Notre Dame Journal of Formal Logic
Volume 60, Number 4, 2019

Coxeter Groups and Abstract Elementary Classes: The Right-Angled Case

Tapani Hyttinen and Gianluca Paolini

Abstract We study classes of right-angled Coxeter groups with respect to the strong submodel relation of a parabolic subgroup. We show that the class of all right-angled Coxeter groups is not smooth and establish some general combinatorial criteria for such classes to be abstract elementary classes (AECs), for them to be finitary, and for them to be tame. We further prove two combinatorial conditions ensuring the strong rigidity of a right-angled Coxeter group of arbitrary rank. The combination of these results translates into a machinery to build concrete examples of AECs satisfying given model-theoretic properties. We exhibit the power of our method by constructing three concrete examples of finitary classes. We show that the first and third classes are nonhomogeneous and that the last two are tame, uncountably categorical, and axiomatizable by a single $L_{\omega_1,\omega}$-sentence. We also observe that the isomorphism relation of any countable complete first-order theory is κ-Borel reducible (in the sense of generalized descriptive set theory) to the isomorphism relation of the theory of right-angled Coxeter groups whose Coxeter graph is an infinite random graph.

1 Introduction

Abstract elementary classes (AECs; see Shelah [18]) are pairs $(\mathbf{K}, \preccurlyeq)$ such that $\mathbf{K}$ is a class of structures of the same similarity type and $\preccurlyeq$ is a partial order on $\mathbf{K}$, often referred to as a *strong submodel relation*, satisfying a certain set of axioms, which generalize some of the properties of the relation of an elementary submodel of first-order logic. Although AECs generalize the first-order setting, the situation in AECs is very different from the one in elementary model theory. In fact, in the latter setting the strong submodel relation is *always fixed*. The same remark holds for the model theory of infinitary languages, since also in this context one tends to use the

Received February 7, 2017; accepted August 19, 2018
First published online September 14, 2019
2010 Mathematics Subject Classification: Primary 03C48; Secondary 05E15
Keywords: classification theory, abstract elementary classes, Coxeter groups
 10.1215/00294527-2019-0027

canonical strong submodel relations (which in this case depend on what the formula is that defines the class under study). On the other hand, in the theory of AECs we are free to choose *any* strong submodel relation, as long as the AEC axioms are satisfied. This choice determines very strongly the model-theoretic properties of the class under analysis. A classical example is when we consider as $\mathbf{K}$ the class of all abelian groups. In this case, letting $\preccurlyeq_0$ be the subgroup relation and $\preccurlyeq_1$ be the pure subgroup relation, we have that $(\mathbf{K}, \preccurlyeq_0)$ is ω-stable, while $(\mathbf{K}, \preccurlyeq_1)$ is not even superstable.

In the context of AECs, when one tries to find examples of various model-theoretic properties, one tends to start from a class $\mathbf{K}$ of structures and then search for a suitable or natural strong submodel relation $\preccurlyeq$. In this paper we conduct an experiment and reverse this process. That is, we first choose the relation $\preccurlyeq$ and then we try to find $\mathbf{K}$ so that $(\mathbf{K}, \preccurlyeq)$ satisfies certain given model-theoretic properties. We hope that in this way we are able to increase our understanding of the vast number of dividing lines that currently dominate the universe of AECs and to generate new (counter)examples for the theory. A similar approach has been pioneered in Hyttinen and Paolini [12], where several well-behaved classes of geometric lattices have been found in this way, when considering as $\preccurlyeq$ the strong submodel relation of principal extension of a combinatorial geometry, arising from the work of Crapo in [8].

In this case study, we consider the strong submodel relation of a *parabolic subgroup*, from *geometric group theory*. The beginning of our study is the search for groups which together with the parabolic subgroup relation are AECs (i.e., the first property we test is the property of being an AEC). We very quickly restricted our attention to classes consisting of so-called *right-angled Coxeter groups*. These groups are in fact the most well-understood structures in geometric group theory. In particular, they satisfy a crucial requirement known as *rigidity* (see Castella [7]; see also below[1]). However, it turns out that rigidity alone is not enough for our purposes. In fact, we will see that the smoothness axiom fails in general, and without additional assumptions, we do not even know whether $\preccurlyeq$ is transitive or not. We get out of this impasse assuming a stronger property, known as *strong rigidity*.

Whereas in the case of finitely generated right-angled Coxeter groups clear necessary and sufficient conditions are known for strong rigidity, not much is known about infinitely generated ones. What is known is basically just that in this more general setting these conditions are only necessary, but not sufficient. Thus, we start our study by giving two combinatorial conditions ensuring the strong rigidity of an arbitrary right-angled Coxeter group. These results will be used to construct three concrete examples of AECs: $(\mathbf{K}_0, \preccurlyeq)$, $(\mathbf{K}_1, \preccurlyeq)$, and $(\mathbf{K}_2, \preccurlyeq)$.

We continue our study by giving some general criteria for a class of strongly rigid right-angled Coxeter groups to be an AEC and for it to satisfy the usual sufficient conditions for the construction of a monster model, that is, amalgamation, joint embedding, and arbitrarily large models. We then turn to notions that describe the behavior of Galois types, namely, *homogeneity*, *finitarity*, and *tameness*. (We will also point out that, excluding the class of infinite vector spaces over the two-element field, classes of infinite right-angled Coxeter groups are not first-order axiomatizable.) Also in this case we give general criteria for the satisfaction of these properties under the assumption of strong rigidity. The underlying theme of these general results is the reduction of model-theoretic properties of a class of right-angled Coxeter groups to *combinatorial conditions* on the associated graphs, the so-called *Cox-*

eter graphs. These conditions are often easy to realize, and paired with our two general results on the strong rigidity of right-angled Coxeter groups, they translate into a machinery to build concrete examples of AECs. The classes $(\mathbf{K}_0, \preccurlyeq)$, $(\mathbf{K}_1, \preccurlyeq)$, and $(\mathbf{K}_2, \preccurlyeq)$ should be considered under this perspective as explicit examples of this machinery.

We conclude the paper with a close analysis of these classes. First, we show that $(\mathbf{K}_0, \preccurlyeq)$, $(\mathbf{K}_1, \preccurlyeq)$, and $(\mathbf{K}_2, \preccurlyeq)$ are finitary. Then, we show that $(\mathbf{K}_0, \preccurlyeq)$ has the independence property (and thus it is unstable), while $(\mathbf{K}_1, \preccurlyeq)$ and $(\mathbf{K}_2, \preccurlyeq)$ are both tame and uncountably categorical (and thus stable in every infinite cardinality). Finally, we show that $(\mathbf{K}_0, \preccurlyeq)$ and $(\mathbf{K}_2, \preccurlyeq)$ are not homogeneous. We leave the tameness of $(\mathbf{K}_0, \preccurlyeq)$ and the homogeneity of $(\mathbf{K}_1, \preccurlyeq)$ as open questions. John Baldwin pointed out to us that, by combining our results with results from Krueker in [14], various definability results can be obtained. For example, the classes $(\mathbf{K}_1, \preccurlyeq)$ and $(\mathbf{K}_2, \preccurlyeq)$ are axiomatizable by a single $L_{\omega_1,\omega}$-sentence, and over strong submodels Galois types and $L_{\omega_1,\omega}$-types coincide in both $(\mathbf{K}_1, \preccurlyeq)$ and $(\mathbf{K}_2, \preccurlyeq)$.

In the process of writing this paper, we also observed that right-angled Coxeter groups provide a way of finding a group whose first-order theory is maximal in the order of complexity that was introduced in the theory of generalized descriptive set theory (see Friedman, Hyttinen, and Kulikov [9]). We will point out how one can see this.

2 Coxeter Groups

Let S be a set. A matrix $m : S \times S \to \{1, 2, \dots, \infty\}$ is called a *Coxeter matrix* if it satisfies

$$m(s, s') = m(s', s),$$
$$m(s, s') = 1 \Leftrightarrow s = s'.$$

Equivalently, m can be represented by a labeled graph Γ, called a *Coxeter graph*, whose node set is S and whose edges are the unordered pairs $\{s, s'\}$ such that $m(s, s') < \infty$, with label $m(s, s')$. (Notice that some authors refer to the Coxeter graph as the graph Γ such that s and s' are adjacent if and only if $m(s, s) > 2$.) Let $S^2_{\text{fin}} = \{(s, s') \in S^2 : m(s, s') < \infty\}$. A Coxeter matrix m determines a group W with presentation

$$\begin{cases} \text{Generators} : S, \\ \text{Relations} : (ss')^{m(s,s')} = e, \quad \text{for all } (s, s') \in S^2_{\text{fin}}. \end{cases} \tag{2.1}$$

If a group W has a presentation such as (2.1), then the pair (W, S) is called a *Coxeter system* of type $m = m_{(W,S)}$ or of type $\Gamma = \Gamma_{(W,S)}$. The group $W = W_\Gamma$ is called a *Coxeter group*, and the set S is a *Coxeter basis* (or *Coxeter generating set*) for W. The cardinality of S is called the *rank* of (W, S). Notice that in the present paper we *do not* assume that our Coxeter groups are of finite rank, as is done in most of the literature on the subject. As is well known, the isomorphism type of $\Gamma_{(W,S)}$ is not determined by the group W alone (see, e.g., Björner and Brenti [4, Chapter 1, Exercise 2]). This motivates the following definition.

Definition 2.1 Let W be a Coxeter group.

(1) We say that W is *rigid* if for any two Coxeter bases S and S' for W there is an automorphism $\alpha \in \text{Aut}(W)$ such that $\alpha(S) = S'$.

(2) We say that W is *strongly rigid* if for any two Coxeter bases S and S' for W there is an *inner* automorphism $\alpha \in \mathrm{Inn}(W)$ such that $\alpha(S) = S'$.

That is, W is rigid if and only if for any two Coxeter bases S and S' for W there exists an isomorphism of labeled graphs between $\Gamma_{(W,S)}$ and $\Gamma_{(W,S')}$. The problem of deciding whether two nonisomorphic Coxeter graphs determine isomorphic Coxeter groups is known as the *isomorphism problem* for Coxeter groups. This problem is highly nontrivial, and it has been solved only partially (see Bahls [1]). The most well-understood class of Coxeter groups in this respect (and any other respect) is the class of so-called *right-angled* Coxeter groups.

Definition 2.2 We say that a Coxeter system (W, S) is *right-angled* if $m_{(W,S)}$ has coefficients in $\{1, 2, \infty\}$ and that a Coxeter group W is right-angled if there exists a right-angled Coxeter system for W.

Theorem 2.3 (Castella [7, Theorem 2]) *The right-angled Coxeter groups are rigid.*

Thus, in the case of right-angled Coxeter systems (W, S) the group W alone determines the isomorphism type of $\Gamma_{(W,S)}$. Consequently, given a right-angled Coxeter group W we denote by Γ_W (or simply Γ) its associated Coxeter graph (unique modulo graph isomorphisms). Given a Coxeter group W there is a special class of subgroups of W, which are called the *parabolic* subgroups of W. These subgroups (and the subgroup relation which they induce) will be the main ingredient in our model-theoretic analysis of right-angled Coxeter groups.

Definition 2.4 Let W be a Coxeter group.

(1) Given a Coxeter basis S for W, we say that W' is an *S-parabolic* subgroup of W if $W' = \langle S' \rangle_W$ for some $S' \subseteq S$, that is, W' is generated by a subset of S. In this case, we denote the subgroup W' as $W_{S'}$.

(2) We say that W' is a *parabolic* subgroup of W, denoted as $W' \preccurlyeq W$, if W' is an S-parabolic subgroup of W for *some* Coxeter basis S of W.

A parabolic subgroup $W' = \langle S' \rangle_W$ of a Coxeter group $W = (W, S)$ is a Coxeter group in its own right, with a Coxeter generating set the induced subgraph determined by S' (see, e.g., [4, Proposition 2.4.1]). As is evident from the definition, the parabolic subgroup relation depends on the particular choice of Coxeter basis S for W. This generates some difficulties in the analysis of this relation, for example, in the proof of very basic properties such as transitivity. To this end, the notion of strong rigidity (see Definition 2.1) is of great help. (Notice for example that in the presence of strong rigidity the transitivity of the parabolic subgroup relation is essentially trivial (see the proof of Theorem 5.2).) For this reason we are interested in sufficient (and possibly necessary) conditions for strong rigidity. The problem of the (strong) rigidity of a Coxeter group W is of course strictly related to our understanding of the corresponding group of automorphisms $\mathrm{Aut}(W)$. In the case of right-angled Coxeter groups a fundamental result of Tits from [20] gives an explicit description of $\mathrm{Aut}(W)$ as a semidirect product of “tame” subgroups of $\mathrm{Aut}(W)$. We describe these two subgroups. Given a right-angled Coxeter group W with Coxeter graph $\Gamma = (S, E)$, let $F(\Gamma)$ be the collection of the S-spheric subgroups of W, that is, the S-parabolic subgroups $W_{S'}$ of W, with S' a finite clique of $\Gamma_{(W,S)}$ (i.e., $m_{(W,S)}(s, s') \in \{1, 2\}$). Then let $\mathrm{Aut}(W, F(\Gamma))$ be the subgroup of $\mathrm{Aut}(W)$ which stabilizes $F(\Gamma)$, and let $\mathrm{Spe}(W)$ be the subgroup of $\mathrm{Aut}(W)$ which stabilizes the conjugacy class of every $s \in S$.

Theorem 2.5 (Tits [20, p. 1]) *Let W be a right-angled Coxeter group. Then*

$$\mathrm{Aut}(W) = \mathrm{Spe}(W) \rtimes \mathrm{Aut}\big(W, F(\Gamma)\big).$$

Evidently,

$$\mathrm{Inn}(W) \subseteq \mathrm{Spe}(W) \qquad \text{and} \qquad \mathrm{Aut}(\Gamma) \subseteq \mathrm{Aut}\big(W, F(\Gamma)\big),$$

where $\mathrm{Aut}(\Gamma)$ denotes the automorphism group of the graph Γ, which is naturally thought of as a subgroup of $\mathrm{Aut}(W)$, since every automorphism of Γ extends canonically to an automorphism of $\mathrm{Aut}(W)$. The next proposition shows the connection between $\mathrm{Inn}(W)$ and $\mathrm{Aut}(\Gamma)$ and the strong rigidity of W.

Proposition 2.6 *Let W be a right-angled Coxeter group. Then*

$$W \text{ is strongly rigid } \Leftrightarrow \mathrm{Inn}(W) = \mathrm{Spe}(W) \text{ and } \mathrm{Aut}(\Gamma) = \mathrm{Aut}\big(W, F(\Gamma)\big). \quad (2.2)$$

Proof See [7, Remarque 5(b)]. □

We are then interested in criteria which ensure that the two containments in (2.2) are equalities. The next theorem recapitulates what is known on the subject. We first introduce some definitions which will be useful for the statement of the theorem.

Definition 2.7 Let $\Gamma = (V, E)$ be a graph.

(1) For $v \in \Gamma$, we let $N(v) = \{v' \in \Gamma : vEv'\}$ and $st(v) = N(v) \cup \{v\}$.
(2) We say that Γ is *star connected* if for every $v \in \Gamma$ we have that $\Gamma - st(v)$ is connected.
(3) We say that Γ has the *star property* if for every $v \neq v' \in \Gamma$ we have that $st(v) \not\subseteq st(v')$.

Theorem 2.8 *Let W be a right-angled Coxeter group.*

(a) $\mathrm{Aut}(W, F(\Gamma)) = \mathrm{Aut}(\Gamma)$ *if and only if* Γ_W *has the star property (see [7, Proposition 7]).*
(b) *If W is of finite rank, then* $\mathrm{Spe}(W) = \mathrm{Inn}(W)$ *if and only if* Γ_W *is star connected (see Mühlher [16, corollary to the main theorem]).*
(c) *If W is of arbitrary rank, then the star-connectedness of* Γ_W *is a necessary but not sufficient condition for* $\mathrm{Spe}(W) = \mathrm{Inn}(W)$.

Proof (c) For the necessity of the condition, see [20, Proposition 5]. The nonsufficiency of the condition is claimed in [20, final remark of Section 3], but the exhibited map is not surjective. We thus show the nonsufficiency of the condition. Let $\Gamma = \bigcup_{i<\omega} \Gamma_i$ be a countably infinite star-connected graph such that for each $i < \omega$ we have that Γ_i is finite and there exists $a_i \neq b_i \in \Gamma_i - \Gamma_{i-1}$ such that a_i is not adjacent to b_i and a_i is adjacent to every element in Γ_{i-1}. Such a $\Gamma = \bigcup_{i<\omega} \Gamma_i$ can easily be found. For example, take the countably infinite random graph. For every $i < \omega$, let $\alpha_i \in \mathrm{Spe}(W_{\Gamma_i})$ be such that for every $x \in \Gamma_i$ we have $\alpha_i(x) = a_0 \cdots a_i x a_i \cdots a_0$. Then for every $i \leqslant j < \omega$ we have that α_j restricted to W_{Γ_i} equals α_i, and so $\alpha = \bigcup_{i<\omega} \alpha_i \in \mathrm{Spe}(W_\Gamma)$. But obviously $\alpha \notin \mathrm{Inn}(W_\Gamma)$. □

Point (c) above was already observed in [20] and also noticed in [7], where it is also shown that the star property is equivalent to one of the two conditions used in Brady, McCammond, Mühlherr, and Neumann[6] to characterize strong rigidity in the finite-rank case. In the case of right-angled Coxeter groups of arbitrary rank a necessary and sufficient condition on Γ_W ensuring $\mathrm{Spe}(W) = \mathrm{Inn}(W)$ is not known.

In the next two theorems, relying on technology from Servatius in [17] and [20], we establish two sufficient conditions for $\mathrm{Spe}(W) = \mathrm{Inn}(W)$. We first need to develop some combinatorics of right-angled Coxeter groups. Let (W, S) be a Coxeter system. Each element $w \in W$ can be written as a product of generators:

$$w = s_1 s_2 \cdots s_k,$$

with $s_i \in S$. (The identity element e is represented by the empty word.) If k is minimal among all such expressions for w, then k is called the *length of* w (written as $|w| = k$) and the word $s_1 s_2 \cdots s_k$ is called a *normal form* (or *reduced word*) for w. We denote by $sp(w)$ the set of letters appearing in *any* normal form for w, and we call it the *support* of W with respect to the Coxeter basis S. This is well defined, since if $s_1 s_2 \cdots s_k$ and $s'_1 s'_2 \cdots s'_k$ are two normal forms for w, then the set of letters appearing in the word $s_1 s_2 \cdots s_k$ equals the set of letters appearing in $s'_1 s'_2 \cdots s'_k$ (see, e.g., [4, Corollary 1.4.8]). We now describe two "moves" which take a word $s_1 s_2 \cdots s_k$ in (W, S) and change it into another word in (W, S) that represents the same elements of W and which is at most as long:

- (M_1) if $s_i = s_{i+1}$, then cancel the letters s_i and s_{i+1};
- (M_2) if $m(s_i, s_{i+1}) = 2$, then exchange s_i and s_{i+1}.

Theorem 2.9 (Tits [19, Theorem 3]) *Let (W, S) be a right-angled Coxeter system. If $s_1 s_2 \cdots s_n$ and $s'_1 s'_2 \cdots s'_m$ are two words representing the same element $w \in W$, then $s_1 s_2 \cdots s_n$ and $s'_1 s'_2 \cdots s'_m$ can be reduced to identical normal forms using moves (M_1) and (M_2).*

Proposition 2.10 *Let $s_1 \cdots s_n$ be a word in the right-angled Coxeter system (W, S). Then $s_1 \cdots s_n$ is a normal form if and only if for every $1 \leqslant i < j \leqslant k$ with $s_i = s_j$, there exists $i < l < j$ such that $s_l \notin st(s_i)$.*

Proof See, for example, Barkauskas [3, Lemma 21]. □

Proposition 2.11 *Let $s_1 \cdots s_n$ be a word in the right-angled Coxeter system (W, S), and suppose that s_i and s_j can be brought next to each other using (M_2) moves in order to use the move (M_1) to shorten the word $s_1 \cdots s_n$. Then s_i and s_j can be brought together using only moves each of which involves either s_i or s_j.*

Proof See, for example, [3, Lemma 18]. □

We now prove some facts about reflections (see Definition 2.12) in right-angled Coxeter groups. In this section we will only use Corollary 2.15, but the rest will be crucial in what follows. Specifically, Lemma 2.14 will be the main ingredient in the proof of Theorem 5.2.

Definition 2.12 Let (W, S) be a Coxeter system. We define the set of *reflections* of (W, S) to be the set $R(W, S) = \{wsw^{-1} : s \in S, w \in W\}$.

Lemma 2.13 *Let (W, S) be a right-angled Coxeter system, let $wsw^{-1} \in R(W, S)$, and let $a_1 \cdots a_k$ be a normal form for w. If $a_1 \cdots a_k s a_k \cdots a_1$ is not a normal form for wsw^{-1}, then there exists $1 \leqslant i \leqslant k$ such that*

(a) *$wsw = a_1 \cdots a_{i-1} a_{i+1} \cdots a_k s a_k \cdots a_{i+1} a_{i-1} \cdots a_1$;*
(b) *a_i commutes with a_j for every $i < j \leqslant k$;*
(c) *a_i commutes with s;*
(d) *$a_1 \cdots a_{i-1} a_{i+1} \cdots a_k$ is a normal form.*

Proof If $a_1 \cdots a_k s a_k \cdots a_1 = b_1 \cdots b_{2k+1}$ is not a normal form for wsw^{-1}, then because of Theorem 2.9 and the fact that $a_k \cdots a_1$ is normal, it must be the case that in any reduction of $a_1 \cdots a_k s a_k \cdots a_1$ to a normal form at some point we use the move (M_1) for the pair (b_x, b_y), where $x < y$ and either

(i) $b_x = b_i$ for $i \leqslant k$ and $b_y = b_{k+1} = s$, or
(ii) $b_y = b_i$ for $k+2 \leqslant i \leqslant 2k+1$ and $b_x = b_{k+1} = s$, or
(iii) $b_x = b_i$ for $i \leqslant k$ and $b_y = b_j$ for $k+2 \leqslant j \leqslant 2k+1$.

Furthermore, because of Proposition 2.11, we can assume that in this reduction we only use moves that involve either b_x or b_y. Now, if we are in case (iii), then it is clear that i is as desired. In fact, it must be the case that $j = (2k+1) - (i-1)$; otherwise, $a_1 \cdots a_k$ would not be normal, and so we satisfy condition (a) because of our assumption that we use only moves that involve either b_x or b_y. Furthermore, conditions (b) and (c) are satisfied because of Proposition 2.10. Finally, it is easy to see that (d) is also satisfied, because otherwise $a_1 \cdots a_k$ would not be normal. Cases (i) and (ii) are symmetric, and so it suffices to analyze case (i). But this is essentially as in case (iii), since after deleting the pair (b_x, b_y) we can move $b_{(2k+1)-(i-1)} = s$ to where $b_y = b_{k+1} = s$ was, that is, in the middle of the word. □

Lemma 2.14 *Let (W, S) be a right-angled Coxeter system, let $T \subseteq S$, and let $wsw^{-1} \in R(W, S) \cap W_T$. Let $a_1 \cdots a_k$ be a normal form for w, and let $a_{q_1} \cdots a_{q_n}$ be the subword of $a_1 \cdots a_k$ obtained by deleting all the occurrences of letters in $S - T$. Then*

$$wsw^{-1} = a_{q_1} \cdots a_{q_n} s a_{q_n} \cdots a_{q_1}.$$

Proof Iterating Lemma 2.13, we get $l \leqslant k$ and a sequence of words $(w_i)_{i \leqslant l}$ such that

(i) $w_0 = a_1 \cdots a_k$;
(ii) for every $i < l$, the word w_{i+1} is a subword of w_i of length $|w_i| - 1$;
(iii) for every $i \leqslant l$, the word w_i is normal;
(iv) for every $i \leqslant l$, $w_i s w_i^{-1} = wsw$;
(v) $w_l s w_l^{-1}$ is normal (and so $sp(w_l), sp(s) \subseteq T$);
(vi) w_l is a subword of $a_{q_1} \cdots a_{q_n}$.

For $i < l$, let a_i be the letter witnessing that w_{l-i} is a subword of $w_{l-(i+1)}$ of length $|w_{l-(i+1)}| - 1$, and consider the sequence $((a_i, a_i))_{i<l}$. Then, because of conditions (b) and (c) of Lemma 2.13, for every $X = \{i_1 < \cdots < i_m\} \subseteq l$, the pairs $((a_i, a_i))_{i \in X}$ can be put back into the word $w_l s w_l^{-1}$ following the order $(a_{i_1}, a_{i_1}) < \cdots < (a_{i_m}, a_{i_m})$. This suffices, since w_l is a subword of $a_{q_1} \cdots a_{q_n}$. □

The following corollary is immediate from Lemma 2.14. This fact is known for any Coxeter group (see, e.g., Gal [10, Corollary 1.4]).

Corollary 2.15 *Let (W, S) be a Coxeter system, and let $T \subseteq S$. Then*

$$R(W, S) \cap W_T = R(W_T, T).$$

We also need an explicit description of centralizers of Coxeter generators.

Lemma 2.16 (Tits [20, Corollaire 3]) *Let W be a right-angled Coxeter group, and let $v \in \Gamma_W$. Then the centralizer $C_W(v)$ of v in W is the parabolic subgroup $W_{st(v)}$.*

Proof See [20, Corollaire 3]. □

We now go back to the main theme of this section, that is, strong rigidity. To this end, we need two lemmas. These lemmas are essentially Theorem 3 and Lemma 4 of [17] proved in the context of Coxeter groups. (Note that [17] proves this fact for Artin groups (also known as graph groups).)

Lemma 2.17 *Let W be a right-angled Coxeter group, let $\alpha \in \mathrm{Spe}(W)$, let $v \in \Gamma_W$, and let Y be a connected component of $\Gamma_W - st(v)$. Then if $v \in sp(\alpha(y))$ for some $y \in Y$, then $v \in sp(\alpha(x))$ for every $x \in Y$.*

Proof We show that $v \in sp(\alpha(x))$ for any x adjacent to y and not adjacent to v. The result follows by the connectedness of Y. Now, $\alpha(y) = wyw^{-1}$ for some $w \in W$, because $\alpha \in \mathrm{Spe}(W)$, and $sp(wyw^{-1}) \subseteq sp(w) \cup sp(y)$ (see Theorem 2.9). By hypothesis $v \in sp(\alpha(y))$, and evidently $v \notin sp(y) = \{y\}$; thus, $v \in sp(w)$. Consider now $\alpha(x)$. As for $\alpha(y)$, there exists $p \in W$ such that $\alpha(x) = pxp^{-1}$. By the choice of x, the element y commutes with x, and so $\alpha(y)$ commutes with $\alpha(x)$. That is, $\alpha(x) \in C_W(\alpha(y))$. By Lemma 2.16

$$C_W(wyw^{-1}) = wC_W(y)w^{-1} = wW_{st(y)}w^{-1},$$

and so $\alpha(x) \in wW_{st(y)}w^{-1}$, that is, $\alpha(x) = wy'w^{-1}$ for some $y' \in st(y)$. Furthermore, with $\alpha(x)$ being conjugate to x, we have $x \in sp(\alpha(x)) = sp(wy'w^{-1})$. We distinguish two cases.

Case 1: $x \in sp(y')$. If this is the case, then $v \in sp(wy'w^{-1})$, because x is not adjacent to v (see Theorem 2.9).

Case 2: $x \notin sp(y')$. We show that this case is not possible. If $x \notin sp(y')$, then $x \in sp(w) - sp(y')$. Thus, for any normal form $w_1 \cdots w_k$ and $y'_1 \cdots y'_m$ for w and y', respectively, we have that x occurs an even number of times in

$$w_1 \cdots w_k y'_1 \cdots y'_m w_k \cdots w_1.$$

Hence, x occurs an even number of times also in $p_1 \cdots p_l x p_l \cdots p_1$, for $p_1 \cdots p_l$ a normal form for p (see, e.g., Meyers [15, p. 14]), but this is obviously absurd. □

Lemma 2.18 *Let W be a right-angled Coxeter group such that Γ_W satisfies the following conditions:*

(a) *Γ_W is star connected;*
(b) *Γ_W is triangle-free;*
(c) *Γ_W contains a copy of P_4 (the path of length 4) as a subgraph (not necessarily induced).*

Then for every $\alpha \in \mathrm{Spe}(W)$ there exists $w \in W$ such that $w\alpha w^{-1}$ fixes P_4 pointwise.

Proof Let $P_4 = aEbEcEd$, and let $\alpha \in \mathrm{Spe}(W)$. Then $\alpha(a) = pap^{-1}$, and so conjugating α by p^{-1} we get $\alpha_1 \in \mathrm{Spe}(W)$ such that $\alpha_1(a) = a$. Now, a and b commute, and so we have $\alpha_1(b) = qbq^{-1}$ with $sp(q) \subseteq N(a)$ (see Lemma 2.16). Thus, conjugating α_1 by q^{-1} we get $\alpha_2 \in \mathrm{Spe}(W)$ such that $\alpha_2(a) = a$ and $\alpha_2(b) = b$. Similarly, b and c commute, and so we have $\alpha_2(c) = rcr^{-1}$ with $sp(r) \subseteq N(b)$. Let $x \in N(b) - \{a, c\}$. Then by the triangle-freeness of Γ_W, x is adjacent neither to a nor to c, and so $a, c \in \Gamma_W - st(x)$. By the star-connectedness of Γ_W, a and c are connected in $\Gamma_W - st(x)$, and so given that $x \notin sp(\alpha_2(a)) = sp(a) = \{a\}$, by Lemma 2.17 we have $x \notin sp(\alpha_2(c))$. Hence, $sp(\alpha_2(c)) \subseteq \{a, c\}$. Then $\langle \alpha_2(a) = a, \alpha_2(c) \rangle_W \subseteq \langle a, c \rangle_W$. On the other hand, $\alpha_2^{-1} \in \mathrm{Spe}(W)$, $\alpha_2^{-1}(a) = a$, and

$\alpha_2^{-1}(b) = b$, and so the same argument used for α_2 shows that $sp(\alpha_2^{-1}(c)) \subseteq \{a, c\}$. Thus, $\alpha_2^{-1}(c) \in \langle a, c\rangle_W$, from which it follows that

$$c \in \alpha_2\big(\langle a, c\rangle_W\big) = \big\langle \alpha_2(a) = a, \alpha_2(c)\big\rangle_W,$$

that is, $\langle a, c\rangle_W \subseteq \langle a, \alpha_2(c)\rangle_W$. Hence,

$$\langle a, c\rangle_W = \big\langle a, \alpha_2(c)\big\rangle_W,$$

that is, α_2 restricted to $\langle a, c\rangle_W = W_{\{a,c\}} \in \mathrm{Aut}(W_{\{a,c\}})$. Furthermore, because of Corollary 2.15 we see that $\alpha_2 \in \mathrm{Spe}(W_{\{a,c\}})$. Also, $(\{a, c\}, E) = (\{a, c\}, \emptyset)$ is star connected, and so by Theorem 2.8(b) we have $\alpha_2 \in \mathrm{Inn}(W_{\{a,c\}})$. But then obviously it must be the case that $\alpha_2(c)$ is either c or aca, because otherwise $\alpha_2(a) \neq a$. It follows that $sp(r) \subseteq \{a\}$, and so conjugating α_2 by r^{-1} we get $\alpha_3 \in \mathrm{Spe}(W)$ such that $\alpha_3(a) = a$, $\alpha_3(b) = b$, and $\alpha_3(c) = c$. Using the same argument for $\alpha_3(d) = tdt^{-1}$, we see that $sp(t) \subseteq \{b\}$, and so conjugating α_3 by t^{-1} we get $\alpha_4 \in \mathrm{Spe}(W)$ such that $\alpha_4(a) = a$, $\alpha_4(b) = b$, $\alpha_4(c) = c$, and $\alpha_4(d) = d$. □

We now arrive at the first sufficient condition for $\mathrm{Spe}(W) = \mathrm{Inn}(W)$. This theorem takes inspiration from [17, Theorem 6], where Servatius's use of [17, Theorem 3 and Lemma 4] is replaced by our Lemmas 2.17 and 2.18.

Theorem 2.19 *Let W be a right-angled Coxeter group such that Γ_W satisfies the following conditions:*

(a) *Γ_W is star connected;*
(b) *Γ_W is triangle-free;*
(c) *Γ_W contains P_4 as a subgraph.*

Then $\mathrm{Spe}(W) = \mathrm{Inn}(W)$.

Proof Let $\alpha \in \mathrm{Spe}(W)$. Then by Lemma 2.18 there exists $w \in W$ such that $w\alpha w^{-1}$ fixes $P_4 = aEbEcEd$ pointwise. We show that $\alpha_1 = w\alpha w^{-1}$ is the identity id_W on W. This of course suffices, since then

$$\alpha = w^{-1}w\alpha w^{-1}w = w^{-1}\,\mathrm{id}_W w = \mathrm{Inn}(w^{-1}),$$

where, for $x \in W$, $\mathrm{Inn}(x)$ denotes the inner automorphism determined by x. To this end, let $y \notin P_4$, and suppose that $\alpha_1(y) \neq gyg^{-1}$. Then there is $v \in sp(g)$ such that $v \neq y$ and v is not adjacent to y. By the triangle-freeness of Γ_W there exists $e \in \{a, b, c, d\} - \{v\}$ such that e is not adjacent to v. It follows that $\Gamma - st(v)$ contains y and e. Furthermore, $v \in sp(\alpha_1(y))$ and so by Lemma 2.17 we have

$$v \in sp\big(\alpha_1(e)\big) = sp(e) = \{e\},$$

which is a contradiction. Thus, we must have $\alpha_1(y) = y$. It follows that $\alpha_1 = \mathrm{id}_W$. □

Corollary 2.20 *Let W be as in Theorem 2.19, and suppose that in addition Γ_W has the star property. Then W is strongly rigid.*

Proof This is immediate from Proposition 2.6 and Theorems 2.8 and 2.19. □

Finally, we arrive at the second sufficient condition for $\mathrm{Spe}(W) = \mathrm{Inn}(W)$. This theorem takes inspiration from [20, Proposition 6], although the setting of the reference is quite different from the one in the theorem.

Theorem 2.21 *Let W be a right-angled Coxeter group such that Γ_W satisfies the following conditions:*

(a) *Γ_W is star connected;*

and either Γ_W is finite or there exists $s, s' \in \Gamma_W$ such that

(b) *$st(s) \cup st(s')$ is finite and star connected (as an induced subgraph);*

(c) *for every $v \in \Gamma_W$ there exists $a \in st(s) \cup st(s')$ such that $a \neq v$ and a is not adjacent to v.*

Then $\mathrm{Spe}(W) = \mathrm{Inn}(W)$.

Proof If Γ_W is finite, then we know that star-connectedness suffices for $\mathrm{Spe}(W) = \mathrm{Inn}(W)$. Suppose then that Γ_W is infinite (and so conditions (b) and (c) hold). Let s, s' be as in the statement of the theorem, and let $\alpha \in \mathrm{Spe}(W)$. We will show that there exists $w \in W$ such that $w\alpha w^{-1}$ is the identity on W. By assumption $\alpha(s) = psp^{-1}$, and so conjugating α by p^{-1} we get $\alpha_1 \in \mathrm{Spe}(W)$ such that $\alpha_1(s) = s$. Now, s and s' commute, and so we have $\alpha_1(s') = qs'q^{-1}$ with $sp(q) \subseteq N(a)$ (see Lemma 2.16). Thus, conjugating α_1 by q^{-1} we get $\alpha_2 \in \mathrm{Spe}(W)$ such that $\alpha_2(s) = s$ and $\alpha_2(s') = s'$. Given that $C_W(s) = W_{st(s)}$ and $C_W(s') = W_{st(s')}$ (see Lemma 2.16) we must have that α_2 fixes $W_{st(s)\cup st(s')}$ setwise, that is, α_2 restricted to $W_{st(s)\cup st(s')}$ is in $\mathrm{Aut}(W_{st(s)\cup st(s')})$. Furthermore, because of Corollary 2.15 we see that $\alpha_2 \in \mathrm{Spe}(W_{st(s)\cup st(s')})$. Also, by assumption $st(s) \cup st(s')$ is finite and star connected, and so we have $\alpha_2 \in \mathrm{Inn}(W_{st(s)\cup st(s')})$ (see Theorem 2.8(b)). Thus, composing α_2 with an inner automorphism, we get $\alpha_3 \in \mathrm{Spe}(W)$, which fixes $W_{st(s)\cup st(s')}$ pointwise. We show that α_3 fixes every element of Γ_W. To this end, let $y \notin st(s) \cup st(s')$, and suppose that $\alpha_3(y) = gyg^{-1}$ is not fixed. Then there is $v \in sp(g)$ such that $v \neq y$ and v is not adjacent to y. Notice that because of (c) there exists $a \neq v \in st(s) \cup st(s')$ such that v is not adjacent to a. It follows that $\Gamma - st(v)$ contains y and a. Furthermore, $v \in sp(\alpha_3(y))$, and so by Lemma 2.17 we have

$$v \in sp\big(\alpha_3(a)\big) = sp(a) = \{a\},$$

which is a contradiction. Thus, we must have $\alpha_3(y) = y$. It follows that $\alpha_3 = \mathrm{id}_W$. □

Corollary 2.22 *Let W be as in Theorem 2.21, and suppose that in addition Γ_W has the star property. Then W is strongly rigid.*

Proof This is immediate from Proposition 2.6 and Theorems 2.8 and 2.21. □

We will refer to groups satisfying the conditions of Corollary 2.22 as *centered* right-angled Coxeter groups (centered because of the s and s').

3 Random Right-Angled Coxeter Groups

Let T_{rg} be the first-order theory of random graphs, and let T_{racg} be $Th(A)$ for A *any* right-angled Coxeter group such that $\Gamma_A \models T_{\mathrm{rg}}$. This does not depend on A, since for all right-angled Coxeter groups B and C such that $\Gamma_B, \Gamma_C \models T_{\mathrm{rg}}$ the two groups B and C are elementary equivalent. This can be seen using, for example, the Ehrenfeucht–Fraïssé game $\mathrm{EF}_\omega(B, C)$ of length ω. (This definitely suffices, since it shows that B and C are elementary equivalent in the infinitary logic $L_{\infty,\omega}$.) We sketch the idea. If in the game $\mathrm{EF}_\omega(B, C)$ Player I plays an element $b_0 \in B$ with normal form $s^0_1 \cdots s^0_n$, then Player II plays the element $c_0 = t^0_1 \cdots t^0_n$, for $t^0_1 \cdots t^0_n$

the answer of Player II to the move $s_1^0 \cdots s_n^0$ of Player I in the game $\mathrm{EF}_\omega(\Gamma_B, \Gamma_C)$, in which, as is well known, Player II has a winning strategy, since $\Gamma_B, \Gamma_C \models T_{\mathrm{rg}}$. (Notice that, in a game of length ω, playing elements or tuples does not matter.) The other moves are played in the same fashion.

We now fix a cardinal $\kappa > \omega$ such that $\kappa^{<\kappa} = \kappa$ and code models A of cardinality κ in a universal countable language L^* (countably many relation symbols for any arity) as elements $\eta(A)$ of 2^κ in the usual fashion (see, e.g., [9]). Given a complete first-order theory T in the language L^*, we define the isomorphism relation $\cong_T$ on $2^\kappa \times 2^\kappa$ as the relation

$$\big\{\big(\eta(A), \eta(B)\big) \in 2^\kappa \times 2^\kappa : A, B \models T, A \cong B\big\} \cup \big\{\big(\eta(A), \eta(B)\big) \in 2^\kappa \times 2^\kappa : A, B \not\models T\big\}.$$

Finally, given two complete first-order theories T_0 and T_1 in the language L^* we can say that the isomorphism relation of T_0 reduces to the isomorphism relation of T_1, denoted as $\cong_{T_0} \leqslant_B \cong_{T_1}$, if the relation $\cong_{T_0}$ is Borel reducible to $\cong_{T_1}$ in the usual sense of generalized descriptive set theory (see, e.g., [9]). Clearly any (complete) countable first-order theory can be thought of canonically as a (complete) theory in the language L^*. (In particular, T_{rg} and T_{racg} can be thought so.) We denote by $\cong_{\mathrm{RACG}}$ the isomorphism relation $\cong_{T_{\mathrm{racg}}}$. Given a graph $\Gamma = (V, E)$ and $X \subseteq V$ we say that V is a clique (resp., an independent set) if for every $x \neq y \in X$ we have xEy (resp., x is not adjacent to y).

Theorem 3.1 *For any countable complete first-order theory T,*

$$\cong_T \leqslant_B \cong_{T_{\mathrm{rg}}}.$$

Proof This is folklore; we sketch a proof for completeness of exposition. As is well known, it suffices to do the following. For every graph Γ of power κ we define a random graph R_Γ of power κ such that $\Gamma \cong \Gamma'$ if and only if $R_\Gamma \cong R_{\Gamma'}$. We do this. Let $\Gamma = (V, E)$ be a graph of power κ with $V \cap E = \emptyset$ (without loss of generality). Define a graph R_Γ^0 on $V \cup E$ by letting a and b be adjacent to $\{a, b\}$, for every $\{a, b\} \in E$. Now, for every $a \in V$ add a clique K_a of size ω_1 such that a is adjacent to co-countably many $x \in K_a$, that is, $K_a - N(a)$ has size ω. Similarly, for every $\{a, b\} \in E$ add a clique of size ω_1 such that $K_{a,b} \cap N(\{a, b\})$ and $K_{a,b} - N(\{a, b\})$ both have size ω_1. Let $R_\Gamma^1(0)$ be the resulting graph, and define $R_\Gamma^2(i+1)$ by closing $R_\Gamma^2(i)$ under the following condition: for every finite X there exists a_X such that $N(a_X) = X$. Then $\bigcup_{i<\omega} R_\Gamma^2(i) = R_\Gamma \models T_{\mathrm{rg}}$ as desired. □

Theorem 3.2 *For any countable complete first-order theory T,*

$$\cong_T \leqslant_B \cong_{\mathrm{RACG}}.$$

Proof Because of Theorem 3.1, it suffices to show that $\cong_T \leqslant_B \cong_{\mathrm{RACG}}$ for $T = T_{\mathrm{rg}}$ the theory of random graphs. But this is immediate since we can define $F : 2^\kappa \to 2^\kappa$ by setting

$$F\big(\eta(\Gamma)\big) = \begin{cases} \eta(A) & \text{if } \Gamma \not\models T_{\mathrm{rg}}, \\ \eta(A_\Gamma) & \text{if } \Gamma \models T_{\mathrm{rg}}, \end{cases}$$

where in the first clause A denotes any fixed right-angled Coxeter group A such that $A \not\models T_{\mathrm{racg}}$, and in the second clause A_Γ is the right-angled Coxeter group of type Γ. The function F is evidently Borel. □

The following result shows that no nontrivial class of right-angled Coxeter groups can be treated from the perspective of first-order model theory. This motivates our use of AECs.

Theorem 3.3 *Let* $\mathbf{K}$ *be a class of right-angled Coxeter groups such that there exists* $A \in \mathbf{K}$ *with* Γ_A *containing two nonadjacent vertices* a *and* b*. Then* $\mathbf{K}$ *is not first-order axiomatizable.*

Proof Let A, a, and b be as in the statement of the theorem. Then for every positive integer n the element $c_n = (ab)^n \in A$ is divisible by n. It follows that in the ultrapower $\prod_{i<\omega} A_i/U$ (U nonprincipal ultrafilter) there exists a divisible element c (i.e., an element divisible by every positive integer n), but a Coxeter group cannot contain such an element c. Thus, $\prod_{i<\omega} A_i/U \notin \mathbf{K}$ (and so $\mathbf{K}$ is not first-order). □

4 Abstract Elementary Classes

In this section we introduce the basics of AECs (see, e.g., [18] and Jarden and Shelah [13]). This machinery will be used in later sections in order to study various classes of right-angled Coxeter groups. As usual in this context, type means Galois type (see, e.g., Baldwin [2, Definition 8.10]). Given a class $\mathbf{K}$ of structures in the vocabulary L, we denote by $\leqslant$ the L-submodel relation on structures in $\mathbf{K}$.

Definition 4.1 ([18]) Let $\mathbf{K}$ be a class of structures in the vocabulary L. We say that $(\mathbf{K}, \preccurlyeq)$ is an *abstract elementary class* (AEC) if the following conditions are satisfied.

(1) $\mathbf{K}$ and $\preccurlyeq$ are closed under isomorphism.
(2) If $A \preccurlyeq B$, then A is a substructure of B ($A \leqslant B$).
(3) The relation $\preccurlyeq$ is a partial order on $\mathbf{K}$.
(4) If $(A_i)_{i<\delta}$ is an increasing continuous $\preccurlyeq$-chain, then:
  (4.1) $\bigcup_{i<\delta} A_i \in \mathbf{K}$;
  (4.2) for each $j < \delta$, $A_j \preccurlyeq \bigcup_{i<\delta} A_i$;
  (4.3) if each $A_j \preccurlyeq B$, then $\bigcup_{i<\delta} A_i \preccurlyeq B$ (smoothness axiom).
(5) If $A, B, C \in \mathbf{K}$, $A \preccurlyeq C$, $B \preccurlyeq C$, and $A \leqslant B$, then $A \preccurlyeq B$ (coherence axiom).
(6) There is a Löwenheim–Skolem number $\mathrm{LS}(\mathbf{K}, \preccurlyeq)$ such that if $A \in \mathbf{K}$ and $B \subseteq A$, then there is $C \in \mathbf{K}$ such that $B \subseteq C$, $C \preccurlyeq A$, and $|C| \leqslant |B| + |L| + \mathrm{LS}(\mathbf{K}, \preccurlyeq)$ (existence of LS-number).

Definition 4.2 If $A, B \in \mathbf{K}$ and $f : A \to B$ is an embedding such that $f(A) \preccurlyeq B$, then we say that f is a $\preccurlyeq$-embedding.

Let λ be a cardinal. We let $\mathbf{K}_\lambda = \{A \in \mathbf{K} \mid |A| = \lambda\}$.

Definition 4.3 Let $(\mathbf{K}, \preccurlyeq)$ be an AEC.

(1) We say that $(\mathbf{K}, \preccurlyeq)$ has the *amalgamation property* (AP) if for any $A, B_0, B_1 \in \mathbf{K}$ with $A \preccurlyeq B_i$, for $i < 2$, there are $C \in \mathbf{K}$ and $\preccurlyeq$-embeddings $f_i : B_i \to C$, for $i < 2$, such that $f_0 \upharpoonright A = f_1 \upharpoonright A$.
(2) We say that $(\mathbf{K}, \preccurlyeq)$ has the *joint embedding property* (JEP) if for any $B_0, B_1 \in \mathbf{K}$ there are $C \in \mathbf{K}$ and $\preccurlyeq$-embeddings $f_i : B_i \to C$, for $i < 2$.
(3) We say that $(\mathbf{K}, \preccurlyeq)$ has *arbitrarily large models* (ALM) if, for every $\lambda \geqslant \mathrm{LS}(\mathbf{K},\preccurlyeq)$, $\mathbf{K}_\lambda \neq \emptyset$.

As is well known, given an AEC, say, $(\mathbf{K}, \preccurlyeq)$, with AP, JEP, and ALM, we can construct a monster model $\mathfrak{M} = \mathfrak{M}(\mathbf{K}, \preccurlyeq)$ for $(\mathbf{K}, \preccurlyeq)$, that is, a κ-model homogeneous and κ-universal (for κ large enough) structure in $\mathbf{K}$. We say that a subset A of $\mathfrak{M}$ is bounded if its cardinality is smaller than κ. Given bounded $A \subseteq \mathfrak{M}$ and $n < \omega$, we denote by $S_n(A)$ the set of Galois types[2] over A of length n and by $S(A)$ the set $\bigcup_{n<\omega} S_n(A)$.

Definition 4.4 Let $(\mathbf{K}, \preccurlyeq)$ be an AEC with AP, JEP, and ALM. We say that $(\mathbf{K}, \preccurlyeq)$ has the *independence property* if there exist finite $A \subseteq \mathfrak{M}$ and $P \subseteq S(A)$ such that for every ordinal $\alpha < |\mathfrak{M}|$ there exist $(a_i)_{i<\alpha} \in \mathfrak{M}$ such that for every $X \subseteq \alpha$ there exists $b_X \in \mathfrak{M}$ such that $tp(b_X a_i/A) \in P$ if and only if $i \in X$.

Definition 4.5 Let $(\mathbf{K}, \preccurlyeq)$ be an AEC with AP, JEP, and ALM. We say that $(\mathbf{K}, \preccurlyeq)$ is *homogeneous* if, for every ordinal $\alpha < |\mathfrak{M}|$ and $(a_i)_{i<\alpha}, (b_i)_{i<\alpha} \in \mathfrak{M}$, it holds that if $tp(a_X) = tp(b_X)$ for every $X \subseteq_{\text{fin}} \alpha$, then $tp((a_i)_{i<\alpha}) = tp((b_i)_{i<\alpha})$.

Definition 4.6 (Hyttinen and Kesälä [11, Definition 2.5]) Let $(\mathbf{K}, \preccurlyeq)$ be an AEC. We say that $(\mathbf{K}, \preccurlyeq)$ has *finite character* if whenever $A \leqslant B$ and for every $X \subseteq_{\text{fin}} A$ there exists a $\preccurlyeq$-embedding $f_X : A \to B$ such that $f \restriction X = \mathrm{id}_X$, then $A \preccurlyeq B$.

Definition 4.7 ([11]) Let $(\mathbf{K}, \preccurlyeq)$ be an AEC. We say that $(\mathbf{K}, \preccurlyeq)$ is *finitary* if the following are satisfied:

(1) $\mathrm{LS}(\mathbf{K}, \preccurlyeq) = \omega$;
(2) $(\mathbf{K}, \preccurlyeq)$ has arbitrarily large models;
(3) $(\mathbf{K}, \preccurlyeq)$ has the amalgamation property;
(4) $(\mathbf{K}, \preccurlyeq)$ has the joint embedding property;
(5) $(\mathbf{K}, \preccurlyeq)$ has finite character.

Definition 4.8 Let $(\mathbf{K}, \preccurlyeq)$ be an AEC with AP, JEP, and ALM. For $\mathrm{LS}(\mathbf{K}, \preccurlyeq) \leqslant \kappa \leqslant \lambda$, we say that $(\mathbf{K}, \preccurlyeq)$ is (κ, λ)-*tame* if, for every $B \in \mathbf{K}$ of power λ and $a, b \in \mathfrak{M}^{<\omega}$, it holds that if $tp(a/B) \neq tp(b/B)$, then there is $A \preccurlyeq B$ of power κ such that $tp(a/A) \neq tp(b/A)$. We say that $(\mathbf{K}, \preccurlyeq)$ is *tame* if it is $(\mathrm{LS}(\mathbf{K}, \preccurlyeq), \lambda)$-tame for every $\lambda \geqslant \mathrm{LS}(\mathbf{K}, \preccurlyeq)$.

As usual, we say that $(\mathbf{K}, \preccurlyeq)$ is *uncountably categorical* if for every uncountable cardinal κ there exists only one model of power κ, up to isomorphism. In later sections we will use the following classical result on AECs.

Theorem 4.9 (see, e.g., [2, Theorem 8.21]) *Let* $(\mathbf{K}, \preccurlyeq)$ *be an* AEC *with* AP, JEP, *and* ALM. *If* $(\mathbf{K}, \preccurlyeq)$ *is uncountably categorical, then* $(\mathbf{K}, \preccurlyeq)$ *is stable*[3] *in every infinite cardinality* $\lambda \geqslant \mathrm{LS}(\mathbf{K}, \preccurlyeq)$.

We will also use the following results connecting finitary AECs with infinitary logic. Given $\theta \in L_{\infty,\omega}$, we let $\mathrm{Mod}(\theta) = \{A : A \models \theta\}$.

Theorem 4.10 ([14, Theorem 3.10]) *Let* $(\mathbf{K}, \preccurlyeq)$ *be a finitary* AEC *with countable vocabulary. If* $\mathbf{K}$ *contains at most* λ *models of cardinality* λ *for some infinite* λ*, then* $\mathbf{K} = \mathrm{Mod}(\theta)$ *for some* $\theta \in L_{\infty,\omega}$. *If in addition* $\mathbf{K}$ *contains at most* λ *models of cardinality less than* λ*, then we can find* $\theta \in L_{\lambda^+,\omega}$.

Definition 4.11 Let $(\mathbf{K}, \preccurlyeq)$ be a finitary AEC with monster model $\mathfrak{M}$. Let also $a \in \mathfrak{M}^{<\omega}$ and $A \preccurlyeq \mathfrak{M}$. Then

$$tp_{\omega_1,\omega}(a/A) = \{\varphi(x, b) : \varphi(x, y) \in L_{\omega_1,\omega},\ b \in A^{<\omega} \text{ and } \mathfrak{M} \models \varphi(a, b)\}.$$

Theorem 4.12 ([14, Remark after Corollary 4.9]) *Let* $(\mathbf{K}, \preccurlyeq)$ *be a finitary and tame* AEC *with countable vocabulary. Assume also that* $(\mathbf{K}, \preccurlyeq)$ *is* ω*-stable. Then for every* $A \preccurlyeq \mathfrak{M}$ *we have* $tp_{\omega_1,\omega}(a/A) = tp_{\omega_1,\omega}(b/A)$ *if and only if* $tp(a/A) = tp(b/A)$.

5 Triangle-Free Right-Angled Coxeter Groups

From now until the end of the paper we denote by $\mathbf{K}$ the class of right-angled Coxeter groups and by $\preccurlyeq$ the parabolic subgroup relation on $\mathbf{K}$ (see Definition 2.4), that is, $A \preccurlyeq B$ if and only if there exists a Coxeter basis S for B such that $A \cap S$ is a Coxeter basis for A. Also, we denote by $\leqslant$ both the subgroup and the induced subgraph relation. Finally, we simply talk of bases instead of Coxeter bases. The next theorem shows that $(\mathbf{K}, \preccurlyeq)$ does not give rise to an AEC. In the rest of the paper we will see that restricting to particular classes of *strongly rigid* right-angled Coxeter groups we *do get* AECs and actually finitary ones (and in some cases also tame).

Theorem 5.1 *The smoothness axiom fails for* $(\mathbf{K}, \preccurlyeq)$.

Proof Let (B, S) be the Coxeter system with $S = \{a_i : i < \omega\} \cup \{b_i : i < \omega_1\}$ such that $\{a_i : i < \omega\}$ is an independent set, $\{b_i : i < \omega_1\}$ is a clique, and a_i commutes with b_j if and only if $j < i$, for every $i < \omega$. For $n < \omega$, let $c_n = a_0 \cdots a_n$, let $e_n = c_n b_n c_n^{-1}$, and let $A_n = \langle e_i : i < n\rangle_B$. Notice that for every $i \leqslant j < \omega$ we have $c_j b_i c_j^{-1} = c_i b_i c_i^{-1}$. It follows that, for every $m < n < \omega$, we have $A_m \preccurlyeq A_n \preccurlyeq B$, as witnessed by the bases $\{e_i : i < m\} \subseteq \{e_i : i < n\} \subseteq c_n S c_n^{-1}$. We claim that $\bigcup_{n<\omega} A_n = A \not\preccurlyeq B$. Suppose not, and let S^* be a basis of A that extends to a basis S' of B. Let $\alpha \in \mathrm{Aut}(B)$ be such that $\alpha(S') = S$. Then $\alpha(S^*) \subseteq \{b_i : i < \omega_1\}$, and so there exists $x \in S - \alpha(S^*)$ such that x commutes with every element of $\alpha(S^*)$. Let $y = \alpha^{-1}(x)$. Then y commutes with every element of A. Let $n < \omega$ be such that if b_i or a_i is in the S-support of y, then either $i \geqslant \omega$ or $i < n$. Also, let $z = c_n^{-1} y c_n$. Now, y commutes with every element of A, and so in particular it commutes with e_n. Thus, $z = c_n^{-1} y c_n$ commutes with $c_n^{-1} e_n c_n = b_n$. Now, if, for some $i \geqslant n$, b_i is in the S-support of z, then also a_n is there and so z does not commute with b_n (see Lemma 2.16). Similarly, for every $i < \omega$, a_i is not in the S-support of z. Thus, $z \in \langle b_i : i < n\rangle_B$ and so $c_n z c_n^{-1} = y \in \langle c_n b_i c_n^{-1} : i < n\rangle_B = A_n$, which is a contradiction, since $y = \alpha^{-1}(x)$, for $x \in S - \alpha(S^*)$. □

Theorem 5.2 *Let* $\mathbf{K}'_*$ *be a class of graphs such that* $(\mathbf{K}'_*, \leqslant)$ *is closed under limits and every* $B \in \mathbf{K}_* = \{A \in \mathbf{K} : \Gamma_A \in \mathbf{K}'_*\}$ *is strongly rigid. Then* $(\mathbf{K}_*, \preccurlyeq)$ *satisfies conditions (1), (2), (3), (4.1), (4.2), and (5) of Definition 4.1. Furthermore,* $\mathrm{LS}(\mathbf{K}_*, \preccurlyeq) = \mathrm{LS}(\mathbf{K}'_*, \leqslant)$*, and if* $(\mathbf{K}'_*, \leqslant)$ *has* AP, JEP*, and* ALM*, then* $(\mathbf{K}_*, \preccurlyeq)$ *does.*

Proof The furthermore part is immediate. For amalgamation, let $A, B, C \in \mathbf{K}_*$ be such that $C \preccurlyeq A, B$ and $A \cap B = C$ (without loss of generality). Then there exist bases S' for A and T' for B such that $S = S' \cap A$ and $T = T' \cap B$ are bases for C. Thus, there exists $g \in C$ such that $gTs^{-1} = S$, and so $gT's^{-1} = S''$ is a basis for B such that $S' \cap S'' = S$. Hence, any amalgam for $(S, E) \leqslant (S', E), (S'', E)$ is an amalgam for $C \preccurlyeq A, B$. Conditions (1) and (2) of Definition 4.1 are clear. We prove (3). Let $A \preccurlyeq B \preccurlyeq C$. Then there exists a basis S' for B such that $S = S' \cap A$ is a basis for A, and there exists a basis T'' for C such that $T' = T'' \cap B$ is a basis for B.

Thus, because of strong rigidity, there exists $g \in B$ such that $S' = gT'g^{-1}$, and so $S'' = gT''g^{-1}$ is a basis for C containing S, that is, $A \preccurlyeq C$.

We prove (4.1) and (4.2). Let $(A_i)_{i<\delta}$ be an increasing continuous $\preccurlyeq$-chain. Using strong rigidity, without loss of generality we can assume that $(\Gamma_{A_i} = (S_i, E))_{i<\alpha}$ is an increasing continuous chain of graphs under the induced subgraph relation. Using the universality property for Coxeter groups (see, e.g., [4, p. 3]) it is immediate to see that $\bigcup_{i<\delta} A_i = A$ is the Coxeter group of type $\bigcup_{i<\alpha} \Gamma_{A_i}$, and so $A \in \mathbf{K}$. This establishes (4.1) and (4.2) at once.

We prove (5). Let $A \preccurlyeq C$, $B \preccurlyeq C$, and $A \leqslant B$. Let S'' be a witness for $A \preccurlyeq C$, and let $S = S'' \cap A$. Let also T'' be a witness for $B \preccurlyeq C$, and let $T' = T'' \cap B$. Now, S'' and T'' are two bases for C and so we can find $g \in C$ such that $S'' = gT''g^{-1}$, that is, for every $s \in S''$ there exists $t_s \in T''$ such that $s = gt_sg^{-1}$. Let $a_1 \cdots a_k$ be a T''-normal form for g. Notice that $S \subseteq A \subseteq B$ and $S \subseteq S''$, and so for every $s \in S$ we have $s = gt_sg^{-1} \in B$. Thus,

$$sp(gt_sg^{-1}) \subseteq T', \tag{5.1}$$

where the support is taken in the basis T''. Let $a_{q_1} \cdots a_{q_n}$ be the subword of $a_1 \cdots a_k$ obtained by deleting all the occurrences of letters in $T'' - T'$. Then because of (5.1) and Lemma 2.14 we have that $a_{q_1} \cdots a_{q_n} = h \in B$ is such that

$$s = gt_sg^{-1} = ht_sh^{-1},$$

for every $s \in S$. Thus, $hT'h^{-1} = S'$ is a basis for B such that $S \subseteq S'$, and so $A \preccurlyeq B$. □

Lemma 5.3 *Let B be a strongly rigid right-angled Coxeter group, and let T_0 and T_1 be bases for B. If $T_0 \cap T_1$ contains $P_4 = s_0Es_1Es_2Es_3$, s_0 is not adjacent to s_2, s_1 is not adjacent to s_3, and there is no $t \in T_1$ such that s_0EtEs_1, then $T_0 = T_1$.*

Proof Let T_0, T_1, and $P_4 = s_0Es_1Es_2Es_3$ be as in the statement of the theorem. Then there exists $g \in B$ such that $T_1 = gT_0g^{-1}$. Let $s \in P_4$. Then $gsg^{-1} = s$, because otherwise we would have $s \neq gsg^{-1}$ both in T_1, contradicting the fact that T_1 is a basis for B (see Bourbaki [5, p. 5]). Suppose now that there exists $t \in sp(g) - \{s_0, s_1\}$, where the support is taken in the basis T_1. Then t commutes with s_0 because otherwise by Theorem 2.9 we would have $s_0 \neq gs_0g^{-1}$. Similarly, t commutes with s_1 because otherwise $s_1 \neq gs_1g^{-1}$. Thus, s_0EtEs_1, which is a contradiction. Hence, $sp(g) \subseteq \{s_0, s_1\}$. On the other hand, $s_0 \notin sp(g)$ and $s_1 \notin sp(g)$, because otherwise $s_2 \neq gs_2g^{-1}$ or $s_3 \neq gs_3g^{-1}$. It follows that $g = 1$, that is, $T_1 = T_0$. □

Theorem 5.4 *Let $\mathbf{K}_*$ be a class of strongly rigid right-angled Coxeter groups such that for every $A \in \mathbf{K}_*$ we have that Γ_A is triangle-free. Suppose further that, whenever $A \preccurlyeq B \in \mathbf{K}_*$ and T is a basis for B such that $S = T \cap A$ is a basis for A, then the basis S contains a copy of $P_4 = s_0Es_1Es_2Es_3$ such that s_0 is not adjacent to s_2 and s_1 is not adjacent to s_3. Then $(\mathbf{K}_*, \preccurlyeq)$ satisfies the smoothness axiom, and it has finite character.*

Proof We show that $(\mathbf{K}_*, \preccurlyeq)$ is smooth. Let $(A_i)_{i<\alpha}$ be an increasing continuous $\preccurlyeq$-chain such that each $A_i \preccurlyeq B$. Using strong rigidity, without loss of generality we can assume that $(\Gamma_{A_i} = (S_i, E))_{i<\alpha}$ is an increasing continuous chain of graphs under the induced subgraph relation and that there are $(T_i)_{i<\alpha}$ bases for B such that

$T_i \cap A_i = S_i$, for every $i < \alpha$. Let $i < \alpha$. Then using the assumption of the theorem for T_i and S_0 we have that $T_0 \cap T_i$ contains $P_4 = s_0 E s_1 E s_2 E s_3$, s_0 is not adjacent to s_1, s_1 is not adjacent to s_3, and there is no $t \in T_i$ such that $s_0 E t E s_1$. Thus, by Lemma 5.3, we have that $T_i = T_0$. Hence, $\bigcup_{i<\alpha} S_i \subseteq T_0$, witnessing that $\bigcup_{i<\alpha} A_i \preccurlyeq B$.

We show that $(\mathbf{K}_*, \preccurlyeq)$ has finite character. Suppose that $A \leqslant B$ and for every $X \subseteq_{\text{fin}} A$ there exists a $\preccurlyeq$-embedding $f_X : A \to B$ such that $f \restriction X = \text{id}_X$. Let S be a basis for A. For every $X \subseteq A$ we have $A \cong f_X(A)$, and so $f_X(S)$ is a basis for $f_X(A)$. It follows that

$$\forall X \subseteq_{\text{fin}} S, \exists T_X \text{ basis of } B \text{ such that } T_X \text{ extends } f_X(S) \text{ and } X \subseteq T_X; \qquad (\star)$$

this is because $f_X(A) \preccurlyeq B$, of course. Fix $Y \subseteq_{\text{fin}} S$. Then $f_Y(A) \preccurlyeq B$, and so using the assumption of the theorem for T_Y and $f_Y(S)$ we get $P'_4 = s'_0 E s'_1 E s'_2 E s'_3$ in $f_Y(S)$ such that s'_0 is not adjacent to s'_2 and s'_1 is not adjacent to s'_3. Now let $f_Y^{-1}(P'_4) = P_4 = s_0 E s_1 E s_2 E s_3$. Then, noticing that $P_4 \subseteq S$ and recalling $(\star)$ and that Γ_B is triangle-free, we have that T_{P_4} is a basis of B such that s_0 is not adjacent to s_1, s_1 is not adjacent to s_3, and there is no $t \in T_{P_4}$ such that $s_0 E t E s_1$. Thus, by Lemma 5.3, for every $P_4 \subseteq X \subseteq_{\text{fin}} S$ we have that $T_X = T_{P_4}$. Hence, for every $X \subseteq_{\text{fin}} S$ we have $X \subseteq T_{P_4}$, and so $S \subseteq T_{P_4}$, that is, $A \preccurlyeq B$. □

Let $\mathbf{K}'_*$ be a class of graphs such that $(\mathbf{K}'_*, \leqslant)$ is an AEC with AP, JEP, and ALM. Suppose that $\mathbf{K}_* = \{A \in \mathbf{K} : \Gamma_A \in \mathbf{K}'_*\}$ is a class of strongly rigid right-angled Coxeter groups, and suppose that $(\mathbf{K}_*, \preccurlyeq)$ is also an AEC (and thus, by Theorem 5.2, it has AP, JEP, and ALM). Notice that, under these conditions, by slightly modifying the construction of $\mathfrak{M}(\mathbf{K}_*, \preccurlyeq)$ we can assume that $\Gamma_{\mathfrak{M}(\mathbf{K}_*, \preccurlyeq)} = \mathfrak{M}(\mathbf{K}'_*, \leqslant)$. In the following theorem we will use this assumption crucially.

Theorem 5.5 *Let $\mathbf{K}'_*$ be a class of graphs such that $(\mathbf{K}'_*, \leqslant)$ is an* AEC *with* AP, JEP, *and* ALM. *Suppose that $\mathbf{K}_* = \{A \in \mathbf{K} : \Gamma_A \in \mathbf{K}'_*\}$ is a class of strongly rigid right-angled Coxeter groups, and suppose that $(\mathbf{K}_*, \preccurlyeq)$ is also an* AEC *(and thus, by Theorem 5.2, it has* AP, JEP, *and* ALM*) with $\mathrm{LS}(\mathbf{K}_*, \preccurlyeq) = \omega$. Suppose further that for every $A \in \mathbf{K}$, $\mathrm{Aut}(\mathfrak{M}/A) \leqslant \mathrm{Aut}(\Gamma_{\mathfrak{M}})$. Then if $(\mathbf{K}'_*, \leqslant)$ is tame, so is $(\mathbf{K}_*, \preccurlyeq)$.*

Proof We show the tameness of $(\mathbf{K}_*, \preccurlyeq)$ for elements; the argument generalizes to tuples. Let $B \in \mathbf{K}_*$, and let a, b be elements in $\mathfrak{M}(\mathbf{K}_*, \preccurlyeq)$. Suppose that $tp(a/B) \neq tp(b/B)$. Notice that for every $\alpha \in \mathrm{Aut}(\Gamma_{\mathfrak{M}})$ the following are equivalent:

(i) $\alpha(a) = b$;

(ii) α restricted to $sp(a)$ is a bijection from $sp(a)$ into $sp(b)$ such that if $a_1 \cdots a_k$ is a normal form for a, then $\alpha(a_1) \cdots \alpha(a_k)$ is a normal form b;

(iii) α restricted to $sp(a)$ is a bijection from $sp(a)$ into $sp(b)$, and there exists a normal form $a_1 \cdots a_k$ for a such that $\alpha(a_1) \cdots \alpha(a_k)$ is a normal form b.

Now, if $|sp(a)| \neq |sp(b)|$, then for any countable $A \preccurlyeq B$ we have that $tp(a/A) \neq tp(b/A)$, since by assumption $\mathrm{Aut}(\mathfrak{M}/A) \leqslant \mathrm{Aut}(\Gamma_{\mathfrak{M}})$. Suppose then that $|sp(a)| = |sp(b)|$, fix a normal form $a_1 \cdots a_k$ for a, and let $\{b^j_1 \cdots b^j_k : j < n\}$ be the set of normal forms for b. For every $j < n$ we must have that

$$tp\big((a_i)_{0<i\leqslant k}/\Gamma_B\big) \neq tp\big((b^j_i)\big)_{0<i\leqslant k}/\Gamma_B),$$

where types are in the sense of $(\mathbf{K}'_*, \leqslant)$. In fact, otherwise there is

$$\alpha \in \mathrm{Aut}\big(\mathfrak{M}(\mathbf{K}'_*, \leqslant)/\Gamma_B\big) = \mathrm{Aut}(\Gamma_{\mathfrak{M}(\mathbf{K}_*, \preccurlyeq)}/B)$$

such that $\alpha(sp(a)) = sp(b)$ and $\alpha(a_1)\cdots\alpha(a_k)$ is a normal form b, and so $tp(a/B) = tp(b/B)$. Thus, by the tameness of $(\mathbf{K}'_*, \leqslant)$, for every $j < n$ there is countable $\Gamma_{A_j} \leqslant \Gamma_B$ such that

$$tp\big((a_i)_{0<i\leqslant k}/\Gamma_{A_j}\big) \neq tp\big((b_i^j)_{0<i\leqslant k}/\Gamma_{A_j}\big).$$

Let $A \preccurlyeq B$ be such that $\bigcup_{j<n} A_j \subseteq A$. Then $tp(a/A) \neq tp(b/A)$. In fact, otherwise there exists $\alpha \in \mathrm{Aut}(\Gamma_{\mathfrak{M}}/A)$ such that $\alpha(sp(a)) = sp(b)$ and $\alpha(a_1)\cdots\alpha(a_k)$ is a normal form b, and so there exists $j < n$ and $\alpha \in \mathrm{Aut}(\mathfrak{M}(\mathbf{K}'_*, \leqslant)/\Gamma_{A_j})$ mapping $(a_i)_{0<i\leqslant k}$ to $(b_i^j)_{0<i\leqslant k}$, which is a contradiction. □

Let $\mathbf{K}'_0$ be the class of graphs satisfying the following requirements:

(1) Γ has the star property;
(2) Γ is star connected;
(3) Γ is triangle-free;
(4) Γ contains C_4 (the cycle of length 4) as an (induced) subgraph.

Then let $\mathbf{K}_0 = \{A \in \mathbf{K} : \Gamma_A \in \mathbf{K}'_0\}$. Notice that, because of Corollary 2.20, every $A \in \mathbf{K}_0$ is strongly rigid. We ask that Γ contains C_4 instead of simply P_4 because C_4 has the star property, while P_4 does not. The fact that C_4 embeds as an induced subgraph in every structure in $\mathbf{K}'_0$ will be useful in proving joint embedding from amalgamation. We need a lemma before proving the main theorem of this section.

Lemma 5.6 *Let Γ be triangle-free and such that it contains C_4 as an induced subgraph. By induction on $i < \omega$, define Γ_i such that:*

(i) $\Gamma_0 = \Gamma$;
(ii) *Γ_{i+1} is the extension of Γ_i following the condition: for every $a \neq b \in \Gamma_i$ if a is not adjacent to b, then add c such that $N(c) = \{a, b\}$.*

Then $\Gamma \leqslant \bigcup_{i<\omega} \Gamma_i = \Gamma^* \in \mathbf{K}'_0$.

Proof Obviously, $C_4 \leqslant \Gamma \leqslant \Gamma^*$ and Γ^* is triangle-free. Regarding the star property, let $a \neq b \in \Gamma^*$. We show that $st(a) \not\subseteq st(b)$. Assume that $a, b \in \Gamma_i$. Then $\Gamma_{i+1} - \Gamma_i$ contains an element x which is not adjacent to a (since C_4 contains two adjacent vertices different from a). Now, $\Gamma_{i+2} - \Gamma_{i+1}$ contains an element c which is adjacent to a and x, but not to b. Hence, $c \in st(a) - st(b)$, as desired. Regarding star-connectedness, let $v \in \Gamma^*$, and let $a \neq b \in \Gamma^* - st(v)$. Assume that $v, a, b \in \Gamma_i$. If a and b are adjacent in Γ^*, then they are connected in $\Gamma^* - st(v)$ (since $a \neq b \in \Gamma^* - st(v)$). If a and b are not adjacent in Γ^*, then they are not adjacent in Γ_i either, and so at stage Γ_{i+1} we have added c such that $N(c) = \{a, b\}$, witnessing the connectedness of a and b in $\Gamma^* - st(v)$. □

Theorem 5.7 *We have that $(\mathbf{K}_0, \preccurlyeq)$ is a finitary* AEC.

Proof As already noticed, because of Corollary 2.20, every $A \in \mathbf{K}_0$ is strongly rigid. Furthermore, obviously $(\mathbf{K}'_0, \leqslant)$ is closed under limits and $\mathrm{LS}(\mathbf{K}'_0, \leqslant) = \omega$. Also, every $A \in \mathbf{K}_0$ is such that Γ_A is triangle-free and contains C_4 as an induced subgraph, and so we can always find a P_4 as in Theorem 5.4. Thus, by Theorems 5.2 and 5.4, in order to conclude it suffices to show that $(\mathbf{K}'_0, \leqslant)$ has joint embedding and amalgamation. Now, $C_4 \in \mathbf{K}'_0$ and C_4 embeds as an induced subgraph in every $A \in \mathbf{K}'_0$; thus, it suffices to prove amalgamation. Then let $A, B, C \in \mathbf{K}'_0$ be such that $C \leqslant A, B$ and $A \cap B = C$ (without loss of generality), and consider $D = (A \cup B)^*$. Then is it easy to see that D is an amalgam of A and B over C. □

Theorem 5.8 *We have the following:*

(a) $(\mathbf{K}_0, \preccurlyeq)$ *is not homogeneous.*
(b) $(\mathbf{K}_0, \preccurlyeq)$ *has the independence property, and thus it is unstable.*

Proof We prove (a). Let $(t_i)_{i<\omega}$ and $(a_i)_{i<\omega}$ in $\Gamma_{\mathfrak{M}}$, for $\mathfrak{M}$ the monster model of $(\mathbf{K}_0, \preccurlyeq)$, be such that the following conditions are met:

(i) $(t_i)_{i<\omega}$ is an independent set;
(ii) $(a_i)_{i<\omega}$ is an independent set;
(iii) for every $i < \omega$, a_i is adjacent to t_j if and only if $j \leqslant i$.

Such sequences $(t_i)_{i<\omega}$ and $(a_i)_{i<\omega}$ can be found in $\Gamma_{\mathfrak{M}}$, for example, using Lemma 5.6. For $i < \omega$, let

$$c_i = a_0 \cdots a_{i-1} t_i a_{i-1} \cdots a_0.$$

Then for every $X \subseteq_{\text{fin}} \omega$ we have $tp(t_X/\emptyset) = tp(c_X/\emptyset)$, as witnessed by the inner automorphism determined by $a_0 \cdots a_{k-1}$, for $k = \max\{i < \omega : i \in X\}$. On the other hand, $tp((t_i)_{i<\omega}/\emptyset) \neq tp((c_i)_{i<\omega}/\emptyset)$ because there is no automorphism of $\mathfrak{M}$ such that $t_i \mapsto c_i$ for every $i < \omega$, as this would contradict the strong rigidity of $\mathfrak{M}$; in fact, no inner automorphism gxg^{-1} (for $g \in \mathfrak{M}$) could serve as witness for this candidate automorphism, since $sp(g)$ is finite. We prove (b). Let

$$P = \left\{p \in S_2(\emptyset) : \forall a, b \in \mathfrak{M},\ \text{if } (a, b) \models p, \text{then } ab = ba\right\},$$

$\alpha < |\mathfrak{M}|$, and $(t_i)_{i<\alpha}$ and $(a_X)_{X\subseteq\alpha}$ in $\Gamma_{\mathfrak{M}}$ be such that the following conditions are met:

(i) $(t_i)_{i<\alpha}$ is an independent set;
(ii) $(a_X)_{X\subseteq\alpha}$ is an independent set;
(iii) for every $X \subseteq \alpha$, a_X is adjacent to t_i if and only if $i \in X$.

Such sequences $(t_i)_{i<\alpha}$ and $(a_X)_{X\subseteq\alpha}$ can be found in $\Gamma_{\mathfrak{M}}$, for example, using Lemma 5.6. Evidently, $tp(a_X t_i/\emptyset) \in P$ if and only if $i \in X$. □

Remark 5.9 The first configuration used in the proof of Theorem 5.8 will play a crucial role also in the proof of Theorem 6.4 (where a similar nonhomogeneity result is proved). It is interesting to notice that the existence of this configuration (on tuples of elements), also known as the *half-graph*, can always be found in a definable way in the monster model of an unstable theory. Thus, we here have an analogy between nonhomogeneity in AECs and instability in first-order theories.

Given a graph $\Gamma = (V, E)$ we define the *barycentric subdivision* of Γ, denoted $\hat{\Gamma}$, to be the graph whose node set is the disjoint union of V and $\{c_{a,b} : a, b \in \Gamma, aEb\}$, and so that $N(c_{a,b}) = \{a, b\}$ and, for $a \in V$, $N(a) = \{c_{a,b} : b \in \Gamma, aEb\}$. Let $\mathbf{K}'_1$ be the class of barycentric subdivisions of clique with at least four elements, and let $\mathbf{K}_1 = \{A \in \mathbf{K} : \Gamma(A) \in \mathbf{K}'_1\}$.

Theorem 5.10 *We have that* $(\mathbf{K}_1, \preccurlyeq)$ *is a finitary* AEC.

Proof Obviously, $(\mathbf{K}'_1, \leqslant)$ is closed under limits, it has AP, JEP, and ALM, and $\mathrm{LS}(\mathbf{K}'_1, \preccurlyeq) = \omega$. Also, it is immediate to see that every $\Gamma \in \mathbf{K}'_1$ is star connected, it has the star property, and it contains P_4, and so, by Corollary 2.20, every $A \in \mathbf{K}_1$ is strongly rigid. Finally, it is obvious from the definition that for any graph Γ the graph $\hat{\Gamma}$ is bipartite (and thus triangle-free). Hence, by Theorems 5.2 and 5.4 we are done. □

Let $\mathbf{K}_1''$ be the class of infinite structures in $\mathbf{K}_1'$. It is immediate to see that the class $\mathbf{K}_1''$ is axiomatizable by the following first-order theory T:

(A) there are infinitely many elements;
(B) every x has either exactly two neighbors or at least three neighbors;
(C) if x has exactly two neighbors y and z, then y and z have at least three neighbors;
(D) if x has at least three neighbors, then each neighbor of x has exactly two neighbors;
(E) if $x \neq y$ have at least three neighbors, then there exists a unique z such that $xEzEy$.

Proposition 5.11 *We have that T is complete and it is model complete.*

Proof This is standard. □

Theorem 5.12 *We have that $(\mathbf{K}_1, \preccurlyeq)$ is tame.*

Proof Obviously, $(\mathbf{K}_1', \leqslant)$ is an AEC with AP, JEP, and ALM. Furthermore, by Lemma 5.3, for every $A \in \mathbf{K}_1$, $\mathrm{Aut}(\mathfrak{M}/A) \leqslant \mathrm{Aut}(\Gamma_{\mathfrak{M}})$. Thus, by Theorem 5.5, it suffices to show that $(\mathbf{K}_1', \leqslant)$ is tame. Clearly, it suffices to prove tameness for the class $\mathbf{K}_1''$ of infinite structures in $\mathbf{K}_1'$. By Proposition 5.11, the class $\mathbf{K}_1''$ is axiomatizable by a complete first-order theory which is model complete. Thus, $(\mathbf{K}_1'', \leqslant) = (\mathbf{K}_1'', \preccurlyeq^*)$, where $\preccurlyeq^*$ denotes the elementary submodel relation of first-order logic, and clearly $(\mathbf{K}_1'', \preccurlyeq^*)$ is tame. □

Theorem 5.13 *We have that $(\mathbf{K}_1, \preccurlyeq)$ is uncountably categorical.*

Proof For uncountable $A, B \in \mathbf{K}_1$, letting $\Gamma_A = \hat{\Gamma}_0$ and $\Gamma_B = \hat{\Gamma}_1$ (for Γ_0 and Γ_1 cliques), we have $|A| = |B|$ if and only if $|\Gamma_A| = |\Gamma_B|$ if and only if $|\Gamma_0| = |\Gamma_1|$ if and only if $\Gamma_0 \cong \Gamma_1$ if and only if $\Gamma_A \cong \Gamma_B$ if and only if $A \cong B$. □

Corollary 5.14 *We have that $(\mathbf{K}_1, \preccurlyeq)$ is stable in every infinite cardinality.*

Proof This is a consequence of Theorems 5.7, 5.13, and 4.9. □

Corollary 5.15 *We have that $\mathbf{K}_1 = \mathrm{Mod}(\theta)$ for some $\theta \in L_{\omega_1,\omega}$. Furthermore, for every $A \preccurlyeq \mathfrak{M}$ we have $tp_{\omega_1,\omega}(a/A) = tp_{\omega_1,\omega}(b/A)$ if and only if $tp(a/A) = tp(b/A)$.*

Proof This is an immediate consequence of Theorems 4.10, 4.12, 5.10, 5.12, and 5.13 together with the easy observation that $\mathbf{K}_1$ has at most countably many countable models. □

6 Centered Right-Angled Coxeter Groups

Theorems 5.8 and 5.13 leave open the question of finding classes of right-angled Coxeter groups which are stable and nonhomogeneous. In this section, we use Corollary 2.22 to achieve this. Let C^* be the graph on vertex set $\{s, s'\} \cup \{t_i : i < 4\}$, with the following edge relation: $t_0 E t_2 E t_3 E t_1 E t_0$, $t_0 E s E t_2$, $t_1 E s' E t_3$, and $s E s'$ (see Figure 1). For every $B_T \models T = Th(\mathbb{N}, s, 0)$ (where s denotes the successor function) we define a graph $\Gamma_{B_T} = (C^* \cup B_T, E)$ in the following way (without loss of generality, we assume that $s^n(0) = n$ in B_T):

(1) C^* is an induced subgraph of Γ_{B_T};

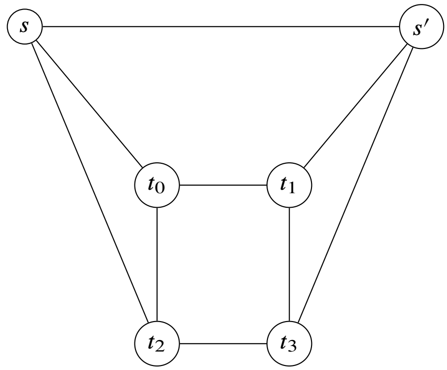

Figure 1 The graph C^*.

(2) t_0 and t_2 are adjacent to all the even numbers in $\mathbb{N}$;
(3) t_1 is adjacent to 2 and to all the odd numbers in $\mathbb{N}$;
(4) t_3 is adjacent to all the odd numbers in $\mathbb{N}$;
(5) $N(0) - \mathbb{N} = \{1\}$ and, for every $0 < n \in \mathbb{N}$, $N(n) \cap \mathbb{N} = \{n-1, n+1\}$;
(6) for every copy Z of $\mathbb{Z}$ in B_T and $b \in Z$, $N(b) \cap Z = \{b-1, b+1\}$;
(7) for every copy Z of $\mathbb{Z}$ in B_T, there exists $0_Z \in Z$ such that for every $\pm n_Z = 0_Z \pm n$ we have $N(\pm n_Z) \cap \mathbb{N} = \{0, \ldots, n\}$;
(8) for every copy Z of $\mathbb{Z}$ in B_T, 0_Z, -1_Z, and 1_Z are adjacent to t_3.

Let $\mathbf{K}_2'$ be the class of graphs Γ' isomorphic to one of the graphs $\Gamma = (C^* \cup B_T, E)$ described above, and let $\mathbf{K}_2 = \{A \in \mathbf{K} : \Gamma_A \in \mathbf{K}_2'\}$.

Remark 6.1 The proof of the theorem below is straightforward, but the details are tiresome. We include them for completeness of exposition.

Theorem 6.2 *We have that* $(\mathbf{K}_2, \preccurlyeq)$ *is a finitary* AEC.

Proof Notice that, for every $\Gamma = (C^* \cup B_T, E) \in \mathbf{K}_2'$, the structure B_T can be recovered from Γ, and so $(\mathbf{K}_2', \leqslant)$ is closed under limits, it has AP, JEP, and ALM, and $\mathrm{LS}(\mathbf{K}_2', \preccurlyeq) = \omega$. Thus, by Theorems 5.2 and 5.4 we are left to show that every $A \in \mathbf{K}_2$ is strongly rigid and that the assumptions of Theorem 5.4 are met. The latter is immediate, since, for every $A \in \mathbf{K}_2$ and basis T of A, the elements $sEs'Et_1Et_0 \in C^*$ (without loss of generality C^* is in T) are such that s is not adjacent to t_1, s' is not adjacent to t_0, and there is no $t \in T$ such that $sEtEs'$. To see strong rigidity we use Corollary 2.22. Let $A \in \mathbf{K}_2$. Then the elements $s, s' \in C^*$ are such that s is adjacent to s' in Γ_A and $st(s) \cup st(s') = C^*$ is finite and star connected, since for every $x \in C^*$ we have $C^* - st(x) = \{y, z\}$, for some $y, z \in C^*$ such that y is adjacent to z. Furthermore, clearly for every $v \in \Gamma_A$ there exists $v \neq a \in st(s) \cup st(s')$ such that v is not adjacent to a. Thus, we are left to show that Γ_A is star connected and it has the star property. For ease of notation, we assume that in Γ_A the copies of C^* and $\mathbb{N}$ are actually C^* and $\mathbb{N}$. (We already did this for C^* above.) Also, we denote by Z, Z', and so on the copies of $\mathbb{Z}$ possibly present in Γ_A. We first show that Γ_A has the star property. Let $a \neq b \in \Gamma_A$.

Case 1: $a, b \in C^*$. This is clear.

Case 2: $a, b \in \mathbb{N}$. Without loss of generality $a < b$. If $a = 0$ and $b = 1$, then $t_0 \in st(a) - st(b)$, and $t_1 \in st(b) - st(a)$. If $a = 0$ and $b = 2$, then $3 \in st(2) - st(0)$ and $t_1 \in st(2) - st(0)$. If $a = 0$ and $b > 2$, then $b + 1 \in st(b) - st(0)$ and $1 \in st(0) - st(b)$. If $a > 0$, then $b + 1 \in st(b) - st(a)$ and $a - 1 \in st(a) - st(b)$.

Case 3: $a, b \in Z$. Without loss of generality $a < b$. We have $b+1 \in st(b) - st(a)$ and $a - 1 \in st(a) - st(b)$.

Case 4: $a \in C^*$ and $b \in \mathbb{N}$. If $a = s$ or $a = s'$, then a is not adjacent to b. Let $a = t_i$, for $i < 4$. If i is even and b is odd, then a is not adjacent to b. If i is odd and b is even, then a is not adjacent to b. If i is even and b is even, then $s \in st(a) - st(b)$ and $b + 1 \in st(b) - st(a)$. If i is odd and b is odd, then $s' \in st(a) - st(b)$ and $b + 3 \in st(b) - st(a)$. (In the case $b = 1$ we have $1 + 1 = 2Et_1$.)

Case 5: $a \in C^*$ and $b \in Z$. In this case a is not adjacent to b, unless $a = t_3$ and $b \in \{0_Z, -1_Z, 1_Z\}$. In this case we have $t_2 \in st(a) - st(b)$ and $3_Z \in st(b) - st(a)$.

Case 6: $a \in \mathbb{N}$ and $b \in Z$. Let $b = \pm n_Z$. If $a > n$, then a is not adjacent to b. If $0 < a \leqslant n$, then $n + 2 \in st(a) - st(b)$ and $a - 1 \in st(b) - st(a)$. If $a = 0 = n$, then $n + 2 \in st(a) - st(b)$ and $t_3 \in st(b) - st(a)$. If $a = 0 < n$, then $n + 2 \in st(a) - st(b)$ and $1 \in st(b) - st(a)$.

Case 7: $a \in Z$ and $b \in Z'$. In this case a is not adjacent to b.

We now show that Γ_A is star connected. Let $v \in \Gamma_A$, and let $a \neq b \in \Gamma_A - st(v)$.

Case A: $v \in C^*$. If $v = s$ or $v = s'$, then it is clear that a is connected to b. Suppose then that $v = t_i$, for $i < 4$.

Case A.1: $a, b \in C^*$. This is clear.

Case A.2: $a, b \in \mathbb{N}$. Then either both a and b are even, or both a and b are odd. In either case we are fine.

Case A.3: $a, b \in Z$. If $i < 3$, then we have $(\Gamma_A - st(v)) \cap Z = Z$. If $i = 3$, then we have $(\Gamma_A - st(v)) \cap Z = Z - \{0_Z, -1_Z, 1_Z\}$. In either case we are fine.

Case A.4: $a \in C^*$ and $b \in \mathbb{N}$. If $a = s$ or $a = s'$, then a is not adjacent to b. Suppose then that $a \notin \{s, s'\}$. If i is even (for $v = t_i$, remember), then $a = t_j$ is such that j is odd and b is odd, and so we are fine. If i is odd, then $a = t_j$ is such that j is even and b is even, and so we are fine.

Case A.5: $a \in C^*$ and $b \in Z$. If i is even, then we can find an odd number $n \in \Gamma_A - st(v)$ that connects what is left of C^* to nEn_ZEb. If i is odd, then we can find an even number that does the same.

Case A.6: $a \in \mathbb{N}$ and $b \in Z$. If i is even, then we can find an odd number $n \in \Gamma_A - st(v)$ such that $aEnEn_ZEb$. If i is odd, then we can find an even number that does the same.

Case A.7: $a \in Z$ and $b \in Z'$. If i is even, then we can find an odd number $n \in \Gamma_A - st(v)$ such that $aEn_ZEnEn_{Z'}Eb$. If i is odd, then we can find an even number that does the same.

Case B: $v \in \mathbb{N}$.

Case B.1: $a, b \in C^*$. If $v = 2$, then $(\Gamma_A - st(v)) \cap C^* = sEs'Et_3$. If $v \neq 2$, then $(\Gamma_A - st(v)) \cap C^*$ is either $sEs'Et_1Et_3Es'$ or $s'EsEt_0Et_2Es$. In all of these cases we are fine.

Case B.2: $a, b \in \mathbb{N}$. If $v = 0$, then $(\Gamma_A - st(v)) \cap \mathbb{N} = \{1\}$, and so this case is not possible, since we are assuming that $a \neq b$. Thus, we must have that $v = n \neq 0$, and so $n - 1 = aEb = n + 1$.

Case B.3: $a, b \in Z$. If $v = 0$ or $v = 1$, then $(\Gamma_A - st(v)) \cap Z$ is either $\emptyset$ or $\{0_Z\}$, and so this case is not possible, since we are assuming that $a \neq b$.

If $v = 2$, then $(\Gamma_A - st(v)) \cap Z = \{0_Z, -1_Z, 1_Z\}$, but $-1_Z E t_3 E 0_Z E t_3 E 1_Z$ and $t_3 \in \Gamma_A - st(2)$, and so we are fine. If $v > 2$, then $(\Gamma_A - st(v)) \cap Z = \{-(v_Z - 1), \ldots, 0_Z, \ldots, (v_Z - 1)\}$ is "long enough," and so it is connected.

Case B.4: $a \in C^*$ and $b \in \mathbb{N}$. If $a = s$ or $a = s'$, then a is not adjacent to b. Suppose then that $a \notin \{s, s'\}$. If v is odd, then b is even and $a \in \{t_0, t_2\}$, and so we are fine. If v is even, then b is odd and $a \in \{t_1, t_3\}$, and so we are fine.

Case B.5: $a \in C^*$ and $b \in Z$. If $v = 0$, then $(\Gamma_A - st(v)) \cap Z = \emptyset$, and so this case is not possible. If $v = 1$, then $(\Gamma_A - st(v)) \cap Z = \{0_Z\}$ and $(\Gamma_A - st(v)) \cap C^* = \{t_0, s, s', t_2\}$, but then we are fine because $0_Z E 0 E t_0 E t_2 E s E s'$ and $0 \in \Gamma_A - st(1)$. If $v = 2$, then $(\Gamma_A - st(v)) \cap Z = \{0_Z, -1_Z, 1_Z\}$ and $(\Gamma_A - st(v)) \cap C^* = \{t_3, s, s'\}$, but $-1_Z E t_3 E 0_Z E t_3 E 1_Z$, and so we are fine. If $v > 2$, then $(\Gamma_A - st(v)) \cap Z$ is connected, and so we can connect it to what is left of C^* via $v - 1 \notin (\Gamma_A - st(v))$.

Case B.6: $a \in \mathbb{N}$ and $b \in Z$. If $v = 0$, then $(\Gamma_A - st(v)) \cap Z = \emptyset$, and so this case is not possible. If $v = 1$, then $(\Gamma_A - st(v)) \cap Z = \{0_Z\}$, $(\Gamma_A - st(v)) \cap \mathbb{N} = \{0, 2\}$, and $0_Z E 0 E 2$. If $v = 2$, then $(\Gamma_A - st(v)) \cap Z = \{0_Z, -1_Z, 1_Z\}$, $(\Gamma_A - st(v)) \cap \mathbb{N} = \{1, 3\}$, and for $x \in \{0_Z, -1_Z, 1_Z\}$ and $y \in \{1, 3\}$ we have $x E t_3 E y$, and so we are fine because $t_3 \in \Gamma_A - st(2)$. If $v > 2$, then $(\Gamma_A - st(v)) \cap Z$ is connected, and $v_Z - 1 E v - i E v + 1$.

Case B.7: $a \in Z$ and $b \in Z'$. If $v = 0$, then $(\Gamma_A - st(v)) \cap Z = \emptyset = (\Gamma_A - st(v)) \cap Z'$, and so this case is not possible. If $v = 1$, then $(\Gamma_A - st(v)) \cap Z = \{0_Z\}$, $(\Gamma_A - st(v)) \cap Z = \{0_{Z'}\}$, and $0_Z E 0 E 0_{Z'}$, and so we are fine because $0 \notin \Gamma_A - st(1)$. If $v = 2$, then $(\Gamma_A - st(v)) \cap Z = \{0_Z, -1_Z, 1_Z\}$, $(\Gamma_A - st(v)) \cap Z' = \{0_{Z'}, -1_{Z'}, 1_{Z'}\}$, and for $x \in \{0_Z, -1_Z, 1_Z\}$ and $y \in \{0_{Z'}, -1_{Z'}, 1_{Z'}\}$ we have $x E t_3 E y$, and so we are fine because $t_3 \in \Gamma_A - st(2)$. If $v > 2$, then $(\Gamma_A - st(v)) \cap Z$ and $(\Gamma_A - st(v)) \cap Z'$ are connected, and so we can connect them via $v - 1 \notin \Gamma_A - st(v)$.

Case C: $v \in Z$. Let $v = \pm n_Z$.

Case C.1: $a, b \in C^*$. We have $(\Gamma_A - st(v)) \cap C^* \subseteq C^* - \{t_3\}$, and so we are fine.

Case C.2: $a, b \in \mathbb{N}$. We have $(\Gamma_A - st(v)) \cap \mathbb{N} = \{m \in \mathbb{N} : n < m\}$, and so we are fine.

Case C.3: $a, b \in Z$. In this case $v - 1 = a E b = v + 1$.

Case C.4: $a \in C^*$ and $b \in \mathbb{N}$. We have $(\Gamma_A - st(v)) \cap C^* \subseteq C^* - \{t_3\}$ and $(\Gamma_A - st(v)) \cap \mathbb{N} = \{m \in \mathbb{N} : n < m\}$, and so we are fine.

Case C.5: $a \in C^*$ and $b \in Z$. We have $(\Gamma_A - st(v)) \cap C^* \subseteq C^* - \{t_3\}$ and $(\Gamma_A - st(v)) \cap Z = \{\pm n_Z - 1, \pm n_Z + 1\}$. Now, $\pm n_Z - 1 E \pm n_Z + 1 E n + 1 E t_j$, for some $j < 3$, and so we are fine because $n + 1 \in \Gamma_A - st(v)$ and $(\Gamma_A - st(v)) \cap C^*$ is connected.

Case C.6: $a \in \mathbb{N}$ and $b \in Z$. We have $(\Gamma_A - st(v)) \cap \mathbb{N} = \{m \in \mathbb{N} : n < m\}$ and $(\Gamma_A - st(v)) \cap Z = \{\pm n_Z - 1, \pm n_Z + 1\}$. Now, $\pm n_Z - 1 E \pm n_Z + 1 E n + 1$, and $n + 1$ is connected in $\Gamma_A - st(v)$ to every $x \in \{m \in \mathbb{N} : n < m\}$.

Case C.7: $a \in Z$ and $b \in Z'$. We have $(\Gamma_A - st(v)) \cap Z = \{\pm n_Z - 1, \pm n_Z + 1\}$ and $(\Gamma_A - st(v)) \cap Z' = Z'$, and so $\pm n_Z - 1 E \pm n_Z + 1 E n + 1 E n_{Z'} + 1$, and $n_{Z'} + 1$ is connected in $\Gamma_A - st(v)$ to every $x \in Z'$. □

Theorem 6.3 *We have that $(\mathbf{K}_2, \preccurlyeq)$ is tame.*

Proof Obviously, $(\mathbf{K}_2', \leqslant)$ is an AEC with AP, JEP, and ALM. Furthermore, by Lemma 5.3, for every $A \in \mathbf{K}_2$, $\mathrm{Aut}(\mathfrak{M}/A) \leqslant \mathrm{Aut}(\Gamma_{\mathfrak{M}})$. Thus, by Theorem 5.5, it suffices to show that $(\mathbf{K}'_2, \leqslant)$ is tame. We show the tameness of $(\mathbf{K}'_2, \leqslant)$ for elements; the argument generalizes to tuples. Let $B \in \mathbf{K}'_2$, and assume that in B the copies of C^* and $\mathbb{N}$ are actually C^* and $\mathbb{N}$. Let $a, b \in \mathfrak{M}(\mathbf{K}'_2, \leqslant) - B$. Then a and b lie in some of the copies of $\mathbb{Z}$ not in B, say, a is in Z and b is in Z'. Let $a = 0_Z \pm n$, and let $b = 0_{Z'} \pm m$. Notice that there is $\alpha \in \mathrm{Aut}(\mathfrak{M}/B)$ mapping a to b if and only if $n = m$ if and only if there is $\alpha \in \mathrm{Aut}(\mathfrak{M}/C^* \cup \mathbb{N})$ mapping a to b. In fact, for every copy Z'' of $\mathbb{Z}$ we have that $0_{Z''} \pm n$ is adjacent to exactly $n + 1$ elements from $\mathbb{N}$. It follows that $tp(a/B) \neq tp(b/B)$ if and only if $tp(a/C^* \cup \mathbb{N}) \neq tp(b/C^* \cup \mathbb{N})$, and so $(\mathbf{K}'_2, \leqslant)$ is tame, because $C^* \cup \mathbb{N} \leqslant B$. □

Theorem 6.4 *We have the following:*

(a) $(\mathbf{K}_2, \preccurlyeq)$ *is not homogeneous.*
(b) $(\mathbf{K}_2, \preccurlyeq)$ *is uncountably categorical.*

Proof The proof of (a) is as in the proof of Theorem 5.8(a). In fact letting $t'_i = i$ and $a_i = i_Z$, for Z a copy of $\mathbb{Z}$, we have that the argument used in the proof of Theorem 5.8(a) works also in this case (where the role of the t_i's there is played by the t'_i's here). Uncountable categoricity is also immediate, since for $C, D \in \mathbf{K}_2$ we have $(A \cup B_T, E) = \Gamma_C \cong \Gamma_D = (A' \cup B'_T, E')$ if and only if $B_T \cong B'_T$ (in the language $\{0, s\}$), and $T = Th(\mathbb{N}, s, 0)$ is well known to be uncountably categorical. □

Corollary 6.5 *We have that* $(\mathbf{K}_2, \preccurlyeq)$ *is stable in every infinite cardinality.*

Proof This is a consequence of Theorems 6.2, 6.4, and 4.9. □

Corollary 6.6 *We have that* $\mathbf{K}_2 = \mathrm{Mod}(\theta)$ *for some* $\theta \in L_{\omega_1,\omega}$. *Furthermore, for every* $A \preccurlyeq \mathfrak{M}$ *we have* $tp_{\omega_1,\omega}(a/A) = tp_{\omega_1,\omega}(b/A)$ *if and only if* $tp(a/A) = tp(b/A)$.

Proof This is an immediate consequence of Theorems 4.10, 4.12, 6.2, 6.3, and 6.4, together with the easy observation that $\mathbf{K}_2$ has at most countably many countable models. □

We conclude the paper with the following open problem.

Open Problem 6.7 Find combinatorial conditions on Γ_A which are necessary and sufficient for the strong rigidity of an arbitrary right-angled Coxeter group A, and use them to develop the model theory of strongly rigid right-angled Coxeter groups, in the style of the present paper.

Notes

1. Notice that here rigidity does not mean what it usually means in model theory.

2. For a definition of Galois type see, for example, [14, beginning of Section 4].

3. The notion of stability in this context is the exact analogue of the notion of stability in the classical context of first-order logic, where we replace the notion of type with the notion of Galois type. For an explicit definition see, for example, [2, Definition 8.20].

References

[1] Bahls, P., *The Isomorphism Problem in Coxeter Groups*, Imperial College Press, London, 2005. Zbl 1113.20034. MR 2162146. DOI 10.1142/9781860947193. 710

[2] Baldwin, J. T., *Categoricity*, vol. 50 of *University Lecture Series*, American Mathematical Society, Providence, 2009. Zbl 1183.03002. MR 2532039. DOI 10.1090/ulect/050. 718, 719, 729

[3] Barkauskas, D. A., "Centralizers in graph products of groups," *Journal of Algebra*, vol. 312 (2007), pp. 9–32. Zbl 1130.20023. MR 2320444. DOI 10.1016/j.jalgebra.2007.01.047. 712

[4] Björner, A., and F. Brenti, *Combinatorics of Coxeter Groups*, vol. 231 of *Graduate Texts in Mathematics*, Springer, New York, 2005. Zbl 1110.05001. MR 2133266. 709, 710, 712, 721

[5] Bourbaki, N., *Lie Groups and Lie Algebras, Chapters 4–6*, Elements of Mathematics (Berlin), Springer, Berlin, 2002. Zbl 0983.17001. MR 1890629. DOI 10.1007/978-3-540-89394-3. 721

[6] Brady, N., J. P. McCammond, B. Mühlherr, and W. D. Neumann, "Rigidity of Coxeter groups and Artin groups," *Geometriae Dedicata*, vol. 94 (2002), pp. 91–109. Zbl 1031.20035. MR 1950875. DOI 10.1023/A:1020948811381. 711

[7] Castella, A., "Sur les automorphismes et la rigidité des groupes de Coxeter à angles droits," *Journal of Algebra*, vol. 301 (2006), pp. 642–69. Zbl 1104.20037. MR 2236762. DOI 10.1016/j.jalgebra.2006.01.007. 708, 710, 711

[8] Crapo, H. H., "Single-element extensions of matroids," *Journal of Research of the National Bureau of Standards, Section B*, vol. 69B (1965), pp. 55–65. Zbl 0141.21701. MR 0190045. 708

[9] Friedman, S.-D., T. Hyttinen, and V. Kulikov, "Generalized descriptive set theory and classification theory," *Memoirs of the American Mathematical Society*, vol. 230 (2014), no. 1081. Zbl 1402.03047. MR 3235820. 709, 717

[10] Gal, Ś. R., "On normal subgroups of Coxeter groups generated by standard parabolic subgroups," *Geometriae Dedicata*, vol. 115 (2005), pp. 65–78. Zbl 1121.20030. MR 2180042. DOI 10.1007/s10711-005-7890-1. 713

[11] Hyttinen, T., and M. Kesälä, "Categoricity transfer in simple finitary abstract elementary classes," *Journal of Symbolic Logic*, vol. 76 (2011), pp. 759–806. Zbl 1250.03055. MR 2849245. DOI 10.2178/jsl/1309952520. 719

[12] Hyttinen, T., and G. Paolini, "Beyond abstract elementary classes: On the model theory of geometric lattices," *Annals of Pure and Applied Logic*, vol. 169 (2018), pp. 117–45. Zbl 06815860. MR 3725203. DOI 10.1016/j.apal.2017.10.003. 708

[13] Jarden, A., and S. Shelah, "Non-forking frames in abstract elementary classes," *Annals of Pure and Applied Logic*, vol. 164 (2013), pp. 135–91. Zbl 1315.03050. MR 3001542. DOI 10.1016/j.apal.2012.09.007. 718

[14] Kueker, D. W., "Abstract elementary classes and infinitary logics," *Annals of Pure and Applied Logic*, vol. 156 (2008), pp. 274–86. Zbl 1155.03016. MR 2484485. DOI 10.1016/j.apal.2008.07.002. 709, 719, 720, 729

[15] Meyers, A., "Rigidity of right-angled Coxeter groups," Ph.D. dissertation, Bucknell University, Lewisburg, Penn., 2009. 714

[16] Mühlherr, B., "Automorphisms of graph-universal Coxeter groups," *Journal of Algebra*, vol. 200 (1998), pp. 629–49. Zbl 0897.20033. MR 1610676. DOI 10.1006/jabr.1997.7230. 711

[17] Servatius, H., "Automorphisms of graph groups," *Journal of Algebra*, vol. 126 (1989), pp. 34–60. Zbl 0682.20022. MR 1023285. DOI 10.1016/0021-8693(89)90319-0. 712, 714, 715

[18] Shelah, S., *Classification Theory for Abstract Elementary Classes*, vol. 18 of *Studies in*

Logic (London), College Publications, London, 2009. Zbl 1225.03036. MR 2643267. 707, 718

[19] Tits, J., "Le problèm des mots dans les groupes de Coxeter" pp. 175–85 in *Symposia Mathematica (INDAM, Rome, 1967/68), Vol. 1*, Academic Press, London, 1969. Zbl 0206.03002. MR 0254129. 712

[20] Tits, J., "Sur le groupe des automorphismes de certains groupes de Coxeter," *Journal of Algebra*, vol. 113 (1988), pp. 346–57. Zbl 0657.20027. MR 0929765. DOI 10.1016/0021-8693(88)90164-0. 710, 711, 712, 713, 715

Hyttinen
Department of Mathematics and Statistics
University of Helsinki
Helsinki
Finland
thyttine@cc.helsinki.fi
https://wiki.helsinki.fi/display/mathstatHenkilokunta/Hyttinen2C+Tapani

Paolini
Einstein Institute of Mathematics
Hebrew University of Jerusalem
Jerusalem
Israel
paolinigianluca@gmail.com