{t,j}-x))^\alpha}-\frac{1}{z^\alpha}\right)\mathrm{d}z\end{aligned}$$

Then, we obtain

$$
\begin{aligned}
&\hat{\ell}_{t,i^*}\Big(\phi_{i^*}(\eta_t\hat{L}_t) - \phi_{i^*}(\eta_t(\hat{L}_t+\hat{\ell}_t))\Big)\\
&\quad= \hat{\ell}_{t,i^*}\int_0^{\hat{\ell}_t} -\frac{\mathrm{d}}{\mathrm{d}x}\phi_{i^*}(\eta_t(\underline{\hat{L}}_t + xe_{i^*}))\mathrm{d}x\\
&\quad\le \hat{\ell}_{t,i^*}\int_0^{\hat{\ell}_t}\int_0^{\infty}\frac{\alpha}{z^{\alpha+1}}\sum_{j\neq i^*}\frac{\alpha\eta_t}{(z+\eta_t(\underline{\hat{L}}_{t,j}-x))^{\alpha+1}}\exp\left(-\sum_{j\neq i^*}\frac{1}{(z+\eta_t(\underline{\hat{L}}_{t,j}-x))^{\alpha}}-\frac{1}{z^{\alpha}}\right)\mathrm{d}z\mathrm{d}x\\
&\quad\le \hat{\ell}_{t,i^*}\int_0^{\hat{\ell}_t}\int_0^{\infty}\frac{\alpha}{z^{\alpha+1}}\sum_{j\neq i^*}\frac{\alpha\eta_t}{(z+\eta_t(\underline{\hat{L}}_{t,j}-x))^{\alpha+1}}\exp\left(-\frac{1}{z^{\alpha}}\right)\mathrm{d}z\mathrm{d}x\\
&\quad\le \hat{\ell}_{t,i^*}\int_0^{\hat{\ell}_t}\int_0^{\infty}\frac{\alpha}{z^{\alpha+1}}\sum_{j\neq i^*}\frac{1}{(1-\zeta)^{\alpha+1}}\frac{\alpha\eta_t}{(\eta_t\underline{\hat{L}}_{t,j})^{\alpha+1}}\exp\left(-\frac{1}{z^{\alpha}}\right)\mathrm{d}z\mathrm{d}x \qquad (\text{by } x\le\zeta/\eta_t,\ \text{and } \hat{L}_{t,i}\ge 1/\eta_t)\\
&\quad= \hat{\ell}_{t,i^*}\int_0^{\hat{\ell}_t}\sum_{j\neq i^*}\frac{1}{(1-\zeta)^{\alpha+1}}\frac{\alpha\eta_t}{(\eta_t\underline{\hat{L}}_{t,j})^{\alpha+1}}\mathrm{d}x\\
&\quad= \hat{\ell}_{t,i^*}^2\sum_{j\neq i^*}\frac{1}{(1-\zeta)^{\alpha+1}}\frac{\alpha\eta_t}{(\eta_t\underline{\hat{L}}_{t,j})^{\alpha+1}}\\
&\quad\le \hat{\ell}_{t,i^*}^2\sum_{j\neq i^*}\frac{1}{(1-\zeta)^{\alpha+1}}\frac{\alpha}{\underline{\hat{L}}_{t,j}},
\end{aligned}
$$

where the last inequality comes from $\underline{\hat{L}}_{t,i}\ge\frac{1}{\eta_t}$. Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}[\hat{\ell}_{t,i^*}\le\zeta/\eta_t]\hat{\ell}_{t,i^*}\Big(\phi_{i^*}(\eta_t\hat{L}_t)-\phi_{i^*}(\eta_t(\hat{L}_t+\hat{\ell}_t))\Big)\middle|\hat{L}_t\right]\\
&\qquad\le \mathbb{E}\left[\mathbb{1}[\hat{\ell}_{t,i^*}\le\zeta/\eta_t]\hat{\ell}_{t,i^*}^2\sum_{j\neq i^*}\frac{\alpha}{(1-\zeta)^{\alpha+1}\underline{\hat{L}}_{t,j}}\middle|\hat{L}_t\right]\\
&\qquad\le \mathbb{E}\left[\frac{2\ell_{t,i^*}^2}{w_{t,i^*}}\sum_{j\neq i^*}\frac{\alpha}{(1-\zeta)^{\alpha+1}\underline{\hat{L}}_{t,j}}\middle|\hat{L}_t\right]\\
&\qquad\le 2\alpha e\sum_{j\neq i^*}\frac{1}{(1-\zeta)^{\alpha+1}\underline{\hat{L}}_{t,j}}.
\end{aligned} \tag{49}
$$

(b) When $\widehat{w_{t,i^*}^{-1}}>\frac{\zeta}{\eta_t}$, by Lemma 23, we have

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{1}[\hat{\ell}_{t,i^*}>\zeta/\eta_t]\hat{\ell}_{t,i^*}\Big(\phi_{i^*}(\eta_t\hat{L}_t)-\phi_{i^*}(\eta_t(\hat{L}_t+\hat{\ell}_t))\Big)\middle|\hat{L}_t\right] &\le \mathbb{E}\left[\mathbb{1}[\hat{\ell}_{t,i^*}>\zeta/\eta_t]\hat{\ell}_{t,i^*}\middle|\hat{L}_t\right]\\
&\le \frac{1}{1-e^{-1}}(1-e^{-1})^{\frac{\zeta}{\eta_t}}\left(\frac{\zeta}{\eta_t}+e\right).
\end{aligned} \tag{50}
$$

Combining (49) and (50) concludes the proof. ■

Lemma 25 *On D_t, for any $\zeta \in (0,1)$, $\mathcal{D}_\alpha \in \mathfrak{D}_\alpha$ and $\alpha > 1$, we have*

$$\mathbb{E}\left[\hat{\ell}_{t,i^*}\left(\phi_{i^*}(\eta_t \hat{L}_t; \mathcal{D}_\alpha) - \phi_{i^*}(\eta_t(\hat{L}_t + \hat{\ell}_t); \mathcal{D}_\alpha)\right) \middle| \hat{L}_t\right] \leq \frac{14.4 A_u^\alpha \rho_1 e(1-e^{-1})}{(1-\zeta)^{\alpha+1}} \sum_{j \neq i^*} \frac{1}{\underline{\hat{L}}_{t,j}} + \frac{1}{1-e^{-1}}(1-e^{-1})^{\frac{\zeta}{\eta_t}}\left(\frac{\zeta}{\eta_t} + e\right).$$

Proof As the proof of Lemma 24, we consider two cases (a) $\widehat{w_{t,i^*}^{-1}} \leq \frac{\zeta}{\eta_t}$ and (b) $\widehat{w_{t,i^*}^{-1}} > \frac{\zeta}{\eta_t}$, separately. For case (b), one can see that Lemma 23 can be directly applied as Lemma 24.

(a) When $\widehat{w_{t,i^*}^{-1}} \leq \frac{\zeta}{\eta_t}$, we have

$$\phi_{i^*}(\eta_t(\hat{L}_t + e_{i^*}x); \mathcal{D}_\alpha) = \int_1^\infty f(z) \prod_{j \neq i^*} F\left(z + \eta_t(\underline{\hat{L}}_{t,j} - x)\right) \mathrm{d}z,$$

which implies for $x \leq \frac{\zeta}{\eta_t}$,

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}x}\phi_{i^*}\left(\eta_t(\hat{L}_t + e_{i^*}x); \mathcal{D}_\alpha\right) \\
&= \int_1^\infty f(z) \sum_{i \neq i^*} \left(\eta_t f\left(z + \eta_t(\underline{\hat{L}}_{t,j} - x)\right) \prod_{j \neq i, i^*} F\left(z + \eta_t(\underline{\hat{L}}_{t,j} - x)\right)\right) \mathrm{d}z \\
&\leq \int_1^\infty f(z) \sum_{i \neq i^*} \left(\eta_t f\left(z + \eta_t(\underline{\hat{L}}_{t,i} - x)\right) \exp\left(-\sum_{j \neq i, i^*} \left(1 - F\left(z + \eta_t(\underline{\hat{L}}_{t,j} - x)\right)\right)\right)\right) \mathrm{d}z \\
&\leq e^2 \int_1^\infty f(z) \sum_{i \neq i^*} \eta_t f\left(z + \eta_t(\underline{\hat{L}}_{t,i} - x)\right) \exp\left(-\sum_{j \neq i^*} \left(1 - F\left(z + \eta_t(\underline{\hat{L}}_{t,j} - x)\right)\right) - (1 - F(z))\right) \mathrm{d}z \\
&\leq e^2 \int_1^\infty f(z) \sum_{i \neq i^*} \eta_t f\left(z + \eta_t(\underline{\hat{L}}_{t,i} - x)\right) \exp(-(1 - F(z))) \mathrm{d}z \\
&= e^2 \int_1^\infty f(z) \sum_{i \neq i^*} \eta_t \frac{S_F(z + \eta_t(\underline{\hat{L}}_{t,i} - x)) \varrho(z + \eta_t(\underline{\hat{L}}_{t,i} - x))}{(z + \eta_t(\underline{\hat{L}}_{t,i} - x))^{\alpha+1}} \exp(-(1 - F(z))) \mathrm{d}z \\
&\leq e^2 \eta_t A_u^\alpha \rho_1 \int_1^\infty f(z) \exp(-(1 - F(z))) \sum_{i \neq i^*} \frac{1}{(z + \eta_t(\underline{\hat{L}}_{t,i} - x))^{\alpha+1}} \mathrm{d}z \qquad (51) \\
&\leq e^2 \eta_t A_u^\alpha \rho_1 \int_1^\infty f(z) \exp(-(1 - F(z))) \sum_{i \neq i^*} \frac{1}{(1-\zeta)^{\alpha+1}(\eta_t \underline{\hat{L}}_{t,i})^{\alpha+1}} \mathrm{d}z \\
&\leq A_u^\alpha \rho_1 e^2 \sum_{i \neq i^*} \frac{\eta_t}{(1-\zeta)^{\alpha+1}(\eta_t \underline{\hat{L}}_{t,i})^{\alpha+1}} (1 - e^{-1}) \\
&\leq A_u^\alpha \rho_1 e^2 (1 - e^{-1}) \sum_{i \neq i^*} \frac{1}{(1-\zeta)^{\alpha+1} \underline{\hat{L}}_{t,i}} \qquad (\text{by } \eta_t \underline{\hat{L}}_{t,i} \geq 1, \forall i \neq i^*),
\end{aligned}$$

where (51) follows from the boundedness of $S_F \leq S$ and Assumption 2. Therefore, we have

$$\mathbb{E}\left[\hat{\ell}_{t,i^*}\left(\phi_{i^*}(\eta_t \hat{L}_t; \mathcal{D}_\alpha) - \phi_{i^*}(\eta_t(\hat{L}_t + \hat{\ell}_t); \mathcal{D}_\alpha)\right)\middle|\hat{L}_t\right] \\ \leq \sum_{i \neq i^*} \frac{14.4 A_u^\alpha \rho_1 e(1-e^{-1})}{(1-\zeta)^{\alpha+1}\underline{\hat{L}}_{t,i}} + \frac{1}{1-e^{-1}}(1-e^{-1})^{\frac{\zeta}{\eta_t}}\left(\frac{\zeta}{\eta_t} + e\right).$$

Here, 14.4 is introduced by $\frac{2}{0.14}$ by following the same steps in (49). ■

E.3. Proof of Theorems 5 and 6

Although the overall proofs are identical for $\mathcal{F}_\alpha$ and $\mathfrak{D}_\alpha$ in essential, we provide the proof of $\mathcal{F}_\alpha$ first and then $\mathfrak{D}_\alpha$ for completeness.

E.3.1. FRÉCHET DISTRIBUTION WITH $\alpha \geq 2$

For simplicity, let $K_\alpha = K^{\frac{1}{\alpha}-\frac{1}{2}}$ so that $\eta_t = \frac{cK_\alpha}{\sqrt{t}}$. Combining the results obtained thus far, the regret is bounded by

$$\begin{aligned}
\mathcal{R}(T) &\leq \sum_{t=1} \mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1}\right\rangle\right] + \sum_{t=1}^{T}\left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right)\mathbb{E}\left[r_{t+1,I_{t+1}} - r_{t+1,i^*}\right] \\
&\quad + \frac{K^{\frac{1}{\alpha}}\Gamma\left(1-\frac{1}{\alpha}\right)}{\eta_1} + \frac{\alpha}{2}\log(T+1) \qquad \text{(by Lemmas 7 and 8)} \\
&\leq \sum_{t=1}^{T}\mathbb{E}\left[\mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1}\right\rangle + \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right)(r_{t+1,I_{t+1}} - r_{t+1,i^*})\middle|\hat{L}_t\right]\right] \\
&\quad + \frac{\sqrt{K}\Gamma\left(1-\frac{1}{\alpha}\right)}{c} + \frac{\alpha}{2}\log(T+1) \\
&\leq \sum_{t=1}^{T}\mathbb{E}\left[\mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1}\right\rangle + \frac{r_{t+1,I_{t+1}} - r_{t+1,i^*}}{2cK_\alpha\sqrt{t}}\middle|\hat{L}_t\right]\right] + \frac{\sqrt{K}\Gamma\left(1-\frac{1}{\alpha}\right)}{c} + \frac{\alpha}{2}\log(T+1),
\end{aligned} \tag{52}$$

where the last inequality follows from

$$\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} = \frac{1}{cK_\alpha}(\sqrt{t+1} - \sqrt{t}) = \frac{\sqrt{t}}{cK_\alpha}(\sqrt{1+1/t} - 1) \leq \frac{1}{2cK_\alpha\sqrt{t}}.$$

Note that $w_t = \phi(\eta_t \hat{L}_t)$ and $w_{t+1} = \phi(\eta_t(\hat{L}_t + \ell_t))$ by definition of ϕ.

On F_t, where $\eta_t \underline{\hat{L}}_{t,j} \geq 1$ for $j \neq i^*$, we have for $\alpha \geq 2$

$$\mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1} \right\rangle + \frac{r_{t+1,I_{t+1}} - r_{t+1,i^*}}{2cK_\alpha\sqrt{t}} \middle| \hat{L}_t \right]$$

$$\leq \sum_{i \neq i^*} \frac{2\alpha}{\underline{\hat{L}}_{t,i}} + \frac{1}{2cK_\alpha\sqrt{t}} \frac{\alpha}{\alpha-1} \frac{1}{(\eta_t \underline{\hat{L}}_{t,i})^{\alpha-1}} + \frac{2\alpha e}{(1-\zeta)^{\alpha+1}} \frac{1}{\underline{\hat{L}}_{t,i}} + \sum_{t=1}^{T} \frac{(1-e^{-1})^{\frac{\zeta}{\eta_t}}}{1-e^{-1}} \left(\frac{\zeta}{\eta_t} + e\right)$$

(by Lemmas 13, 11, and 24)

$$\leq \sum_{i \neq i^*} \frac{2\alpha}{\underline{\hat{L}}_{t,i}} + \frac{1}{2cK_\alpha\sqrt{t}} \frac{\alpha}{\alpha-1} \frac{1}{(\eta_t \underline{\hat{L}}_{t,i})} + \frac{2\alpha e}{(1-\zeta)^{\alpha+1}} \frac{1}{\underline{\hat{L}}_{t,i}} + \sum_{t=1}^{T} \frac{(1-e^{-1})^{\frac{\zeta}{\eta_t}}}{1-e^{-1}} \left(\frac{\zeta}{\eta_t} + e\right) \tag{53}$$

$$\leq \sum_{i \neq i^*} \frac{2\alpha}{\underline{\hat{L}}_{t,i}} + \frac{1}{2(cK_\alpha)^2} \frac{\alpha}{\alpha-1} \frac{1}{\underline{\hat{L}}_{t,i}} + \frac{2\alpha e}{(1-\zeta)^{\alpha+1}} \frac{1}{\underline{\hat{L}}_{t,i}} + \sum_{t=1}^{T} \frac{(1-e^{-1})^{\frac{\zeta}{\eta_t}}}{1-e^{-1}} \left(\frac{\zeta}{\eta_t} + e\right)$$

$$\leq \sum_{i \neq i^*} \frac{2\alpha + \frac{2\alpha e}{(1-\zeta)^\alpha} + \frac{\alpha}{2(cK_\alpha)^2(\alpha-1)}}{\hat{L}_{t,i}} + \mathcal{O}(c^2 K_\alpha^2) \qquad \text{(by Lemma 23)}$$

$$= \sum_{i \neq i^*} \frac{2\alpha(1+e^2) + \frac{\alpha}{2(cK_\alpha)^2(\alpha-1)}}{\hat{L}_{t,i}} + \mathcal{O}(c^2 K_\alpha^2) \tag{54}$$

where (53) follows from $\eta_t \underline{\hat{L}}_{t,i} \geq 1$ for all $i \neq i^*$ on F_t and $\alpha \geq 2$ and we chose $\zeta = 1 - e^{-\frac{1}{\alpha}} \in (0,1)$ for simplicity.

On F_t^c, we have

$$\mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1} \right\rangle + \frac{r_{t+1,I_{t+1}} - r_{t+1,i^*}}{2cK_\alpha\sqrt{t}} \middle| \hat{L}_t \right]$$

$$\leq \frac{2\alpha(\alpha+1)}{\alpha-1} \Gamma\left(1 + \frac{1}{\alpha}\right) K^{1-\frac{1}{\alpha}} \eta_t + \frac{K^{\frac{1}{\alpha}}}{2cK_\alpha\sqrt{t}} \frac{\alpha^2(2\alpha+e-2)}{(\alpha-1)(2\alpha-1)e}$$

(by Lemmas 13 and 12)

$$= \left(\frac{2c\alpha(\alpha+1)}{\alpha-1} \Gamma\left(1+\frac{1}{\alpha}\right) + \frac{\alpha^2(2\alpha+e-2)}{2ce(\alpha-1)(2\alpha-1)}\right) \sqrt{\frac{K}{t}}. \tag{55}$$

Combining (54) and (55) with (52) provides

$$\mathcal{R}(T) \leq \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{1}[F_t] \sum_{i \neq i^*} \frac{2\alpha(1+e^2) + \frac{\alpha}{2(cK_\alpha)^2(\alpha-1)}}{\hat{L}_{t,i}} \right.$$

$$\left. + \mathbb{1}[F_t^c] \left(\frac{2c\alpha(\alpha+1)}{\alpha-1} \Gamma\left(1+\frac{1}{\alpha}\right) + \frac{\alpha^2(2\alpha+e-2)}{2ce(\alpha-1)(2\alpha-1)}\right) \sqrt{\frac{K}{t}} \right]$$

$$+ \frac{\sqrt{K}\Gamma\left(1-\frac{1}{\alpha}\right)}{c} + \frac{\alpha}{2}\log(T+1) + \mathcal{O}(c^2 K_\alpha^2). \tag{56}$$

On the other hand, by Lemma 21, we have

$$\mathcal{R}(T) \geq \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{1}[F_t] c_{s,1}(\mathcal{F}_\alpha) \frac{\Delta_i t^{\frac{\alpha}{2}}}{(cK_\alpha \underline{\hat{L}}_{t,i})^\alpha} + \mathbb{1}[F_t^c] \frac{\Delta}{2^{\alpha+1}+1} \right]. \tag{57}$$

By applying self-bounding technique, (56) - (57)/2, we have

$$\begin{aligned}\frac{\mathcal{R}(T)}{2} \leq \sum_{t=1}^{T} \mathbb{E}&\left[\mathbb{1}[F_t] \sum_{i \neq i^*} \left(\frac{2\alpha(1+e^2) + \frac{\alpha}{2(cK_\alpha)^2(\alpha-1)}}{\hat{L}_{t,i}} - c_{s,1}(\mathcal{F}_\alpha) \frac{\Delta_i t^{\frac{\alpha}{2}}}{2(cK_\alpha \underline{\hat{L}}_{t,i})^\alpha} \right) \right] \\ &+ \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{1}[F_t^c] \left(\left(\frac{2c\alpha(\alpha+1)}{\alpha-1} \Gamma\left(1+\frac{1}{\alpha}\right) + \frac{\alpha^2(2\alpha+e-2)}{2ce(\alpha-1)(2\alpha-1)} \right) \sqrt{\frac{K}{t}} - \frac{\Delta}{2^{\alpha+1}+1} \right) \right] \\ &\quad + \frac{\sqrt{K}\Gamma\left(1-\frac{1}{\alpha}\right)}{c} + \frac{\alpha}{2}\log(T+1) + \mathcal{O}\left(c^2 K_\alpha^2\right). \end{aligned} \tag{58}$$

For the first term of (58), we have

$$\begin{aligned} &\left(\frac{2\alpha(1+e^2) + \frac{\alpha}{2(cK_\alpha)^2(\alpha-1)}}{\hat{L}_{t,i}} - c_{s,1}(\mathcal{F}_\alpha) \frac{\Delta_i t^{\frac{\alpha}{2}}}{2(cK_\alpha \underline{\hat{L}}_{t,i})^\alpha} \right) \\ &\leq \left(2\alpha(1+e^2) + \frac{\alpha}{2(cK_\alpha)^2(\alpha-1)} \right) \frac{\alpha-1}{\alpha} \left(\frac{4\alpha(1+e^2) + \frac{\alpha}{(cK_\alpha)^2(\alpha-1)}}{\alpha c_{s,1}(\mathcal{F}_\alpha)\Delta_i} \right)^{\frac{1}{\alpha-1}} \left(\frac{cK_\alpha}{\sqrt{t}} \right)^{\frac{\alpha}{\alpha-1}} \\ &= \left(4(\alpha-1) + \frac{1}{2(cK_\alpha)^2} \right) \left(\frac{4\alpha(1+e^2) + \frac{\alpha}{(cK_\alpha)^2(\alpha-1)}}{\alpha c_{s,1}(\mathcal{F}_\alpha)\Delta_i} \right)^{\frac{1}{\alpha-1}} \frac{(cK_\alpha)^{\frac{\alpha}{\alpha-1}}}{t^{\frac{\alpha}{2(\alpha-1)}}} \\ &= \mathcal{O}\left(\frac{1}{\Delta_i^{\frac{1}{\alpha-1}} K^{\frac{\alpha-2}{2(\alpha-1)}} t^{\frac{\alpha}{2(\alpha-1)}}} \right), \end{aligned} \tag{59}$$

since $Ax - Bx^\alpha \leq A\frac{\alpha-1}{\alpha}\left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha-1}}$ holds for $A, B > 0$ and $\alpha > 1$.

For the second term of (58), we have

$$\begin{aligned} &\sum_{t=1}^{T} \left(\frac{2c\alpha(\alpha+1)}{\alpha-1} \Gamma\left(1+\frac{1}{\alpha}\right) + \frac{\alpha^2(2\alpha+e-2)}{2ce(\alpha-1)(2\alpha-1)} \right) \sqrt{\frac{K}{t}} - \frac{\Delta}{2^{\alpha+1}+1} \\ &\leq \sum_{t=1}^{T} \max\left\{ \left(\frac{2c\alpha(\alpha+1)}{\alpha-1} \Gamma\left(1+\frac{1}{\alpha}\right) + \frac{\alpha^2(2\alpha+e-2)}{2ce(\alpha-1)(2\alpha-1)} \right) \sqrt{\frac{K}{t}} - \frac{\Delta}{2^{\alpha+1}+1}, 0 \right\} \\ &\leq \left(\frac{\frac{2c\alpha(\alpha+1)}{\alpha-1}\Gamma\left(1+\frac{1}{\alpha}\right) + \frac{\alpha^2(2\alpha+e-2)}{2ce(\alpha-1)(2\alpha-1)}}{\frac{\Delta}{2^{\alpha+1}+1}} \right)^2 K = \mathcal{O}(K) \end{aligned} \tag{60}$$

Therefore, by combining (59) and (60) with (58), we obtain

$$\begin{aligned}\frac{\mathcal{R}(T)}{2} &\leq \mathcal{O}\left(\sum_{i\neq i^*}\sum_{t=1}^{T}\frac{1}{\Delta_i^{\frac{1}{\alpha-1}}K^{\frac{\alpha-2}{2(\alpha-1)}}t^{\frac{\alpha}{2(\alpha-1)}}}\right)+\mathcal{O}(K)+\frac{\sqrt{K}\Gamma\left(1-\frac{1}{\alpha}\right)}{c}+\frac{\alpha}{2}\log(T+1)+\mathcal{O}\left(c^2K_\alpha^2\right)\\ &\leq \mathcal{O}(K)+\frac{\sqrt{K}\Gamma\left(1-\frac{1}{\alpha}\right)}{c}+\frac{\alpha}{2}\log(T+1)\\ &\qquad+\mathcal{O}\left(c^2K_\alpha^2\right)+\begin{cases}\mathcal{O}\left(\sum_{i\neq i^*}\frac{\log T}{\Delta_i}\right), & \text{if } \alpha=2\\ \mathcal{O}\left(\sum_{i\neq i^*}\frac{1}{(\alpha-2)}\frac{T^{\frac{\alpha-2}{2(\alpha-1)}}}{\Delta_i^{\frac{1}{\alpha-1}}K^{\frac{\alpha-2}{2(\alpha-1)}}}\right), & \text{if } \alpha>2,\end{cases}\end{aligned}$$

which concludes the proof for $\mathcal{F}_\alpha$ with $\alpha \geq 2$.

E.3.2. FRÉCHET DISTRIBUTION WITH $\alpha \in (1,2)$

The proof for $\alpha \in (1,2)$ begin by modifying (53), where we obtain

$$\begin{aligned}&\mathbb{E}\left[\left\langle\hat{\ell}_t, w_t-w_{t+1}\right\rangle+\frac{r_{t+1,I_{t+1}}-r_{t+1,i^*}}{2cK_\alpha\sqrt{t}}\middle|\hat{L}_t\right]\\ &\leq\sum_{i\neq i^*}\frac{2\alpha}{\hat{\underline{L}}_{t,i}}+\frac{1}{2cK_\alpha\sqrt{t}}\frac{\alpha}{\alpha-1}\frac{1}{(\eta_t\hat{\underline{L}}_{t,i})^{\alpha-1}}+\frac{2\alpha e}{(1-\zeta)^{\alpha+1}}\frac{1}{\hat{\underline{L}}_{t,i}}+\sum_{t=1}^{T}\frac{(1-e^{-1})^{\frac{\zeta}{\eta_t}}}{1-e^{-1}}\left(\frac{\zeta}{\eta_t}+e\right)\\ &\leq\sum_{i\neq i^*}\frac{2\alpha+\frac{2\alpha e}{(1-\zeta)^\alpha}}{\hat{L}_{t,i}}+\frac{1}{2cK_\alpha\sqrt{t}}\frac{\alpha}{\alpha-1}\frac{1}{(\eta_t\hat{\underline{L}}_{t,i})^{\alpha-1}}+\mathcal{O}\left(c^2K_\alpha^2\right) \qquad \text{(by Lemma 23)}\\ &=\sum_{i\neq i^*}\frac{2\alpha+\frac{2\alpha e}{(1-\zeta)^\alpha}}{\hat{L}_{t,i}}+\frac{1}{2(cK_\alpha)^\alpha}\frac{\alpha}{\alpha-1}\frac{1}{t^{1-\frac{\alpha}{2}}(\hat{\underline{L}}_{t,i})^{\alpha-1}}+\mathcal{O}\left(c^2K_\alpha^2\right).\end{aligned}$$

By following the same steps from (55), one can obtain

$$\begin{aligned}\frac{\mathcal{R}(T)}{2} &\leq \sum_{t=1}^{T}\mathbb{E}\left[\mathbb{1}[F_t]\sum_{i\neq i^*}\left(\frac{2\alpha(1+e^2)}{\hat{L}_{t,i}}+\frac{1}{2(cK_\alpha)^\alpha}\frac{\alpha}{\alpha-1}\frac{1}{t^{1-\frac{\alpha}{2}}(\hat{\underline{L}}_{t,i})^{\alpha-1}}-c_{s,1}(\mathcal{F}_\alpha)\frac{\Delta_i t^{\frac{\alpha}{2}}}{2(cK_\alpha\hat{\underline{L}}_{t,i})^\alpha}\right)\right]\\ &\quad+\sum_{t=1}^{T}\mathbb{E}\left[\mathbb{1}[F_t^c]\left(\left(\frac{2c\alpha(\alpha+1)}{\alpha-1}\Gamma\left(1+\frac{1}{\alpha}\right)+\frac{\alpha^2(2\alpha+e-2)}{2ce(\alpha-1)(2\alpha-1)}\right)\sqrt{\frac{K}{t}}-\frac{\Delta}{2^{\alpha+1}+1}\right)\right]\\ &\qquad+\frac{\sqrt{K}\Gamma\left(1-\frac{1}{\alpha}\right)}{c}+\frac{\alpha}{2}\log(T+1)+\mathcal{O}\left(c^2K_\alpha^2\right).\end{aligned}$$

Here, the first term can be written as

$$\begin{aligned}&\frac{2\alpha(1+e^2)}{\hat{L}_{t,i}}+\frac{1}{2(cK_\alpha)^\alpha}\frac{\alpha}{\alpha-1}\frac{1}{t^{1-\frac{\alpha}{2}}(\hat{\underline{L}}_{t,i})^{\alpha-1}}-c_{s,1}(\mathcal{F}_\alpha)\frac{\Delta_i t^{\frac{\alpha}{2}}}{2(cK_\alpha\hat{\underline{L}}_{t,i})^\alpha}\\ &=\left(\frac{2\alpha(1+e^2)}{\hat{L}_{t,i}}-\frac{c_{s,1}(\mathcal{F}_\alpha)}{2}\frac{\Delta_i t^{\frac{\alpha}{2}}}{(cK_\alpha\hat{\underline{L}}_{t,i})^\alpha}\right)+\frac{1}{2(cK_\alpha)^\alpha}\left(\frac{\alpha}{\alpha-1}\frac{1}{t^{1-\frac{\alpha}{2}}(\hat{\underline{L}}_{t,i})^{\alpha-1}}-c_{s,1}(\mathcal{F}_\alpha)\frac{\Delta_i t^{\frac{\alpha}{2}}}{(\hat{\underline{L}}_{t,i})^\alpha}\right)\end{aligned} \tag{61}$$

The first term of (61) can be bounded in the same way of (59) and the second term is bounded as

$$\frac{\alpha}{\alpha-1}\frac{1}{t^{1-\frac{\alpha}{2}}(\hat{\underline{L}}_{t,i})^{\alpha-1}} - c_{s,1}(\mathcal{F}_\alpha)\frac{\Delta_i t^{\frac{\alpha}{2}}}{(\hat{\underline{L}}_{t,i})^{\alpha}} \leq \frac{1}{\alpha-1}\frac{1}{t^{1-\frac{\alpha}{2}}}\left(\frac{1}{\Delta_i t c_{s,1}(\mathcal{F}_\alpha)}\right)^{\alpha-1} = \frac{1}{\alpha-1}\frac{1}{(\Delta_i c_{s,1}(\mathcal{F}_\alpha))^{\alpha-1}}\frac{1}{t^{\frac{\alpha}{2}}},$$

by $Ax^{\alpha-1} - Bx^{\alpha} \leq \frac{A}{\alpha}\left(\frac{\alpha-1}{\alpha}\frac{A}{B}\right)^{\alpha-1}$ for $A, B > 0$ and $\alpha > 1$. Therefore, (61) is bounded by

$$\left(\frac{2\alpha(1+e^2)}{\hat{L}_{t,i}} - \frac{c_{s,1}(\mathcal{F}_\alpha)}{2}\frac{\Delta_i t^{\frac{\alpha}{2}}}{(cK_\alpha \hat{\underline{L}}_{t,i})^{\alpha}}\right) + \frac{1}{2(cK_\alpha)^{\alpha}}\left(\frac{\alpha}{\alpha-1}\frac{1}{t^{1-\frac{\alpha}{2}}(\hat{\underline{L}}_{t,i})^{\alpha-1}} - c_{s,1}(\mathcal{F}_\alpha)\frac{\Delta_i t^{\frac{\alpha}{2}}}{(\hat{\underline{L}}_{t,i})^{\alpha}}\right)$$
$$\leq (4(\alpha-1))\left(\frac{2\alpha(1+e^2)}{\alpha c_{s,1}(\mathcal{F}_\alpha)\Delta_i}\right)^{\frac{1}{\alpha-1}}\frac{(cK_\alpha)^{\frac{\alpha}{\alpha-1}}}{t^{\frac{\alpha}{2(\alpha-1)}}} + \frac{1}{2(cK_\alpha)^{\alpha}}\frac{1}{\alpha-1}\frac{1}{(\Delta_i c_{s,1}(\mathcal{F}_\alpha))^{\alpha-1}}\frac{1}{t^{\frac{\alpha}{2}}}. \tag{62}$$

Since $\frac{\alpha}{2(\alpha-1)} > 1$, the summation over the first term in (62) is constant. Therefore, by following the same steps from (60), we can obtain for $\alpha \in (1, 2)$ that

$$\mathcal{R}(T) \leq \mathcal{O}\left(\sum_{i \neq i^*}\frac{1}{2-\alpha}\frac{1}{c^{\alpha}K^{1-\frac{\alpha}{2}}}\frac{T^{1-\frac{\alpha}{2}}}{\Delta_i^{\alpha-1}}\right) + \mathcal{O}(K) + \frac{\sqrt{K}\Gamma\left(1-\frac{1}{\alpha}\right)}{c} + \frac{\alpha}{2}\log(T+1),$$

which concludes the proof.

E.3.3. FRÉCHET-TYPE DISTRIBUTIONS WITH BOUNDED SLOWLY VARYING FUNCTION

Let us begin by replacing terms in (52) with the corresponding terms for $\mathfrak{D}_\alpha$, which gives

$$\mathcal{R}(T) \leq \sum_{t=1}^{T}\mathbb{E}\left[\mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1}\right\rangle + \frac{r_{t+1,I_{t+1}} - r_{t+1,i^*}}{2cK_\alpha\sqrt{t}}\middle|\hat{L}_t\right]\right] + \frac{MA_u\sqrt{K}}{c} + \frac{\rho_1(e^2+1)}{2}\log(T+1). \tag{63}$$

On D_t, where $\eta_t \underline{\hat{L}}_{t,j} \geq 1$ for $j \neq i^*$, we have for $\alpha \geq 2$

$$\begin{aligned}
&\mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1} \right\rangle + \frac{r_{t+1,I_{t+1}} - r_{t+1,i^*}}{2cK_\alpha\sqrt{t}} \middle| \hat{L}_t \right] \\
&\leq \sum_{i \neq i^*} \frac{\frac{2e\alpha A_u \rho_2}{A_l(\alpha+1)}}{\underline{\hat{L}}_{t,i}} + \frac{1}{2cK_\alpha\sqrt{t}} \frac{e\rho_1 A_u^\alpha}{\alpha - 1} \frac{1}{(\eta_t \underline{\hat{L}}_{t,i})^{\alpha-1}} + \frac{14.4 A_u^\alpha \rho_1 e(1-e^{-1})}{(1-\zeta)^{\alpha+1}} \frac{1}{\underline{\hat{L}}_{t,i}} \\
&\qquad + \sum_{t=1}^T \frac{(1-e^{-1})^{\frac{\zeta}{\eta_t}}}{1-e^{-1}} \left(\frac{\zeta}{\eta_t} + e\right) \qquad \text{(by Lemmas 11, 13 and 25)} \\
&\leq \sum_{i \neq i^*} \frac{\frac{2e\alpha A_u \rho_2}{A_l(\alpha+1)}}{\underline{\hat{L}}_{t,i}} + \frac{1}{2cK_\alpha\sqrt{t}} \frac{e\rho_1 A_u^\alpha}{\alpha - 1} \frac{1}{\eta_t \underline{\hat{L}}_{t,i}} + \frac{14.4 A_u^\alpha \rho_1 e(1-e^{-1})}{(1-\zeta)^{\alpha+1}} \frac{1}{\underline{\hat{L}}_{t,i}} \\
&\qquad + \sum_{t=1}^T \frac{(1-e^{-1})^{\frac{\zeta}{\eta_t}}}{1-e^{-1}} \left(\frac{\zeta}{\eta_t} + e\right) \qquad (64) \\
&\leq \sum_{i \neq i^*} \frac{\frac{2e\alpha A_u \rho_2}{A_l(\alpha+1)}}{\underline{\hat{L}}_{t,i}} + \frac{1}{2(cK_\alpha)^2} \frac{e\rho_1 A_u^\alpha}{\alpha - 1} \frac{1}{\underline{\hat{L}}_{t,i}} + \frac{14.4 A_u^\alpha \rho_1 e(1-e^{-1})}{(1-\zeta)^{\alpha+1}} \frac{1}{\underline{\hat{L}}_{t,i}} + \mathcal{O}(c^2 K_\alpha^2) \\
&\qquad \text{(by Lemma 23)} \\
&= \sum_{i \neq i^*} \frac{\frac{2e\alpha\rho_2 A_u}{A_l(\alpha+1)} S^{\frac{1}{\alpha}} + 14.4 A_u^\alpha \rho_1 e^2 (1-e^{-1}) + \frac{e\rho_1 A_u^\alpha}{2(cK_\alpha)^2(\alpha-1)}}{\hat{L}_{t,i}} + \mathcal{O}(c^2 K_\alpha^2), \qquad (65)
\end{aligned}$$

where (64) follows from $\eta_t \underline{\hat{L}}_{t,i} \geq 1$ for all $i \neq i^*$ on D_t and $\alpha \geq 2$ and we chose $\zeta = 1 - e^{\frac{-1}{\alpha+1}} \in (0,1)$ in (65) for simplicity.

On D_t^c, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\langle \hat{\ell}_t, w_t - w_{t+1} \right\rangle + \frac{r_{t+1,I_{t+1}} - r_{t+1,i^*}}{2cK_\alpha\sqrt{t}} \middle| \hat{L}_t \right] \\
&\qquad \leq \frac{2\alpha\rho_1 m A_u}{A_l(\alpha-1)} K^{1-\frac{1}{\alpha}} \eta_t + \frac{K^{\frac{1}{\alpha}} C_{1,1}(\mathcal{D}_\alpha)}{2cK_\alpha\sqrt{t}} \qquad \text{(by Lemmas 12 and 13)} \\
&\qquad = \left(\frac{2\alpha\rho_1 m A_u c}{A_l(\alpha-1)} + \frac{C_{1,1}(\mathcal{D}_\alpha)}{2c}\right)\sqrt{\frac{K}{t}}. \qquad (66)
\end{aligned}$$

Combining (65) and (66) with (63) provides

$$\begin{aligned}
\mathcal{R}(T) \leq \sum_{t=1}^T \mathbb{E}\Bigg[&\mathbb{1}[F_t] \frac{\frac{2e\alpha\rho_2 A_u}{A_l(\alpha+1)} S^{\frac{1}{\alpha}} + 14.4 A_u^\alpha \rho_1 e^2(1-e^{-1}) + \frac{e\rho_1 A_u^\alpha}{2(cK_\alpha)^2(\alpha-1)}}{\hat{L}_{t,i}} \\
&+ \mathbb{1}[F_t^c]\left(\frac{2\alpha\rho_1 m A_u c}{A_l(\alpha-1)} + \frac{C_{1,1}(\mathcal{D}_\alpha)}{2c}\right)\sqrt{\frac{K}{t}}\Bigg] \\
&+ \frac{M A_u \sqrt{K}}{c} + \frac{\rho_1(e^2+1)}{2}\log(T+1) + \mathcal{O}(c^2 K_\alpha^2). \qquad (67)
\end{aligned}$$

On the other hand, by Lemma 22, we have

$$\mathcal{R}(T) \geq \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{1}[F_t] c_{s,1}(\mathcal{D}_\alpha) \frac{\Delta_i t^{\frac{\alpha}{2}}}{(cK_\alpha \hat{\underline{L}}_{t,i})^\alpha} + \mathbb{1}[F_t^c] c_{s,2}(\mathcal{D}_\alpha)\Delta \right]. \tag{68}$$

By applying self-bounding technique, (67) - (68)/2, we have

$$\begin{aligned}\frac{\mathcal{R}(T)}{2} \leq \sum_{t=1}^{T} \mathbb{E}\Bigg[\mathbb{1}[F_t] \sum_{i \neq i^*} \Bigg(&\frac{\frac{2e\alpha\rho_2 A_u}{A_l(\alpha+1)} S^{\frac{1}{\alpha}} + 14.4 A_u^\alpha \rho_1 e^2 (1-e^{-1}) + \frac{e\rho_1 A_u^\alpha}{2(cK_\alpha)^2(\alpha-1)}}{\hat{L}_{t,i}} \\ &- c_{s,1}(\mathcal{D}_\alpha) \frac{\Delta_i t^{\frac{\alpha}{2}}}{2(cK_\alpha \hat{\underline{L}}_{t,i})^\alpha} \Bigg) \Bigg] \\ + \sum_{t=1}^{T} \mathbb{E}\Bigg[\mathbb{1}[F_t^c] \Bigg(\left(\frac{2\alpha\rho_1 m A_u c}{A_l(\alpha-1)} + \frac{C_{1,1}(\mathcal{D}_\alpha)}{2c} \right) &\sqrt{\frac{K}{t}} - c_{s,2}(\mathcal{D}_\alpha)\Delta \Bigg) \Bigg] \\ &+ \frac{MA_u\sqrt{K}}{c} + \frac{\rho_1(e^2+1)}{2} \log(T+1) + \mathcal{O}\big(c^2 K_\alpha^2\big).\end{aligned}$$

Therefore, following the same steps as the Fréchet distribution from (58) concludes the proof. For $\alpha \in (1,2)$, one can follow the same steps in the Fréchet case.

Appendix F. Numerical validation

This section presents simulation results to verify our theoretical findings. Following Zimmert and Seldin (2021) and Honda et al. (2023), we consider the stochastically constrained adversarial setting. The results in this section are the averages of 100 independent trials. Following Honda et al. (2023), we consider FTPL with a stable variant of geometric resampling (GR 10). In the stable variant, resampling (Lines 7–9 in Algorithm 1) is iterated ten times, and the mean is calculated, leading to a reduction in the variance of $\widehat{w_{t,i}^{-1}}$. We consider this stable variant to examine the effect of perturbations in FTPL more accurately.

Since K perturbations are independently generated from a common distribution, the behavior of FTPL is influenced by the distribution of maximum perturbations. Therefore, in this experiment, we consider perturbations whose limiting distribution converges to the same Fréchet distribution with shape α. Since one can rewrite (2) as

$$\Pr[M_K/a_K \geq x] \overset{K\to\infty}{\to} \mathbb{1}[x \geq 0] \exp\big(-x^{-\alpha}\big),$$

for $a_K = \inf\{x : F(x) \geq 1 - 1/K\}$, we use denormalized perturbations $X = ra_K$ instead of r generated from a common distribution $\mathcal{D}_\alpha$. This ensures that normalized block maxima of different perturbations converge to the same extreme distribution as K increases.

Figures 1, 2, and 3 are the results examining the behavior of FTPL in the adversarial setting using distributions from FMDA with index $\alpha = 2$. In these figures, the legends represent the original perturbations denoted by r, while FTPL employs denormalized perturbations X. Despite the absence of variance in r for $\alpha = 2$, the behavior of FTPL is almost the same as K becomes sufficiently large. This experimental observation supports our theoretical findings, demonstrating that the dominating factor in the behavior of FTPL is determined by the limiting distributions.

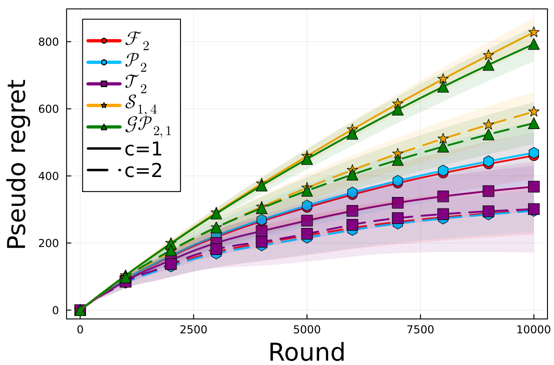

Figure 1: Adversarial setting with $K = 8$.

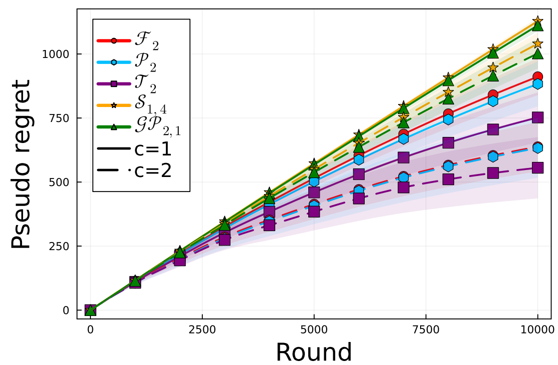

Figure 2: Adversarial setting with $K = 16$.

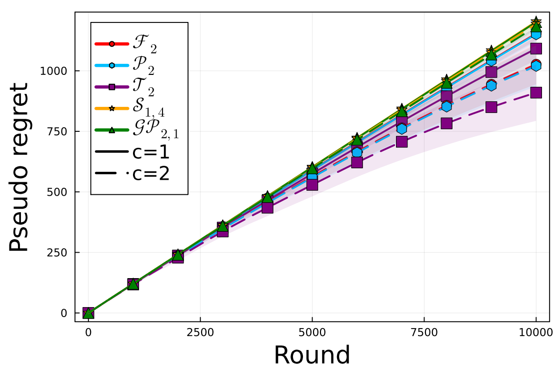

Figure 3: Adversarial setting with $K = 32$.

Appendix G. Technical lemmas

Lemma 26 (Equation 8.10.2 of Olver et al. (2010)) *For $x > 0$ and $a \geq 1$,*

$$\gamma(a, x) \leq \frac{x^{a-1}}{a}(1 - e^{-x}).$$

Lemma 27 (Gautschi's inequality) *For $x > 0$ and $s \in (0, 1)$,*

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}.$$

Lemma 28 *For any $\alpha > 1$, $\frac{B\left(x;1+\frac{1}{\alpha},i\right)}{B(x;1,i)}$ is monotonically increasing with respect to $x \in (0, 1]$.*

Proof From the definition of the incomplete Beta function, $B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t$, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{B\left(x;1+\frac{1}{\alpha},i\right)}{B(x;1,i)} &= \frac{1}{B^2(x;1,i)}\left(x^{\frac{1}{\alpha}}(1-x)^{i-1}\int_0^x(1-t)^{i-1}\mathrm{d}t - \int_0^x t^{\frac{1}{\alpha}}(1-t)^{i-1}\mathrm{d}t(1-x)^{i-1}\right) \\ &= \frac{(1-x)^{i-1}}{B^2(x;1,i)}\left(\int_0^x x^{\frac{1}{\alpha}}(1-t)^{i-1}\mathrm{d}t - \int_0^x t^{\frac{1}{\alpha}}(1-t)^{i-1}\mathrm{d}t\right) \geq 0, \end{aligned}$$

which concludes the proof. ■

Lemma 29 (Potter bounds (Beirlant et al., 2006)) *Let $S(x)$ be a slowly varying function. Given $A > 1$ and $\delta > 0$, there exists a constant x_0 such that*

$$\frac{S(y)}{S(x)} \leq A \max\left\{\left(\frac{y}{x}\right)^{\delta}, \left(\frac{x}{y}\right)^{\delta}\right\}, \qquad \forall x, y \geq x_0.$$