On the representation theory of persistence modules

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Abstract

In this thesis we study representation theoretic aspects of persistence modules. The first subject is an investigation of the spectrum of the category of vector space representations of a totally ordered set T: We show the existence of and determine the spectrum of this category in the sense of Krause and Herzog, and then show that it is homeomorphic to the space of ideals of T equipped with the order topology. Using this homeomorphism we then study its further properties.

The second subject concerns the theory of middle exact representations: We generalise and discuss the notions of middle exactness and block representations from two- to three-parameter persistence modules and generalise the decomposition theorem of Botnan and Crawley-Boevey to this case.

Both of this is complemented by a discussion of the representation theoretic foundations of persistence modules in terms of functor categories, reviewing their calculus, decomposition theory and homological properties.

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As a mathematician with classical school education I should possibly refer to Pythagoras for the role of friendship, here. Anyway, I would like to thank Willem de Muinck Keizer, Yanik-Pascal Förster and Marko A. Kalajdžić for their feedback to drafts of my thesis, and also in place of all the other friends I could rely on during my studies.

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Contents

Introduction

The topic of this thesis is the theory of *persistence modules*. These are linear representation of a partially ordered set. Persistence modules have been studied intensively in recent years, as they are the central algebraic objects of *topological data analysis (TDA)*, see [EH10; Oud15; CVJ22]. While in most of the literature these are studied as a tool to solve TDA motivated problems, we study the representation theory of these algebraic structures themselves in terms of functor categories. Nonetheless, in Chapter 1 we briefly motivate the main notions of persistent homology.

The first aspect we are illuminating is the spectrum of the category of linear representations of a totally ordered set T. The spectrum is an invariant of locally coherent Grothendieck categories introduced in [Her97] and [Kra97], independently. As a generalisation of the Ziegler spectrum of a ring it encodes multiple homological properties of such a category. The spectrum of a category was recently used to study mutations of silting and cosilting t-structures in [HLŠV22].

Based on our discussion of the representation theory of persistence modules in Chapters 2 and 3, we prove that the spectrum is well-defined in Chapter 4 and also show the first main result there, after characterising the finitely presented and the indecomposable injective objects.

Theorem A (Theorem 4.7.2)**.** *Let k be a field and T a totally ordered set. Then the spectrum of the category of k-linear representations of T is homeomorphic to the space of ideals of T equipped with the order topology.*

The proof of this makes use of the well-known barcode theorem by Botnan–Crawley-Boevey, see Theorem 3.4.3. Thereafter, we study a few properties of this topology; as an example we compare a finite set with the continuum $\mathbb{R}$ and end the chapter with a comparison of the topology on the spectrum with the topology induced from the interleaving distance. Most of this can already be found in [Ler22].

In Chapter 5 we investigate middle exact persistence modules, which were introduced in [CS10] as two-parameter persistence modules that satisfy a property naturally occuring in zigzag persistence homology. First in [CO20] and then in [BCB20], this notion was further generalised and it was shown that every pointwise finite-dimensional vector space representation of the product

$P = R \times S$ of totally ordered sets decomposes into a direct sum of block modules if it is middle exact. These block modules are of interest, because they are a generalisation of one-parameter interval modules to two parameters and have a remarkably simple structure. In particular, they are indecomposable. Our second main contribution is a generalisation of first, the notions of middle exactness and block modules to three parameters, and second, the decomposition theorem.

Theorem B (Theorem 5.4.2)**.** *Let k be a field and R, S and T totally ordered sets. If M is a pointwise finite-dimensional k-linear representation of $R \times S \times T$ which is 3-middle exact and 2-middle exact, then M decomposes into a direct sum of block modules.*

In this thesis, we also pursue a further objective: Since the topic of persistence homology is very interdisciplinary, bringing together researchers from many different areas of mathematics, we are explicitly reviewing technical details and the used representation theoretic notions in Chapter 2 in order to provide the necessary information for readers which are not coming from representation theory. This also comprises the possibly less prominent non-unital analogues of well-known theorems about module categories over a ring. We add a brief discussion of various structural results about persistence modules, including their decomposition theory and their homological properties.

We end with some open questions in Chapter 6.

Chapter 1

Persistence modules and topological data analysis

In this chapter we motivate the main concepts which underly the topic of persistence modules. We briefly look into persistence modules along with their main application, persistent homology, their stability theory and the role of representation theory, which is illustrated on an example in Section 1.2. The chapter is meant to also address the non-expert in applied topology or representation theory.

1.1 Data analysis, topology and representation theory

For a long time data analysis and statistics did not have a lot in common with more abstract disciplines like algebra and topology. This division is crumbling now, as the trend prevails that more and more methods from the latter fields are applied to problems in data analysis, including methods from algebraic topology, differential or algebraic geometry, or sheaf theory, see e.g. [Car09; Cur14; MHM20].

One strategy is to first find an interpretation of the data as a sampling of a geometric object and then to study mathematical invariants of this underlying structure, see for example Section 1.2. An invariant in this context is a function on these objects which is constant even under admissible transformations. Such could be for example a reflection at a point or a translation. These invariants can take integral values, as for example the dimension or the curvature of a manifold, or other mathematical objects like isomorphism classes of abelian groups, or a graph or simplicial complex. The next requirement is *stability*, meaning that for a candidate of a technique it must be granted that a slight variation of the initial data only leads to a small variation of the outcome. In the final step one can then for example use statistical machinery to analyse the data in terms of these numerical invariants.

One of the most prominent examples is likely persistent homology, which studies the homology of a space subject to a filtration. Its application in data analysis goes back to the 1990s, but its roots are much older. The idea of this technique is to study filtrations on spaces. The homology of the filtration of a space is a functor from an indexing set to vector spaces, which can then be decomposed into *interval modules* which have a stable and complete invariant, the *barcode*.

Certainly there are many possible techniques to interpret data, which usually require a set of initial parameters. So, it is no surprise that finding such a technique is a non-linear process and that the application of such techniques to research data is often conducted in interdisciplinary teams of mathematicians and applied scientists. On one side, problems arising in such applications generate abstract mathematical problems and fertilise mathematical research, and on the other side, new (or rediscovered) mathematical methods open new opportunities in applications. Consequently, the study of methods in topological data analysis attracts applied and discrete topologists as well as algebraists, discrete mathematicians and computer scientists, but also physicists and even life scientists or engineers, see for example [NLC11; PABEJ+19; KOTMH18].

Our approach is that of *representation theory.* Roughly speaking, representation theory studies the properties of algebraic objects by representing them as linear transformations of vector spaces. An example well-known in areas outside pure mathematics is the representation theory of finite groups. If the objects studied have further underlying structure, then it is desirable for an invariant to also reflect these properties. For example, if a space has a symmetry, then an invariant admitting a non-trivial induced action of the symmetry group comprises simultaneously information about the space and the symmetry. If the invariant is an algebraic structure, then it can often be considered as a representation of that group. Here, representation theory can help to derive simpler, possibly numerical invariants from more complicated invariants. In the example of persistent homology of data points, this is the decomposition into interval modules, where these intervals again yield pairs of real numbers, which can then be processed numerically.

But a representation theoretic proposition itself can also be of interest, because of its abstract nature: On one hand it might be applicable to plenty of problems from different areas. On the other hand it can help to understand the mathematical constructions better and to reveal universal concepts which are shared by various branches of mathematics.

1.2 An example of persistent homology

Starting with a finite data set of points in a Euclidean space it is a common assumption that they are sampled from a geometric object or a function with a not necessarily known underlying geometry. For example, it could be random points along a circle or sphere. If we only consider this as a discrete topological space, then there is no more structure than the number of points. So this is not interesting at all. If instead we consider the Euclidean metric on this space, we can measure the distances of each pair of points. Now, the set of distances contains information about the geometry of the underlying structure of the data, but it forgets the information about neighbourhoods of points.

To visualise the advantage of persistence homology, we use the following example: Suppose we have three circles of different radii placed next to each other. From each circle we take a uniformly distributed subset of points such that the distance between two neighbours shrinks with decreasing radius of the circle, as illustrated in Figure 1.1. The healthy human eye immediately recognises the underlying structure of three circles. But for a computer this is not so easy.[1] An obvious idea is to connect points which are close to each other with straight line segments. If we connect the neighboured points in the big circle then we obtain something which looks very close to a circle, but we also define the length of these line segments to be a close distance. Doing the same with the medium circle, we obtain no more a circle- but an annulus-like shape, because not only the neighbours but also next neighbours are close, now. So this approach works here, too. But at the chosen distance, all points of the small circle are close to each other, so we cannot not recognise the circle here, anymore, but only a disk. Conversely, if we start with the small circle, then we cannot recognise the medium and big circle as circles, because in the size scale of the small circle those circles have no neighboured points.

So, we see that the image contains different information on different size scales, which is not simultaneously present at a single size scale. Now, one of the ideas of persistence homology in topological data analysis, which goes back to[Vie27], is to skip the choice of a single size scale and to consider all possible or at least 'many' different size scales at once, while keeping track of how the topology changes. This can be realised by generating a filtration of simplicial complexes out of the data and then taking simplicial homology. In categorical terms we obtain a diagram of functors

$$P \xrightarrow{S_\bullet} \mathrm{SimpComplexes} \xrightarrow{H} \operatorname{mod} k,$$

where P is the category defined by a partially ordered set, the parameter set of the filtration, SimpComplexes the category of simplicial complexes and $\operatorname{mod} k$ the category of finite-dimensional vector spaces over a field k. In simpler words, this means that for each point p in P there is a k-vector space V_p and for two comparable points $p \leq q$ in P there is a matrix which defines a vector space map from V_p to V_q. The set P could for example be the set of real numbers $\mathbb{R}$, but for analysing point clouds it makes sense to restrict to a finite subset of $\mathbb{R}$.

For the simplicial complex there are multiple choices with the most prominent being the *Čech complex* and the *Vietoris–Rips complex*. The Čech complex is

[1] Certainly one could divide our picture into the three subimages containing only one of each circles and do the analysis for each subimage then. But this might not be possible for more complicated setups, and when analysing information locally one can also loose global information.

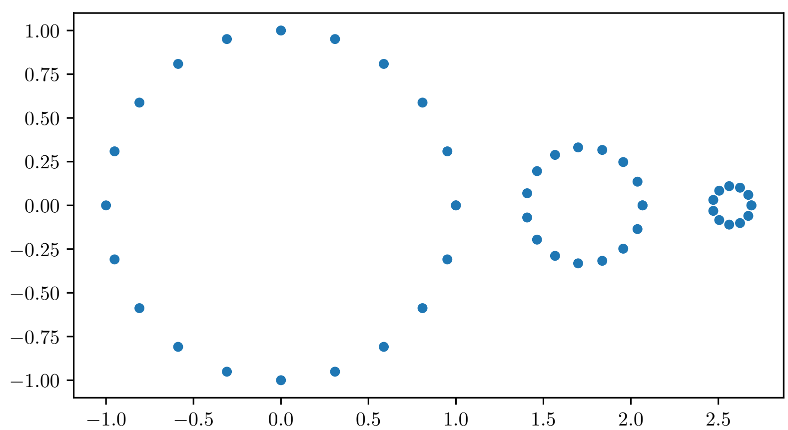


Figure 1.1: Three circular sets of different radii with different densities of points.

the filtered simplicial complex where for each parameter ε the topological nerve is taken of the space where all points are replaced by ε-disks. The Vietoris–Rips complex is a simplicial complex which is filtered by a real parameter t and whose 0-simplices are the points of the data set and whose n-simplices in filtration step t correspond to sets of $n+1$ points which have pairwise distance of less than t. Unlike the Čech complex, the Vietoris–Rips-complex only requires a distance matrix an no euclidean embedding. For more details on this, see [ELZ02] or for an extensive introduction the text books [CVJ22; Oud15].

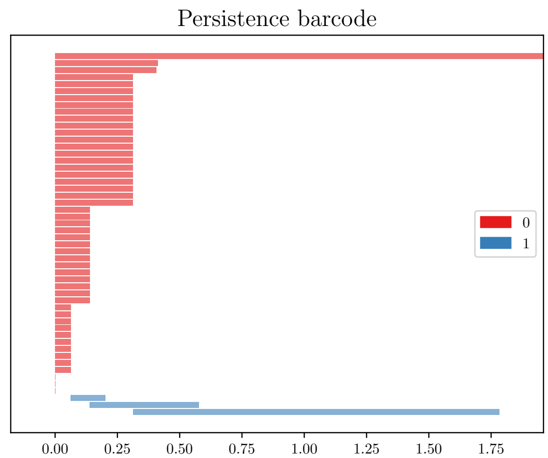
Now, let P be a finite linearly ordered parameter set of n points. Then, categorically, P is the same as a quiver $1 \to 2 \to ... \to n$. If the homology is taken over a field k, then the composition $H \circ S_\bullet$ is a functor $P \to \operatorname{mod} k$, or in other words a finite-dimensional k-linear representation of P. By the well-known *Barcode Theorem 3.4.3* every such functor has a decomposition into indecomposable summands, which are completely parametrised by a *birth* and *death parameter* in P. One can express these parameters graphically as a bar which starts at the birth-parameter and ends at the death parameter. The multiset of all bars is called the *barcode*. One usually visualises the barcode by stacking the bars sorted first by their birth parameter and second by their length, see Figure 1.2.(a).

A different way to express this information is to just consider the pair of coordinates in the real plane. If the horizontal axis represents the birth parameter and the vertical axis represents the death parameter, then the pairs representing indecomposable summands are points in the upper half-plane. This graphical representation is called *persistence diagram*, see Figure 1.2.(b). Note that the persistence diagram also is a multiset, which has to be taken into account for interpreting the visualisation. All calculations were conducted in GUDHI 3.8.0 for Python 3.11 [GUD23].

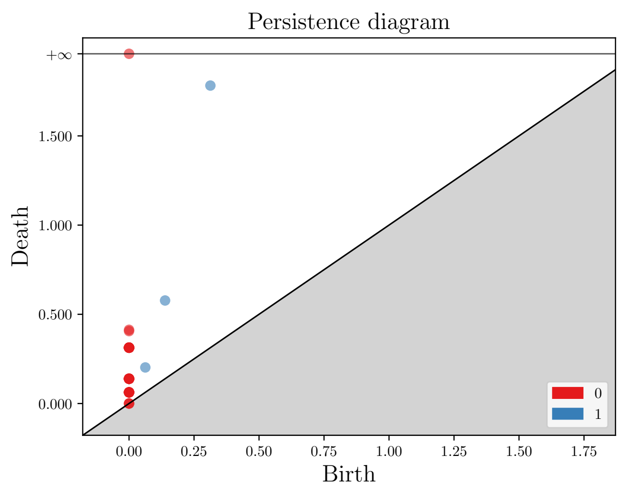
1.3 Stability of persistence modules

Now, if we want to analyse data, no matter if it comes from theoretical studies or if it is real world data like a point cloud, we are usually only interested in a small but hidden part of it which is relevant for answering our questions. Since usually we cannot directly identify and extract the relevant the part from the data, we want to simplify each record of the data set into numbers or vectors in a way such that all essential information is still preserved but most of the non-essential information is not. Based on this one can then classify given data or make predictions about unknown variables. But for each such method we need to ensure that the differences between these 'simplifications' of two data records are small if the data records are almost the same: Removing or adding a few out of many points or wiggling any points slightly should not substantially change the overall outcome!

Essentially, algebraic topology is about finding and calculating invariants of spaces, which on one hand are fine enough to distinguish sufficiently different spaces, and which are on the other hand coarse enough to really simplify



(a)



(b)

Figure 1.2: (a) The barcode diagram of the Vietoris-Rips-complex for the example in Figure 1.1. The bars in dimension zero correspond to the connected components and are all born at parameter zero. All but one die eventually, as the components are merged into a single component when the filtration parameter increases. In dimension one, the birth of the first bar is reflected by the filtration parameter exceeding the distance between to neighboured points of the small circle. The death of this bar occurs when the filtration parameter outreaches the diameter of the smallest circle. The first and third bar not overlapping reflects that the big and small circles cannot be compared at a single size scale. (b) The persistence diagram can be obtained by plotting the birth and death coordinate of each bar over the upper diagonal.

classification problems and to be stable under admissible variations. Examples of such are singular or simplicial homology or just the rank of it, the *Betti-numbers*, see for example [Hat02]. Persistence diagrams are an invariant of filtered spaces. Because for each filtration parameter they contain the entire simplicial homology of that filtration step, they are fine enough. But it is a problem of *stability theory* to show that they are also coarse enough, see for example [CVJ22, Section 4.5.5].

Now, this was shown in [CSEH07]: The original idea was to establish a meaningful metric on a space of persistence diagrams, the *bottleneck distance*, and then to show that this is bounded by a metric on the data set, in this case given by the L^∞-norm. In the example above, this means that the diagram in Figure 1.2.(b) looks almost the same if there is only a very small variation of the image in Figure 1.1.

Chapter 2

Representation theory of generalised modules

In this chapter we discuss the representation theoretic concepts used for our further study of persistence modules, requiring only basic knowledge of categories, abelian categories and homological algebra. For this, see for example [ML98; Wei94].

The central concept of this chapter is the generalised module category of a preadditive small category. We treat these and the necessary concepts in Sections 2.1 to 2.3, followed by a discussion of their (co)limits, projective generators and why they are Grothendieck categories, in Section 2.4. Our approach is strongly influenced by the theory of *rings with several objects* by Mitchell, hence we discuss in Section 2.5 how this notion enables us to work with generalised modules as if they were modules over a ring. In Section 2.6 we briefly look at finitely presented objects and in Section 2.7 we address weak limits and colimits, because they are scarce in literature and are used later. The chapter ends with a technical discussion of additivisations of preadditive categories in Section 2.8.

2.1 Preadditive categories

Let $\mathcal{C}$ be a category. We say $\mathcal{C}$ is *preadditive* if its morphism sets are abelian groups, such that composition is bilinear. Furthermore, if k is a commutative ring, then $\mathcal{C}$ is called *k-linear* if the morphism sets are k-modules and if the composition is k-bilinear. As there is always a canonical ring homomorphism $\mathbb{Z} \to k$, every k-category is also preadditive. A preadditive or k-linear category is called *additive* if it further has all finite coproducts.

For two categories $\mathcal{C}$ and $\mathcal{D}$ we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, or shorthand $\mathcal{D}^{\mathcal{C}}$, the category which has (covariant) functors as object and natural transformations between functors as morphisms. A functor $F\colon \mathcal{C} \to \mathcal{D}$ of k-linear categories is called *k-linear* if for all c and c' in $\mathcal{C}$ the map

$$\operatorname{Hom}_{\mathcal{C}}(c, c') \to \operatorname{Hom}_{\mathcal{D}}(F(c), F(c'))$$

is k-linear. F is called *additive* if it is $\mathbb{Z}$-linear. For the full subcategory of additive functors we write $\operatorname{Add}(\mathcal{C}, \mathcal{D})$. Note that if $\mathcal{C}$ and $\mathcal{D}$ are additive, then F is additive if and only if it preserves finite direct coproducts. If $\mathcal{C}$ is a small preadditive category we write $\operatorname{Mod}\mathcal{C}$ for $\operatorname{Add}(\mathcal{C}, \mathrm{Ab})$. This is called the category of *(generalised left) modules* of $\mathcal{C}$. It can also be seen as the ind-completion of $\mathcal{C}^{\mathrm{op}}$, that is the universal category containing $\mathcal{C}^{\mathrm{op}}$ which is closed under direct limits and minimal with this property, see [Sga, Section 8.2].

In this context we define the *contravariant Yoneda embedding*

$$\begin{aligned} h_-\colon\ \mathcal{C}^{\mathrm{op}} &\to \operatorname{Mod}\mathcal{C} \\ a &\mapsto \operatorname{Hom}_{\mathcal{C}}(a, -) \end{aligned}$$

and the *covariant Yoneda embedding*

$$\begin{aligned} h^-\colon\ \mathcal{C} &\to \operatorname{Mod}\mathcal{C}^{\mathrm{op}} \\ a &\mapsto \operatorname{Hom}_{\mathcal{C}}(-, a), \end{aligned}$$

which are fully faithful by the following additive version of Yoneda's Lemma, see for example [Bor94b, Proposition 1.3.7].

Lemma 2.1.1 (Yoneda)**.** *Let $\mathcal{C}$ be a small preadditive category and $F\colon \mathcal{C} \to \mathrm{Ab}$ an additive functor. Then there is a group isomorphism*

$$\mathcal{Y}\colon \operatorname{Nat}(h_a, F) \to F(a),$$

for every a in $\mathcal{C}$, where $\operatorname{Nat}(h_a, F)$ denotes the additive group of natural transformations from h_a to F. This isomorphism is natural in a and F. In particular, if $F = h_a$ we obtain a ring isomorphism

$$\operatorname{Nat}(h_a, h_a) \to \operatorname{End}_{\mathcal{C}}(a)^{\mathrm{op}}.$$

The analogue holds for contravariant functors, and similarly we get an isomorphism

$$\operatorname{Nat}(h^a, h^a) \to \operatorname{End}_{\mathcal{C}}(a).$$

Sketch of proof. For the definition of the morphism, recall that $\mathcal{Y}$ maps every natural transformation $\Phi\colon h_a \to F$ to the image of id_a under $\Phi_a\colon \operatorname{Hom}_{\mathcal{C}}(a, a) \to F(a)$. Conversely, an element u in $F(a)$ determines such a natural transformation uniquely up to isomorphism, because for each b in $\mathcal{C}$ and $f\colon a \to b$ we obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}}(a,a) & \xrightarrow{h_a(f)} & \operatorname{Hom}_{\mathcal{C}}(a,b) \\ \downarrow{\scriptstyle \Phi_a} & & \downarrow{\scriptstyle \Phi_b} \\ F(a) & \xrightarrow{F(f)} & F(b), \end{array}$$

where $\Phi_b(f)$ is the homomorphism $F(f) \circ \Phi_a(\mathrm{id}_a)$.

For the second statement observe, that an inverse transformation for the case $F = h_a$ is given by:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(a,a) &\to \mathrm{Nat}\,(h_a, h_a)^{\mathrm{op}} \\ f &\mapsto h_f \end{aligned}$$

This reverses the order of factors by functoriality of h_-. The remaining is similar. □

Remark 2.1.2. Note that in other places in literature, see for example [Kra22], the notation 'Mod $\mathcal{C}$' is used for the category of additive contravariant functors $\mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, instead of covariant functors. In this case one speaks of *right modules* of $\mathcal{C}$. This has the advantage that the Yoneda embedding

$$h^-\colon\, \mathcal{C} \to \mathrm{Add}\,(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$$

is covariant. Since taking the opposite preserves additivity, one can switch between both conventions without problems. Also, the category $\mathcal{C}$ can be assumed to be additive instead of being just preadditive, but these notions are compatible, see Section 2.8.

2.2 Constructions of abelian categories

Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{C} \subseteq \mathcal{A}$ is called *Serre* if for every exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\mathcal{A}$ the terms M' and M'' are in $\mathcal{C}$ if and only if M is in $\mathcal{C}$.

In this setting it is possible to form the *quotient category* $\mathcal{A}/\mathcal{C}$. The objects are the same as in $\mathcal{A}$ and the morphisms between two objects X and Y are given by the formula

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X,Y) = \varinjlim \mathrm{Hom}_{\mathcal{A}}\,(X', Y/Y')\,,$$

where the filtered colimit is taken over the direct system of pairs of objects $X' \subseteq X$ and $Y \subseteq Y'$ in $\mathcal{C}$ with respect to the order $(X', Y') \leq (X'', Y'')$ if and only if $X'' \subseteq X'$ and $Y' \subseteq Y''$. This gives rise to a localisation functor $T\colon\, \mathcal{A} \to \mathcal{A}/\mathcal{C}$, which is the identity on the objects and which maps every morphism to its image in the filtered colimit. The quotient category of an abelian category by a Serre subcategory is abelian and T is an *exact functor*, i.e. it preservers short exact sequences, see [Gab62, III.1 Proposition 1]. A Serre subcategory $\mathcal{C} \subseteq \mathcal{A}$ is called *localising* if the functor T has a right adjoint S.

Recall that a *torsion pair* is defined as a pair of two full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ such that $\mathcal{T}$ consists of exactly those elements without non-trivial morphisms into $\mathcal{F}$, and $\mathcal{F}$ consists of exactly those elements without non-trivial morphisms from $\mathcal{T}$ into it, and where each object M in $\mathcal{A}$ admits a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

with M' in $\mathcal{T}$ and M'' in $\mathcal{F}$. This also gives rise to a right adjoint functor t of the inclusion $\mathcal{T} \subseteq \mathcal{A}$. A torsion pair is called *hereditary* if $\mathcal{T}$ is closed under subobjects or equivalently if t is exact. A torsion pair or a localising subcategory is *of finite type* if the respective right adjoints of their inclusions preserve direct limits.

Note that for a collection of objects $\mathcal{C}$ of an abelian category $\mathcal{A}$ the *right perpendicular category of* $\mathcal{C}$ *in* $\mathcal{A}$ is the full subcategory $\mathcal{C}^{\perp}$ of objects M of $\mathcal{A}$ satisfying

$$\operatorname{Hom}_{\mathcal{A}}(C, M) = \operatorname{Ext}^1_{\mathcal{A}}(C, M) = 0$$

for all C in $\mathcal{C}$. If $\mathcal{C}$ is localising subcategory of $\mathcal{A}$, then the right adjoint functor $S\colon \mathcal{A}/\mathcal{C} \to \mathcal{A}$ is fully faithful with essential image $\mathcal{C}^{\perp}$, see for example [Kra22, Corollary 2.2.10].

2.3 Grothendieck categories

Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}$ is called *Grothendieck* if it satisfies the following conditions:

AB3 The category $\mathcal{A}$ is closed under small coproducts.

AB4 There is a generator object G in $\mathcal{A}$, meaning that for every object X in $\mathcal{A}$ the canonical homomorphism

$$\coprod_{\varphi \in \operatorname{Hom}(G,X)} G \xrightarrow{\sum \varphi} X$$

is an epimorphism.

AB5 Filtered colimits are exact functors.

Typical examples for Grothendieck categories are module categories $\operatorname{Mod} R$ for a ring R, in particular the category of abelian groups Ab, or the categories of sheaves of abelian groups over a topological space, or the category of quasi-coherent sheaves over a scheme, see [Gro57; Gab62].

Grothendieck categories generalise many properties of these examples and have proven to be a good framework to work with. In particular, they are closely related to module categories as they embed into such. This is the famous theorem by Gabriel and Popescu.

Theorem 2.3.1 ([PG64])**.** *Let $\mathcal{A}$ be a Grothendieck category with generator G and let $R = \operatorname{End}_{\mathcal{A}}(G)$ denote its endomorphism ring. Then the functor*

$$\operatorname{Hom}_{\mathcal{A}}(G, -)\colon \mathcal{A} \to \operatorname{Mod} R$$

is fully faithful and embeds $\mathcal{A}$ as the orthogonal subcategory $\mathcal{C}^{\perp}$ for a localising subcategory $\mathcal{C}$ of $\operatorname{Mod} R$ into the category of R-modules. □

2.4 Generalised module categories

The study of *generalised module categories* as defined in Section 2.1 goes back to work of Gabriel and Mitchell in [Gab62; Mit65], but this concept was already introduced as *diagrammes de schèma* in different terminology by Grothendieck in [Gro57].

Note that module categories for rings with several objects are ubiquitous, as the following Theorem of Freyd and Mitchell suggests.

Theorem 2.4.1 ([Mit72, Theorem 3.1])**.** *Let $\mathcal{A}$ be an additive category. Then $\mathcal{A}$ is equivalent to a module category* $\operatorname{Mod}\mathcal{C}$ *for a small preadditive category $\mathcal{C}$ if and only if $\mathcal{A}$ is abelian, closed under (small) coproducts and if it has a set $\mathcal{P}$ of compact projective generators, where compactness means that* $\operatorname{Hom}_{\mathcal{A}}(P,-)$ *commutes with all coproducts for each P in $\mathcal{P}$.* □

The proof works by showing that if such a set of compact projectives is considered as full preadditive subcategory $\mathcal{P}$ of $\mathcal{A}$, then the functor

$$\begin{aligned} \mathcal{A} &\longrightarrow \operatorname{Mod}\left(\mathcal{P}^{\mathrm{op}}\right), \\ A &\longmapsto \left(P \mapsto \operatorname{Hom}_{\mathcal{A}}(P, A)\right), \end{aligned}$$

embedding $\mathcal{A}$ into the module category of the opposite category of $\mathcal{P}$ is an equivalence.

Let us discuss several constructions and properties of these categories in the following.

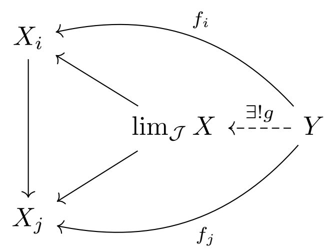
2.4.2 Limits and colimits

Since limits and colimits in module categories of small categories play a crucial role in this thesis, we briefly discuss their theory, in particular why they can be calculated pointwise.

To see this, first recall that for small categories $\mathcal{J}$ and $\mathcal{C}$ and a category $\mathcal{D}$ closed under limits the existence of the limit of a $\mathcal{J}$-indexed system of functors X_j in $\mathcal{D}^{\mathcal{C}}$, or in other words, of a functor X in $\operatorname{Fun}\left(\mathcal{J}, \mathcal{D}^{\mathcal{C}}\right)$, is exactly the same as finding a universal arrow from the constant functor $\Delta_{\mathcal{J}}$ to X. This means that there is an object $\lim_{\mathcal{J}} X$ of $\mathcal{D}^{\mathcal{C}}$ and a morphism $\varepsilon_X \colon \Delta_{\mathcal{J}}(\lim_{\mathcal{J}} X) \to X$ such that for all $f \colon \Delta_{\mathcal{J}} Y \to X$ there is a unique morphism $g \colon Y \to \lim_{\mathcal{J}} X$, making the following diagram commute:

$$\begin{array}{ccc} \Delta_{\mathcal{J}} Y & & \\ {\scriptstyle \Delta_{\mathcal{J}}(g)}\big\downarrow & \overset{f}{\searrow} & \\ \Delta_{\mathcal{J}}\left(\lim_{\mathcal{J}} X\right) & \xrightarrow[\varepsilon_X]{} & X \end{array}$$

Or with other words, for each arrow $i \to j$ in $\mathcal{J}$, we have a commutative diagram:



Now, set $\lim_{\mathcal{J}} X \coloneqq (c \mapsto \lim_{\mathcal{J}} X(-, c))$. From the pointwise universal property for every $c \in \mathcal{C}$ and naturality in this variable follows that this is the limit of X. Since the choice of X was arbitrary, it follows from general nonsense, see [ML98, IV.1 Theorem 2], that there is a right adjoint $\lim_{\mathcal{J}}$ of $\Delta_{\mathcal{J}}$.

In particular, the category $\operatorname{Mod} \mathcal{C}$ inherits to be abelian from the category of abelian groups Ab. Similarly, all filtered colimits are exact: The evaluation functor for an object in $\mathcal{C}$ is a left adjoint (confer Theorem 3.2.2), so it commutes with colimits. Calculating the filtered colimit pointwise in Ab is exact, and by the arguments above, pointwise exactness yields global exactness.

We end this digression with a collection of important results. More details can be found in [Pop73, §3.4].

Theorem 2.4.3. *Let $\mathcal{C}$ be a small category. Then the following hold:*

- *Limits and colimits in* $\operatorname{Mod} \mathcal{C}$ *are calculated pointwise.*
- *The category* $\operatorname{Mod} \mathcal{C}$ *is closed under all limits and colimits.*
- *The category* $\operatorname{Mod} \mathcal{C}$ *is abelian.*
- *All filtered colimits of exact sequences in* $\operatorname{Mod} \mathcal{C}$ *are exact.* □

2.4.4 Projective Generators

The remaining requirement for $\operatorname{Mod} \mathcal{C}$ to be a Grothendieck category is the existence of a generator. Conveniently, this category is equipped with a set of projective generators, corresponding to the objects of $\mathcal{C}$.

Consider the contravariant Yoneda embedding $h_- \colon \mathcal{C}^{op} \to \operatorname{Mod} \mathcal{C}$, which maps every object x in $\mathcal{C}$ to the representable functor $h_x = \operatorname{Hom}_{\mathcal{C}}(x, -)$, and every morphism $f \colon x \to y$ in $\mathcal{C}$ to the morphism

$$h_f = \operatorname{Hom}_{\operatorname{Mod} \mathcal{C}}(f, -) \colon \operatorname{Hom}_{\mathcal{C}}(y, -) \to \operatorname{Hom}_{\mathcal{C}}(x, -).$$

It is not hard to see, that h_x is projective for every x in $\mathcal{C}$: The functor h_x is projective if and only if the homomorphism of functors

$$f_* \colon \operatorname{Hom}_{\operatorname{Mod} \mathcal{C}}(h_x, F) \to \operatorname{Hom}_{\operatorname{Mod} \mathcal{C}}(h_x, G)$$

is an epimorphism for every epimorphism $f\colon F \to G$. But by the Yoneda Lemma 2.1.1 and its naturality in F we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{C}}(h_x, F) & \xrightarrow{f_*} & \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{C}}(h_x, G) \\ \sim\downarrow & & \downarrow\sim \\ F(x) & \xrightarrow{f(x)} & G(x), \end{array}$$

where the vertical maps are the Yoneda maps. Now suppose we have a non-zero object M in $\mathrm{Mod}\,\mathcal{C}$. Then there is an x in $\mathcal{C}$ such that $M(x) \neq 0$. Using the Yoneda Lemma again, we obtain

$$\mathrm{Hom}_{\mathrm{Mod}\,\mathcal{C}}(h_x, M) \xrightarrow{\sim} M(x) \neq 0.$$

Therefore, the representable functors form a set of projective generators of $\mathrm{Mod}\,\mathcal{C}$. Thus, combined with Theorem 2.4.3 this implies the following.

Theorem 2.4.5. *The category* $\mathrm{Mod}\,\mathcal{C}$ *is Grothendieck.* □

2.4.6 Submodules and generators

Next, we are studying the structure of subobjects and elements of a generalised module.

First, fix the notation $|\,\mathcal{C}\,|$ for the set of objects of $\mathcal{C}$. Moreover, for $x \in M(p)$ and $p \in |\,\mathcal{C}\,|$ just write $x \in M$ and define $|x| := p$. This way we can conveniently identify M with $\bigoplus_{c\in|\mathcal{C}|} M(c)$. Note that $M(p)$ is a k-module and $M(f)\colon M(p) \to M(q)$ is a k-module homomorphism for any arrow $f\colon p \to q$ in $\mathcal{C}$. Later we also need the *support* function of a module:

$$\mathrm{supp}\colon\ \mathrm{Mod}\,\mathcal{C} \to \mathcal{P}(|\,\mathcal{C}\,|)\,,\ \mathrm{supp}\,M = \{c \in |\,\mathcal{C}\,| : M(c) \neq 0\}\,,$$

where $\mathcal{P}(-)$ denotes the power set.

As $\mathrm{Mod}\,\mathcal{C}$ is a Grothendieck category, having the representable functors as projective generators, for every object M of $\mathrm{Mod}\,\mathcal{C}$ there is a generating epimorphism

$$\gamma\colon \bigoplus_{j\in J} h_{c_j} \to M,$$

for a set of objects $c_j \in \mathcal{C}$ indexed by a set J. For every $j \in J$ fix an $x_j \in M(c_j)$, the image of the identity element in $\mathrm{Hom}_{\mathcal{C}}(c_j, c_j)$ under the morphism $\gamma(c_j)$. These elements x_j associated with a generating epimorphism γ are called *generators* of M and they are uniquely defined by the Yoneda Lemma. For a subset $J' \subseteq J$ write $\langle x_j \mid j \in J' \rangle$ for the subrepresentation generated by the images of the summands $h_{c_j}, j \in J'$ under γ.

Conversely, every element $x \in M$ defines a unique homomorphism $h_{|x|} \to \langle x \rangle$. In this notation, an element $x \in M(c)$ for $c \in \mathcal{C}$ can be expressed in terms of those generators with $\langle x_j \rangle(c) \neq 0$. So, for adequate morphisms $f_j \in \mathrm{Hom}_{\mathcal{C}}(c_j, c)$, we can write

$$x = \sum_{j \in J} M(f_j) x_j,$$

where only finitely many f_j are non-trivial. A subset of a set of generators is a set of generators, again, if all elements in M can be expressed as a linear combination in terms of these generators, as above. A set of generators is called *linearly independent* if there are no non-trivial linear combinations of zero in terms of them. Equivalently, M is the direct sum of the images of h_{c_j} under γ for an adequate γ.

2.5 Rings with several objects

In this section we discuss how preadditive categories and generalised module categories generalise rings and their modules in a very useful and applicable way, which makes them our tool of choice.

Though the theory of rings and modules has proven to be a useful tool in algebra for more than a century, it has its limits: Let $Q = 1 \to 2 \to \ldots$ be the diagram category (also called *quiver*) corresponding to the totally ordered set of natural numbers. Moreover, for a field k let kQ be the path algebra of Q, that is the non-unital k-algebra freely generated by all paths in Q with multiplication defined by composition of paths.

Then kQ has a set of orthogonal idempotents $E \subseteq kQ$, that is $e = e^2$ and $e'e \neq 0$ if and only if $e = e'$ for all $e, e' \in E$. These are given by the trivial paths at each vertex, and therefore there is a *local unit*, that is for each element $x \in kQ$ there is an element $e_x \in kQ$ with $e_x^2 = e_x$ and $e_x x = x e_x = x$. However, the algebra kQ has no unit, so we cannot directly use the classical approach for quiver representations.

Non-unital rings, and by this we mean a set with two binary operations that obey the ring axioms except the unitality axiom raise a lot of problems, as many basic results require the existence of a unit element. A standard mitigation method suggested in [Dor32] is to embed the ring without unit into a unital ring, the *unitalisation*. Namely, given a non-unital ring $(R, +, \cdot)$ one defines a new unital ring on the additive abelian group $R \oplus \mathbb{Z}$ with the product defined by $(r, a) \cdot (s, b) = (rs + rb + as, ab)$ and the unit given by $(0, 1)$. But the category of modules over this unitalisation turns out to be too big, so Quillen studied subcategories and quotients of this in [Qui96], which is very technical.

Another approach, which unlike Quillen's directly reflects the structure of the underlying diagram, was suggested by Mitchell in [Mit72], based on observations of Freyd in [Fre60]: Instead of considering a (unital) ring as a set, we can also interpret it as a category $\mathcal{R}$ which has only a single object and in which the arrows are given by the elements of the ring with the composition coinciding with the ring multiplication, so $\mathrm{End}(\mathcal{R}) \cong R$.

Then the category of left modules $\operatorname{Mod} R^{\mathrm{op}}$ is equivalent to the category of additive covariant functors from $\mathcal{R}$ to Ab, the category of abelian groups: Evaluated at the only object $*$ of $\mathcal{R}$, every such functor M is an abelian group $M(*)$. Moreover, left multiplication of this by an element $r \in R$ is the same as applying the induced map $M(r) \in \operatorname{Hom}_{\mathcal{R}}(*, *)$ to $M(*)$ from the left. Associativity, unitality and distributivity all follow from the functoriality and linearity.

Now, there is no reason to consider only categories with one element, but this works for arbitrary small preadditive categories. These were called *rings with several objects* in [Mit72]:

Module calculus over rings with several objects

Let k be a commutative ring and let $\mathcal{C}$ be a small k-category. Then we can observe the following generalisation of the ring axioms:

- For all objects i, j of $\mathcal{C}$ the morphisms $\operatorname{Hom}_{\mathcal{C}}(i, j)$ form a k-module by definition.
- The composition of morphisms is associative, as $\mathcal{C}$ is a category.
- The distributivity law holds for all composable morphisms φ, ψ, χ, by k-linearity of $\mathcal{C}$.

$$\chi(\varphi + \psi) = \chi\varphi + \chi\psi$$
$$(\varphi + \psi)\chi = \varphi\chi + \psi\chi$$

- Unitality holds for all morphisms φ in $\mathcal{C}$, i.e. $\varphi \operatorname{id} = \varphi = \operatorname{id} \varphi$, as $\mathcal{C}$ is a category.

This also admits a generalisation of left module theory: Let M be in $\operatorname{Mod} \mathcal{C}$. Then:

- $M(c)$ is a k-module for each c in $\mathcal{C}$, as it is an abelian group by definition and it admits a multiplication by elements λ of k, defined by restriction along the ring homomorphism $k \to \operatorname{Hom}_{\mathcal{C}}(c, c)$. So $\lambda \cdot x := M(\lambda \cdot \operatorname{id}_c)(x)$ for all x in $M(c)$.

Moreover, let x, y be in $M(c)$ and r, s in $\operatorname{Hom}_{\mathcal{C}}(c, c')$ and t in $\operatorname{Hom}_{\mathcal{C}}(c', c'')$ for c, c', c'' in $\mathcal{C}$. Then:

- $M(r)(x + y) = M(r)(x) + M(r)(y)$, as $M(r)$ is a group homomorphism for all r.
- $M(r + s)(x) = M(r)(x) + M(s)(x)$, by additivity of M.
- $M(t \cdot s)(x) = M(t)\left(M(s)(x)\right)$, by functoriality of M.
- $M(\operatorname{id}_c)(x) = x$, by functoriality of M.

Thus, it is natural to write $M(s)(x) = s \cdot x$ if unambiguous, and we can multiply by arrows as we do with ring elements in classical module theory.

2.6 Finitely presented objects

Another central concept for this thesis is the notion of finitely presented objects, as these are crucial for the definition of the spectrum.

Let $\mathcal{A}$ be a cocomplete additive category. An object X in $\mathcal{A}$ is *finitely presented* if the functor $\operatorname{Hom}_{\mathcal{A}}(X,-)$ preserves filtered colimits. For the full subcategory of finitely presented objects we write $\operatorname{fp}\mathcal{A}$. The category $\mathcal{A}$ is called *locally finitely presented* if $\operatorname{fp}\mathcal{A}$ is essentially small and every object of $\mathcal{A}$ is a filtered colimit of objects in $\operatorname{fp}\mathcal{A}$.

In case that $\mathcal{A}$ is the module category $\operatorname{Mod}\mathcal{C}$ for a small preadditive category $\mathcal{C}$, then the representable functors $\operatorname{Hom}_{\mathcal{C}}(x,-)$ are finitely presented: This follows from the Yoneda Lemma and the pointwise calculation of colimits. Also, the five lemma implies that M is finitely presented if it fits into an exact sequence

$$\bigoplus_{i=1}^{m} \operatorname{Hom}_{\mathcal{C}}(x_i,-) \to \bigoplus_{j=1}^{n} \operatorname{Hom}_{\mathcal{C}}(y_j,-) \to M \to 0 \tag{2.1}$$

with x_i, y_j in $\mathcal{C}$. In fact, this condition is an equivalence.

Theorem 2.6.1 ([CB94, 1.2 Theorem])**.** *Let $\mathcal{C}$ be a small preadditive category. Then the category $\operatorname{Mod}\mathcal{C}$ is locally finitely presented. Moreover, an object M is finitely presented if and only if it appears in an exact sequence (2.1).* □

We write $\operatorname{mod}\mathcal{C} \coloneqq \operatorname{fp}\operatorname{Mod}\mathcal{C}$ for the full subcategory of finitely presented objects. Moreover, we write $\operatorname{proj}\mathcal{C}$ for the full subcategory of finitely generated projective objects of $\operatorname{Mod}\mathcal{C}$.

2.7 Weak limits and colimits

Limits are ubiquitous in the study of abelian categories. But there are many additive categories which do not admit limits in a strong sense, just think of triangulated categories, but sometimes admit weak limits. As we discuss in Chapter 5, these are also related with homotopy theory.

Definition 2.7.1. Let $\mathcal{C}$ be a category, $\mathcal{I}$ a small category and $F\colon \mathcal{I} \to \mathcal{C}$ a functor. A *weak limit* of F is an object L together with a natural transformation $\Delta_{\mathcal{I}}L \to F$ which factors every natural transformation $\Delta_{\mathcal{I}}N \to F$, where $\Delta_{\mathcal{I}}L$ is the constant functor and N any object of $\mathcal{C}$.

Note that neither this factorisation nor the object L is required to be unique. Dually, we can define *weak colimits*. The most important weak limits and colimits are the following constructions: Let $\mathcal{C}$ be additive. Then a *weak kernel* of a morphism $f\colon X \to Y$ in $\mathcal{C}$ is a morphism $W \to X$ which factors every morphism $g\colon Z \to X$ with $f \circ g = 0$. So a weak kernel of f has the same definition as the kernel except the uniqueness, and it is a weak limit for the same diagram as the equaliser, consisting of f and the parallel zero-morphism.

A *weak pullback* is a weak limit of the diagram:

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

It satisfies the properties of a pullback with the exception of uniqueness. Dually, one can define a *weak pushout* to be a weak colimit of the opposite diagram: Note that a weak kernel of a morphism f is a weak pullback of diagram:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & Y \end{array}$$

and a weak cokernel of f is a weak pushout of the opposite diagram.

Recall that in an abelian category the image of a morphism f is defined as $\operatorname{im} f := \ker(\operatorname{coker} f)$.

Lemma 2.7.2. *Consider a sequence of vector spaces*

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Then the following are equivalent:

(1) $\ker g = \operatorname{im} f$.

(2) f *is a weak kernel of* g.

(3) g *is a weak cokernel of* f.

Proof. First, we show that (1) implies (2): Suppose $\ker g = \operatorname{im} f$ holds and let $t\colon T \to Y$ be a morphism such that $g \circ t = 0$. Then t factors through $\ker g = \operatorname{im} f$. But the canonical quotient $X \to \operatorname{Im} f$ is a split epimorphism, so $\operatorname{im} f$ factors through f and therefore t factors through f. Next, we show that (1) implies (3): Let $t\colon Y \to T$ be a morphism such that $t \circ f = 0$. In particular, $t \circ \operatorname{im} f = 0$ as $\operatorname{im} f$ factors through f, again, and therefore t factors through $\operatorname{coker} \operatorname{im} f = \operatorname{coker} \ker g$. But $\operatorname{coker} \ker g$ factors through g as $\operatorname{im} g$ is a split monomorphism. Thus, t factors through g.

Now, we show that (2) implies (1): Let f be a weak kernel of g. In particular $gf = 0$ and therefore there is an embedding $\iota\colon \operatorname{Im} f \to \operatorname{Ker} g$ such that $\operatorname{im} f = \iota \circ \ker g$. By the weak kernel property of f, there is a morphism $k\colon \operatorname{Ker} g \to X$ with $f \circ k = \ker g$. Now, $\operatorname{coker} f \circ \ker g = \operatorname{coker} f \circ f \circ k = 0$, hence there is a morphism $s\colon \operatorname{Ker} g \to \operatorname{Im} f$ such that $\ker g = \operatorname{im} f \circ s$. Thus, $\operatorname{im} f = \ker g$.

To see that (3) implies (1), note that by the weak cokernel property of g there is a morphism $m\colon Z \to \operatorname{Coker} f$ such that $\operatorname{coker} f = m \circ g$. Then $\operatorname{coker} f \circ \ker g = m \circ g \circ \ker g = 0$, so there is a morphism $s\colon \operatorname{Ker} g \to \operatorname{Ker} \operatorname{coker} f = \operatorname{Im} f$ with $\ker g = \operatorname{im} f \circ s$. So, the morphisms $\ker g$ and $\operatorname{im} f$ factor each other and consequently $\ker g = \operatorname{im} f$. □

Note, that this does not hold for more general abelian categories.

Example 2.7.3. Consider the chain complex of abelian groups

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z}/(6) \xrightarrow{g} \mathbb{Z}/(2)$$

defined by $f(x) = \overline{2x}$ and $g(x) = \overline{3x}$. Then $\operatorname{Ker} g = 2\,\mathbb{Z}/(6) = \operatorname{Im} f$. But there is a nonzero map $t\colon \mathbb{Z}/(3) \to \mathbb{Z}/(6)$ with $g \circ t = 0$, which certainly not factors through f.

2.8 Additivisation

Many constructions for additive categories like the generalised module category also work for preadditive categories. But several characterisations of this module category are formulated on an additive structure, see e.g. the proof of Lemma 4.3.2. The technical Lemma 2.8.3, which we prove, illustrates that there is not so much difference between the preadditive and additive notions of modules.

Definition 2.8.1. Let k be a commutative ring and $\mathcal{C}$ a small category. The *k-linearisation* $k\,\mathcal{C}$ of $\mathcal{C}$ is the uniquely defined k-linear category which has the same objects as $\mathcal{C}$ and where the morphism set between two objects x and y is given as the free k-module generated by the elements in $\operatorname{Hom}_{\mathcal{C}}(x, z)$, such that the composition is k-bilinear.

Note that the category of modules $\operatorname{Mod}(k\,\mathcal{C})$ is equivalent to $\operatorname{Fun}(\mathcal{C}, \operatorname{Mod} k)$, the category of all functors from $\mathcal{C}$ to the category of k-modules. This follows quickly from the observations in Section 2.5. The construction above does not yield additive categories in general, but this is the case for the following construction.

Definition 2.8.2. Let k be a field and $\mathcal{C}$ a small k-category. The *matrix ring category* of $\mathcal{C}$ is the k-linear category $\operatorname{Mat}\mathcal{C}$ having as objects finite sequences $(x_1, ..., x_n)$ of objects of $\mathcal{C}$. A morphism to another sequence $(y_1, ..., y_m)$ is given by a matrix having entries $m_{ij} \in \operatorname{Hom}_{\mathcal{C}}(x_i, y_j)$.

$$(x_1, ..., x_n) \xrightarrow{\begin{pmatrix} m_{11} & m_{12} & ... & m_{1n} \\ m_{21} & m_{22} & ... & m_{2n} \\ ... & ... & ... & ... \\ m_{m1} & m_{12} & ... & m_{mn} \end{pmatrix}} (y_1, ..., y_m).$$

The direct sum of two objects is defined as the concatenation of sequences.

There is a canonical inclusion functor $\iota_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mat}\mathcal{C}$, sending a morphism f to a 1×1 matrix.

$$\begin{aligned} \mathcal{C} &\to \operatorname{Mat}\mathcal{C}, \\ x &\mapsto (x), \\ f &\mapsto (f). \end{aligned}$$

The matrix category satisfies the universal property of the additivisation of k-linear categories. Namely, let $\mathcal{D}$ be any additive k-linear category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a k-linear functor. Then there exists an additive k-linear functor $F'\colon \operatorname{Mat}\mathcal{C} \to \mathcal{D}$ factoring F as $F' \circ \iota_{\mathcal{C}}$, which is unique up to natural equivalence:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle \iota_{\mathcal{C}}}\downarrow & \nearrow_{F'} & \\ \operatorname{Mat}\mathcal{C} & & \end{array}$$

Moreover, every object in $\operatorname{Mod}\mathcal{C}$ extends to an object in $\operatorname{Mod}\operatorname{Mat}\mathcal{C}$ by applying the functor element-wise, and a morphism in $\operatorname{Mod}\mathcal{C}$ extends to a morphism in $\operatorname{Mod}\operatorname{Mat}\mathcal{C}$ uniquely up to isomorphism. Therefore, the induced functor

$$\begin{aligned} \operatorname{Add}(\operatorname{Mat}\mathcal{C}, \mathrm{Ab}) &\to \operatorname{Mod}\mathcal{C} \\ F &\mapsto F \circ \iota_{\mathcal{C}} \end{aligned}$$

is an equivalence of k-categories.

The next lemma is a preadditive analogue of a well-known result from Morita theory.

Lemma 2.8.3. *Let $\mathcal{C}$ be a small k-category. Then the categories* $\operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$ *and* $\operatorname{Mod}\operatorname{proj}\mathcal{C}$ *are equivalent.*

Proof. We provide a proof which emphasises the connection between the objects of $\mathcal{C}$ and the finitely generated projectives in $\operatorname{Mod}\mathcal{C}$: Consider the functor $F = h_{-}\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{proj}\mathcal{C}$, which is the contravariant Yoneda embedding. By the properties discussed above, the functor h_{-} factors through a functor $F'\colon \operatorname{Mat}(\mathcal{C}^{\mathrm{op}}) \to \operatorname{proj}\mathcal{C}$. This F' maps every sequence $(x_1, ..., x_n)$ to the direct sum $\bigoplus_{i=1}^{n} h_{x_i}$ and every matrix $(m_{ij})_{ij}$, representing a morphism from $(x_1, ..., x_n)$ to $(y_1, ..., y_m)$, to a morphism represented by the matrix $(h_{m_{ij}})_{ji}$:

$$\bigoplus_{j=1}^{m} h_{y_j} \xrightarrow{\begin{pmatrix} h_{m_{11}} & h_{m_{21}} & \cdots & h_{m_{n1}} \\ h_{m_{12}} & h_{m_{22}} & \cdots & h_{m_{n2}} \\ \cdots & \cdots & \cdots & \cdots \\ h_{m_{1m}} & h_{m_{2m}} & \cdots & h_{m_{nm}} \end{pmatrix}} \bigoplus_{i=1}^{n} h_{x_i}$$

Applying the Yoneda Lemma on the components of the matrix reveals that the functor F' is fully faithful and thus we can treat $\operatorname{Mat}(\mathcal{C}^{\mathrm{op}})$ as a full subcategory of $\operatorname{proj}\mathcal{C}$.

So, we obtain the following sequence of functors:

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{\iota_{\mathcal{C}^{\mathrm{op}}}} \operatorname{Mat}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{F'} \operatorname{proj}\mathcal{C}$$

Now, the first functor $\iota_{\mathcal{C}^{\mathrm{op}}}$ induces an equivalence of the module categories $\operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$ and $\operatorname{Mod}(\operatorname{Mat}(\mathcal{C}^{\mathrm{op}}))$ as discussed previously to this lemma.

The functor F' identifies $\mathrm{Mat}\,(\mathcal{C}^{\mathrm{op}})$ with the full subcategory of $\mathrm{proj}\,\mathcal{C}$ consisting of the finite direct sums of representable functors. Now, since every finitely generated projective is a direct summand of a free object, every functor in $\mathrm{Mod}\,\mathrm{proj}\,\mathcal{C}$ is completely determined by its restriction to the finitely generated free objects, since Ab is idempotent complete. Similarly, every morphism in $\mathrm{Mod}\,\mathrm{Mat}\,\mathcal{C}^{\mathrm{op}}$ extends to a morphism in $\mathrm{Mod}\,\mathrm{proj}\,\mathcal{C}$ uniquely up to isomorphism. Again, it follows that F' induces an equivalence of categories of additive functors $\mathrm{Mod}\,(\mathrm{Mat}\,(\mathcal{C}^{\mathrm{op}}))$ and $\mathrm{Mod}\,(\mathrm{proj}\,\mathcal{C})$, see discussion after [Mit72, Lemma 1.1].

Taken all together, we obtain an equivalence $\mathrm{Mod}\,(\mathcal{C}^{\mathrm{op}}) \cong \mathrm{Mod}\,(\mathrm{proj}\,\mathcal{C})$. □

Since $\mathrm{Mod}\,(\mathcal{C}^{\mathrm{op}})$ is a Grothendieck category with a set of projective generators given by the objects of $\mathrm{proj}\,(\mathcal{C}^{\mathrm{op}})$, the assertion also follows from Theorem 2.4.1 using the following.

Lemma 2.8.4. *Let $\mathcal{C}$ be additive. Then there is an equivalence*

$$\mathrm{proj}\,(\mathcal{C}^{\mathrm{op}}) \cong (\mathrm{proj}\,\mathcal{C})^{\mathrm{op}}\,.$$

Proof. The category $\mathrm{proj}\,\mathcal{C}$ is exactly the idempotent completion of the image of $\mathcal{C}^{\mathrm{op}}$ under the contravariant Yoneda embedding h_{-}. Similarly, $\mathrm{proj}\,(\mathcal{C}^{\mathrm{op}})$ is exactly the idempotent completion of the image of $\mathcal{C}$ under the covariant Yoneda embedding h^{-}. The opposite of an idempotent complete additive category is idempotent complete additive, again. But the idempotent closure is universal with this property, so $(\mathrm{proj}\,(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$ must be a full subcategory of $\mathrm{proj}\,\mathcal{C}$, and vice versa. □

Chapter 3

Categories of persistence modules

This chapter deals with categories of persistence modules. First, the notions of persistence modules and interval modules are defined in Section 3.1, whereafter the structure of projective and injective objects is delved into in Section 3.2 for the general and the pointwise finite-dimensional case. In Section 3.3 and Section 3.4 we briefly review results on the homological dimension and the decomposition theory of persistence modules. This chapter ends with a concise introduction to Kan extensions by the example of persistence modules in Section 3.5, as it is used frequently in Chapter 5.

3.1 Representations of posets

Let $(P, \leq)$ be any partially ordered set. This intrinsically carries the structure of a small category: The objects are the elements of P and there is a unique morphisms between two elements $x, y \in P$ for every relation $x \leq y$. We identify P with its category. Further, let k be a field and let kP denote the k-linearisation of $(P, \leq)$. It is then clear that for p, q in P the k-dimension of $\mathrm{Hom}_{kP}(p, q)$ is at most 1.

Consider the category $\mathrm{Mod}\, kP$ of representations of kP as in Section 2.1. Then a representation of M is equivalent to a family of vector spaces indexed by P, and vector space homomorphism between them indexed by the relation $\leq$. In the context of persistent homology such representations M are called *persistence module* and the morphisms assigned to the relations are called the *structure maps of* M.

An *interval* I in P is a subset which is convex and connected. Note that a subset of a poset is *convex* if for all $x, y, z \in P$ with $x \leq y \leq z$ follows that $y \in I$ if $x, z \in I$. The set I is *connected* if every two points in P are connected by a (finite) zig zag. This means that for all $x, y \in I$ there are elements $x = x_0, x_1, ..., x_n = y$, such that $x \leq x_1 \geq x_2 \leq x_3 ... \geq y$.

If $P = T$ is a totally ordered set, we use the familiar notation for intervals: For $a < b \in T$ the interval $(-\infty, b]$ (or $(-\infty, b)$) denotes the subset of elements which are (strictly) smaller than b. The intervals $[a, \infty)$ and (a, ∞) are defined dually. We define the intervals (a, b), $[a, b)$ $(a, b]$ and $[a, b]$ by taking the intersections $(a, \infty) \cap (-\infty, b)$, $[a, \infty) \cap (-\infty, b)$ and so on.

Let $J \subseteq P$ be an interval and $\mathcal{A}$ an abelian category. For an object M in $\mathcal{A}$ we define $\Delta_J^P M$ to be the the functor in $\mathrm{Fun}(P, \mathcal{A})$ which constantly takes the value M on J and 0 otherwise, and which has the structure maps $\Delta_J^P M(x \leq y) = \mathrm{id}_M$, whenever $x \leq y$ in J, and zero otherwise. If $\mathcal{A}$ is the category of k-modules for a commutative ring k, we also write k_J for $\Delta_J^P k$ as object in $\mathrm{Mod}\, kP$. This is called the *interval representation* of J.

3.2 The structure of injective and projective objects

Let k be commutative and $\mathcal{C}$ a small k-category. In the following we analyse several properties of injective, projective and flat objects in $\mathrm{Mod}\,\mathcal{C}$ for any such $\mathcal{C}$ in general and for partially ordered sets in particular.

3.2.1 Projective objects

As discussed earlier, the representable functors h_x for x in $\mathcal{C}$ form a set of projective generators of the module category $\mathrm{Mod}\,\mathcal{C}$. By general theory, a projective object of $\mathrm{Mod}\,\mathcal{C}$ is always a direct summand of a coproduct of representable functors. An object is called *free* if it is a coproduct of representable objects. Moreover, if $\mathcal{C}$ has finite coproducts, then every finitely generated projective is a summand of a representable functor. Furthermore, given that $\mathcal{C}$ has split idempotents, that is every idempotent e in an endomorphism ring $\mathrm{End}_{\mathcal{C}}(c)$ has a kernel and gives rise to a direct sum decomposition $c \cong \mathrm{Ker}\, e \oplus \mathrm{Ker}(1-e)$, then every finitely generated projective is representable. Recall that by Lemma 2.1.1 there is a ring isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}\,\mathcal{C}}\,(h_c, h_c) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}\,(c, c)^{\mathrm{op}} = \mathrm{End}_{\mathcal{C}}(c)^{\mathrm{op}}\,.$$

Since $\mathrm{Mod}\,\mathcal{C}$ is abelian, every non-trivial idempotent of $\mathrm{End}_{\mathcal{C}}(c)$ yields a non-trivial direct sum decomposition of h_c. Therefore, h_c is indecomposable if c is.

But how about projectives which are not necessarily finitely generated? Generalising Kaplansky's Theorem [Kap58], this was answered in a special case for $\mathrm{Mod}\, kP$ if k is a division ring, see [Mit78, Corollary 9.2], and further generalised by Höppner and Lenzing.

Theorem 3.2.2 ([HL81, Proposition 5]). *Let P be a poset and $\mathcal{A}$ an abelian category with a set of projective generators and all coproducts. Then every* *projective object M in* $\operatorname{Fun}(P,\mathcal{A})$ *is free, that is*

$$M \cong \bigoplus_{i\in I} S_{c_i}(P_i),$$

where S_{c_i} denotes the left adjoint of the evaluation functor $\mathrm{ev}_{c_i}\colon \operatorname{Fun}(P,\mathcal{A}) \to \mathcal{A}$ *at c_i, or the left Kan extension of P_i along the embedding of c_i into P, for an index set I and objects c_i in P and projective objects P_i in $\mathcal{A}$.* □

Remark 3.2.3. Note that in this notation we are not immediately considering a module category in the sense of Section 2.4. Therefore, the meaning of freeness is a little bit subtle and to recognise the summands in the above decomposition as representable it might be necessary to consider the following: By Theorem 2.4.1 we have an equivalence $\mathcal{A} \cong \operatorname{Mod}((\operatorname{proj}\mathcal{A})^{\mathrm{op}})$, hence

$$\operatorname{Fun}(P,\mathcal{A}) \cong \operatorname{Fun}(P, \operatorname{Mod}((\operatorname{proj}\mathcal{A})^{\mathrm{op}})) \cong \operatorname{Add}(\mathbb{Z}\,P \otimes_{\mathbb{Z}} (\operatorname{proj}\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}),$$

by the tensor calculus of preadditive categories, see [Mit72, Section 2]. Now, a representable functor in the latter category is a functor of the shape

$$\operatorname{Hom}_{\mathbb{Z}\,P}(c,-) \otimes \operatorname{Hom}_{\operatorname{proj}\mathcal{A}}(-,Q),$$

where c is in P and Q is an object in $\operatorname{proj}\mathcal{A}$. Under the isomorphism above, this corresponds to $S_c(Q)$.

3.2.4 Injective objects

Now, we take a closer look at the injective objects. Dually to what we discussed in Section 2.4.4, the representable functor $\operatorname{Hom}_{\mathcal{C}}(-,c)$ is injective in $(\operatorname{Mod}\mathcal{C})^{\mathrm{op}}$. If we could take recourse to a duality functor $D\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$, see Section 3.2.10, then this section could be completely symmetric to the section on projectives. But this is not the case, as exposed there.

Nonetheless, we can use another symmetry, namely between restrictions and extensions, see Section 3.5.7. In particular, for a small k-category $\mathcal{C}$ the functor $c\colon * \to \mathcal{C}$ that chooses a single point c in $\mathcal{C}$ induces an exact restriction $c^*\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}k$ which has a right adjoint Ran_c, dually to Section 3.2.1. Because right adjoints of exact functors preserve injective objects, we see that $T_cQ \coloneqq \mathrm{Ran}_c Q$ is an injective object in $\operatorname{Mod}\mathcal{C}$ if Q is an injective k-module. If $\mathcal{C} = kP$ for a poset P, then T_cQ is exactly the representation that is the functor which is constantly Q on the lower-or-equal-set (or ideal) generated by c, and zero otherwise.

Though the injective objects have no such classification like Theorem 3.2.2 for the projectives, it is still possible to describe some of their properties. The next result by Höppner is of important role in the analysis of the spectrum of a totally ordered set, see Chapter 4.

Theorem 3.2.5 ([Höp83, Proposition 1.1])**.** *Let P be a partially ordered set, R any (not necessarily commutative) ring and consider the functor category* $\operatorname{Fun}(P, \operatorname{Mod}R)$*. Then for any F in* $\operatorname{Fun}(P,\operatorname{Mod}R)$*, the following are equivalent:*

(1) F is injective and indecomposable.

(2) $F \cong \Delta_J^P Q$ for an injective and indecomposable R-module Q and $J \subseteq P$ a subset which is a filtered ideal, that is a subset closed under smaller elements.

Proof. We only show that (2) implies (1). Let ι denote the inclusion $J \hookrightarrow P$ and consider the diagram of functors

$$\operatorname{Mod} k \underset{\varinjlim_J}{\overset{\Delta_J}{\rightleftarrows}} \operatorname{Mod} kJ \underset{\iota^*}{\overset{\operatorname{Ran}_\iota}{\rightleftarrows}} \operatorname{Mod} kP$$

By definition of colimit and right Kan extension there are adjoint pairs $\varinjlim_J \dashv \Delta_J$ and $\iota^* \dashv \operatorname{Ran}_\iota$, and so is

$$\varinjlim_J \circ \iota^* \dashv \operatorname{Ran}_\iota \circ \Delta_J.$$

But the left term equals $\varinjlim_J$ and the right term equals Δ_J^P. As the first functor is exact and the second its right adjoint, it follows that Δ_J^P preserves injective objects. The indecomposability is immediate. □

One might be tempted to believe that there is a dual of Theorem 3.2.2, asserting that for any injective object M in $\operatorname{Mod} kP$ there is a product decomposition

$$M \cong \prod_i T_{c_i} Q_i$$

for i in an indexing set I, objects c_i in $\mathcal{C}$ and injective k-modules Q_i. But this is not true in general, consider for example the set of natural number $\mathbb{N}$:

Example 3.2.6 ([Höp83, Counterexampel 2.1]). Let Q be a non-trivial injective. Then the injective hull $E = E\left(\bigoplus_{n\in\mathbb{N}} T_n Q\right)$ is not of this form: Suppose

$$E = \Delta_{\mathbb{N}} M \oplus \prod_{n\in\mathbb{N}} T_n M_n$$

for injective k-modules M and M_n for all $n \in \mathbb{N}$. We can assume $E \subseteq \prod_{n\in\mathbb{N}} T_n Q$, so $Q = \operatorname{Ker}(E(n) \to E(n+1)) \cong M_n$ for all $n \in \mathbb{N}$. Now, each non-torsion elements of the product, for example the diagonal element $(x)_{n\in\mathbb{N}}$ for any nonzero element $x \in Q$, generates a proper injective subobject of the product which does not intersect with $\bigoplus_{n\in\mathbb{N}} T_n Q$, so it splits and E cannot have the desired shape.

But it was shown that there is in fact such a decomposition if and only if we restrict to representations of posets which are Noetherian [Höp81, 3.2.2 Theorem].

3.2.7 Structure maps of projectives and injectives over posets

Let P be a poset and let k be a commutative ring. Following [Höp81, 2.2.3 Satz] we are examining the structure maps of the projectives in $\operatorname{Mod} kP$. For convenience of the reader, we deliver a self-contained proof.

Theorem 3.2.8 (Höppner)**.** *Let $i \in P$ and let J be a subset of P which is closed under smaller elements and bounded above by i. If M is a projective object of $\mathcal{A} = \operatorname{Mod} kP$, then the following holds:*

(1) $M(i)$ is projective.

(2) the canonical morphism

$$\varphi\colon \varinjlim_{j \in J} M(j) \to M(i)$$

is a split monomorphism.

Dually, if M is an injective object and J is a subset of P which is closed under larger elements and which is bounded below by i, then

(1) $M(i)$ is injective.

(2) the canonical morphism

$$\psi\colon M(i) \to \varprojlim_{j \in J} M(j)$$

is a split epimorphism.

Proof. For the first claim, note that the right adjoint T_i of the evaluation ev_i is exact, since this holds pointwise, and therefore ev_i preserves projectives.

For the second part, note that these adjunctions have the following unit and counit:

$$\begin{aligned} \eta\colon 1_{\operatorname{Mod} kP} &\to T_i \circ \operatorname{ev}_i, \\ \varepsilon\colon \operatorname{ev}_i \circ T_i &\to 1_{\operatorname{Mod} k}. \end{aligned}$$

Here the unit is given pointwise by the canonical morphism $T_i(M(i)) \to N$, and the counit just acts as identity on an abelian group N.

Then note that J-colimits are left adjoint to the diagonal functor Δ_J^P, so there is another pair of unit and counit

$$\begin{aligned} \eta'\colon 1_{\operatorname{Mod} kP} &\to \Delta_J^P \circ \varinjlim_{j \in J}, \\ \varepsilon'\colon \varinjlim_{j \in J} \circ \Delta_J^P &\to 1_{\operatorname{Mod} k}, \end{aligned}$$

where the counit is determined by the canonical maps $M(j') \to \varinjlim_{j \in J} J(j)$ into the filtered colimit and the unit acts as the identity.

Let N be any k-module. For the notation $i\downarrow = \{x \in J \mid x \le i\}$, we consider the short exact sequence

$$0 \to \Delta^P_{i\downarrow \setminus J} N \to T_i N \to \Delta^P_J N \to 0,$$

where $i\downarrow \setminus J$ denotes the difference of sets, and apply $\mathrm{Hom}_{\mathcal{A}}(M, -)$ to yield an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}\left(M, T_i N\right) \xrightarrow{f} \mathrm{Hom}_{\mathcal{A}}\left(M, \Delta^P_J N\right) \to \mathrm{Ext}^1_{\mathcal{A}}\left(M, \Delta^P_{i\downarrow \setminus J} N\right),$$

where the last term vanishes if M is projective.

Now, observe that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}\left(M, T_i N\right) & \xrightarrow{\quad f \quad} & \mathrm{Hom}_{\mathcal{A}}\left(M, \Delta^P_J N\right) \\ {\scriptstyle T_i(-)\circ \eta_M}\uparrow & & \uparrow{\scriptstyle \Delta^P_J(-)\circ \eta'_M} \\ \mathrm{Hom}_k\left(M(i), N\right) & \xrightarrow{\ \varphi^* \ } & \mathrm{Hom}_k\left(\varinjlim_{j \in J} M(j), N\right), \end{array}$$

where the vertical maps are the isomorphisms from the adjunction. This can be checked element-wise and by using the universal property of the colimit. Thus, φ^* is an epimorphism for every k-module N and therefore φ is a split monomorphism.

The other statement follows dually. ☐

From this and the basic properties of direct and inverse limits it follows immediately:

Corollary 3.2.9. *Let M be a projective (injective) object in $\mathrm{Mod}\, kP$. Then all structure maps $M(i) \to M(j)$ are monomorphisms (epimorphisms) of projective (injective) k-modules.*

3.2.10 Pointwise finite-dimensional objects

Let k be a field and P a partially ordered set. Here, we consider the full subcategory of pointwise finite-dimensional modules $\mathrm{Mod}^{\mathrm{pfd}}\, kP$, that are these objects M such that $\dim_k M(p) < \infty$ for all p in P, and explore the symmetry between projective and injective objects.

For finite-dimensional vectors spaces there is a duality

$$\mathrm{Hom}_k(-, k)\colon\ \mathrm{mod}\, k \to \mathrm{mod}\, k,$$

which extends to a duality of persistence modules.

$$\begin{aligned} D\colon\ \mathrm{Mod}^{\mathrm{pfd}}\, kP &\to \mathrm{Mod}^{\mathrm{pfd}}\, kP^{\mathrm{op}}, \\ M &\mapsto \left(\mathrm{Hom}_k\left(M(p), k\right)\right)_{p \in P}, \end{aligned}$$

where P^{op} denotes the same set P subject to the opposite order. It is then clear that D maps all injective objects to projective objects and vice versa. Therefore, we have a symmetry between these notions. But restricting to this subcategory may also yield a set of new projective objects which are not projective in the surrounding category $\operatorname{Mod} kP$. For example, consider $P = \mathbb{R}$.

Then the indicator module $k_{(-\infty,a)}$ is an injective representation of $\mathbb{R}$ for every a in $\mathbb{R}$, confer Corollary 4.4.3. The dual of this is isomorphic to the representation $k_{(a,\infty)}$, which is not projective, see also [IRT23, Proposition 2.5.3]:

Lemma 3.2.11. *The representation $k_{(a,\infty)}$ is not projective in $\operatorname{Mod} kP$.*

Proof. Let a_i be a real strictly monotonous sequence converging against a from above. Then we can write $k_{(a,\infty)} \cong \varinjlim k_{[a_i,\infty)}$ and there is a canonical epimorphism

$$\pi\colon M = \bigoplus_{i\in\mathbb{N}} k_{[a_i,\infty)} \to k_{(a,\infty)}.$$

If we assume projectivity, then there is a section of π, so there is a nonzero monomorphism

$$f\colon k_{(a,\infty)} \to \bigoplus k_{[a_i,\infty)}.$$

Evaluated at a_i and a_{i-1} we get

$$f(a_i)(1_k) = (r_0, r_1, ..., r_i, 0, ...),$$
$$f(a_{i-1})(1_k) = (s_0, s_1, ..., s_{i-1}, 0, ...),$$

where $r_j, s_j \in k_{[a_j,\infty)}$. Since f is a morphism, for all $i \in \mathbb{N}$ we have

$$f(a_{i-1}) \circ k_{(a,\infty)}(a_i \leq a_{i-1}) = M(a_i \leq a_{i-1}) \circ f(a_i).$$

But this implies $r_i = 0$ for all i, as all structure maps of M are injective, so $f = 0$. □

Alternatively, we can see this from Theorem 3.2.2.

We make use of another property of pointwise finite-dimensional representations of posets.

Lemma 3.2.12 ([BCB20, Lemma 2.3])**.** *Suppose P is a filtered and cofiltered partially ordered set. Let M be a pointwise finite-dimensional k-representation of P, such that $M(p) \neq 0$ for all $p \in P$ and such that all structure maps are epimorphisms. Then M has a direct summand of type k_P.*

Proof. Let $p \in P$ be such that M_p has maximal dimension. Then $M(x \leq p)$ is an isomorphism for all $x \leq p$. Now we define a subrepresentation N of M as follows: For any $q \in P$ there is some $c \in P$ such that $c \leq p, q$. Then we set $N_q \coloneqq M(c \leq q) \circ M(c \leq p)^{-1}(m_p)$. As P is codirected, any two choices for c have a minimum and therefore N is well defined. All structure morphisms of N are isomorphisms and it is one-dimensional everywhere, so $N \cong k_P$. Since P is also directed it follows from the proof of Theorem 3.2.5 that $N \cong k_P$ is injective. □

3.3 Homological dimension

In this section we go over some known results on the homological dimension of persistence modules. In particular we observe that for a field k the global dimension of $\operatorname{Mod} k\mathbb{R}$ is 2 and we suggest a technique to estimate the global dim for more complicated posets.

Let $\mathcal{A}$ be an abelian category. Recall that the projective dimension of an object M in $\mathcal{A}$ is the supremum of the set of integers n such that $\operatorname{Ext}^n_{\mathcal{A}}(M, -) \neq 0$. For the definition of $\operatorname{Ext}_{\mathcal{A}}$ in terms of abelian categories, see [ML63, §3.5] or [Wei94, Vista 3.4.6]. Moreover, the *global dimension* of an abelian category $\mathcal{A}$ is defined as the supremum of the set of projective dimensions of every object. The global dimension has a huge impact on the homological properties of the category.

In view of categories of additive functors, an important question to ask is, how the homological algebra of the functor category is determined by the domain and codomain of the functors. In particular, if the functors are equivalent to representations of a diagram category, one would like to understand the impact of the geometry of the diagram on the global dimension of the functor category. One approach is to study the Hochschild–Mitchell dimension:

Definition 3.3.1. Let $\mathcal{C}$ be a small k-category and let $\mathcal{C}^e = \mathcal{C}^{\mathrm{op}} \otimes_k \mathcal{C}$ denote the *enveloping category.* Then the *Hochschild–Mitchell dimension* of $\mathcal{C}$ is defined as

$$\dim_k \mathcal{C} = \sup \left\{ n \geq 0 \mid \operatorname{Ext}^n_{\mathcal{C}^e} \left(\operatorname{Hom}_{\mathcal{C}}, - \right) \neq 0 \right\},$$

where the k-linear bifunctor $\operatorname{Hom}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \otimes_k \mathcal{C} \to \mathrm{Ab}$ is considered as an object of $\operatorname{Mod} \mathcal{C}^e$.

Then there is the following relation between this dimension and the global dimension of the module category of $\mathcal{C}$.

Theorem 3.3.2 ([Mit72, Corollary 13.4']). *Let $\mathcal{A}$ be a Grothendieck k-category and let $\mathcal{C}$ be a small k-category. Then for every M in* $\operatorname{Fun}(\mathcal{C}, \mathcal{A})$ *we have*

$$\operatorname{proj.dim}_{\mathcal{A}^{\mathcal{C}}} M \leq \dim_k \mathcal{C} + \sup_{p \in |\,\mathcal{C}\,|} \operatorname{proj.dim}_{\mathcal{A}} M(c),$$

and therefore we get

$$\operatorname{gl.dim} \mathcal{A}^{\mathcal{C}} \leq \dim_k \mathcal{C} + \operatorname{gl.dim} \mathcal{A}\,.$$

□

In some cases one has an upper bound for the Hochschild–Mitchell dimension.

Theorem 3.3.3 ([Mit72, Corollary 37.6]). *Let P be a poset of cardinality $\aleph_n$, $-1 \leq n \leq \infty$. Then*

$$\dim_{\mathbb{Z}} P \leq n + 1 + \delta(P),$$

where $\delta(P)$ is a non-negative integer. It assumes the value zero if and only if P is discrete, and it is smaller or equal one if and only if it does not contain the commutative square $[2]^2$ as subposet. Moreover, it is smaller or equal to two if and only if P does not contain a suspended crown *as subset.* □

For more details see [Mit72, Theorem 35.7]. In particular $\mathbb{R}$ is not a free category of a graph; neither is $\mathbb{N}^2$. If P is totally ordered, it then immediately follows:

$$\dim_{\mathbb{Z}} P \leq n + 2.$$

In his previous work Mitchell already studied inequations similar to Theorem 3.3.2 for the special case of finite posets.

Theorem 3.3.4 ([Mit68, Section 4])**.** *Let $\mathcal{A}$ be a non-trivial abelian category and let P be a finite poset. Then we have the equation*

$$\operatorname{gl.dim} \mathcal{A}^P = n + \operatorname{gl.dim} \mathcal{A}$$

- *with $n = 0$ if and only if no points in P are comparable.*
- *with $n = 1$ if and only if there are at least two comparable points, but P has no commutative square $[2]^2$ as a full subset.*
- *with $n = 2$ if and only if it contains a full subset $[2]^2$ but no* uncrossed, suspended crown *(for example a cube $[2]^3$).* □

These results can also be used to provide lower bounds for the global dimensions of infinite posets, making use of the following well-known result on the dimension for quotient categories.

Proposition 3.3.5 ([Gab62, 3.3, Corollaire 5])**.** *Let $\mathcal{C}$ be a localising subcategory of a Grothendieck category $\mathcal{G}$. Then for the global dimension of the quotient $\mathcal{G} / \mathcal{C}$ there is the upper bound*

$$\operatorname{gl.dim} \mathcal{G} / \mathcal{C} \leq \operatorname{gl.dim} \mathcal{G}\,.$$

□

Using both Theorem 3.3.4 and Proposition 3.3.5 we see that if we can restrict $\operatorname{Fun}(P, \mathcal{A})$ along an embedding $P' \to P$ for $P = [2], [2]^2$ or $[2]^3$, the global dimension of the module category increases at least by one, two or three, compared with the category $\mathcal{A}$. We also know the following about a change of the underlying commutative ring.

Proposition 3.3.6 ([Mit72, Corollary 14.3])**.** *Suppose L is a commutative k-algebra. Then for every small L-category we have*

$$\dim_L \leq \dim_k .$$

In particular, we always have

$$\dim_k \leq \dim_{\mathbb{Z}}$$

for every commutative ring k. □

But what is the global dimension of $\operatorname{Mod} k\mathbb{R}$? Assume the *continuum hypothesis* holds, that is that the cardinality of the real numbers is the first uncountable cardinal number, so

$$|\,\mathbb{R}\,| = \aleph_1.$$

Then Theorem 3.3.3 and Proposition 3.3.6 yield for the Hochschild–Mitchell dimension:

$$\dim_k \mathbb{R} \leq \dim_{\mathbb{Z}} \mathbb{R} \leq 1+1+1=3.$$

Husainov proved equality:

Theorem 3.3.7 ([Hus95])**.** *Independent of the continuum hypothesis we have*

$$\dim_{\mathbb{Z}} \mathbb{R} = 3.$$

□

If we combine this with the subadditivity Theorem 3.3.2, we obtain

$$\operatorname{gl.dim} \operatorname{Mod} k\mathbb{R} \leq 3+1=4.$$

On the other side, consider the representation $k_{(0,\infty)}$ in $\operatorname{Mod} k\mathbb{R}$. This clearly is a subobject of the representable functor $k_{[0,\infty)}$, but it is not projective, see Lemma 3.2.11. Therefore $\operatorname{Ext}^2_{\operatorname{Mod} k\mathbb{R}}$ cannot vanish and the global dimension of $\operatorname{Mod} k\mathbb{R}$ must be at least 2. So

$$2 \leq \operatorname{gl.dim} \operatorname{Mod} k\mathbb{R} \leq 4.$$

Brune proved a better upper bound for the dimension, which does not require the continuum-hypothesis. Recall for this that a subset J of a poset P is coinitial if for every x in P there is an element y in J with $y \leq x$.

Theorem 3.3.8 ([Bru78, Satz 6])**.** *Let T be totally ordered and let k be a commutative Noetherian ring. Suppose that every bounded below subset of T has a coinitial subset of cardinality smaller or equal $\aleph_n$. Then for the global dimension there is an upper bound:*

$$\operatorname{gl.dim} \operatorname{Mod} kT \leq 2+n+\operatorname{gl.dim} k.$$

□

The set of natural numbers $\mathbb{N}$ is a coinitial subset of $\mathbb{R}$. Thus, we conclude that for a field k we have:

$$\operatorname{gl.dim} \operatorname{Mod} k\mathbb{R} = 2.$$

3.4 Decomposition of Persistence Modules

Let k be a field and let $\mathcal{C}$ be a small k-category. Recall that an object M of $\operatorname{Mod}\mathcal{C}$ is called *pointwise finite-dimensional* if for every object c of $\mathcal{C}$ we have $\dim M(c) < \infty$. We collect several decomposition results.

The following theorem by Botnan–Crawley-Boevey [BCB20, Theorem 1.1] is a reformulation of a result of Crawley-Boevey [CB94, §3.5, Theorem 2], which generalises work of Zimmermann-Huisgen [ZH79]; see also [JL89, Theorem 8.1] and [GR97, § 3.6].

Recall that a commutative ring is *local* if it satisfies $1 \neq 0$ and if the sum of two non-units is a non-unit.

Theorem 3.4.1. *Any pointwise finite-dimensional object in* $\operatorname{Mod}\mathcal{C}$ *is isomorphic to a direct sum of indecomposable objects with local endomorphism ring.* □

This decomposition is unique up to isomorphisms and reordering of the summands by the Krull–Remak–Schmidt–Azumaya theorem [Azu50]. So, there is a good decomposition theory whenever a reasonable finiteness condition applies. However, note that in general there is no hope to parametrise representations of an arbitrary partially ordered set: the problem is wild, that is not classifiable even for finitely generated representations.

A classical result of Gabriel characterises those small k-categories $\mathcal{C}$ which are k-categories freely generated by a finite directed graph and which have a finite list of indecomposable pointwise finite-dimensional representations, see [Gab72]. Such categories $\mathcal{C}$ are called *of finite representation type.*

Theorem 3.4.2 (Gabriel)**.** *Let k be a field and $\mathcal{C}$ the free k-category generated by a directed connected graph. Then the category* $\operatorname{Mod}\mathcal{C}$ *has finitely many pointwise finite-dimensional indecomposable objects if and only if the underlying graph is of Dynkin type A_n, D_n, E_6, E_7 or E_8.* □

But there are also categories $\mathcal{C}$ which are not of finite representation type but still have a similarly useful decomposition theory in the sense that all indecomposable object can be parametrised. One class of such is given by totally ordered sets:

Theorem 3.4.3 ([BCB20, Theorem 1.2])**.** *Let k be a field and T a totally ordered set. Then every pointwise finite-dimensional object in* $\operatorname{Mod} kT$ *is isomorphic to a direct sum of interval representations. This decomposition is unique up to isomorphism.* □

Remark 3.4.4. This is the most general version of the famous *barcode theorem*, so far. As further generalisation to specific two-parameter representations can be seen the decomposition Theorem 5.2.4 for middle exact modules by Botnan and Crawley-Boevey.

Unlike many other results cited here, the barcode theorem does not readily generalise to partially ordered sets. The quiver D_4 for example already has an indecomposable representation which is not an interval module. The requirement of pointwise finite dimension can also not be lifted, because there are modules which are not pointwise finite-dimensional and not interval decomposable. See the following example, which is inspired by [Höp81, 3.2.1. Beispiele].

Example 3.4.5. Let $M = \prod_{i\in\mathbb{N}_0} k_{[-i,\infty)}$ in $\operatorname{Mod} k\,\mathbb{R}$. This representation is not decomposable into interval modules.

Proof. Assume there is a direct sum of interval modules $S = \bigoplus_{j\in J} k_{I_j}^{(\nu_j)}$ with cardinals ν_j such that there is an isomorphism $\varphi\colon S \to M$.

The structure maps of M are all injective and they are the identity on the positive real numbers and on every interval $[-i, -i+1)$ for $i \in \mathbb{N}$. Therefore, the intervals I_j must all be of the shape $I_j = [j, \infty)$ for $j \in \mathbb{Z}_{\leq 0}$ or $I_\infty = \mathbb{R}$. Because $\operatorname{Coker} M(-i \leq -i+1) \cong k$ for all $i \in \mathbb{N}$, we get $\nu_j = 1$ for all $j \in \mathbb{Z}_{\leq 0}$ by the isomorphism. So we get

$$S \cong \bigoplus_{i\in\mathbb{N}_0} k_{[-i,\infty)} \oplus k_{\mathbb{R}}^{(\nu_\infty)}.$$

It follows that ν_∞ must equal $2^{\aleph_0}$, because $\dim M_0 = 2^{\aleph_0}$, while the left summand has only point wise countable dimension at 0. In particular, the right summand is nonempty.

Now choose any nonzero $x \in k_{\mathbb{R}}^{(\nu_\infty)}(r)$ for an $r \in \mathbb{R}$. Since all structure maps of this summand are identities, for every $s \in \mathbb{R}$ with $s < r$ there is a unique lift $x_s \in k_{\mathbb{R}}^{(\nu_\infty)}(s)$ along the structure map: $k_{\mathbb{R}}^{(\nu_\infty)}(s \leq r)(x_s) = x$. We fix a non positive integer $s < r$. Let $y = \varphi_r^{-1}(x)$ and $y_s = \varphi_s^{-1}(x_s)$.

By definition of M, the projection on the pth factor $\operatorname{pr}_p(M(t)) = 0$ vanishes for all $p > t \in \mathbb{Z}_{\leq 0}$. So, if $p > s$ is a non positive integer, then $\operatorname{pr}_p(y_s) = 0$. By construction we have $\operatorname{pr}_t(M(s \leq r)(y_s)) = \operatorname{pr}_t(y)$ for all $t \in \mathbb{Z}_{\leq 0}$, therefore $\operatorname{pr}_t(y) = 0$ for all $t \in \mathbb{Z}_{\leq 0}$ with $t > s$. But since $s \in \mathbb{Z}_{\leq 0}$ can be chosen arbitrarily smaller than r, this yields $\operatorname{pr}_i(y) = 0$ for all $i \in \mathbb{Z}_{\leq 0}$ and therefore $y = 0$ and thus $x = 0$, leading to a contradiction. □

Next, we would like to mention another family of small categories $\mathcal{C}$ of finite representation type. Let $\mathcal{A}_n$ be a category generated by a graph of Dynkin type A_n and let $\mathcal{A}_m$ be generated by a graph of Dynkin type A_m. The following was shown by Escolar and Hiraoka, based on [LS00, Theorem 4]:

Theorem 3.4.6 ([EH16, Theorems 4, 5])**.** *The category $\mathcal{A}_n \times \mathcal{A}_m$ is of finite representation type if and only if $1 \leq n \leq 4$ and $1 \leq m \leq 2$, or reversely.*

3.5 Extension of the parameter set

In this section we go through the concept of Kan extensions in view of the restriction and extension of parameter set for persistence modules. The theory is illustrated on the comparison of a finite totally ordered set with the set of reals $\mathbb{R}$.

3.5.1 Kan extensions

We review several properties of Kan extensions which we use, following [ML98, Section X.3]; for the analogue statements for k-categories and other enrichments, see: [Bor94b, Section 6.7].

Kan extensions are universal solutions to the following extension problems: Let $F\colon \mathcal{A} \to \mathcal{C}$ and $j\colon \mathcal{A} \to \mathcal{B}$ be functors of categories. Can we extend F to a functor $F'\colon \mathcal{B} \to \mathcal{C}$? There are two canonical answers to this, which are dual to each other:

Definition 3.5.2. The *left Kan extension of F along j* is a functor $\mathrm{Lan}_j F$ such that there is a natural isomorphism

$$\Phi\colon \operatorname{Nat}(\mathrm{Lan}_j F, -) \to \operatorname{Nat}(F, j^*(-)) .$$

The *right Kan extension of F along j* is defined dually.

For categories with (co)limits we can explicitly describe Kan extensions:

Lemma 3.5.3 ([ML98, Ch. X.3, Theorem 1])**.** *Let $F\colon \mathcal{A} \to \mathcal{C}$ and $j\colon \mathcal{A} \to \mathcal{B}$ be functors for a small category $\mathcal{A}$. Moreover assume that $\mathcal{C}$ has all colimits. Then the left Kan extension $\mathrm{Lan}_j F$ of F along j exists and is given by the pointwise colimits*

$$(\mathrm{Lan}_j F)(b) = \operatorname*{colim}_{ja \to b} F(a).$$

Moreover, if $\mathcal{C}$ has all limits, then the right Kan extension $\mathrm{Ran}_F X$ of X along F exits and is given by the pointwise limit

$$(\mathrm{Ran}_j F)(b) = \lim_{b \to ja} F(a).$$

□

Applying the previous result can be simplified if we consider cofinal subcategories. Note that a subset $J \subseteq P$ of a poset P is cofinal if for each $p \in P$ there is a $q \in J$ with $p \leq q$.

Theorem 3.5.4 ([ML98, IX.3, Theorem 1])**.** *Let $J' \subseteq J$ be a cofinal subcategory of a small category and $F\colon J \to X$ a functor such that* $\operatorname{colim} FL$ *exists. Then* $\operatorname{colim} F$ *exists and the canonical morphism*

$$\operatorname{colim} FL \to \operatorname{colim} F$$

is an isomorphism. □

Lemma 3.5.5 ([ML98, X.3, Corollary 3])**.** *Suppose j is fully faithful. Then the unit $\eta_F = \Phi\left(1_{\mathrm{Lan}_j F}\right) : 1_F \to \mathrm{Lan}_j F \circ j$ is a natural isomorphism. Dually, the counit $\varepsilon_F\colon \mathrm{Ran}_j F \circ j \to F$ is an isomorphism.* □

Lemma 3.5.6. *Let $j\colon \mathcal{C} \hookrightarrow \mathcal{D}$ be a functor of small categories and let $\mathcal{A}$ be a a complete category. Then the right Kan extension* Ran_j *defines a functor* $\mathrm{Fun}(\mathcal{C},\mathcal{A}) \to \mathrm{Fun}(\mathcal{D},\mathcal{A})$ *which is right adjoint to j^*. If j is fully faithful, then* Ran_j *is fully faithful.*

Proof. The right Kan extension exists by Lemma 3.5.3 for each $F \in \mathrm{Fun}(\mathcal{C},\mathcal{A})$. The natural isomorphism of right Kan extensions is equivalent to an universal arrow $(\mathrm{Ran}_j F, \varepsilon_F)$, with $\varepsilon_F\colon j^*\mathrm{Ran}_\iota F \to F$, from j^* to F.

Since this holds for any F, by general nonsense, we obtain a right adjoint Ran_j of j^*, which assigns to each object $F \in \mathrm{Fun}(\mathcal{C},\mathcal{A})$ the object $\mathrm{Ran}_\iota F$, see [ML98, IV.1 Theorem 2]. For the action on morphisms consider a natural transformation $f\colon F \to G$ for $F, G \in \mathrm{Fun}(\mathcal{C},\mathcal{A})$. Both functors have universal morphisms $(\varepsilon_F, \mathrm{Ran}_j F)$ from j^* to F and $(\varepsilon_G, \mathrm{Ran}_j G)$ from j^* to G, and by universality there is a unique morphisms $\mathrm{Ran}_j(f)$ in $\mathrm{Fun}(\mathcal{D},\mathcal{A})$ such that the following diagram commutes:

$$\begin{array}{ccc} j^*\mathrm{Ran}_j F & \xrightarrow{\varepsilon_F} & F \\ {\scriptstyle j^*(\mathrm{Ran}_j f)}\downarrow & & \downarrow{\scriptstyle f} \\ j^*\mathrm{Ran}_j G & \xrightarrow{\varepsilon_G} & G \end{array}$$

If j is fully faithful, then the counit ε_F is an isomorphism by Lemma 3.5.5 and so the first map is an isomorphism as well. ☐

The analogue lemma for left Kan extensions is dual.

3.5.7 Change of categories

To compare the representations of a small category with those of a full subcategory, we consider the *change of categories-functor*. This way we can directly relate the respective functor categories.

Suppose we have small preadditive categories $\mathcal{C}$ and $\mathcal{D}$ and a fully faithful functor $j\colon \mathcal{C} \hookrightarrow \mathcal{D}$. This induces a canonical functor of module categories $j^*\colon \mathrm{Mod}\,\mathcal{D} \to \mathrm{Mod}\,\mathcal{C}$, the restriction functor, where $j^*(F) = F \circ j$ for an object F in $\mathrm{Mod}\,\mathcal{D}$.

Note that j^* has left and right adjoints Lan_j and Ran_j by Lemma 3.5.6, and therefore preserves all limits and colimits. In particular it is exact, and we obtain the following diagram:

$$\mathrm{Mod}\,\mathcal{D} \quad \begin{array}{c} \xleftarrow{\mathrm{Lan}_j} \\ \xrightarrow{j^*} \\ \xleftarrow{\mathrm{Ran}_j} \end{array} \quad \mathrm{Mod}\,\mathcal{C}$$

By Lemma 3.5.6, Ran_ι is fully faithful (by the dual of the Lemma, Lan_ι is, either). Now consider the full subcategory $\mathrm{Ker}\, j^*$ of $\mathrm{Mod}\,\mathcal{D}$. This is a Serre subcategory, as j^* is exact. It follows from [Gab62, III.2 Proposition 5] that the functor j^* identifies the quotient $\mathrm{Mod}\,\mathcal{D}\,/\,\mathrm{Ker}\, j^*$ with $\mathrm{Mod}\,\mathcal{C}$. Thus, $\mathrm{Ker}\, j^*$

is a localising subcategory and therefore a Grothendieck category [Gab62, III.4, Proposition 9]. So, we can extend this diagram to:

$$\operatorname{Ker} j^* \xrightarrow{i_*} \operatorname{Mod} \mathcal{D} \begin{array}{c} \xleftarrow{\operatorname{Lan}_j} \\ \xrightarrow{j^*} \\ \xleftarrow{\operatorname{Ran}_j} \end{array} \operatorname{Mod} \mathcal{C}$$

Now, by [Kra22, Lemma 2.2.10] and its dual, the functor i_* admits a left adjoint i^* and a right adjoint $i^!$, and the right adjoint of j^* identifies $\operatorname{Mod} \mathcal{C}$ with $(\operatorname{Ker} j^*)^{\perp}$. So there is even a recollement:

$$\operatorname{Ker} j^* \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \operatorname{Mod} \mathcal{D} \begin{array}{c} \xleftarrow{j_! = \operatorname{Lan}_j} \\ \xrightarrow{j^*} \\ \xleftarrow{j_* = \operatorname{Ran}_j} \end{array} \operatorname{Mod} \mathcal{C}$$

The left term $\operatorname{Ker} j^*$ consists of exactly these functors in $\operatorname{Mod} \mathcal{D}$ which vanish when restricted to $\mathcal{C}$. This category might be a module category itself, depending on the choice of $\mathcal{D}$ and $\mathcal{C}$, but this is not clear a priori.

3.5.8 Discrete versus continuous parameter sets

Next we want to compare k-linear representations of a finite totally ordered set of n points with the representations of $\mathbb{R}$ in the fashion of the previous discussion. So, let k be a field (or commutative ring), and set $\mathcal{C} = kA_n$ and $\mathcal{D} = k\,\mathbb{R}$, where by A_n we mean a linearly oriented quiver of Dynkin type A_n.

For an embedding $A_n \to \mathbb{R}$ we fix an arbitrary choice of strictly ascending elements $a_1 < ... < a_n$ and define a functor $j \colon A_n \to \mathbb{R}, i \mapsto a_i$. This is fully faithful. Note that j has left and right adjoint functors which assign to each real number its respective supremum or infimum in the image of j.

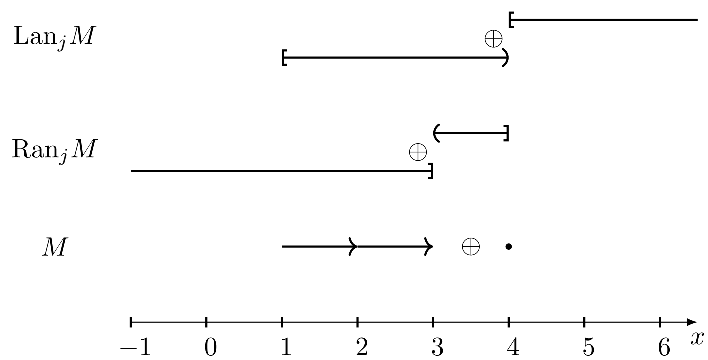
Now, we can specify the recollement from the previous Section 3.5.7 for this example: For the restriction functor j^* there is nothing more to say. The objects of $\operatorname{Ker} j^*$ are exactly those functors which vanish on $j(A_n)$. Thus, every such object decomposes into a direct sum of representations which are supported in between the points of $j(A_n)$. So we can also describe them as representations of the disjoint union of real intervals, whereby we mean the coproduct of small categories:

$$\mathcal{K} = \coprod_{i=1}^{n} (a_i, a_{i+1}) \cong \coprod_{i=1}^{n} \mathbb{R}\,.$$

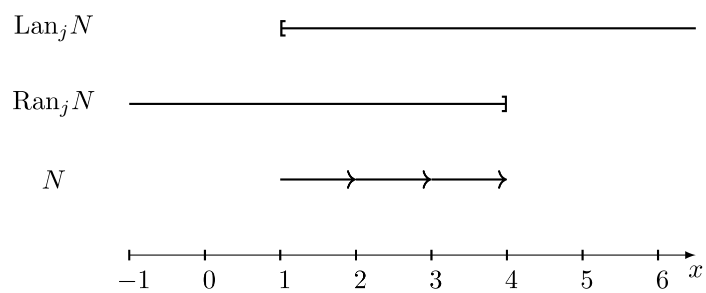
Here, we use the notation $a_0 = -\infty$ and $a_{n+1} = \infty$. This reveals the formula

$$\operatorname{Ker} j^* \cong \operatorname{Mod} k\,\mathcal{K}\,.$$

Now we take a look at the adjoints of j^*. For an impression of the difference of the right and left adjoint or Kan extension in this setting, we consider the example $A_n = \{1, 2, ..., n\} \to \mathbb{R}$ defined by the mapping $i \mapsto i$.



(a) Kan extensions of M.



(b) Kan extensions of N

Figure 3.1: Left- and right Kan extensions of the representations M and N of A_4 to the real line in interval notation. The arrows denote that the intervals of M and N are not continuous but discrete.

Example 3.5.9. Suppose $n = 4$ and consider the representation $M = k_{[1,3]} \oplus k_{\{4\}}$ in $\mathrm{Mod}\, kA_n$. We go through the construction of both left and right Kan extensions of M to the real line, see Figure 3.1. By Lemma 3.5.3 the functor $\mathrm{Ran}_\iota M$ is obtained from extending the value at i constantly to the interval $[i, i+1)$ for $1 \leq i \leq n-1$, and from n to $[n, \infty)$. Moreover, it is zero before 1. Dually, the extension $\mathrm{Lan}_\iota M$ is obtained from extending the value at i constantly to the interval $[i, i+1)$ for $1 \leq i \leq n-1$, and from 1 to $(-\infty, 1]$. It is zero everywhere after n. All together, we get

$$\mathrm{Ran}_j M \cong k_{(-\infty,3]} \oplus k_{(3,4]} \quad \text{and} \quad \mathrm{Lan}_j M \cong k_{[1,4)} \oplus k_{[4,\infty)}.$$

Now consider the representation $N = k_{[1,4]}$. Then we get

$$\mathrm{Ran}_j N \cong k_{(-\infty,4]} \quad \text{and} \quad \mathrm{Lan}_j N \cong k_{[1,\infty)},$$

which are an injective and a projective module.

Chapter 4

The spectrum of persistence modules

In this chapter we investigate the spectrum $\operatorname{Sp}\operatorname{Mod} kT$ of the category of k-linear representations of a totally ordered set T for any field k. Our main result Theorem 4.7.2 simplifies the problem to determine the topology of the spectrum of this category to the calculation of the order topology of a totally ordered set.

After a short overview on notions of spectra in algebra in Section 4.1, we discuss the definition of the spectrum for locally coherent Grothendieck categories and its uses in Section 4.2. In Sections 4.3 to 4.5 we classify the indecomposable injectives and how they are related with order ideals, the finitely presented objects and the relation between both types of representations. Moreover we distinguish different types of ideals and differentiate their properties. This distinction is used in the characterisation of closed sets of the spectrum in Section 4.6, where we proof the central Proposition 4.6.8.

Now, with the characterisation of the topology available we prove the main Theorem 4.7.2 and use it to investigate some basic properties of the space in Section 4.7. In Section 4.8 we return to the setting of Section 3.5.8 and examine the relations between the spectra for a finite totally ordered set and for $\mathbb{R}$.

In Section 4.9 we study the big category $\operatorname{Mod} kT$, in view of the structure of its injective objects, whereas the Krull–Gabriel filtration of this category and the Cantor–Bendixson filtration of its spectrum are compared in Section 4.10. Finally, in Section 4.11 we compare the topology of the spectrum with the topology coming from the interleaving distance on persistence modules. It turns out that the Ziegler topology on the spectrum refines the topology induced by the interleaving distance if one distinguished point is removed.

4.1 Spectra in algebra

First, we review several of the related notions of spectra. As invariants they help to classify or distinguish objects in various fields. In algebra the most prominent example of a spectrum is the prime spectrum $\operatorname{Spec} A$ of a commutative ring A. Since studying rings themselves can become complicated quickly, the spectrum helps to understand their structure better. Particularly, its topology, the Zariski topology, is an essential tool to understand the module category of a ring.

In this context, Matlis made the observation in his thesis [Mat58] that for a given commutative Noetherian rings the prime spectrum $\operatorname{Spec} A$ is in bijection with the isomorphism classes of indecomposable injective modules, where E denotes the injective envelope operator:

$$\begin{aligned} \operatorname{Spec} A &\to \{\text{indecomposable injective } A\text{-modules}\} \\ I &\mapsto E(A/I) \end{aligned}$$

Gabriel generalised Matlis' results in his thesis [Gab62], where he defined the *spectrum* of a Grothendieck category $\mathcal{A}$ to be the set $\operatorname{Sp} \mathcal{A}$ of isomorphism classes of indecomposable injective objects, but without defining a topology, yet.

Later, Ziegler introduced the *spectrum of pure-injective modules* in his model theoretic paper [Zie84], where he introduced a topology on this set. For a ring R he defined the spectrum to be the set of isomorphism classes of indecomposable pure-injective modules, which is now known as the *Ziegler spectrum*.

There is a bijection between isomorphism classes of indecomposable pure-injective objects in $\operatorname{Mod} R$ for a ring R and isomorphism classes of indecomposable injective objects in $\operatorname{Mod}(\operatorname{mod}(R^{\mathrm{op}}))$, the generalised modules of the category of finitely presented left R-modules, see [GJ73]. The topology on $\operatorname{Sp}\operatorname{Mod}(\operatorname{mod}(R^{\mathrm{op}}))$ is then the topology induced by this bijection and a basis is given by the sets

$$\{E \in \operatorname{Sp}\operatorname{Mod}(\operatorname{mod}(R^{\mathrm{op}})) : \operatorname{Hom}(C, E) \neq 0\}$$

for all finitely presented objects C in $\operatorname{Mod}(\operatorname{mod}(R^{\mathrm{op}}))$.

Spectra and duality

Note that there is an extension of Matlis' observation to the topology of the spectrum: The Zariski spectrum $\operatorname{Spec} A$ is a closed subset of the Ziegler spectrum of indecomposable pure-injectives, see [Kra22, Corollary 12.4.18], and therefore $\operatorname{Spec} A$ also has a Ziegler topology. As $\operatorname{Spec} A$ is a *Stone space* we can construct its *Hochster dual* as the topological space which has as basis of closed sets the compact open sets in the Zariski topology. Then, as discussed in [Pre93], the spectrum $\operatorname{Sp}\operatorname{Mod} A$ with the Ziegler topology is the Hochster dual of the Zariski topology.

4.2 The spectrum of a category

Next we discuss the generalisation of the spectrum to locally coherent categories introduced by [Her97] and [Kra97], independently. Note that there is also a generalisation of the Ziegler spectrum of pure injectives, which is related via an embedding into functor categories, see [CB94].

4.2.1 The spectrum of a locally coherent category

Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. It is known that the isomorphism classes of indecomposable objects of $\mathcal{A}$ form a set, as every indecomposable injective is isomorphic to the injective envelope of a quotient of a generator of $\mathcal{A}$, and because for Grothendieck categories the class of subobjects forms a set. To simplify notation we define the *spectrum* $\operatorname{Sp}\mathcal{A}$ of $\mathcal{A}$ to be any set of representatives for every isomorphism class of indecomposable injectives in $\mathcal{A}$.

If $\mathcal{A}$ is locally coherent, meaning that the category $\operatorname{fp}\mathcal{A}$ of finitely presented objects of $\mathcal{A}$ admits kernels (and therefore is abelian), the *Ziegler topology* on $\operatorname{Sp}\mathcal{A}$ is given by the Kuratowski closure operator $\mathcal{U} \mapsto \overline{(\mathcal{U})} := \Upsilon \circ \Sigma(\mathcal{U})$, where

$$\Upsilon(\mathcal{C}) = \{X \in \operatorname{Sp}\mathcal{A} \mid \operatorname{Hom}_{\mathcal{C}}(C, X) = 0 \text{ for all } C \in \mathcal{C}\} \text{ and}$$
$$\Sigma(\mathcal{U}) = \{C \in \operatorname{fp}\mathcal{A} \mid \operatorname{Hom}_{\mathcal{C}}(C, X) = 0 \text{ for all } X \in \mathcal{U}\}$$

for all subsets of objects $\mathcal{C} \subseteq \operatorname{fp}\mathcal{A}$ and $\mathcal{U} \subseteq \operatorname{Sp}\mathcal{A}$. See e.g. [Kra22, Lemma 12.1.12] for this. Note that the bijection extends to another incarnation of *Stone duality*, as the lattice of Serre subcategories forms a *frame*. For the details, see [Joh82].

In the following, we fix the notation $\operatorname{Sp} T := \operatorname{Sp}\operatorname{Mod} kT$ for the spectrum.

4.2.2 Correspondence theorem

Next, we briefly discuss what structures the spectrum of a locally coherent category encodes and how they are related. For this we follow [Kra97].

First, note that a localising subcategory $\mathcal{C} \subseteq \mathcal{A}$ of a Grothendieck category gives rise to a hereditary torsion pair $(\mathcal{C}, \{X \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(\mathcal{C}, X) = 0\})$, and conversely, every torsion class of a hereditary torsion pair of finite type is a localising subcategory of finite type.

Now, let $\mathcal{S}$ be a Serre subcategory of the category $\operatorname{fp}\mathcal{A}$ of finitely presented objects of $\mathcal{A}$. Moreover, let $\overrightarrow{\mathcal{S}}$ denote the closure of $\mathcal{S}$ under all filtered colimits in $\mathcal{A}$. Then $\overrightarrow{\mathcal{S}}$ is a localising subcategory of finite type of $\mathcal{A}$. Conversely, given a localising subcategory $\mathcal{C}$ we obtain a Serre subcategory of $\operatorname{fp}\mathcal{A}$ by taking the restriction $\mathcal{S} = \mathcal{C} \cap \operatorname{fp}\mathcal{A}$ and so we have $\overrightarrow{\mathcal{S}} \cong \mathcal{C}$.

The last necessary observations for the correspondence theorem are the following: The set $\Sigma_{\mathcal{A}}(\mathcal{S})$ is a closed set of $\operatorname{Sp}\mathcal{A}$, and for every closed set $V \subseteq \operatorname{Sp}\mathcal{A}$ the category $\mathcal{Y}(V)$ is a Serre subcategory of $\operatorname{fp}\mathcal{A}$.

All in all we end up with the following theorem.

Theorem 4.2.3 (Herzog, Krause)**.** *Let $\mathcal{A}$ be a locally coherent Grothendieck category. Then the Ziegler closed subsets of* $\operatorname{Sp}\mathcal{A}$ *are in order reversing bijection to the:*

- *Serre subcategories of* $\operatorname{mod}\mathcal{A}$*,*
- *Localising subcategories of finite type of $\mathcal{A}$,*
- *Hereditary torsion pairs of finite type of $\mathcal{A}$.* □

4.2.4 The spectrum and extensions

In this section we scrutinise how the spectra behave under a change of categories, filling in more details into the original presentations. So assume the setting of Section 3.5.7.

Suppose $\operatorname{Mod}\mathcal{D}$ and $\operatorname{Mod}\mathcal{C}$ are locally coherent. Moreover, assume that $j\colon \mathcal{C} \to \mathcal{D}$ has a left adjoint. Then j^* preserves finitely presented objects: Consider a finitely presented functor $F\colon \mathcal{D} \to \mathrm{Ab}$. This means that there is a presentation

$$\operatorname{Hom}_{\mathcal{D}}(Y,-) \to \operatorname{Hom}_{\mathcal{D}}(Y,-) \to F \to 0.$$

Applying j^* yields an exact sequence

$$\operatorname{Hom}_{\mathcal{D}}(Y,j(-)) \to \operatorname{Hom}_{\mathcal{D}}(Y,j(-)) \to j^*F \to 0,$$

and by adjointness the first two terms are isomorphic to representable functors in $\operatorname{Mod}\mathcal{C}$.

Thus, we have $\operatorname{Ker} j^* \cong \overrightarrow{\mathcal{S}}$ for $\mathcal{S} = \operatorname{Ker} j^* \cap \operatorname{mod}\mathcal{D}$ and that this is a localising subcategory of finite type, by [Kra97, Corollary 2.9]. Therefore it is also locally coherent by [Kra97, Theorem 2.6], so the spectrum is well-defined as topological space for all terms of the recollement in Section 3.5.7. This naturally raises the question how the spectra of the respective terms are related with each other.

Set-wise, this was already answered by Gabriel [Gab62, III.3, Corollaire 2]: There is a bijection of sets, expressing the spectrum of $\operatorname{Mod}\mathcal{D}$ as disjoint union:

$$\operatorname{Sp}\operatorname{Mod}\mathcal{D} \cong \operatorname{Sp}\operatorname{Ker} j^* \sqcup \operatorname{Sp}\operatorname{Mod}\mathcal{C} \tag{4.1}$$

Here, the inclusion $\operatorname{Sp}\operatorname{Ker} j^* \hookrightarrow \operatorname{Sp}\operatorname{Mod}\mathcal{D}$ is induced by the map which sends every element to the injective hull in $\operatorname{Mod}\mathcal{D}$. The other inclusion

$$\operatorname{Sp}\operatorname{Mod}\mathcal{C} \cong \operatorname{Sp}(\operatorname{Mod}\mathcal{D} / \operatorname{Ker} j^*) \hookrightarrow \operatorname{Sp}\operatorname{Mod}\mathcal{D}$$

is induced by $j_* = \operatorname{Ran}_j$. Since this is the right adjoint of the exact functor j^*, it preserves injective objects .

As discussed above, the functor Ran_j induces the equivalence

$$\operatorname{Mod}\mathcal{C} \to (\operatorname{Ker} j^*)^{\perp}.$$

Then we have $\mathcal{S}^{\perp} = \operatorname{Sp}(\operatorname{Ker} j^*)^{\complement} = \operatorname{Sp}\operatorname{Mod}\mathcal{C}$, so $\operatorname{Sp}\operatorname{Ker} j^*$ is an open subset of $\operatorname{Sp}\operatorname{Mod}\mathcal{D}$ and therefore $\operatorname{Sp}\operatorname{Mod}\mathcal{C}$ is closed. By $A^{\complement}$, for a subset $A \subseteq X$, we denote the set complement.

Moreover, both terms in (4.1) have the subspace topology with respect to $\operatorname{Sp}\operatorname{Mod}\mathcal{D}$, see [Kra97, Corollary 4.4]:

First, let $U \subseteq \operatorname{Sp}\operatorname{Ker} j^*$ be open. By Theorem 4.2.3 this means that we have $U = (\Sigma_{\operatorname{Sp}\operatorname{Ker} j^*}\mathcal{S})^{\complement}$ for a Serre subcategory $\mathcal{S} \subseteq \operatorname{Ker} j^* \cap \operatorname{mod}\mathcal{D}$. Now, note that a Serre subcategory of a Serre subcategory of an abelian category $\mathcal{A}$ is a Serre subcategory of $\mathcal{A}$, again. Also, we have $\operatorname{Hom}_{\operatorname{Mod}\mathcal{D}}(M,X) = 0$ for all $M \in \mathcal{S}$ and $X \in (\operatorname{Sp}\operatorname{Ker} j^*)^{\complement}$, therefore $U = (\Sigma_{\operatorname{Mod}\mathcal{D}}\mathcal{S})^{\complement}$, so U is open in $\operatorname{Sp}\operatorname{Mod}\mathcal{D}$.

Second, let $V \subset \mathrm{Sp}\,\mathrm{Mod}\,\mathcal{D} \,/\, \mathrm{Ker}\, j^*$. Then $\mathrm{Ran}_j V \subset \mathrm{Sp}\,(\mathrm{Ker}\, j^*)^{\perp}$ is closed: The set V is closed if and only if $V = \Sigma_{\mathrm{Sp}(\mathrm{Ker}\, j^*)^{\perp}}\, \mathcal{S}$ for a Serre subcategory $\mathcal{S} \subseteq (\mathrm{fp}\,(\mathrm{Mod}\,\mathcal{D} \,/\, \mathrm{Ker}\, j^*))$. By [Kra97, Theorem 2.6], the latter identifies with the quotient $\mathrm{mod}\,\mathcal{D} \,/\, (\mathrm{Ker}\, j^* \cap \mathrm{mod}\,\mathcal{D})$. Also, note that every Serre subcategory of this quotient comes from a Serre subcategory $\tilde{\mathcal{S}}$ of $\mathrm{mod}\,\mathcal{D}$ containing $\mathrm{Ker}\, j^* \cap \mathrm{mod}\,\mathcal{D}$, that is $j^* \tilde{\mathcal{S}} \cong \mathcal{S}$ [Kra22, Proposition 2.2.8]. So, we get

$$\begin{aligned}
\mathrm{Ran}_j V &= \{\mathrm{Ran}_j X \mid X \in \mathrm{Sp}\,\mathrm{Mod}\,\mathcal{D} \,/\, \mathrm{Ker}\, j^*, \\
&\qquad \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{D} \,/\, \mathrm{Ker}\, j^*}(M, X) = 0 \text{ for all } M \in \mathcal{S}\Big\} \\
&= \{\mathrm{Ran}_j X \mid X \in \mathrm{Sp}\,\mathrm{Mod}\,\mathcal{D} \,/\, \mathrm{Ker}\, j^*, \\
&\qquad \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{D}}(\tilde{M}, \mathrm{Ran}_j X) = 0 \text{ for all } \tilde{M} \in \tilde{\mathcal{S}}\Big\} \\
&= \Big\{X \in \mathrm{Sp}\,(\mathrm{Ker}\, j^*)^{\perp} \mid \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{D}}(\tilde{M}, X) = 0 \ \forall \tilde{M} \in \tilde{\mathcal{S}}\Big\} \\
&= \Big\{X \in \mathrm{Sp}\,\mathrm{Mod}\,\mathcal{D} \mid \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{D}}(\tilde{M}, X) = 0 \ \forall \tilde{M} \in \tilde{\mathcal{S}}\Big\} \\
&= \Sigma_{\mathrm{Mod}\,\mathcal{D}} \tilde{\mathcal{S}}
\end{aligned}$$

for a Serre subcategory $\tilde{\mathcal{S}}$ with $\mathrm{Ker}\, j^* \cap \mathrm{mod}\,\mathcal{D} \subseteq \tilde{\mathcal{S}} \subseteq \mathrm{Ker}\, j^*$.

Nonetheless, the bijection in (4.1) yields only a partition of $\mathrm{Sp}\,\mathrm{Mod}\,\mathcal{D}$ into an open and a closed set, from which it is a priori not possible to reconstruct the topology on $\mathrm{Sp}\,\mathrm{Mod}\,\mathcal{D}$.

4.3 Finitely presented objects in $\mathrm{Mod}\, kT$

Now, we proof the technical Lemma 4.3.2, asserting that the subcategory $\mathrm{mod}\, kT$ of finitely presented objects of $\mathrm{Mod}\, kT$ is abelian. This is necessary for the spectrum of $\mathrm{Mod}\, kT$ to be well-defined. Then we classify all finitely presented objects. We start showing that the finitely generated projectives are closed under finitely generated subobjects. This is does not hold for arbitrary small categories: In fact, representations of finite quivers with non-trivial admissible relations do not have this property if they are not representation equivalent to a finite connected acyclic quiver, see for example [ASS06, VII 1.7 Theorem].

Lemma 4.3.1. *Every finitely generated submodule of a finitely generated projective module over kT is projective.*

Proof. Let P be finitely generated projective and $M \subseteq P$ a finitely generated submodule, having generators $x_1, ..., x_n$. From this we can find a linearly independent generating set of M in the following way:

If the given set is not linearly independent, we can express the zero element in $M(p)$ for a $p \in P$ as a non-trivial linear combination

$$0 = \sum_{i=1}^{n} \mu_i \cdot M(|x_i| \leq p)(x_i)$$

with μ_i in k. Among all indices with nonzero coefficient μ_i, choose an index m, with $|x_m|$ maximal. Then, since all structure maps are injective by Theorem 3.2.8, the equation above lifts uniquely to

$$-\mu_m x_m = \sum_{i \neq m} \mu_i \cdot M(|x_i| \leq |x_m|)(x_i),$$

and thus

$$x_m = \frac{-1}{\mu_m} \sum_{i \neq m} \mu_i \cdot M(|x_i| \leq |x_m|)(x_i).$$

So we see that the generator x_m can be expressed in terms of the other generators, and therefore can be deleted. This procedure must terminate after finitely many steps. Because the structure maps of M are all pointwise injective, the same holds for all submodules of M. Thus, $\langle x_j \rangle \cong h_{|x_j|}$ for $1 \leq j \leq n$, so M is a finite direct sum of representable modules. □

Now we show that $\operatorname{mod} kT$ is abelian.

Lemma 4.3.2. *The category* $\operatorname{mod} kT$ *of finitely presented objects is abelian.*

Proof. It only remains to prove the existence of kernels. For an additive category $\mathcal{C}$ the full subcategory of finitely presented objects $\operatorname{mod} \mathcal{C}$ of $\operatorname{Mod} \mathcal{C}$ is always closed under cokernels. Moreover, it is abelian if and only if all morphisms in $\mathcal{C}$ have weak cokernels, see [Kra97, Proposition 2.12].

Using Theorem 2.4.1 yields $\operatorname{Mod} kT \cong \operatorname{Mod}((\operatorname{proj} kT)^{\mathrm{op}})$. So it is enough to show that $\operatorname{proj} kT$ has weak kernels. In fact, $\operatorname{proj} kT$ has proper kernels: For every morphism φ of finitely generated projectives, the image $\operatorname{Im} \varphi$, taken in the surrounding abelian category $\operatorname{Mod} kT$, is certainly finitely generated and a subrepresentation of a finitely generated projective. By Lemma 4.3.1 it is projective. Now, the image morphism $\operatorname{im} \varphi$ is an epimorphism and the domain of φ decomposes into the direct sum $\operatorname{Ker} \varphi \oplus \operatorname{Im} \varphi$. Both summands are in $\operatorname{proj} kT$, as this category is closed under direct summands. □

Remark 4.3.3. This can also be proven without Theorem 2.4.1: by applying Lemma 2.8.3 to $\mathcal{C} = (kT)^{\mathrm{op}}$, we obtain an equivalence of $\operatorname{Mod} kT$ and $\operatorname{Mod} \operatorname{proj}((kT)^{\mathrm{op}})$. So we only need to check that $\operatorname{proj}((kT)^{\mathrm{op}})$ has weak kernels. The opposite category of a k-linearisation of a totally ordered set is just the k-linearisation of the totally ordered set T^{op}, which is obtained from T by reverting all relations. Now one can proceed as in the foregoing proof.

Explicit description of $\operatorname{mod} kT$

Next, we explicitly describe the objects in $\operatorname{mod} kT$.

Lemma 4.3.4. *Every finitely presented indecomposable object in* $\operatorname{Mod} kT$ *is isomorphic to an interval representation* $k_{[a,b)}$ *for* $a, b \in T \cup \{\infty\}$ *with* $a < b$.

Proof. Let M be a finitely presented indecomposable representation of kT. Then M is pointwise finite-dimensional and thus isomorphic to an interval module k_J for an interval $J \subseteq T$, by Theorem 3.4.3. For an indecomposable finitely generated representation of kT one can find a single generator: Take a finite presentation

$$\bigoplus_j h_{b_j} \to \bigoplus_i h_{a_i} \xrightarrow{\pi} M \to 0$$

for a b_j and a_i in T. Then the generators of the representable modules h_{a_i} are mapped to elements x_i in M. But since M is pointwise at most one-dimensional, they are all contained in the subrepresentation of M generated by one generator with minimal support, say it has index l. So, the restriction $\pi_{|_{h_{a_l}}}$ is surjective and M must be a quotient of h_{a_l}, so an interval module for an interval $[a_l, b)$ or $[a_l, b]$ for a b in T. The kernel of π is finitely generated and therefore projective by Lemma 4.3.1. So it must be a direct sum of representable objects and we can assume the projective presentation to be short exact. Comparing the pointwise dimensions of the terms, it becomes apparent that M must be of the form $k_{[a_l,b)}$ for $b \in T \cup \{\infty\}$ and $a < b$. □

So, we see that

$$\operatorname{mod} kT = \operatorname{add}\left(k_{[a,b)} \mid a, \in T, b \in T \cup \{\infty\}\right),$$

where add denotes the closure under finite direct sums.

4.4 The indecomposable injectives

In this section we first analyse the indecomposable injective objects of $\operatorname{Mod} kT$ and therefore the underlying set of the spectrum $\operatorname{Sp} T$. As an important step on the way to determining the spectrum, we then investigate the morphisms from finitely presented objects.

4.4.1 Indecomposable injective objects

To parametrise the indecomposable injectives we use the notion of ideals:

Definition 4.4.2. Let T be a totally ordered set. A nonempty subset $I \subseteq T$ is called *ideal* if it is closed under smaller elements, meaning that whenever $x \leq y$ and $y \in I$, then also $x \in I$. Dually, a nonempty subset $F \subseteq T$ is called a *filter* if it is closed under all greater elements. The set of all ideals of T is denoted by $\operatorname{Idl} T$.

Let further a be an element of T. The *principal ideal generated by* a, denoted $a\downarrow$, is the smallest ideal containing a. Dually, the *principal filter* $a\uparrow$ is the smallest filter containing a. We further write $\widehat{a}\downarrow$ for the principal ideal generated by a with its generator removed: $a\downarrow \setminus \{a\}$.

The set of ideals $\operatorname{Idl} T$ is a partially ordered set with respect to inclusion. In fact, $\operatorname{Idl} T$ is totally ordered by inclusion, as every two non-empty ideals intersect non trivially, and then one already contains the other.

Note that in this notation we have that the relation $\widehat{x}{\downarrow} \subsetneq I$ for $x \in T$ and an ideal $I \subseteq T$ is equivalent to $x{\downarrow} \subseteq I$, because $x{\downarrow}$ is the next greater element after $\widehat{x}{\downarrow}$. Similarly, $I \subsetneq x{\downarrow}$ is equivalent to $I \subseteq \widehat{x}{\downarrow}$.

Next we classify all indecomposable injective in terms of ideals. From Theorem 3.2.5 follows immediately

Corollary 4.4.3. *The indecomposable injective objects of* $\operatorname{Mod} kT$ *are exactly those representations which are isomorphic to interval representations* k_I *for an ideal* I *of* T. □

This implies that every isomorphism class of indecomposable injectives in $\operatorname{Mod} kT$ is uniquely determined by its support, so the support map restricts to the bijection

$$\operatorname{supp}\colon \operatorname{Sp} T \to \operatorname{Idl} T.$$

There is the canonical choice for representatives of $\operatorname{Sp} T$ consisting of the interval representations k_I for all $I \in \operatorname{Idl} T$, so we can explicitly describe the inverse Δ of supp by $\Delta(I) = k_I$. The set $\operatorname{Idl} T$ can be considered as a topological space with the topology being induced by the bijection supp, since working with ideals instead of modules can simplify several proofs. In this case we also speak of the Ziegler topology on $\operatorname{Idl} T$.

Example 4.4.4. In the totally ordered set $\mathbb{R}$ the ideals are exactly the right closed or right open half-lines, that is the intervals $x{\downarrow} = (-\infty, x]$, $\widehat{x}{\downarrow} = (-\infty, x)$ for $x \in \mathbb{R}$. Then the canonical choice for the spectrum is:

$$\operatorname{Sp} \mathbb{R} = \left\{ k_{(-\infty,x)} \mid x \in \mathbb{R} \cup \{\infty\} \right\} \cup \left\{ k_{(-\infty,y]} \mid y \in \mathbb{R} \right\}.$$

The inclusion of $\operatorname{Idl} \mathbb{R}$ into the power set $\mathcal{P}(\mathbb{R})$ equips $\operatorname{Idl} \mathbb{R}$ with a total order. Furthermore, we define the double line with infinity, see Figure 4.1:

$$D_{\mathbb{R}}^{\infty} = \mathbb{R} \times \{0, 1\} \cup \{(\infty, 0)\}.$$

Then there is a bijection $\operatorname{Idl} \mathbb{R} \to D_{\mathbb{R}}^{\infty}$,

$$\begin{aligned} \operatorname{Idl} \mathbb{R} &\longrightarrow D_{\mathbb{R}}^{\infty} \\ (-\infty, x) &\longmapsto (x, 0) \\ (-\infty, y] &\longmapsto (y, 1), \end{aligned}$$

with $x \in \mathbb{R} \cup \{\infty\}$ and $y \in \mathbb{R}$.

On $D_{\mathbb{R}}^{\infty}$ we have the lexicographic order, that is the total order induced by the underlying total order of $\mathbb{R} \cup \{\infty\}$ and $\{0 < 1\}$. This means we have $(x, i) < (y, j)$ whenever $x < y$ and $(x, i) < (x, j)$ whenever $i = 0$ and $j = 1$. It is clear that the map above identifying $\operatorname{Idl} \mathbb{R}$ and $D_{\mathbb{R}}^{\infty}$ preserves the order.

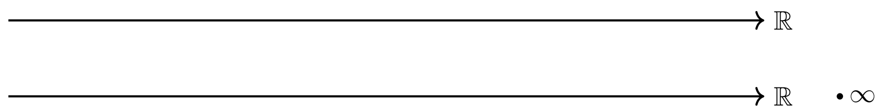


Figure 4.1: Depiction of the set $D_{\mathbb{R}}^{\infty}$. It consists of two copies of $\mathbb{R}$ and a point at infinity. The lower line represents the open intervals and the upper line the half-closed intervals.

4.4.5 Morphisms to indecomposable injectives

With the parametrisation of the spectrum available, we can provide a simple formula to determine if there are non trivial maps from finitely presented objects to indecomposable injective objects in $\operatorname{Mod} kT$.

Lemma 4.4.6. *Let I be an ideal of T and let $a < b \in T \cup \{\infty\}$. Then*

$$\operatorname{Hom}_{\operatorname{Mod} kT}(k_{[a,b)}, k_I) \neq 0$$

holds if and only if $a{\downarrow} \subseteq I$ and additionally $I \subseteq \widehat{b}{\downarrow}$ if $b \neq \infty$. In this case the dimension of this vector space is 1.

Conversely, there are no nonzero morphisms if and only if $I \subsetneq a{\downarrow}$ or $\widehat{b}{\downarrow} \subsetneq I$ for $b \neq \infty$.

Proof. Assume that $\operatorname{Hom}_{\operatorname{Mod} kT}(k_{[a,b)}, k_I)$ is nonzero. Then the supports of $k_{[a,b)}$ and k_I overlap: $[a, b) \cap I \neq 0$. This is the case if and only if $a{\downarrow} \subseteq I$. Now, if $\widehat{b}{\downarrow} \subseteq I$, then there is a point r in the ideal I, which is not in $[a, b)$. In this case there are elements $q \in [a, b) \cap I$ and $r \in I \cap b{\uparrow}$ and there is a diagram

$$\begin{array}{ccc} \left(k_{[a,b)}\right)(q) = k & \longrightarrow & 0 = \left(k_{[a,b)}\right)(r) \\ \big\downarrow \lambda\cdot & & \big\downarrow \\ \left(k_I\right)(q) = k & \xrightarrow{\operatorname{id}_k} & k = \left(k_I\right)(r) \end{array}$$

for all λ in k. This diagram commutes only for $\lambda = 0$, therefore there cannot be any nonzero maps. Conversely it is not hard to see that there is a nonzero map defined by pointwise multiplication by a unit λ in k if the condition on the ideal is satisfied.

If it is nonzero, the space of homomorphisms has dimension 1 because every homomorphism in this context is completely defined by its restriction to the point a. □

4.5 The ideals of a totally ordered set

Next, we analyse the structure of the set $\operatorname{Idl} T$ of order ideals of T. As discussed in Section 4.4.1, the set $\operatorname{Idl} T$ inherits the structure of a totally ordered set.

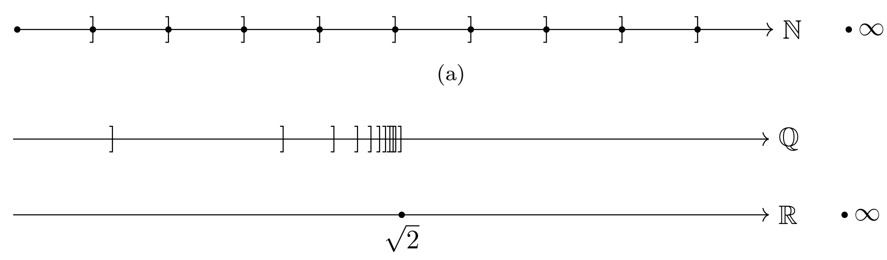


Figure 4.2: Graphic (a) depicts a strictly increasing sequence of ideals in $\operatorname{Idl}\mathbb{N}$ parametrised by all natural numbers. The union of this is the maximal interval $\mathbb{N}$. In (b) the set $\operatorname{Idl}\mathbb{Q}$ is pictured. Moreover, there is an increasing set of principal ideals $a_n\!\downarrow$, with a_n a rational sequence converging against $\sqrt{2}$ from below. The union of these is parametrised by $\sqrt{2}$ in the copy of the real line.

Now, there is a canonical order preserving embedding $T \to \operatorname{Idl} T$, sending each element a of T to the principal ideal $a\!\downarrow$. Moreover, the set of ideals $\operatorname{Idl} T$ can be viewed as a completion of a totally ordered set in the sense that every subset of $\operatorname{Idl} T$ has a supremum, even if this does not hold for T itself.

Example 4.5.1. For $T = \mathbb{N}$, the ideals are all principal ideals $n\downarrow$ for $n \in \mathbb{N}$ and $\mathbb{N}$ itself. Therefore $\operatorname{Idl}\mathbb{N} \xrightarrow{\sim} \mathbb{N} \cup \{\infty\}$ as totally ordered sets. Note that the maximal ideal $\mathbb{N}$ is neither of type $n\!\downarrow$ nor $\widehat{n}\!\downarrow$ for any $n \in \mathbb{N}$.

For $T = \mathbb{R}$, the supremum of the set of real ideals $(-\infty, -\frac{1}{n}]$ for all natural numbers n is $\widehat{0}\downarrow = (-\infty, 0)$, which is not a principal ideal but can still be parametrised in terms of T.

For $T = \mathbb{Q}$, there are even bounded subsets of $\operatorname{Idl} T$ for which the supremum does not exist in T, see Figure 4.2: Let S be the subset of rational numbers $x \in \mathbb{Q}$, such that $x^2 < 2$. Then S has no supremum in T, but the supremum of this set in $\mathbb{R}$ is $\sqrt{2}$. However, embedded into the set of ideals $\operatorname{Idl}\mathbb{Q}$, S has a supremum, which equals the interval $\left(-\infty, \sqrt{2}\right) \cap \mathbb{Q}$.

This 'completion' is not idempotent, though, and, as insinuated in Example 4.4.4, it is much more than the completion of $\mathbb{Q}$ by Dedekind cuts or in the Euclidean metric, which is $\mathbb{R}$. Namely, $\operatorname{Idl}\mathbb{Q}$ is parametrised by a copy of $\mathbb{Q}$ for the principal ideals, a copy of $\mathbb{R}$ for the bounded above non-principal ideals and a point at infinity for the entire set.

In the preceding example there are three different types of ideals: principal ideals, those which are strict lower sets and those which are neither. More generally, there is a trichotomy of ideals in any totally ordered set T.

Definition 4.5.2. An ideal $I \in \operatorname{Idl} T$ is of

- Type 1, if I is a principal ideal: $I = x\!\downarrow$ for $x \in T$.
- Type 2, if I is not a principal ideal and has a supremum in T.

– Type 3, if I is not a principal ideal and has *no* supremum in T.

It is immediate that an ideal of type 2 but not of type 3 is of the form $\widehat{a}\downarrow$. In the examples above, the ideal $(-\infty, 0)$ is of type 2 and $\mathbb{R}$ is of type 3 in $\operatorname{Idl}\mathbb{R}$, while $\left(-\infty, \sqrt{2}\right) \cap \mathbb{Q}$ is of type 3 in $\operatorname{Idl}\mathbb{Q}$.

In the following, let $A^{\complement} = X \setminus A$ denote the set complement for a subset $A \subseteq X$.

Lemma 4.5.3. *Let $I \in \operatorname{Idl} T$ be a non-principal ideal. Then:*

(1) I is of type 2 if and only if $I \subsetneq \bigcap_{x \in I^{\complement}} x\downarrow$.

(2) I is of type 3 if and only if $I = \bigcap_{x \in I^{\complement}} x\downarrow$.

Proof. First, we note that for every $I \in \operatorname{Idl} T$ we have that

$$I \subseteq \bigcap_{x \in I^{\complement}} x\downarrow \ .$$

Suppose that I is of type 2. Then $I^{\complement}$ has as minimal element the supremum of I, which is a for $I = \widehat{a}\downarrow$. Thus, the intersection yields $a\downarrow$ and the inclusion is proper.

Suppose I is of type 3 and assume that there is an element y in the intersection which is not in I. Then it is not minimal with this property, since otherwise y would be the supremum of I and therefore I would be of type 2. So, there exists an element $z \in I^{\complement}$ with $z < y$. Thus, y is not contained in $z\downarrow$ and therefore cannot be in the intersection.

Conversely, if equality holds, then I cannot have a supremum in T because otherwise it would be principal. So, it is of type 3. □

Remark 4.5.4. Following [Joh82, Section II.3.2], taking the ideals of a distributive lattice can be considered a completion to a *coherent locale*.

4.6 Characterisation of the closed sets

In this section we study the closure operator on the spectrum $\operatorname{Sp} T$. First, several examples for Ziegler closed sets are discussed. Based on these and further general properties of closures, the closed sets are then characterised.

As mentioned in Section 4.4, it can be easier to work with ideals than with modules, so we use the identification via the support map here.

4.6.1 Closed intervals in $\operatorname{Idl} T$

Since $\operatorname{Idl} T$ is a totally ordered set, we can consider intervals of $\operatorname{Idl} T$ and start with calculating the closures for several standard intervals.

Lemma 4.6.2. *Any point in $\operatorname{Idl} T$ is (Ziegler) closed. For all $x \in T$ the intervals $[x\downarrow, \infty)$ and $(-\infty, \widehat{x}\downarrow]$ are closed.*

Proof. The interval $[x{\downarrow},\infty)$ is closed: The Serre category $\Sigma\circ\Delta\left([x{\downarrow},\infty)\right)$ consists of these sums of finitely presented representations $k_{[a,b)}$ from which there are no nonzero maps to injectives k_I with I in $[x{\downarrow},\infty)$. By Lemma 4.4.6 this holds for $a<b$ in $T\cup\{\infty\}$ if and only if

(1) $I \subsetneq a{\downarrow}$, or

(2) $\widehat{b}{\downarrow}\subsetneq I$, if $b\neq\infty$,

for all ideals I in $[x{\downarrow},\infty)$. Suppose there are such $a<b$ satisfying the first condition, namely that $I\subsetneq a{\downarrow}$ for all I such that $x{\downarrow}\subseteq I$. Then, in particular, we have $x{\downarrow}\subsetneq a{\downarrow}$, or equivalently $x<a$. We also have $y{\downarrow}\subsetneq a{\downarrow}$ for all $y\geq x$, so in particular for $y=a$, and therefore $a{\downarrow}\subsetneq a{\downarrow}$, which is a contradiction.

The second case is equivalent to the condition $\widehat{b}{\downarrow}\subsetneq x{\downarrow}$. But this is the case if and only if $b\leq x$. So:

$$\Sigma\circ\Delta\left([x{\downarrow},\infty)\right)=\operatorname{add}\left(k_{[a,b)}\mid a,b\in T: a<b\leq x\right).$$

Next we calculate the subset of the spectrum associated with this. It consists of all k_I such that $I\subsetneq a{\downarrow}$ or $\widehat{b}{\downarrow}\subsetneq I$ for all $a<b\leq x$. There are no ideals I satisfying the first condition, as no ideal is properly contained in $m{\downarrow}$ if m is an arbitrarily small element of T, or a minimum if it exists.

The second case holds if and only if $\widehat{x}{\downarrow}\subsetneq I$, which holds exactly if $x{\downarrow}\subseteq I$. So we see that the closure of $[x{\downarrow},\infty)$ is itself.

Next, we show that $(-\infty,\widehat{x}{\downarrow}]$ is closed: The associated Serre subcategory of this consists of all these pairs a,b with $a<b\in T\cup\{\infty\}$ such that $I\subsetneq a{\downarrow}$ or, if $b\neq\infty$, $\widehat{b}{\downarrow}\subsetneq I$ for all $I\subseteq\widehat{x}{\downarrow}$. The first condition is equivalent to $\widehat{x}{\downarrow}\subsetneq a{\downarrow}$, wich is again equivalent to $x\leq a$. The second condition $\widehat{b}{\downarrow}\subsetneq I$ for all $I\subseteq\widehat{x}{\downarrow}$ implies $\widehat{b}{\downarrow}\subsetneq\widehat{x}{\downarrow}$ and thus $\widehat{b}{\downarrow}\subseteq\widehat{x}{\downarrow}\subsetneq\widehat{b}{\downarrow}$, hence such a and b do not exist. Thus

$$\Sigma\circ\Delta\left((-\infty,\widehat{x}{\downarrow}]\right)=\operatorname{add}\left(k_{[a,b)}\mid a,b\in T\cup\{\infty\}: x\leq a<b\right).$$

For the subset of the spectrum associated with this we obtain the indecomposable injectives parametrised by the ideals $I\in\operatorname{Idl}T$ such that for all a and b as above either $I\subsetneq a{\downarrow}$ or $\widehat{b}{\downarrow}\subsetneq I$ if $b\neq\infty$. The first condition is equivalent to $I\subsetneq x{\downarrow}$, which again is equivalent to $I\subseteq\widehat{x}{\downarrow}$. The second condition is never satisfied, because $b=\infty$ is allowed here. So the closure consists of exactly those ideals satisfying $I\subseteq\widehat{x}{\downarrow}$ and thus $(-\infty,\widehat{x}{\downarrow}]$ is closed.

To see that a point k_I is closed in $\operatorname{Sp}T$, we use the same technique and obtain

$$\Sigma\left(\{k_I\}\right)=\operatorname{add}\left(\left\{k_{[a,b)}\mid a<b, b\in I\right\}\cup\left\{k_{[a,b)}\mid a<b\in T\cup\{\infty\}, a\in I^{\complement}\right\}\right).$$

Then

$$\begin{aligned}\overline{\{k_I\}}&=\Upsilon\left(\operatorname{add}\left(\left\{k_{[a,b)}\mid a<b, b\in I\right\}\right)\right)\\&\quad\cap\Upsilon\left(\operatorname{add}\left(\left\{k_{[a,b)}\mid a<b\in T\cup\{\infty\}, a\in I^{\complement}\right\}\right)\right).\end{aligned}$$

The left term of the intersection is parametrised by all ideals J with $J \subsetneq a{\downarrow}$ or $\widehat{b}{\downarrow} \subsetneq J$ for all $a < b \in I$. Thus, the first condition vanishes and the second is equivalent to $I \subseteq J$.

The right term in the intersection is parametrised by all ideals J with $J \subsetneq a{\downarrow}$ or $\widehat{b}{\downarrow} \subsetneq J$ for all $a < b \in T \cup \{\infty\}$ and $a \in I^{\complement}$. The second condition vanishes because b can take the value ∞, while the first condition is equivalent to $J \subseteq I$. It follows that $\overline{\{I\}} = \{I\}$. □

Corollary 4.6.3. *For* $\operatorname{Idl} T$ *in the Ziegler topology we have the following examples of closed and corresponding open sets for all* $x \in T$ *and* $I, J \in \operatorname{Idl} T$*, where* I *is any ideal of type 3:*

	Closed interval	Open complement
1.	$(-\infty, \widehat{x}{\downarrow}]$	$[x{\downarrow}, \infty)$
2.	$[x{\downarrow}, \infty)$	$(-\infty, \widehat{x}{\downarrow}]$
3.	$\{J\}$	$\{J\}^{\complement}$
4.	$(-\infty, x{\downarrow}]$	$(x{\downarrow}, \infty)$
5.	$[\widehat{x}{\downarrow}, \infty)$	$(-\infty, \widehat{x}{\downarrow})$
6.	$(-\infty, I]$	(I, ∞)
7.	$[I, \infty)$	$(-\infty, I)$

Proof. 1.-3. were covered in the previous Lemma.

4. We have $(-\infty, x{\downarrow}] = (-\infty, \widehat{x}{\downarrow}] \cup \{x{\downarrow}\}$, which is closed by the cases 1. and 3.

5. We have $[\widehat{x}{\downarrow}, \infty) = \{\widehat{x}{\downarrow}\} \cup [x{\downarrow}, \infty)$, which is closed by the cases 2. and 3.

6. We can write $(-\infty, I] = \bigcap_{x \in I} (-\infty, x{\downarrow}]$, which is an intersection of closed intervals, see case 1, and therefore closed.

7. We can write $[I, \infty) = \bigcap_{x \in I} [x{\downarrow}, \infty)$, which is an intersection of closed intervals, see case 2, and therefore closed.

□

4.6.4 Topological closures

The following simple topological lemmata are used for the characterisation of the closure. We provide a proof for completeness.

Lemma 4.6.5. *Let* X *be a topological space,* $\mathcal{O}$ *an open cover of* X *and* $S \subseteq X$*. Then* S *is closed in* X *if and only if* $S \cap U$ *is closed in* U *for all* $U \in \mathcal{O}$*.*

Proof. Let S be closed in X. Then $X \setminus S$ is open in X and so is every intersection $(X \setminus S) \cap U$ with an open set U. Therefore, $S \cap U = U \setminus ((X \setminus S) \cap U)$ is closed in U.

For the converse, let $S \cap U$ be closed in U for every $U \in \mathcal{O}$. Equivalently, $U \setminus (S \cap U)$ is open in U. Since U is open, by the subspace topology this implies that $U \setminus (S \cap U)$ is also open in X. Now observe that $X \setminus S = \bigcup_{U \in \mathcal{O}} (U \setminus (S \cap U))$, which is open by the foregoing. Thus, S is closed in X. $\square$

We use the following notation for the closure of a set S in X

$$\mathrm{cl}_X(S) = \bigcap_{\substack{C \text{ closed in} X \\ S \subseteq C}} C.$$

Lemma 4.6.6. *Let X be a topological space with open cover $\mathcal{O}$ and $S \subseteq X$ any subset. Then*

$$\mathrm{cl}_X(S) = \bigcup_{U \in \mathcal{O}} \mathrm{cl}_U(S \cap U).$$

Proof. First, note that $\bigcup_{U \in \mathcal{O}} \mathrm{cl}_U(S \cap U)$ is closed in X by Lemma 4.6.5, as it is locally closed in each $U \in \mathcal{O}$.

Now, let $C \subseteq X$ be any closed set containing S. It suffices to show

$$\bigcup_{U \in \mathcal{O}} \mathrm{cl}_U(S \cap U) \subseteq C,$$

because then we have

$$\bigcup_{U \in \mathcal{O}} \mathrm{cl}_U(S \cap U) \subseteq \bigcap_{\substack{C \text{ closed in} U \\ S \subseteq C}} C = \mathrm{cl}_X(S),$$

and equality follows from the minimality of the closure. So observe that C is closed in X and therefore $C \cap U$ is closed in every open subspace $U \in \mathcal{O}$. Clearly, $S \cap U \subseteq C \cap U$ and therefore $\mathrm{cl}_U(S \cap U) \subseteq C \cap U$. We take the union on both sides and get

$$\bigcup_{U \in \mathcal{O}} \mathrm{cl}_U(S \cap U) \subseteq C. \qquad \square$$

4.6.7 Characterisation of the closure of $\mathrm{Sp}\, T$

The following proposition is the key in the proof of Theorem 4.7.2. Note that for a non-empty subset of elements of $\mathrm{Idl}\, T$ their union is in $\mathrm{Idl}\, T$ and that their intersection is in $\mathrm{Idl}\, T$ if and only if this subset has a lower bound in T.

Proposition 4.6.8. *Let $C \subseteq \mathrm{Idl}\, T$. Then C is (Ziegler) closed in $\mathrm{Idl}\, T$ if and only if any ideal of the form $I = \bigcap_{J \in V} J$ or $I = \bigcup_{J \in V} J$ for any non-empty $V \subseteq C$ is contained in C.*

The idea of the proof is first to show that a closed set is closed under these admissible intersections and unions, and then to show that there are no other points in the closure. For the second step the idea is to differentiate between the types of ideals and use that ideals of type 1 (resp. type 2) do not appear as unions (resp. intersections) of a strictly smaller (resp. greater) sequence of ideals, while ideals of type 3 can be both, an intersection and a union. For an ideal which is neither an intersection nor a union of elements of C in a non-trivial way, we then choose open intervals which are part of a disjoint open cover of $\operatorname{Idl} T$ and do not intersect C. Applying the closure operator locally we find that such ideals cannot lie in the closure. For convenience we use the notation $\widehat{\infty}{\downarrow}= T$.

Proof. We begin the proof with a reformulation of the condition

$$\begin{aligned} I \in C &\Leftrightarrow \operatorname{Hom}_{\operatorname{Mod} kT}(k_{[a,b)}, k_I) = 0 \text{ for all } k_{[a,b)} \in \Sigma \circ \Delta(C) \\ &\Leftrightarrow \left(\operatorname{Hom}_{\operatorname{Mod} kT}(k_{[a,b)}, k_I) \neq 0 \Rightarrow k_{[a,b)} \notin \Sigma \circ \Delta(C)\right) \\ &\Leftrightarrow \left(a{\downarrow}\subseteq I \subseteq \widehat{b}{\downarrow}\Rightarrow k_{[a,b)} \notin \Sigma \circ \Delta(C)\right), \end{aligned}$$

where the last equivalence comes from Lemma 4.4.6.

In the first step we show that a closed subset is closed under these intersections and unions and thereafter we show that no other points are added by the closure operator. Let $C = \overline{C}$ be Ziegler closed. Now, let $I = \bigcap_{J\in V} J$ for a subset $V \subseteq C$. For any $a \in T$ with $a{\downarrow}\subseteq I$ we have that $a{\downarrow}\subseteq J$ for all $J \in V \subseteq C$. Thus, $k_{[a,\infty)}$ cannot be an element of $\Sigma \circ \Delta(C)$. Similarly, let $a, b \in T$ satisfy $a{\downarrow}\subseteq \bigcap_{J\in V} J \subseteq \widehat{b}{\downarrow}$. Then there is $J \in V$ with $J \subseteq \widehat{b}{\downarrow}$, so we have $a{\downarrow}\subseteq J \subseteq \widehat{b}{\downarrow}$. But this implies that $k_{[a,b)}$ cannot be in $\Sigma \circ \Delta(C)$. Thus I is in C.

Let now $I = \bigcup_{J\in V} J$ for a subset $V \subseteq C$. Then for any $a \in T$ with $a{\downarrow}\subseteq I$, there is $J \in V$ with $a{\downarrow}\subseteq J$. Thus, $k_{[a,\infty)}$ cannot be in $\Sigma \circ \Delta(C)$. Moreover, let $a, b \in T$ satisfy $a{\downarrow}\subseteq I \subseteq \widehat{b}{\downarrow}$. Then there is $J \in V \subseteq C$ with $a{\downarrow}\subseteq J$. Therefore, we have $a{\downarrow}\subseteq J \subseteq \widehat{b}{\downarrow}$, hence $k_{[a,b)}$ cannot be in $\Sigma \circ \Delta(C)$. Thus I is in C.

It remains to show the converse: for all $I \in \operatorname{Idl} T$ such that I is not a union or non empty intersection of ideals in C, we have that I is not contained in the closure $\overline{C}$. Since every element in C trivially is a union or intersection and $C \subseteq \overline{C}$, we may assume that I is not in C. For this, we differentiate between the three types of ideals I.

If I is of type 1, so $I = a{\downarrow}$ for a in T, then it contains its supremum and therefore cannot be a union of strictly smaller ideals. If a is not maximal in T, there is an element $x \in I^{\complement}$, such that there is no ideal $J \in C$ with the property that $I \subsetneq J \subsetneq x{\downarrow}$: Let $V \coloneqq \{J \in C \mid I \subsetneq J\}$, then $I \subsetneq \bigcap_{J\in V} J$ by assumption. So we can take a $x \in \left(\bigcap_{J\in V} J\right) \setminus I$.

Now consider the interval $[a{\downarrow}, \widehat{x}{\downarrow}]$, which is an intersection of two sets that are both open and closed by Corollary 4.6.3. For a maximal in T, choose the interval $\{a{\downarrow}\}$ instead. The interval certainly contains the ideal I, but does not intersect C. This interval together with its complement is an open covering of $\operatorname{Idl} T$ and therefore closures can be calculated locally in this interval by Lemma 4.6.6. But the closure of the empty set is the empty set, therefore I is not contained in C.

If I is of type 2, so $I = \widehat{a}{\downarrow}$ for $a \in T$, then it cannot be a proper intersection of elements in C. Again, we find a $y \in I$ such that there is no $J \in C$ satisfying the property $y{\downarrow} \subsetneq J \subsetneq I$: Let $V := \{J \in C \mid J \subsetneq I\}$, then $\bigcup_{J \in V} J \subsetneq I$ by assumption. So we can take $y \in I \setminus \left(\bigcup_{J \in V} J\right)$.

Now, we can take the interval $[y{\downarrow}, I]$, which is closed, open and by construction has no intersection with C, and proceed as for type 1.

If I is of type 3, then it can be a union or an intersection or both by Lemma 4.5.3. In the first case, we can proceed as for type 1, in the second case as for type 2 and in the third case we can combine both: we find elements $x, y \in T$ and obtain a closed and open interval $[x{\downarrow}, \widehat{y}{\downarrow}]$, which does not intersect C. Again, the proof is completed as for type 1. □

This translates immediately to

Corollary 4.6.9. *Let $\mathcal{U} \subseteq \operatorname{Sp} T$. Then $\mathcal{U}$ is closed in $\operatorname{Sp} T$ if and only if for any non-empty $\mathcal{V} \subseteq \mathcal{U}$, the intersection $\bigcap_{M \in \mathcal{V}} \operatorname{supp} M$ and the union $\bigcup_{M \in \mathcal{V}} \operatorname{supp} M$ are in $\operatorname{supp} \mathcal{U}$ when non-empty.* □

4.7 Comparison with the order topology

As discussed in Section 4.4, there is a total order on $\operatorname{Idl} T$. Since we have the support map from $\operatorname{Sp} T$ to $\operatorname{Idl} T$ and because a total order induces the order topology, it is natural to ask the question whether the support map is continuous for this topology, or how both topologies are related. In fact, this map is a homeomorphism as we show in the following.

Definition 4.7.1. Let T be a totally ordered set. Then the *order topology on T* is the topology generated by the subbasis given by the *open rays* $\{b \in T \mid b > a\}$ and $\{b \in T \mid b < a\}$ for all $a \in T$.

Theorem 4.7.2. *The support map $\operatorname{supp}\colon \operatorname{Sp} T \to \operatorname{Idl} T$ is a homeomorphism, when $\operatorname{Idl} T$ is considered to have the order topology.*

Proof. The subbasis of the order topology on $\operatorname{Idl} T$ consists of the open rays. The image of these open sets under Δ is open in $\operatorname{Sp} T$, see Corollary 4.6.3.

Conversely, let $U \subseteq \operatorname{Idl} T$ be any Ziegler open set, so $U = C^{\complement}$ for $C := \operatorname{supp} \overline{\mathcal{U}}$ and a subset $\mathcal{U} \subseteq \operatorname{Sp} T$. We claim that every $I \in U$ has a neighbourhood $V \subseteq U$ open in the order topology. Suppose the contrary holds and that there is an $I \in U$ which has no neighbourhood $V \subseteq U$ open in the order topology. This is the case if and only if for all such neighbourhoods V of I we have $V \cap C \neq \emptyset$. As in the proof of Proposition 4.6.8 we proceed with a case distinction about the types of ideals.

If $I = a\downarrow$ is of type 1, then every non-empty interval $[a\downarrow, \infty)$ or $[a\downarrow, b\downarrow)$ for $b > a$ contains a $J \in C$. This yields a set of ideals $W \subseteq C$ with $\bigcap_{J \in W} J = I$, so $I \in C$ by Proposition 4.6.8, leading to a contradiction.

If $I = \widehat{a}\downarrow$ is of type 2, then I is a union of smaller ideals in C: By assumption, every non-empty open interval $V = (b\downarrow, \widehat{a}\downarrow]$ for $b \in T$ contains a $J \in C$. But this implies:

$$\bigcup_{J \in C \cap V} J = \bigcup_{\substack{J' \in \operatorname{Idl} T \\ J' \subsetneq I}} J' = I.$$

As before, this yields $I \in C$, which is a contradiction!

Lastly, if I is of type 3 and $I \neq T$, we can combine both previous cases and observe that for every $a, b \in T$ with $a\downarrow \subsetneq I \subsetneq b\downarrow$, the interval $(a\downarrow, b\downarrow)$ contains an element $J \in C$. If $I = T$ we can take an interval $(a\downarrow, T]$, instead. This yields either a family of smaller ideals of which I is the union of, or a family of greater elements of which I is the intersection, or both. Any way, we have $I \in C$ and the contradiction, again.

This implies that the support map induces a bijection of open sets and finishes the proof. ☐

Properties of the topology

From the equivalence to an order topology and from the study of the closure we can obtain several properties of the spectrum.

Corollary 4.7.3. *The space* $\operatorname{Sp} T$ *is Hausdorff.*

Proof. This follows from Lemma 4.6.2 and Corollary 4.6.3 or from the equivalence in Theorem 4.7.2, since the order topology is Hausdorff. ☐

For the special case $T = \mathbb{R}$ we have the following properties.

Corollary 4.7.4. *The Ziegler topology on* $\operatorname{Sp} \mathbb{R}$ *is not discrete, that is not every one point set is open.* ☐

This phenomenon is not visible for example in the case of finite linearly ordered sets, because then the spectrum is a discrete topological space.

Corollary 4.7.5. *The space* $\operatorname{Sp} \mathbb{R}$ *is not compact in the Ziegler topology.*

Proof. We have an open cover (which is even closed):

$$\operatorname{Sp} \mathbb{R} = \bigcup_{n \leq 0} \Delta\left([(n-1)\downarrow, \widehat{n}\downarrow]\right) \cup \Delta\left([0\downarrow, \mathbb{R}]\right).$$

This is a disjoint union of infinitely many sets and there certainly is no finite subcover. ☐

Remark 4.7.6. In the context of duality as mentioned in the introduction, one is tempted to ask if one can find an interesting duality here, as well. But taking the Hochster dual requires the underlying space to be spectral. Spectral spaces are compact sober spaces, that is that every irreducible closed set is the closure of a unique point. The latter property holds for $\operatorname{Sp}\mathbb{R}$ as it is Hausdorff. But it is not compact, so the known approach fails to work in this context.

4.8 From finite to continuous sets

We return to the setting of Section 3.5.8, considering the difference between the spectra of the real line $\mathbb{R}$ and finite subquivers of type A_n, to relate their spectra with each other. According to Equation (4.1) in Section 4.2.4 we can write the spectrum as a disjoint union

$$\operatorname{Sp}\operatorname{Mod} k\,\mathbb{R} \cong \operatorname{Sp} k\,\mathcal{K} \sqcup \operatorname{Sp}\operatorname{Mod} kA_n,$$

where the left term is an open and the right term a closed subset of $\operatorname{Sp}\operatorname{Mod} k\,\mathbb{R}$. Since $\mathcal{K}$ decomposes into the direct sum

$$\mathcal{K} \cong \coprod_{i=1}^{n} (a_i, a_{i+1}),$$

the spectrum of $\operatorname{Mod} k\,\mathcal{K}$ decomposes as

$$\operatorname{Sp}\operatorname{Mod} k\,\mathcal{K} \cong \coprod_{i=1}^{n} \operatorname{Sp}\operatorname{Mod} k(a_i, a_{i+1}),$$

where for each term there is a canonical decomposition

$$\operatorname{Sp} k(a_i, a_{i+1}) \cong (a_i, a_{i+1}) \sqcup (a_i, a_{i+1}) \sqcup \{a_{i+1}\}.$$

Here, the first two terms represent the principal and non-principal ideals and the right term represents the entire interval. On the other side, the spectrum of $\operatorname{Mod} kA_n$ is equivalent to the discrete set $[n]$ of n ordered points.

Now, compare the totally ordered set $\mathbb{N}$ and its compactification $\overline{\mathbb{N}} = \mathbb{N} \cup \{\mathbb{N}\}$. The point $\mathbb{N}$ can be seen as the limit of all sequences of $\mathbb{N}$ in the order topology. In the spectrum $\operatorname{Sp}\mathbb{N}$ this is a non-isolated point. As opposed to this, in case that the limit is inside the set, namely for $\overline{\mathbb{N}}$, then the limit point also appears as a principal ideal in the spectrum, a maximal point which is isolated.

Similarly, for every $a \in \mathbb{R}$, the principal ideal $a{\downarrow}$ is isolated in the subset of the spectrum $\{I \in \operatorname{Idl}\mathbb{R} \mid I \leq a{\downarrow}\}$.

4.9 Injective objects and the spectral category

As mentioned in Section 4.1, the Zariski spectrum parametrises all indecomposable injective modules of a commutative Noetherian ring. Gabriel and Oberst generalised this observation with their construction of the *spectral category* in [GO66], see also [Ste75, §6]. This category is obtained by localising a Grothendieck category $\mathcal{A}$ at its essential monomorphisms, which identifies every object with its injective hull. In consequence, the isomorphism classes of objects in the spectral category are in one to one correspondence with the isomorphism classes of injective objects in $\mathcal{A}$. Moreover, every object in the spectral category decomposes into the sum of a semisimple object, the *discrete* part, and a summand without simple subobjects, the *continuous* part. Thus, every injective object in $\mathcal{A}$ decomposes into the injective hull of a direct sum of indecomposable injectives and a *superdecomposable* summand without indecomposable summands.

The following theorem of Höppner implies that there are no non-trivial superdecomposable injective objects in $\operatorname{Mod} kT$. Recall that a representation is called *uniform* if every non-trivial subrepresentation is essential, i.e. intersects with every other subrepresentation non-trivially, and that the injective hull of a uniform representation is indecomposable.

Theorem 4.9.1 ([Höp81, 3.3.6 Satz]). *Let R be any ring and P a partially ordered set. Then every representation in* $\operatorname{Fun}(P, \operatorname{Mod} R)$ *has a uniform subobject if and only if the following hold:*

(1) Every non-trivial R-module has a uniform submodule.

(2) The partially ordered set P does not contain the infinite binary tree, which is inductively obtained by starting with a single point and adding two distinct greater elements for every point.

(3) Every non-trivial representation has a non-trivial subrepresentation of the form $\Delta_J^P M/\Delta_L^P M$, where M is an R-module, $J = i\uparrow$ is the principal filter generated by an $i \in P$ and $L \subseteq J$ a subset closed under greater elements. □

The category of vector spaces over a field is semisimple, a linearly ordered set does not contain any branches and the non-trivial image of a generating indecomposable representable object satisfies the third condition. So:

Corollary 4.9.2. *Every injective object in* $\operatorname{Mod} kT$ *is isomorphic to the injective envelope of a direct sum of indecomposable injectives. Equivalently, the spectral category of* $\operatorname{Mod} kT$ *is discrete.* □

This means that all – even the large injectives – can be parametrised in terms of the spectrum $\operatorname{Sp} T$.

4.10 Cantor–Bendixson and Krull–Gabriel analysis

The study of the Cantor–Bendixson and the Krull–Gabriel dimension for the case $T = \mathbb{R}$ is subject of this section. From the previous analysis of the topology follows that $\operatorname{Idl} \mathbb{R}$ has no isolated points, that are points which are open as one point set. Consequently, its Cantor–Bendixson rank is ∞. Since the Cantor–Bendixson filtration is a subfiltration of the *fin-filtration* of the spectrum, which corresponds to the Krull–Gabriel filtration of $\operatorname{mod} k\,\mathbb{R}$, this implies that $\operatorname{KGdim}(\operatorname{mod} k\,\mathbb{R}) = \infty$, confer [Kra98, Section 12].

Alternatively, we can calculate the KG-dimension by an estimate: Let M be in $\operatorname{mod} k\,\mathbb{R}$ and let $\operatorname{Latt}^{\mathrm{fg}} M$ denote the lattice of finitely presented subobjects of M. Since $\operatorname{mod} k\,\mathbb{R}$ is abelian, this is a modular lattice (see [Ste75, Proposition IV.5.3]). Then we have the inequation

$$\operatorname{Kdim}\left(\operatorname{Latt}^{\mathrm{fg}} M\right) \leq \operatorname{mdim}\left(\operatorname{Latt}^{\mathrm{fg}} M\right) = \operatorname{KGdim}(M),$$

see for example [Pre09, Propositions 7.2.3, 13.2.1]. Here Kdim denotes the Krull-dimension of the lattice, mdim denotes the m-dimension and KGdim denotes the Krull–Gabriel dimension of M. For the definitions of these dimensions, see ibidem.

Now take $M = k_{[x,\infty)}$ for any $x \in \mathbb{R}$. For every $y > x$, the representation M has a subobject $k_{[y,\infty)}$. Thus, the ordered set $\mathbb{R}$ embedds as a sublattice into $\operatorname{Latt}^{\mathrm{fg}} M$, and so does in particular $\mathbb{Q}$. By [Pre09, Propositions 7.1.4], this implies that $\operatorname{Kdim}\left(\operatorname{Latt}^{\mathrm{fg}} M\right) = \infty$, and therefore also $\operatorname{KGdim}(M) = \infty$. But this, however, implies $\operatorname{KGdim}(\operatorname{mod} k\,\mathbb{R}) = \infty$ and thus there is no exhausting Krull–Gabriel filtration of this category. Also, observe that no non-trivial finitely presented k-linear representation of $\mathbb{R}$ is of finite length, in particular not its quotients. Therefore a Krull–Gabriel filtration does not yield any further insights.

Note that, nonetheless, there are no superdecomposable injectives, see Section 4.9.

4.11 Interleaving distance

Next, we compare the topology on the spectrum with another topology known on the category of representations of the real line: the interleaving distance, see for example [Oud15, Definition 3.3]. For any real number ε, let $[\varepsilon]$ denote the *ε-shift functor*

$$\begin{aligned} \operatorname{Mod} k\,\mathbb{R} &\to \operatorname{Mod} k\,\mathbb{R}, \\ M &\mapsto M[\varepsilon], \\ f &\mapsto f[\varepsilon], \end{aligned}$$

where

$$\begin{aligned} M[\varepsilon](r) &= M(r+\varepsilon), \\ f[\varepsilon](r \leq s) &= f(r+\varepsilon \leq s+\varepsilon), \end{aligned}$$

for all morphisms of representations $f\colon M \to N$. Moreover, for any $k\,\mathbb{R}$-module M and any $\varepsilon \geq 0$, let $\mathrm{id}_M^{\varepsilon}\colon M \to M[\varepsilon]$ denote the morphism of representations of $\mathbb{R}$ defined by $\mathrm{id}_M^{\varepsilon}(x) = M(|x| \leq |x| + \varepsilon)(x)$, for $x \in M$.

Definition 4.11.1. An *ε-interleaving* between two $k\,\mathbb{R}$-modules M and N is a pair of maps $\phi\colon M \to N[\varepsilon]$ and $\psi\colon N \to M[\varepsilon]$, such that $\psi[\varepsilon] \circ \phi = \mathrm{id}_M^{2\varepsilon}$ and $\phi[\varepsilon] \circ \psi = \mathrm{id}_N^{2\varepsilon}$. We call M and N *ε-interleaved* if there is an interleaving between them. The *interleaving distance* of M and N is

$$d_I(M, N) = \inf \{\varepsilon \geq 0 \mid M \text{ and } N \text{ are } \varepsilon\text{-interleaved}\}.$$

The definition implies that the interleaving distance assumes the value ∞ if there is no ε-interleaving for any nonnegative ε. This is one obstruction for being a metric. In fact, d_i is an extended pseudometric; see for example the discussion after [Oud15, Definition 3.3 (rephrased)].

It immediately follows that there is another topology on $\mathrm{Sp}\,\mathbb{R}$ induced by d_I. It is also clear that it is not Hausdorff (T2) if d_I is only a pseudometric when restricted to $\mathrm{Sp}\,\mathbb{R}$. To find a basis for this topological space, we calculate the interleaving distance on $\mathrm{Sp}\,\mathbb{R}$ explicitly.

Lemma 4.11.2. *Let $M = k_I$ and $N = k_J$ for $I, J \in \mathrm{Sp}\,T$ with $I = x\!\downarrow$ or $I = \widehat{x}\!\downarrow$ and $J = y\!\downarrow$ or $J = \widehat{y}\!\downarrow$. Then $d_I(M, N) = |x - y|$.*

Proof. We derive a criterion for the existence of maps $\phi\colon M \to N[\varepsilon]$ and $\psi\colon N \to M[\varepsilon]$, such that $\psi[\varepsilon] \circ \phi = \mathrm{id}_M^{2\varepsilon}$ and $\phi[\varepsilon] \circ \psi = \mathrm{id}_N^{2\varepsilon}$. Since the defining intervals of the interval modules M and N are left unbounded, the maps id_M^{δ} and id_N^{δ} are non-trivial for every $\delta \geq 0$. This implies that the maps $\phi, \phi[\varepsilon], \psi$ and $\psi[\varepsilon]$ must be non-trivial. Note that $\phi \neq 0$ if and only if $\phi[\varepsilon] \neq 0$, and similarly for ψ.

Since the structure maps are all 0 or the identity, every map between M, N or their shifts can be expressed as a pointwise multiplication by an element of k. So, if there is a nonzero map between them, there also is a (nonzero) map defined by pointwise multiplication by 1. Then, given nonzero maps $\phi\colon M \to N[\varepsilon]$ and $\psi\colon N \to M[\varepsilon]$ which are pointwise multiplication by 1, the concatenations $\psi[\varepsilon] \circ \phi\colon M \to M[2\varepsilon]$ and $\phi[\varepsilon] \circ \psi\colon N \to N[2\varepsilon]$ are represented by pointwise multiplication by 1 and therefore coincide with the maps $\mathrm{id}_M^{2\varepsilon}$, respectively $\mathrm{id}_N^{2\varepsilon}$.

So it is enough to determine the parameters ε for which such nonzero maps φ and ψ exist. Given the case that $x < y$, then on one hand there is a nonzero map $M \to N[\varepsilon]$ if $\varepsilon > y - x$, but not if $\varepsilon < y - x$, by Lemma 4.4.6. On the other hand there is always a map $N \to M[\varepsilon]$ for any $\varepsilon \geq 0$. Thus, there is an ε-interleaving whenever $\varepsilon > y - x$, but not if $\varepsilon < y - x$. So the interleaving distance of M and N is:

$$d_I(M, N) = \inf \{\varepsilon \geq 0 \mid \varepsilon > y - x\} = y - x.$$

Note that we did not cover the case $\varepsilon = y - x$ because it does not change the infimum, but requires a case distinction.

In the case $x > y$ there is an ε-interleaving if $\varepsilon > x - y$, but not if $\varepsilon < x - y$, by symmetry. Combining both cases, we obtain

$$d_I(M, N) = |x - y| \ . \qquad \square$$

As an extended pseudometric d_I induces a topology $\mathcal{T}^{\mathrm{d_I}}$ on $\operatorname{Sp}\mathbb{R}$, in contrast to the topology $\mathcal{T}^{\mathrm{cl}}$ coming from the closure operation, with basis the open balls

$$B_\varepsilon^{d_I}(k_I) = \{L \in \operatorname{Idl}\mathbb{R} \,|\, d_I(k_I, k_L) < \varepsilon\} = \Delta\left((x-\varepsilon)\downarrow, \widehat{(x+\varepsilon)}\downarrow\right)$$

for every $\varepsilon \geq 0$ and $I = x\downarrow$ or $I = \widehat{x}\downarrow \in \operatorname{Idl}\mathbb{R}$. We observe that the interleaving distance cannot distinguish between modules of the form $k_{\widehat{x}\downarrow}$ and $k_{x\downarrow}$ in the spectrum $\operatorname{Sp}\mathbb{R}$, and therefore the induced topological space is not Hausdorff. Note that $d_I(k_\mathbb{R}, M) = \infty$ for all $M \in \operatorname{Sp}\mathbb{R} \setminus \{k_\mathbb{R}\}$, so $B_\varepsilon^{d_I}(k_\mathbb{R}) = \{k_\mathbb{R}\}$.

The open ball $B_\varepsilon^{d_I}(k_I)$ is the complement of the closed set

$$\Delta\left((-\infty, (x-\varepsilon)\downarrow]\right) \cup \Delta\left([(x+\varepsilon)\downarrow, \infty]\right)$$

if $I \neq \mathbb{R}$, so these open balls in the interleaving topology are also Ziegler open. But for any $\varepsilon > 0$ the complement of $B_\varepsilon^{d_I}(k_\mathbb{R})$ is $\Delta\left(\{\mathbb{R}\}\right)$, so $\{k_\mathbb{R}\}$ is open in the interleaving topology.

But it is not open in the Ziegler topology. Hence, the topology induced by the interleaving distance is only refined by the Ziegler topology if we exclude the point at infinity $k_\mathbb{R}$.

Corollary 4.11.3. *The restricted identity morphism*

$$(\operatorname{Sp}\mathbb{R} \setminus \{k_\mathbb{R}\}, \mathcal{T}^{\mathrm{cl}}|_{\operatorname{Sp}\mathbb{R}\setminus\{k_\mathbb{R}\}}) \to (\operatorname{Sp}\mathbb{R}, \mathcal{T}^{\mathrm{d_I}})$$

is continuous.

Chapter 5

Middle exact representations

The main result of this chapter is Theorem 5.6.1 which generalises Theorem 5.2.4 to the setting of three-parameter persistence modules. It asserts that so-called middle exact persistence modules can be decomposed into a direct sum of indecomposable block modules.

After a motivation of two-parameter middle exact persistence modules in Section 5.1 comes the generalisation of this concept to three parameters: We develop and explore the notion of 3-middle exact representations in Section 5.2 and introduce three-parameter block modules in Section 5.3. Then, in Section 5.4 we prove the finite case of the decomposition theorem. The special case of this concerning 2-short exact representations has been relocated to Section 5.5 because of its technical importance. Finally, we use some cases of the finite case in the proof of the general case in Section 5.6. Due to its length the proof of the latter is divided into several cases. Subsequently, in Section 5.7 we discuss the meaning of middle exactness and its relations with homotopy theory. This also contains a brief discussion of excisive functors in the sense of Lurie.

Sections 5.2 to 5.6 are work in progress joint with Vadim Lebovici and Steve Oudot, with the intention of a subsequent publication.

5.1 Interlevelset persistence and middle exact representations

Let X be a compact topological space over $\mathbb{R}$, meaning that there is a continuous function $f\colon X \to \mathbb{R}$. An idea from Morse theory, which was seized by persistence homology, is to study a filtration on X by taking *sublevels* (or *superlevels*), that is sets $X_a(f) = f^{-1}((-\infty, a])$ for $a \in \mathbb{R}$. This yields an $\mathbb{R}$-indexed filtration of topological spaces X_a. If we consider for example simplicial, singular or cellular homology, we then obtain induced graded morphisms of the graded homology groups $H(X_r) \to H(X_s)$ whenever $r \leq s$ in $\mathbb{R}$. Parameters which have an open neighbourhood in which the induced morphisms are isomorphisms are called *regular*, the others are called *critical*. The function f is called *tame* if there are only finitely many critical values.

Let f be tame, having n critical values $r_1 < r_2 ... < r_n$ and regular values s_i with

$$s_0 < r_1 < s_1 < ... < r_n < s_n.$$

Moreover, write $X_i = X_{s_i}(f) = f^{-1}((-\infty, s_i])$. Then all homological information of this filtration is already contained in the diagram of graded groups

$$0 = H(\emptyset) \to H(X_1) \to H(X_2) \to ... \to H(X_n) = H(X).$$

For homology with coefficients in a field k, this clearly is a persistence module in $\operatorname{Mod} kA_n$ for A_n linearly oriented, as defined in Section 2.1.

Usually it is not the case that $H(X)$ vanishes, so there can be *essential* elements in $H_i(X_-)$ when considered as objects in $\operatorname{Mod} kA_n$, which *never die*, that is their support is not strictly smaller than any parameter.

This motivates the notion of *extended persistence* introduced in [CSEH08]. A more general approach in [BEMP13] extends this notion by making use of *relative homology*, which is defined for a pair of spaces (X, A) with $A \subseteq X$. For more details on the latter, see for example [Hat02]. Note that there is a canonical map

$$H(X, B) \to H(X, A)$$

for all spaces $B \subseteq A \subseteq X$. Then classical homology is the same as considering the pair $(X, \emptyset)$. Dually to considering the sublevel set filtration in sublevel set persistence one can consider *superlevel sets*, that is taking the superlevel filtration sets $X^r(f) = f^{-1}([r, \infty))$. Let $s'_1 < ... < s'_m$ denote a set of regular values between all critical values for the superlevel filtration and fix the notation $X^i = X^{s'_{m-i}}(f) = f^{-1}\left([s'_{m-i}, \infty)\right)$. Then $X^0 = \emptyset$ and $X^m = X$. Note that both approaches are not equivalent. Instead of only considering each of these individually, one can combine them into a single object of investigation, using relative homology:

$$\begin{aligned} 0 \xrightarrow{\sim} H(X_0) \to H(X_1) \to H(X_2) \to ... \to H(X_n) \xrightarrow{\sim} H(X) \\ \xrightarrow{\sim} H(X, X^0) \to H(X, X^1) \to ... \to H(X, X^m) \xrightarrow{\sim} 0 \end{aligned} \tag{5.1}$$

This guarantees that every generator eventually dies.

Now, note that there is a relative version of the Mayer–Vietoris exact sequence: Let (X, Y) be a pair of spaces $X \subseteq Y$ and let $C \subseteq A$ and $D \subseteq B$ such that $X = A \cup B = A^\circ \cup B^\circ$ and $Y = C \cup D = C^\circ \cup D^\circ$, where Z° denotes the interior of a subspace $Z \subseteq X$. Then the relative Mayer–Vietoris sequence is the long exact sequence in homology

$$... \to H_n(A \cap B, C \cap D) \to H_n(A, C) \oplus H_n(B, D) \to H_n(X, Y) \xrightarrow{\partial_n} ...,$$

where $\partial_n \colon H_n(X,Y) \to H_{n-1}(A \cap B, C \cap D)$ denotes the boundary map and where the other maps are induced by inclusions. For each degree n this sequence gives rise to the following commutative square:

$$\begin{array}{ccc} H_n(A,C) & \longrightarrow & H_n(X,Y) \\ \uparrow & & \uparrow \\ H_n(A \cap B, C \cap D) & \longrightarrow & H_n(B,D) \end{array}$$

Such squares form the blue print of middle exact representations, which we further investigate in the following, see Section 5.2.

Extended persistence diagrams

We discussed persistence diagrams briefly in Section 1.2. Considering extended persistence, the diagrams reveal more structure: The *ordinary* persistence diagram encodes the birth and death of a generator in sublevel set persistence. We can extend it by the *extended diagram* which represents those generators which are born during the first half of sequence (5.1) and die during the second half. Moreover, the *relative diagram* represents all generators which only exist in the second half of the sequence.

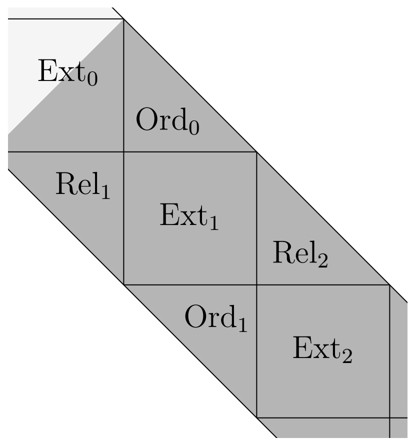
Now, consider pairs (I, C), where I is an open interval of $\mathbb{R}$ and C is the complement of a closed interval in I. Then for different choices of I and C one can obtain all pairs which are involved in the relative interlevel set persistent (co)homology, confer [CSM09; BBF22]. Gluing this parametrisation in the different degrees along the boundary maps, one gets the *Mayer–Vietoris strip*, see Figure 5.1. For more details, see [BBF22]. The relation between extended persistence and block decomposition of interlevel-set persistence is discussed ibidem: The maximal vertex of an indecomposable rectangle in the Mayer–Vietoris strip corresponds to the birth-death-parameters of a topological feature.

5.2 Middle exact representations

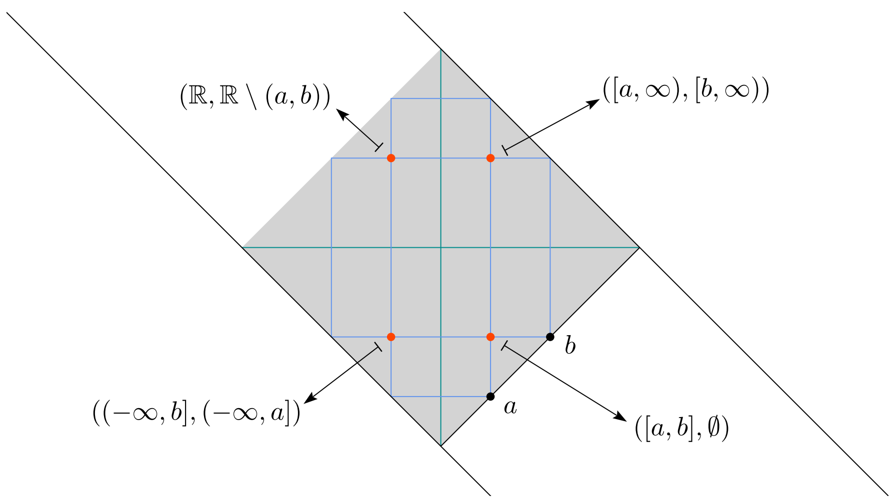
Let k be a field. In the following we use complexes of k-vector spaces, which are indexed in homological or decreasing convention. For the canonical finite totally ordered set with n elements we write $[n] = \{1, ..., n\}$. Then a commutative square is the poset $[2] \times [2]$ or, equivalently, the power set $\mathcal{P}([2])$:

$$Q = \begin{array}{ccc} \{2\} & \xrightarrow{\delta_1^{12}} & \{1,2\} \\ {\scriptstyle \delta^2}\uparrow & & \uparrow{\scriptstyle \delta_2^{12}} \\ \emptyset & \xrightarrow{\delta^1} & \{1\} \end{array}$$

Now, a k-representation of Q is nothing else than a commutative square of k-vector spaces.



(a) The extended Mayer–Vietoris Strip.



(b) Relative pairs as points in the strip.

Figure 5.1: Visualisations of the extended Mayer–Vietoris strip after [BBF22, Figures 2.5, 3.6], with kind permission. It is a subposet of $\mathbb{R}^{\mathrm{op}} \times \mathbb{R}$. The real numbers a and b lie on the embedding of a copy of the real line into the strip.

$$\begin{array}{ccc} M_2 & \xrightarrow{d_1^{12}} & M_{12} \\ {\scriptstyle d^2}\uparrow & & \uparrow{\scriptstyle d_2^{12}} \\ M_\emptyset & \xrightarrow{d^1} & M_1 \end{array} \tag{5.2}$$

The classical definition of middle exactness in two parameters is the following, see [CS10, Definition 5.4]:

Definition 5.2.1. A pointwise finite-dimensional object M in $\operatorname{Mod} kQ$ is called 2-*middle exact* if the complex

$$M_\emptyset \xrightarrow{\binom{d^1}{d^2}} M_1 \oplus M_2 \xrightarrow{\left(d_2^{12} \;\; -d_1^{12}\right)} M_{12} \tag{5.3}$$

is exact in the middle, or short: $H_1 M = 0$. Moreover, we call M *left exact* (resp. *right exact* or *short exact*) if the complex is also left exact (resp. right exact or both). In this notation, M_{12} is in degree 0.

We observe that the complex (5.3) is also the mapping cone of the morphism of two-term complexes d^1 and d_1^{12} defined by the remaining arrows d^2 and d_2^{12} in the diagram (5.2).

We recall a well-known result from homological algebra.

Lemma 5.2.2. *Let $\mathcal{A}$ be an abelian category and consider a commutative square (5.2) in $\mathcal{A}$. Then, this is a pushout if and only if the complex (5.3) is right exact. Dually, it is a pullback if and only if the complex (5.3) is left exact.* □

Definition 5.2.3. Let $S_1, ..., S_n$ be totally ordered sets. An object of the category $\operatorname{Mod} k\left(\prod_{i=1}^n S_i\right)$ is called *n-parameter persistence module.*

Suppose we have totally ordered sets $S_1, ..., S_m$ and $T_1, ..., T_n$. A map of posets $j\colon S = \prod_{i=1}^m S_i \to \prod_{l=1}^n T_l$ is called *biCartesian* if it maps all axis parallel squares to (possibly degenerate) axis parallel squares.

Let M be in $\operatorname{Mod} kP$ for $P = T_1 \times ... \times T_n$. The image of an injective biCartesian map $j\colon Q \to P$ is called a *biCartesian square of* P and the restriction j^*M is called a *square of* M. Similarly, the image of an injective biCartesian map $j\colon C \to P$ is called a *biCartesian cube of* P and the restriction j^*M is called a *cube of* M.

A two-parameter persistence module M in $\operatorname{Mod}(k(S \times T))$ is called 2-middle exact (resp. 2-short exact, 2-left exact, 2-right exact, 2-short exact) if it has this property when restricted to every square of M.

For middle exact representations we have the following decomposition theorem:

Theorem 5.2.4 ([BCB20, Theorem 1.3])**.** *Let M be a pointwise finite-dimensional 2-middle exact representation in* $\operatorname{Mod} kP$ *for $P = R \times S$. Then M is isomorphic to the direct sum of interval modules k_J for block intervals J. A* block (interval) *is a subset $B \subseteq P$ of one of the following types:*

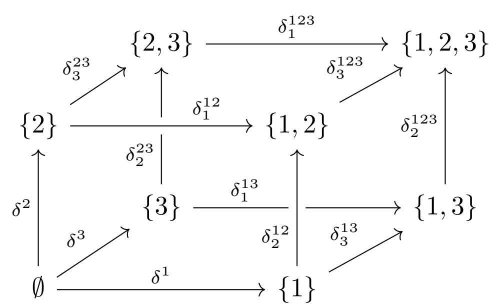
(a) Birth block. (b) Death block. (c) Horizontal strip. (d) Vertical strip.

Figure 5.2: The different types of two-parameter blocks. The dashed line denotes the boundary and may or may not be part of the block.

- *$B = I_R \times I_S$, where $I_R \subseteq R$ and $I_S \subseteq S$ are ideals.*
- *$B = F_R \times F_S$, where $F_R \subseteq R$ and $F_S \subseteq S$ are filters.*
- *$B = I_R \times S$ or $B = R \times I_S$ for intervals $I_R \subseteq R$ or $I_S \subseteq S$.* □

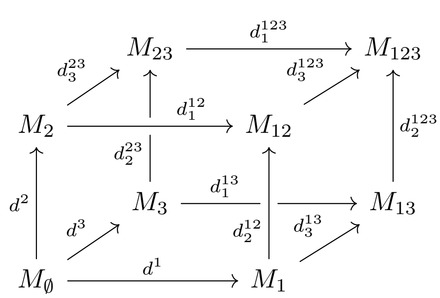
We want to adapt this to higher dimensions. So consider the three dimensional commutative cube as poset C:

$C =$



For a representation M in $\operatorname{Mod} kC$ we use the notation:

$M =$ (5.4)



Given such a representation M, we consider any pair of opposite faces, e.g. front and rear. If $\mathcal{F}$ is the front (resp. left or bottom) face of C, and if $\mathcal{R}$ is the rear (resp. the right or top) face of P, we write $M^\bullet_{\mathcal{F}}$ and $M^\bullet_{\mathcal{R}}$ for the three-term complexes corresponding to these faces. The remaining arrows, connecting both sides, define a chain morphism $\Phi\colon M^\bullet_{\mathcal{F}} \to M^\bullet_{\mathcal{R}}$. So we get a triangle in the homotopy category

$$M^\bullet_{\mathcal{F}} \xrightarrow{\Phi} M^\bullet_{\mathcal{R}} \to \mathrm{Cone}(\Phi) \to M^\bullet_{\mathcal{F}}[1] \tag{5.5}$$

and we define $M^\bullet = \mathrm{Cone}(\Phi)$.

Example 5.2.5. Let $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ be the front and $\mathcal{R} = \{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ the rear face of C. Then:

$$\Phi = \begin{array}{ccccc} M_\emptyset & \xrightarrow{\begin{pmatrix} d^1 \\ d^2 \end{pmatrix}} & M_1 \oplus M_2 & \xrightarrow{\begin{pmatrix} d_2^{12} & -d_1^{12} \end{pmatrix}} & M_{12} \\ \downarrow d^3 & & \downarrow d_3^{13} \oplus d_3^{23} & & \downarrow d_3^{123} \\ M_3 & \xrightarrow{\begin{pmatrix} d_1^{13} \\ d_2^{23} \end{pmatrix}} & M_{13} \oplus M_{23} & \xrightarrow{\begin{pmatrix} d_3^{123} & -d_1^{123} \end{pmatrix}} & M_{123} \end{array}$$

From the commutativity of the cube follows that both squares commute. So we have the four-term complex

$$M^\bullet = \ M_\emptyset \xrightarrow{\begin{pmatrix} d^1 \\ d^2 \\ d^3 \end{pmatrix}} M_1 \oplus M_2 \oplus M_3 \xrightarrow{A} M_{12} \oplus M_{13} \oplus M_{23} \xrightarrow{\begin{pmatrix} d_3^{123} \\ -d_2^{123} \\ d_1^{123} \end{pmatrix}^t} M_{123} \tag{5.6}$$

where the M_{123} is in degree 0 and A is the matrix:

$$A = \begin{pmatrix} d_2^{12} & -d_1^{12} & 0 \\ d_3^{13} & 0 & -d_1^{13} \\ 0 & d_3^{23} & -d_2^{23} \end{pmatrix}$$

Lemma 5.2.6. *For any representation $M \in \mathrm{Mod}\, kP$ all constructions of $M^\bullet$ are isomorphic. This means that the complex and its exactness conditions are independent of the choice of opposite faces for the cone construction, up to isomorphism.*

Proof. For this, note that a commutative square/cube can be interpreted as double/triple complex. Then the complexes (5.3) and (5.6) are isomorphic to the respective total complexes for a choice of order on the axes. But the isomorphism types of the total complexes are independent of the choice of order on the axes, see [Sta23, Section 0FNB]. □

Definition 5.2.7. A k-linear representation M of the cube in $\mathrm{Mod}\, kC$ is called 3-*middle exact* if the complex (5.6) is middle exact, that is the homology vanishes in degrees 1 and 2, but not necessarily in degrees 0 and 3. It is called 3-*left exact* (resp. 3-*right exact* or 3-*properly exact*) if $H_3(M^\bullet) = 0$ (resp. if $H_0(M^\bullet) = 0$ or both).

There is an equivalent element-wise characterisation of 3-middle exactness.

Lemma 5.2.8. *Let M be a k-representation of C. Then M is 3-middle exact if and only if the following hold:*

- $H_1 M = 0$*: For all* $x \in M_{12}, y \in M_{13}, z \in M_{23}$ *with* $d_3^{123}(x) - d_2^{123}(y) + d_1^{123}(z) = 0$ *there are* $\alpha \in M_1, \beta \in M_2, \gamma \in M_3$ *with*

$$\begin{aligned} d_2^{12}(\alpha) - d_1^{12}(\beta) &= x \\ d_3^{13}(\alpha) - d_1^{13}(\gamma) &= y \\ -d_2^{23}(\gamma) + d_3^{23}(\beta) &= z. \end{aligned}$$

- $H_2 M = 0$*: For all* $\alpha \in M_1, \beta \in M_2, \gamma \in M_3$ *such that*

$$\begin{aligned} d_2^{12}(\alpha) &= d_1^{12}(\beta) \\ d_3^{13}(\alpha) &= d_1^{13}(\gamma) \\ d_2^{23}(\gamma) &= d_3^{23}(\beta), \end{aligned}$$

  there is an element $w \in M_\emptyset$ *such that* $d^1(w) = \alpha, d^2(w) = \beta$ *and* $d^3(w) = \gamma$.

Now, let $P = R \times S \times T$ be the direct product of three totally ordered sets in the product order.

Definition 5.2.9. A representation M of P in $\operatorname{Mod} kP$ is called 3-*middle exact* (resp. 3-*left exact*, 3-*right exact* or 3-*properly exact*) if every cube of M has this property. Moreover, it is called 2-*middle exact* (resp. 2-*left exact*, 2-*right exact* or 2-*short exact*) if every square of M has this property.

Note that the sequence (5.5) induces a long exact sequence in homology:

$$\cdots \to H_{i+1}(M) \to H_i(M_{\mathcal{F}}) \to H_i(M_{\mathcal{R}}) \to H_i(M) \to \dots \tag{5.7}$$

where $H_i(M) \coloneqq H_i(M^\bullet)$. We can immediately conclude the following lemma.

Lemma 5.2.10. *Let M be a representation in* $\operatorname{Mod} kP$ *and suppose i^*M is 2-short exact for all biCartesian maps $i\colon Q \to R \times S \times T$ which have a fixed coordinate in one of these, say in T. Then M is 3-properly exact.*

Proof. Let $j\colon C \to P$ be a biCartesian map. We can assume that j is injective, because otherwise one obtains the cone of the identity and all homology vanishes. Then $j(C)$ has two different faces, the face $\mathcal{F}$ with smaller T-coordinate and the face $\mathcal{R}$ with greater T-coordinate. Now, by assumption we have $H_i(M_{\mathcal{F}}) = H_i(M_{\mathcal{R}}) = 0$ for all i. From the long exact sequence it then follows that $H_i(M) = 0$ for all i. □

5.3 Blocks

For $P = R \times S \times T$ as above we generalise the notion of two-parameter blocks to three parameters as follows.

Definition 5.3.1. Let $B \subset P$ be an interval. We call it a *block* if it satisfies one of the following properties:

(1) $B = F_R \times F_S \times F_T$ is the cartesian product of a filter in each axis.

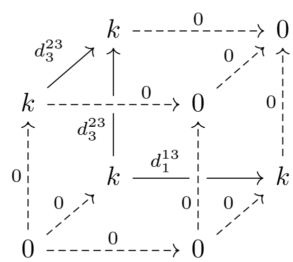
(2) $B = I_R \times I_S \times I_T$ is the cartesian product of an ideal in each axis.

(3) B is a product of a two-parameter block in tow axes and the remaining axis, e.g. $B = B' \times T$ for a block $B' \subseteq R \times S$.

Clearly, this coincides with the two-parameter notion of blocks if we restrict to any axis parallel plane. Therefore, this is a good candidate for a three-parameter generalisation of two-parameter blocks.

Let us now examine what conditions on an interval representation of the cube C are necessary in order to only obtain intervals which are blocks in the above sense. We consider the following examples.

Example 5.3.2. The interval representation of $\{\{2\}, \{3\}, \{13\}, \{2, 3\}\}$ is not a block in the above sense. It can be drawn as in the following diagram:



It is 3-middle exact, but not 2-middle exact: The four-term complex degenerates to

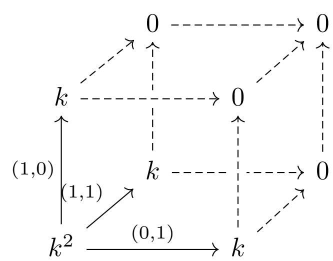
$$0 \to k^2 \xrightarrow{\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}} k^2 \to 0,$$

which is an isomorphism in the middle. But for the front face this yields the three-term complex

$$0 \to k \to 0,$$

which is not middle exact. Hence, we see that 3-middle exactness alone is not sufficient for a good generalisation to three parameters, as it is satisfied also for non-blocks.

Example 5.3.3. On the other hand, the representation



is indecomposable, as it is an embedding of the non-thin indecomposable of the outward oriented D_4 quiver to the cube. It is not difficult to check that it is not 3-middle exact, but 2-middle exact on each face: The associated four-term complex

$$k^2 \to k^3 \to 0 \to 0$$

cannot be middle exact due to the dimensions, and the other complexes are

$$k^2 \xrightarrow{\sim} k^2 \to 0 \quad \text{and} \quad k \to 0 \to 0.$$

But most importantly, this representation is not an interval representation. In Section 5.5.6 we see that it is a pathological example.

Both examples led to the expectation that we have a block decomposition when both properties are satisfied. We successfully tested this hypothesis on the smallest non-trivial cube which led to the development the following proof for the case of an arbitrary finite cube.

5.4 The finite case

Let us now consider the case of finite cubes $P = R \times S \times T$ with $R = S = T = [n] = \{1, ..., n\}$. We use the following notation for subrepresentations for any M in $\operatorname{Mod} kP$.

Definition 5.4.1. By $\operatorname{Ker} M^{\rightarrow}$ we denote the subfunctor of M which consists of all elements which are eventually sent to zero by maps in increasing R-direction. Similarly, define $\operatorname{Ker} M^{\uparrow}$ and $\operatorname{Ker} M^{\nearrow}$ for the S- and T-direction. Then, write for the intersections:

$$\begin{aligned}
\operatorname{Ker} M^{\uparrow\rightarrow} &= \operatorname{Ker} M^{\uparrow} \cap \operatorname{Ker} M^{\rightarrow}, \\
\operatorname{Ker} M^{\nearrow\rightarrow} &= \operatorname{Ker} M^{\nearrow} \cap \operatorname{Ker} M^{\rightarrow}, \\
\operatorname{Ker} M^{\uparrow\nearrow} &= \operatorname{Ker} M^{\nearrow} \cap \operatorname{Ker} M^{\uparrow}, \\
\operatorname{Ker} M^{\uparrow\nearrow\rightarrow} &= \operatorname{Ker} M^{\nearrow} \cap \operatorname{Ker} M^{\rightarrow} \cap \operatorname{Ker} M^{\uparrow}.
\end{aligned}$$

We prove the following special case of our main theorem in this chapter; be advised that an important step of case 1 was moved to the next section.

Theorem 5.4.2. *Let M be a pointwise finite-dimensional linear representation in $\operatorname{Mod} kP$, such that M is 3-middle exact and 2-middle exact. Then M decomposes into the direct sum of block modules. This decomposition is unique up to isomorphism.*

Proof. Generally, there is a decomposition of M into indecomposable objects unique up to isomorphism by [BCB20, Theorem 1.1]. So we can assume that M is indecomposable. We distinguish multiple cases:

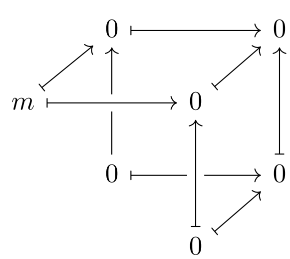
Case 1: M is 2-short exact. By Lemma 5.2.10, this implies 3-proper exactness. For the decomposition result in this case, see Proposition 5.5.1. So we assume in the following that M is not 2-short exact.

Case 2: M is not 3-left exact. We claim that M is a death block module. The representation M is not left exact if and only if the subrepresentation $\operatorname{Ker} M^{\llcorner}$ is nontrivial. We consider the set $\operatorname{supp} \operatorname{Ker} M^{\llcorner}$ (this is a finite nonempty set, of course) and take a maximal element $x = (x_1, x_2, x_3)$. Then $M(x)$ contains a nonzero element m which is in the kernel of all non-trivial structure maps starting in x.

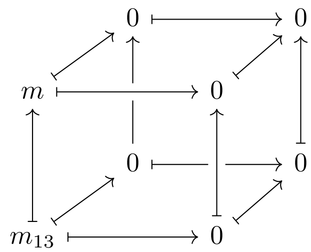
If x is the minimum of P already, then m spans an injective subrepresentation and there is nothing to do. Otherwise consider a cube $j(C)$ of M with $j(\emptyset) = (1,1,1)$ the origin and $j(\{1,2,3\}) = x$, that is the maximum edge is at the vertex x.

Next, we show that m lifts to the origin. To do so, we first show that the element m has a preimage along all edges adjacent to x. For this we use the notation of the diagram (5.4), in which these edges are d_1^{123}, d_2^{123} and d_3^{123}.

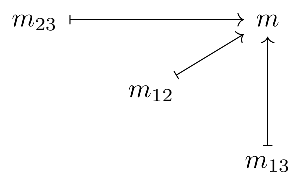
Starting with m at position 2, we consider the following segment of a cube of M:



By the 3-middle exactness of M and in particular due to $H_2(M) = 0$, there is a preimage m_{13} of m under d_2^{123}:

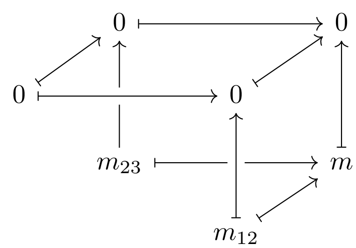


In a symmetric way, starting at positions 1 and 3, we obtain preimages m_{12} and m_{23} under d_3^{123} and d_1^{123}:

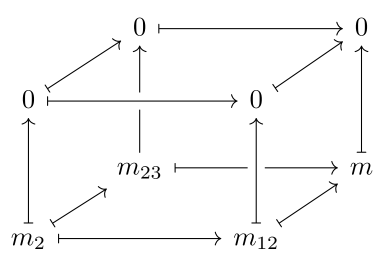


(5.8)

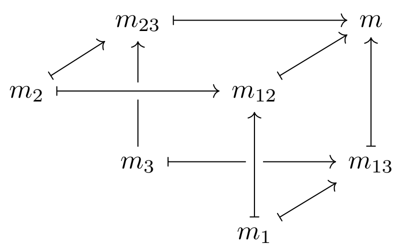
Next, we complete this to a rectangle: For example, consider a diagram of elements of a cube of M having the top face of (5.8) as its bottom face:



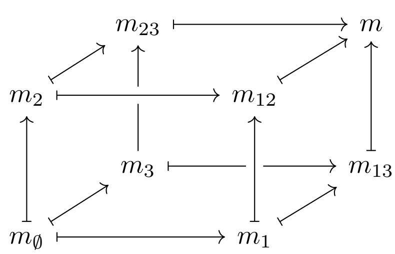
By 3-middle exactness of M, in particular due to $H_2(M) = 0$, this cube segment can be completed to the diagram:



Doing the analogous on the other faces yields a diagram:



Using the same argument again, we can complete this to a commutative diagram of elements:

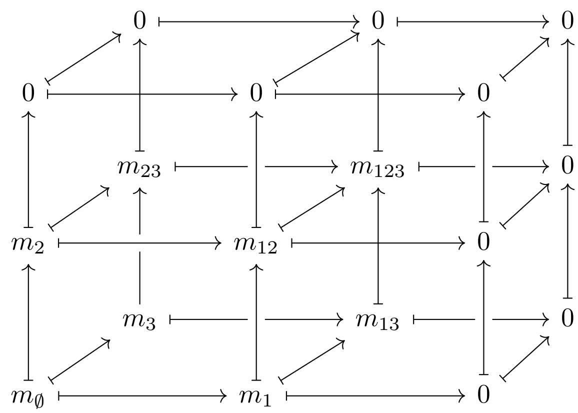


By construction, the subrepresentation spanned by $m_\emptyset$ is isomorphic to an interval representation for the interval of elements greater or equal the origin and less than or equal to m. But this is an injective subobject of M by Theorem 3.2.5 and therefore isomorphic to M.

The case 2* that M is not 3-right exact follows from the previous case and duality, see Section 3.2.10.

Case 3: M is not 2-left exact but 3-properly exact. This means that we can find a plane $W \subsetneq P$ parallel to, say, the $R \times S$ plane, so $W = R \times S \times \{t\}$ for a $t \in T$, such that the restriction $M_{|W}$ of M to W has a direct summand k_B, where $B \subsetneq W$ is a death-block. So $k_B \hookrightarrow \operatorname{Ker} M^{\llcorner}_{|W}$. By the left exactness of 3-proper exactness we have $\operatorname{Ker} M^{\llcorner\!\nearrow} = \operatorname{Ker} M^{\llcorner} \cap \operatorname{Ker} M^{\nearrow} = 0$. This implies that k_B can be lifted to a subrepresentation of M by injectively extending it in increasing T-direction.

In the next step we show that this can also be lifted in decreasing direction of T: Consider an injective biCartesian map $j\colon Q \to P$ such that $j(\emptyset) = (1,1,1)$ and $j(\{1,2,3\}) = |m|$, where m is a nonzero element of k_B with maximal support. We then proceed similarly to the previous step, extending this cube by zero and finding an element $m_\emptyset$ as in the following:



Given $m = m_{123} \in k_B$, where k_B is considered as a subobject of $M_{|L}$, with preimages m_3, m_{13} and m_{23} in k_B, we can find elements m_{12} and m_2 by 3-middle exactness of the upper left and upper right cubes and m_1 and $m_\emptyset$ by 3-middle exactness of the lower right and lower left cube.

Thus we obtain a subrepresentation $k_{\tilde{B}}$ generated by $m_\emptyset$ for an interval $\tilde{B} = \{(r, s, t') \mid (r, s, t) \in B, t' \in T\} \subsetneq P$, which is the extension of B along all of the T axis. Since $\tilde{B}$ is a filtered order ideal, this subrepresentation is injective by Theorem 3.2.5 and therefore $M \cong k_{\tilde{B}}$.

The case 3* that M is not 2-right exact but 3-properly exact follows from the previous case and duality, see Section 3.2.10. □

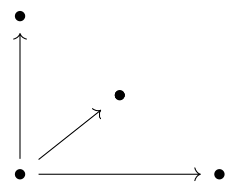
Remark 5.4.3. Bear in mind that the symmetry in the setup of the theorem is not important! We could also assume to have a poset $P = [l] \times [m] \times [n]$ for different $l, m, n \in \mathbb{N}$. In fact, every finite rectangular cuboid set can be extended to a cube, and representations of it can be extended to representations of a cube by identity.

5.5 The 2-short exact case

Here we prove the special case of 2-short exact representations for a finite cube, which is part of the proof of Theorem 5.4.2.

Proposition 5.5.1. *Let M be a pointwise finite-dimensional representation of P. If M is 2-short exact, then M can be decomposed into interval modules for* block intervals *which are the direct product of two axes and an interval in the third set.*

Recall that a 2-*span* (or span) is a diagram of shape $\bullet \leftarrow \bullet \rightarrow \bullet$. Note that a pushout is defined to be the colimit of a span. Dually we define the 2-cospan and note that a pullback is the limit of the 2-cospan. Moreover we define a 3-*span* to be the analogue diagram having three outgoing arrows:

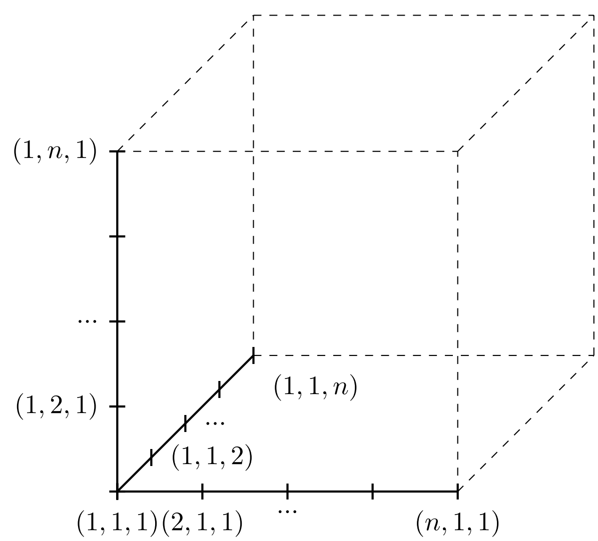


In P, the colimit of a 2-span or 3-span is given as the supremum of its outer points and the limit of a cospan is the infimum. In our setting these are well-defined, as P is finite and filtered.

Now consider a particular subsets for any product of totally ordered spaces $P = R \times S \times T$ which has an infimum (i_R, i_S, i_T), generalising the span: The *claw of* P is the union L of the axes starting in the infimum, or in other terms:

$$L = R \times \{i_S\} \times \{i_T\} \cup \{i_R\} \times S \times \{i_T\} \cup \{i_R\} \times \{i_S\} \times T.$$

For $P = [n] \times [n] \times [n]$, this is the following subset of P:



If $[n] = 2$, then the claw coincides with $\mathcal{P}_{\leq 1}([3])$, which is the subset of subsets of cardinality at most two.

We require another property of Kan extensions, see also [Bor94a, Proposition 3.7.4]:

Lemma 5.5.2. *Let* $\mathcal{C}_0 \xrightarrow{\iota_0} \mathcal{C}_1 \xrightarrow{\iota_1} \mathcal{C}_2$ *be functors of small categories and let* $F\colon \mathcal{C}_0 \to \mathcal{D}$ *be a functor into a category which is closed under colimits. Then* *we have*

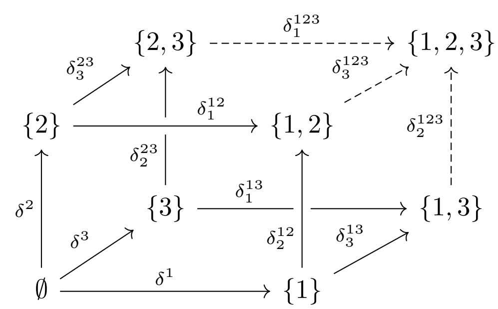
$$\mathrm{Lan}_{\iota_1}\left(\mathrm{Lan}_{\iota_0} F\right) \cong \mathrm{Lan}_{\iota_1 \circ \iota_0} F$$

Proof. Note that the left Kan extension Lan_{ι_i} is the left adjoint of the restriction functor ι_i^* by the dual of Lemma 3.5.6. So we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}^{C_2}}\left(\mathrm{Lan}_{\iota_1}\left(\mathrm{Lan}_{\iota_0} F\right), G\right) &\cong \mathrm{Hom}_{\mathcal{D}^{C_1}}\left(\mathrm{Lan}_{\iota_0} F, \iota_1^* G\right) \\ &\cong \mathrm{Hom}_{\mathcal{D}^{C_0}}\left(F, \iota_0^* \iota_1^* G\right) \\ &\cong \mathrm{Hom}_{\mathcal{D}^{C_0}}\left(F, \left(\iota_1 \iota_0\right)^* G\right) \\ &\cong \mathrm{Hom}_{\mathcal{D}^{C_2}}\left(\mathrm{Lan}_{\iota_1 \iota_0} F, G\right) \end{aligned}$$

for any G in $\mathrm{Mod}\, kP_2$. Now the assertion follows from Yoneda's Lemma 2.1.1. □

Lemma 5.5.3. *Consider the* almost complete cube*, which is the subset* $K = \mathcal{P}_{\leq 2}([3])$ *of subsets of cardinality at most three, in the following drawn in solid lines:*

$K =$ 

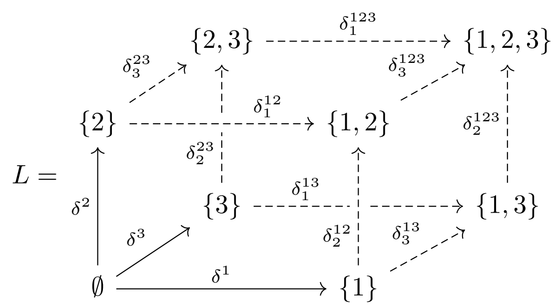
Let M be a 2-right exact representation of K. Then $\mathrm{Lan}_{K \hookrightarrow C} M$ *is a pushout on each face.*

Proof. We prove this claim for the right face. The other cases follow by symmetry.

Any pair (p, q) of morphisms with $p\colon M_{12} \to T$ and $q\colon M_{13} \to T$ and $pd = qe$ for a test object T extends via f and h to morphisms $r\colon M_2 \to T$ and $s\colon M_3 \to T$. By the pushout property on the left face, there is a unique morphism $u\colon M_{23} \to T$ such that $u \circ d_3^{23} = r$ and $u \circ d_2^{23} = s$. In particular, there is an (up to isomorphisms) unique morphism $t\colon (\mathrm{Lan}_{K \hookrightarrow C} M)(\{1,2,3\}) \to T$ which factors u, and especially p and q. □

This holds in an even stronger form:

Corollary 5.5.4. *Consider the claw subset $L = \mathcal{P}_{\leq 1}([3])$ of the cube C, which is drawn in solid lines:*



Then for any representation M of L, the left Kan extension $\mathrm{Lan}_{L \hookrightarrow C} M$ is 2-right exact on every face.

Proof. The left Kan extension restricted to the front, left or bottom face yields a pushout square. Next, use Lemma 5.5.3. □

Proposition 5.5.5. *Let M be a pfd. representation of $P = [n]^3$ which is 2-short exact and let $\iota \colon L \hookrightarrow P$ denote the embedding of the claw of P. Then we have*

$$\mathrm{Lan}_{\iota}\left(\iota^* M\right) \cong M.$$

Proof. First note that it is possible to complete L to P by successively completing 2-spans to squares or almost complete cubes to cubes. We choose any such filtration $j_i \colon L_i \hookrightarrow L_{i+1}$ with $0 \leq i \leq m-1$ and $L_0 = L, L_m = P$. For the embedding into P we use the notation $\iota_i \colon L_i \hookrightarrow P$. Then, by Lemma 5.5.2 it is enough to check that $\mathrm{Lan}_{j_i}\left(\iota_i^* M\right) \cong \iota_{i+1}^* M$ for all i. Note, that in each step we either add a single point p and either two arrows if we just complete a 2-span to a square or three arrows if we complete an almost complete cube $\mathcal{P}_{\leq 2}[n]$ as in Lemma 5.5.3.

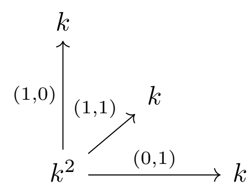
Now, the calculation of this Kan extension at the additional point p can be expressed in terms of colimits by Lemma 3.5.3. We can then reduce this to taking the colimit over a 2-span or an almost complete cube. The cofinal sets in the under category $L_i \uparrow p$, which is given by the set of elements $l \in L_i$ with arrows $\iota_i(l) \to p$, are of this shape. Then the desired colimit is isomorphic to the colimit taken only over these subsets by Theorem 3.5.4.

For the case of a span, it is clear that this is a pushout. For the other type of subset, note that by Corollary 5.5.4 and the preceding argument all restrictions of $\mathrm{Lan}_{j_i}\left(\iota_i^* M\right)$ to squares are pushouts. Since by assumption all such restrictions of $\iota_{i+1}^* M$ are pushouts either, there is an isomorphism $\mathrm{Lan}_{\iota_i}\left(\iota_i^* M\right) \cong \iota_{i+1}^* M$ by the uniqueness. □

The dual results are all true for right Kan extensions!

5.5.6 Proof of the Proposition Proposition 5.5.1

We consider the subset $L \subseteq [n]$ as a star shaped quiver with outbound arrows. The decomposition theory for representations of such quivers is usually inconvenient, as there are non-thin indecomposable representations. For example, in the case $n = 2$ this is a quiver of type D_4, which has a unique non-thin indecomposable whose isomorphism class is represented by:



The left Kan extension of this along the embedding into the cube is trivial at every point which is not on this 3-span, as the pushout on the 2-spans already vanishes. Therefore, it cannot be 2-short exact and by the classification of indecomposables so far, this representation is not allowed to be the restriction of an indecomposable representation of C to the 3-span. Thus, in this case we can rule out non-thin indecomposable summands for the decomposition of the restriction of a 2-short exact representation of the cube to this 3-span.

But even if we only consider the set $[2] \times [3]^2$, we run into a problem: The related poset L we consider is a quiver of type $\tilde{E}_6$, which is of infinite type but still tame [EH18, Section 12.2.1]. For cubes of shape $[3]^3$, the appropriate set L is of wild representation type. In particular in these cases we cannot use a classification of indecomposable representations of these sets L to check whether they are compatible with the exactness properties imposed on the three-parameter lattices they are meant to be restrictions of.

Instead we use a different argument.

Lemma 5.5.7. *Let L be the claw of P with embedding $\iota\colon L \hookrightarrow P$ and let M be 2-short exact representation of P. Then the endomorphism rings of M and its restriction $\iota^* M$ are isomorphic.*

In particular, if M is pointwise finite-dimensional, the restriction induces a one to one correspondence between the indecomposable summands of M and those of $\iota^* M$.

Proof. From Lemma 3.5.6 we know that Lan_ι is the left adjoint of the restriction functor ι^*. So we get:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}\, kL}\left(\iota^* M, \iota^* M\right) &\cong \mathrm{Hom}_{\mathrm{Mod}\, kP}\left(\mathrm{Lan}_\iota\left(\iota^* M\right), M\right) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}\, kP}\left(M, M\right),\end{aligned}$$

where the first isomorphism comes from the adjunction formula. For the second isomorphism, note that the counit $\varepsilon\colon \mathrm{Lan}_\iota \circ \iota^* \to 1$ is an isomorphism on the essential image of Lan_ι, by the triangle identity of the adjunction and

Lemma 3.5.5. This is even an isomorphism of k-algebras, as the involved morphisms are natural.

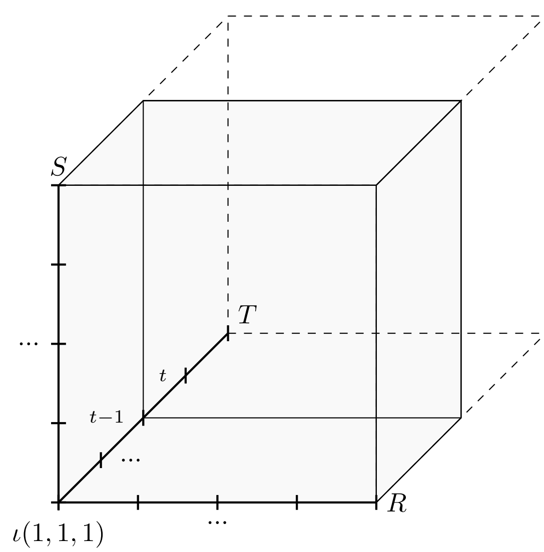
Now, if M is pointwise finite-dimensional, then the idempotents of the endomorphism ring correspond to the indecomposable summands of M. □

Lemma 5.5.8. *Let M be a pointwise finite-dimensional 2-short exact representation of $[n]^3$ and let $\iota\colon L \hookrightarrow P$ be the inclusion of the claw L into P. Then $\iota^* M$ decomposes into a direct sum of thin indecomposable representations.*

Proof. By Theorem 3.4.1 we can assume M to be indecomposable. Moreover, by Lemma 5.5.7 this is the case if and only if its restriction $\iota^* M$ is indecomposable.

If $\iota^* M$ is properly supported on only one or two arms of L, then it is thin by the decomposition theory of A_n. So assume that the support of $\iota^* M$ contains a subquiver of type D_4. All structure maps of $\iota^* M$ must be surjective, as otherwise one could split off a summand on an arm. Then there is a nonzero element $x \in M(\iota(1,1,1))$, where $(1,1,1)$ denotes the minimal point of L.

Suppose that one of the structure maps of $\iota^* M$ is not injective on an arm. If x is sent to zero somewhere on one arm, then it must be mapped injectively along the other arms: By 2-short exactness of M, in particular by left exactness, the kernels in any two different directions intersect trivially. Thus, the submodule $\langle x \rangle$ is isomorphic to k_I, where $I \subseteq P$ is a set of all points below a plane in P. E.g. if x is mapped to zero along the T-axis for the first time at the parameter t, then $I = R \times S \times \{x \in T \mid x < t\}$.



This is an injective subrepresentation and therefore $M \cong \langle x \rangle$. Clearly, the restriction $\iota^* M$ must be thin, then.

So we can assume that $\iota^* M$ is injective on the arms. But then all structure maps are isomorphisms, so $\iota^* M$ is indecomposable if and only if $M(\iota(1,1,1))$ is indecomposable, that is one-dimensional. Thus, $\iota^* M$ is isomorphic to k_L. □

Now we can proof the proposition:

Proof of Proposition 5.5.1. Let $\iota\colon L \hookrightarrow P$ denote the claw of P. By Lemma 5.5.8 the restriction $\iota^* M$ decomposes into a direct sum of interval modules

$$\iota^* M \cong \bigoplus_{I \in \mathcal{I}} k_I .$$

for a set of intervals $\mathcal{I}$. By additivity of Kan extensions and by Proposition 5.5.5 we get

$$\bigoplus_{I \in \mathcal{I}} (\mathrm{Lan}_\iota k_I) \cong \mathrm{Lan}_\iota \bigoplus_{I \in \mathcal{I}} k_I \cong \mathrm{Lan}_\iota \iota^* M \cong M.$$

For any interval $I \in \mathcal{I}$ the representation $\mathrm{Lan}_\iota k_I$ is isomorphic to an interval representation of the specified kind, as it is a pushout in every square and pushouts of identity- and zero morphisms only consist of such. □

5.6 General case

Let $P = R \times S \times T$ a product of arbitrary totally ordered sets. We have finally arrived to state the main theorem:

Theorem 5.6.1. *Let M be a pointwise finite-dimensional representation of P which is 2-middle exact and 3-middle exact. Then M decomposes as a direct sum of block modules.*

The proof of this is a generalisation of the original proof of Botnan–Crawley-Boevey. It is divided into several steps.

5.6.2 Extension Lemmata

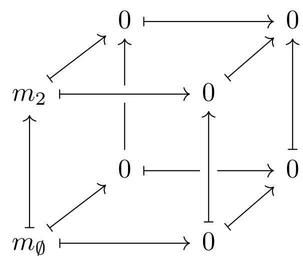
Let M be a 2-middle exact and 3-middle exact representation of P. First, we study how interval modules on the restriction to a lower dimensional subset of P can be lifted to higher dimensions. This extends [BCB20, Lemma 5.4].

Lemma 5.6.3. *Suppose there is a monomorphism $h\colon k_{\{r\}\times J_S \times J_T} \hookrightarrow M_{|\{r\}\times S \times T}$, where $J_S \subseteq S$ and $J_T \subseteq T$ are intervals bounded above in $S \setminus J_S$, respectively $T \setminus J_T$. Then there is a lift to a monomorphism $k_{(-\infty,r]\times J_S \times J_T} \hookrightarrow M_{|(-\infty,r]\times S\times T}$.*

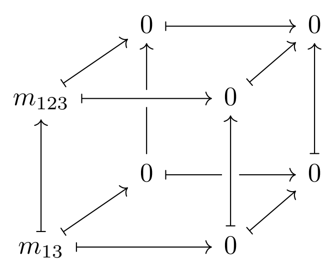
Proof. Let ε_S and ε_T be the respective bounds. Moreover, let π denote the projection $(-\infty, r] \times J_S \times J_T \to \{r\} \times J_S \times J_T$ and let $\alpha_p^{\varepsilon_S,\varepsilon_T}$ denote the arrow $p = (p_R, p_S, p_T) \mapsto (p_R, \varepsilon_S, \varepsilon_T)$ in P. We define:

$$E_p^{\varepsilon_S,\varepsilon_T} = M(\pi)^{-1}(\mathrm{Im}\, h(r, p_S, p_T)) \cap \mathrm{Ker}\, M\left(\alpha_p^{\varepsilon_S,\varepsilon_T}\right).$$

This is clearly nonzero for each $p \in \{r\} \times J_S \times J_T$. Moreover, it is nonzero for every $p \in (-\infty, r] \times J_S \times J_T$: We consider the following cube where the top face is in $\{r\} \times S \times T$ with only $|m_2| \in \{r\} \times J_S \times J_T$, and the front lower left corner at p. Then m_2 has a lift $m_\emptyset$ by Lemma 5.2.8:

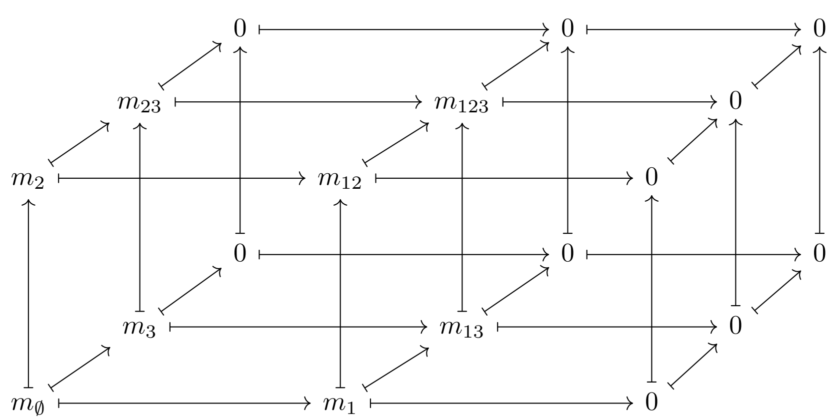


Similarly we see that for all $p \leq q \in (\infty, r] \times J_S \times J_T$ the canonical map $E_p^{\varepsilon_S,\varepsilon_T} \to E_q^{\varepsilon_S,\varepsilon_T}$ coming from restriction of the structure map is surjective: Given an element m_{123} in $E_q^{\varepsilon_S,\varepsilon_T}$, there is a cube of the following kind by 3-middle exactness:



(5.9)

Next, consider the diagram:



Here, $|m_{123}| = q$ and $|m_\emptyset| = p$. Given any $m_{123} \in E_q^{\varepsilon_S,\varepsilon_T}$ we want to find a preimage $m_\emptyset \in E_p^{\varepsilon_S,\varepsilon_T}$. Note that m_2, m_{12} and m_{23} exist by assumption and are in the image of h. 3-middle exactness of the rear right cube yields a preimage m_{13} of m_{123}. Now 3-middle exactness of the front right cube yields an element m_1, on the rear left cube it yields an element m_3 and on the front left cube this then yields the desired element $m_\emptyset$.

Now consider the intersection

$$E_p = \bigcap_{(\varepsilon_S,\varepsilon_T) > J_S \times J_T} E_p^{\varepsilon_S,\varepsilon_T}$$

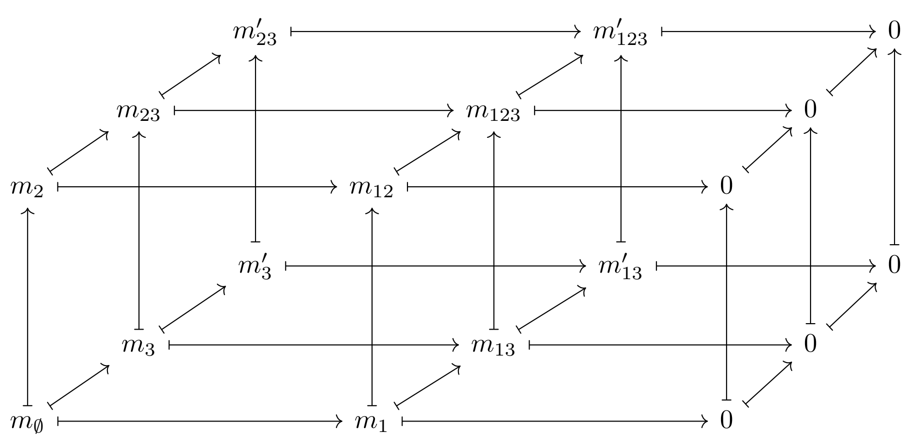
By pointwise finite-dimensionality there is a pair $(\varepsilon_S, \varepsilon_T)$ such that $E_p = E_p^{\varepsilon_S,\varepsilon_T}$. So the E_p define a representation of $(-\infty, r] \times S \times T$, with all structure maps being epimorphisms. Now we can apply Lemma 3.2.12 to obtain a direct summand $k_{(-\infty,r]\times J_S\times J_T}$. Multiplying the embedding by an appropriate scalar, we obtain a lift of h. □

Lemma 5.6.4. *Suppose there is a monomorphism* $h\colon k_{R\times J_S\times\{t\}} \hookrightarrow M_{|R\times S\times\{t\}}$, *where* J_S *is an interval bounded above in* $S \setminus J_S$. *Then there is a lift to a monomorphism* $k_{R\times J_S\times(-\infty,t]} \hookrightarrow M_{|R\times S\times(-\infty,t]}$.

Proof. This is proven similarly to the previous Lemma. For $(p_R, p_S, p_T) = p \in R\times J_S\times(-\infty, t]$ we define $E_p^{\varepsilon_S}$ to be the subspace of M_p consisting of all elements mapped into $\operatorname{Im} h(p_R, p_S, t)$ which are mapped to zero in $M_{(p_R,\varepsilon_S,p_T)}$. This is nonzero by 2-middle exactness as we can always consider the following elements in a square of M with constant T-coordinate t, where $m \in \operatorname{Im} h(p_r, p_S, t)$ and m' is in $E_p^{\varepsilon_S}$:

$$\begin{array}{ccc} m & \longmapsto & 0 \\ \uparrow & & \uparrow \\ m' & \longmapsto & 0 \end{array}$$

Now we show that all restricted structure maps $E_p^{\varepsilon_S} \to E_q^{\varepsilon_S}$ are epimorphisms for $p \leq q$ by considering the diagram:



Given an element $m_8 \in E_q^{\varepsilon_S}$ we want to find a lift $m_1 \in E_p^{\varepsilon_S}$. First note that there are (unique) elements $m_3', m_{13}', m_{23}', m_{123}' \in \operatorname{Im} h$ by assumption. By 2-middle exactness of the top rear left square we obtain m_{23} and by 3-middle exactness of the rear right cube we obtain m_{13}. Then by 3-middle exactness of the rear left cube we obtain m_3. By 2-middle exactness of the top front squares we obtain m_{12} and m_2. Then by 3-middle exactness of the front cubes we obtain m_1 and $m_\emptyset$.

The rest follows analogously to the proof of the previous lemma. □

5.6.5 The kernel subrepresentation

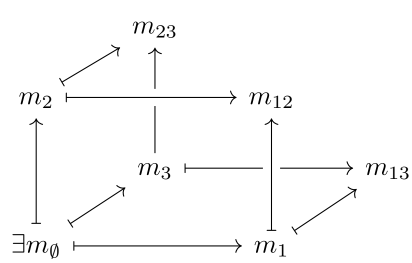
Let M be a 2-middle exact and 3-middle exact representation of P. We discuss some properties of the kernel subrepresentation $\operatorname{Ker} M^{\llcorner}$, which was defined in Definition 5.4.1.

Lemma 5.6.6. *In* $\operatorname{Ker} M^{\llcorner}$ *all structure maps are epimorphisms.*

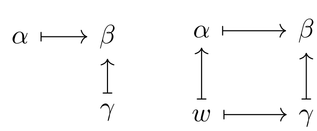
Proof. Every element can be lifted along the axes as in diagram (5.9). Finding a preimage works analogously as in the proof of Lemma 5.6.3. □

Lemma 5.6.7. *The representation* $\operatorname{Ker} M^{\llcorner}$ *is* 3*-middle exact and* 2*-right exact. This also implies that* $\operatorname{Ker} M^{\llcorner}$ *is* 3*-right exact.*

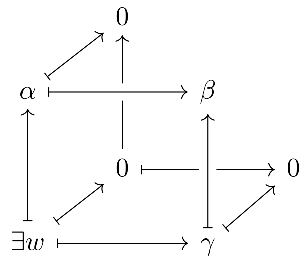
Proof. First, note that $H_2 \operatorname{Ker} M^{\llcorner} = 0$: By the vanishing of $H_2 M$ we can find an element $m_\emptyset$ that fits into the following diagram, where m_1, m_2 and m_3 are assumed to lie in $\operatorname{Ker} M^{\llcorner}$ (and therefore also m_{12}, m_{13} and m_{23}). But this implies that $m_\emptyset$ lies in there, either:



Next, we show 2-middle exactness: Without limitation of generality we show this in a subset of P with fixed T-coordinate, as the other cases are symmetric. Suppose there are elements α, β and γ in $\operatorname{Ker} M^{\llcorner}$ in a square of $\operatorname{Ker} M^{\llcorner}$ as in the left diagram. We show that we can complete it to the right one in $\operatorname{Ker} M^{\llcorner}$:



For this, consider the following diagram, which can be extended to a mapping of elements in a cube of M by 3-middle exactness:



As above, it follows that w is in $\operatorname{Ker} M^{\llcorner}$.

The 2-right exactness then follows from Lemma 5.6.6. Moreover, the 3-right exactness is then a consequence of the long exact sequence (5.7) in Section 5.2. □

5.6.8 Case: M is not 3-properly exact

Lemma 5.6.9. *Let $I \subseteq P$ be a principal filter, that is an upset $a\!\uparrow$ for an element $a \in P$, and let M be a pointwise finite dimensional object in $\operatorname{Mod} kI$ which is 3-middle exact and 2-right exact. Moreover assume that all structure maps are epimorphisms. Then M is completely determined by a finite subposet $I \supseteq I' \cong [n]^3$ and can be recovered by extending the restriction $M_{|I'}$ by isomorphisms.*

Proof. Let $L \hookrightarrow a\!\uparrow$ denote the claw of $I = a\!\uparrow$. We write L_R, L_S, L_T for the arms of L, so we have $L = L_R \cup L_S \cup L_T$. Because M is surjective on each arm of L and since M is pointwise finite-dimensional, each arm of L decomposes into a disjoint union of finitely many intervals of R, S or T on which $M_{|L}$ is of constant dimension. We denote these by $I_R^i \subseteq L_R$, $I_S^j \subseteq L_S$ and $I_T^k \subseteq L_T$ for indices i, j, k, so $L_R = \coprod_i L_R^i$, etc.

Now consider the subsets of shape $I_R^i \times I_S^j \times I_T^k$. We want to prove: If $p \leq q$ are elements of this set, then the structure map from p to q is an isomorphism. By symmetry, it is sufficient to show this only for $p, q \in I_R^i \times I_S^j \times \{t\}$ for a fixed $t \in I_T^k$. In fact, we show this for the front face $L_R \times L_S \times \{t\}$ of I, as the remaining is similar. So consider the following commutative diagram in M, where $r, s \in I$, $m, n, \in I_S^j$ and $u, v \in I_R^i$:

$$\begin{array}{ccccc} M_v & \longrightarrow & M_r & \xrightarrow{\beta''} & M_q \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle\alpha'}\uparrow & & {\scriptstyle\alpha''}\uparrow \\ M_u & \longrightarrow & M_p & \xrightarrow{\beta'} & M_s \\ \uparrow & & \uparrow & & \uparrow \\ M_a & \longrightarrow & M_m & \xrightarrow{\beta} & M_n \end{array}$$

By assumption the maps α and β are isomorphisms. Since all squares involved are pushouts by assumption, all the parallel maps $\alpha', \alpha'', \beta'$ and β'' are isomorphisms, too.

Let $L'_R \subseteq L_R, L'_S \subseteq L_S$ and $L'_T \subseteq L_T$ be finite subsets which all contain the element a and at least one point of each interval I^i_R, I^j_S or I^k_T. Write $I' = L'_R \times L'_S \times L'_T$ and $L' = L'_R \cup L'_S \cup L'_T$. We can assume that $I' \cong [n]^3$ for $n \in \mathbb{N}$.

From the arguments above it follows that M is up to isomorphism completely determined by its restriction $M_{|L'}$. But this is equivalent to a representation of $[n]^3$. □

Lemma 5.6.10. *Let M be an indecomposable pointwise finite-dimensional representation in* $\operatorname{Mod} kP$ *which is* 3*-middle exact and not* 3*-left exact. Then M is isomorphic to a death-block summand.*

Proof. By assumption we have $\operatorname{Ker} M^{\measuredangle} \neq 0$. We choose a nonzero element m from this. Let L be the union of rays in P which start in $p = |m|$ and are parallel to the axes and let $I = p \uparrow$ be the ideal generated by p. By Lemma 5.6.6, all structure maps of $\operatorname{Ker} M^{\measuredangle}$ are epimorphisms. Thus, by pointwise finite-dimensionality it follows from Lemma 5.6.9 that $(\operatorname{Ker} M^{\measuredangle})_{|I}$ is an extension of a representation of $[n]^3$ by isomorphisms, which is isomorphic to a direct sum of death blocks inside I by Case 2 of Theorem 5.4.2. Then we can extend this to a death-block in P, using Lemma 5.6.3 once along each axis. □

Lemma 5.6.11. *Let M be an indecomposable pointwise finite-dimensional representation in* $\operatorname{Mod} kP$ *which is* 3*-middle exact and not* 3*-right exact. Then M is isomorphic to a birth-block summand.*

Proof. This follows from Lemma 5.6.10 by pointwise duality. □

5.6.12 Case M is 3-properly exact, but not 2-short exact

Lemma 5.6.13. *Let M be a pointwise finite-dimensional representation in* $\operatorname{Mod} kP$ *which is* 3*-properly exact, but not* 2*-left exact. Then M has an injective direct summand with the shape of a two-parameter death block times an axis.*

Proof. Since M is not 2-left exact, there is a plane restricted to which M has a death block, say there is a death block $N \cong k_{|J_R \times J_S \times \{t\}}$ of $M_{R \times S \times \{t\}}$. By 3-left exactness we have $\operatorname{Ker} M^{\measuredangle} = 0$, so $N \subseteq (\operatorname{Ker} M^{\llcorner})_{|J_R \times J_S \times \{t\}}$ can be injectively extended in positive direction of the T axis. Thus, N extends to a submodule $\tilde{N}$ of $M_{|R \times S \times [t, \infty)}$. Now we can use Lemma 5.6.3 and obtain a submodule of M which is a block of shape $J_R \times J_S \times T$.

The other cases are symmetric. □

Dually, we get:

Lemma 5.6.14. *Let M be pointwise finite dimensional and* 3*-properly exact, but not* 2*-right exact. Then M has as direct summand a block representation with the shape of a* 2*-birth block times an axis.*

5.6.15 Case: M is 2-short exact

Lemma 5.6.16. *Let M be a pointwise finite-dimensional representation in* $\operatorname{Mod} kP$ *which is* 2*-short exact. Furthermore, suppose that one of* $\operatorname{Ker} M^{\rightharpoonup} \cap \operatorname{Im} M^{\rightharpoonup} \neq 0$, $\operatorname{Ker} M^{\upharpoonleft} \cap \operatorname{Im} M^{\upharpoonleft} \neq 0$, *or* $\operatorname{Ker} M^{\nearrow} \cap \operatorname{Im} M^{\nearrow} \neq 0$ *holds. Then M has a block summand for a block which is the product of an ideal and two axes.*

Proof. Suppose $\operatorname{Ker} M^{\rightharpoonup} \cap \operatorname{Im} M^{\rightharpoonup} \neq 0$ holds. Then for $t \in T$, the restriction $M_{|R\times S\times\{t\}}$ has a two-parameter block module N of type proper ideal times axis by the 2-short exact case of the two-parameter decomposition Theorem 5.2.4. Say this is $N \cong k_B$ for $B = R \times J_S \times \{t\}$, with J_S bounded by an element $\varepsilon \in S \setminus J_S$. By 2-left exactness, N extends monomorphically in T-direction to a submodule $\tilde{N} \cong k_{R\times J_S\times[t,\infty)}$. Then we can extend this in decreasing T-direction with Lemma 5.6.4 to obtain an injective submodule of isomorphism type $k_{R\times J_S\times T}$ and therefore a direct summand.

The other cases are symmetric. □

Now, we cover the last remaining case.

Lemma 5.6.17. *Let M be a pointwise finite-dimensional representation in* $\operatorname{Mod} kP$ *which is* 2*-short exact. Furthermore, assume that* $\operatorname{Ker} M^{\rightharpoonup} \cap \operatorname{Im} M^{\rightharpoonup} = 0$, $\operatorname{Ker} M^{\upharpoonleft} \cap \operatorname{Im} M^{\upharpoonleft} = 0$ *and* $\operatorname{Ker} M^{\nearrow} \cap \operatorname{Im} M^{\nearrow} = 0$ *hold. Then M has a block summand of shape interval times axis times axis.*

Proof. By 2-right exactness we have

$$M = \operatorname{Im} M^{\upharpoonleft} + \operatorname{Im} M^{\rightharpoonup} = \operatorname{Im} M^{\upharpoonleft} + \operatorname{Im} M^{\nearrow} = \operatorname{Im} M^{\nearrow} + \operatorname{Im} M^{\rightharpoonup}.$$

Suppose we have $\operatorname{Im} M^{\upharpoonleft} \cap \operatorname{Im} M^{\rightharpoonup} = 0$. Then $M = \operatorname{Im} M^{\rightharpoonup}$ or $M = \operatorname{Im} M^{\upharpoonleft}$. If the first case holds, then in M all arrows parallel to the R-axis are isomorphisms. Now, take for any $r \in R$ a block decomposition of $\{r\} \times S \times T$ according to Theorem 5.2.4 and extend it isomorphically along R. This yields a block decomposition of M. The other cases are symmetric.

We remain with the case that all pairwise intersections of the image submodules are nontrivial. If $\operatorname{Im} M^{\upharpoonleft} \cap \operatorname{Im} M^{\nearrow} \cap \operatorname{Im} M^{\rightharpoonup} \neq 0$, then all structure maps of this intersection are epimorphisms, thus there is a direct summand of type k_P by Lemma 3.2.12. So we assume in the following for this intersection that $\operatorname{Im} M^{\upharpoonleft} \cap \operatorname{Im} M^{\nearrow} \cap \operatorname{Im} M^{\rightharpoonup} = 0$.

We claim that $M = \operatorname{Im} M^{\upharpoonleft} \cap \operatorname{Im} M^{\rightharpoonup} + \operatorname{Im} M^{\upharpoonleft} \cap \operatorname{Im} M^{\nearrow} + \operatorname{Im} M^{\nearrow} \cap \operatorname{Im} M^{\rightharpoonup}$. Explicitly, this means: For all $m \in M$ and for all $p \in P$ with $p \leq |m|$ there are elements m_1, m_2 and m_3 in M, indexed according to the cube (5.4) with minimum p and maximum $|m|$, such that

$$m = d_2^{123} d_3^{13}(m_1) + d_3^{123} d_1^{12}(m_2) + d_1^{123} d_2^{23}(m_3)$$

Now, choose any $m \in M$ and $p \leq |m|$ in P and let $\iota\colon C \to P$ be a cube such that $\iota(\emptyset) = p$ and $\iota(\{1,2,3\}) = |m|$. By Case 1 of Theorem 5.4.2 in the special case of $n = 2$ there is a block decomposition of the restriction $\iota^* M$, which is just a representation of a finite cube. Since $\iota^* M$ is short exact, it decomposes into block modules corresponding to the faces of ιC. The blocks cannot be the entire interval ιC, as $\operatorname{Im} M^{\uparrow} \cap \operatorname{Im} M^{\nearrow} \cap \operatorname{Im} M^{\rightarrow} = 0$. But from this decomposition it follows that we can find such elements m_1, m_2 and m_3 as above.

Note that the vanishing of this three-fold intersection also implies that we have a direct sum

$$M = \operatorname{Im} M^{\uparrow} \cap \operatorname{Im} M^{\rightarrow} \oplus \operatorname{Im} M^{\uparrow} \cap \operatorname{Im} M^{\nearrow} \oplus \operatorname{Im} M^{\nearrow} \cap \operatorname{Im} M^{\rightarrow}.$$

So, if M is indecomposable, then it equals one of the summands, without limitation of generality, let $M = \operatorname{Im} M^{\uparrow} \cap \operatorname{Im} M^{\rightarrow}$. In this case, since by assumption we have $\operatorname{Im} M^{\uparrow} \cap \operatorname{Ker} M^{\uparrow} = 0$ and $\operatorname{Im} M^{\rightarrow} \cap \operatorname{Ker} M^{\rightarrow} = 0$, all morphisms of M in R- and S-direction are subsequently isomorphisms. So taking an interval decomposition of M restricted to a line parallel to the T-axis, see Theorem 3.4.3, extends to a direct sum decomposition of M, where all terms occurring are of type interval times axis times axis, that is $R \times S \times I$ for an interval $I \subseteq T$. ☐

Now, we conclude the proof of the main theorem.

Proof of Theorem 5.6.1. The preceding lemmata cover all possible cases on the exactness conditions, which proves the assertion. ☐

5.7 Middle exactness revisited

In this section we delve into the definition of middle exactness, again. First, we notice the connections with the notion of weak (co)limits. Thereafter we briefly mention in Section 5.7.7, why 3-middle exactness does not appear as easily as 2-middle exactness in topology. In the remaining subsections we present our current state of work, associating the theory of middle exactness with the theory of excisive functors.

5.7.1 Middle exactness and weak (co)limits

Another way to express the properties of Section 5.2 makes use of the language of weak limits and weak colimits. Recall that for an ordinary category $\mathcal{C}$ a *weak (co-)limit* is defined like a (proper) (co-)limit, but without demanding the uniqueness of the universal property, see Section 2.7.

Lemma 5.7.2. *For the commutative square of vector spaces (5.2) the following are equivalent:*

- *It is a weak pushout square.*
- *It is a weak pullback square.*
- *The complex (5.3) is middle exact.*

Proof. This is an immediate consequence of Lemma 2.7.2. ☐

Lemma 5.7.3. *For the commutative cube M of vector spaces (5.4) the following are equivalent:*

- *$M_\emptyset$ is a weak limit of the restriction of the cube to all other vertices and M_{123} is a weak colimit of the restriction of the cube to all other vertices.*
- *The complex (5.6) is middle exact.*

Proof. The leftmost arrow of the complex is a weak kernel if and only if $H_2 M = 0$, by Lemma 2.7.2. Similarly, the rightmost arrow is a weak cokernel if and only if $H_1 M = 0$.

Now, observe that a cone to the restriction of the cube M to $\mathcal{P}([3]) \setminus \{\emptyset\}$ is a morphism $t \colon T \to M_1 \oplus M_2 \oplus M_3$ with $\operatorname{Im} t \subseteq \operatorname{Ker} A$, and therefore, all such morphisms factor through $M_\emptyset$ if and only if the leftmost morphism is a weak kernel. The other case is dual. □

As an immediate consequence we characterisations of middle exactness.

Corollary 5.7.4. *Let $P = R \times S$ be a direct product of totally ordered sets and M in $\operatorname{Mod} kP$. Then M is 2-middle exact if and only if for all squares $\iota \colon Q \to P$ the following equivalent conditions hold:*

- *$\iota^* M$ is a weak pullback square.*
- *$\iota^* M$ is a weak pushout square.* □

Corollary 5.7.5. *Let $P = R \times S \times T$ be a direct product of totally ordered sets and M in $\operatorname{Mod} kP$. Then M is 3-middle exact if and only if for all cubes $\iota \colon C \to P$ the following holds:*

- *The minimal object $\iota^* M(\emptyset)$ is a weak limit of the restriction of the cube $\iota^* M$ to all other vertices.*
- *The maximal object $\iota^* M(\{1,2,3\})$ is a weak colimit of the restriction of the cube $\iota^* M$ to all other vertices.* □

In categories defined by posets, there is at most one morphism between two objects. As a consequence, we have

Lemma 5.7.6. *Let P be a partially ordered set. Then weak (co)limits in P are exactly the (co)limits in P.* □

5.7.7 Topological motivation

With the 2-middle exactness stemming from the Mayer–Vietoris sequence, it suggest itself to ask if there is a similar motivation for 3-middle exactness. The obvious culprit is that there is no Mayer–Vietoris type long exact sequence for a 3-subspace open covering (U_1, U_2, U_3) of a space $X = U_1 \cup U_2 \cup U_3$. Calculating the homology of X can be approached by using the *Mayer–Vietoris spectral sequence*, which is the spectral sequence coming from the double complex which is the Čech-complex in one direction and the chain complex of a homology theory in the other direction. Though it seems to be of great importance, there is only little coverage in literature, see for example [BT82; GS18].

But deriving a long exact sequence from this spectral sequence which incorporates a middle exact complex like (5.6) seems to be unlikely for a four-term complex, as spectral sequences are rather a tool to calculate the homology of a double complex, while we are interested in the structural properties of this double complex. Also, there are even very simple examples for which a canonical inclusion-exclusion complex fails to be exact. Assuming exactness of inclusion-exclusion complexes for more than two subobjects is actually an infamous fallacy.

5.7.8 (Weakly) n-excisive functors

Only within this section we require the theory of ∞-categories. For an exhaustive treatments of this topic, see [Cis19; Lur09; Lur23].

Let $\mathcal{C}$ be an ordinary category. We consider the *categorical nerve* functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$ from the category of small categories to simplicial sets. Then the nerve $N(\mathcal{C})$ of $\mathcal{C}$ is an ∞-category, see for example [Lur09, Proposition 1.2.3.1].

For the set of subsets of $[n]$ of cardinality greater than or equal to k (smaller than or equal to k) we write $\mathcal{P}_{\geq k}([n])$ ($\mathcal{P}_{\leq k}([n])$). Note that the cube C is the same as the power set $\mathcal{P}([3])$.

The following definitions are based on Goodwillie's work [Goo03]. The generalisation to infinity categories is from [Lur17, Section 6.1.1], while the dual and the weak notion are from [Eld16] and [Wal20].

Definition 5.7.9. Let $\mathbf{C}$ be an ∞-category. An *n-cube* in $\mathbf{C}$ is a functor $X\colon N(\mathcal{P}([n])) \to \mathbf{C}$. We call X *Cartesian* if it is a limit diagram, meaning that there is an equivalence

$$X(\emptyset) \to \varprojlim_{\emptyset \neq S \subseteq [n]} X(S).$$

The n-square X is called *strongly Cartesian* if X is a right Kan extension of its restriction to $\mathcal{P}_{\geq n-1}([n])$. Dually, we define a *coCartesian* and a *strongly coCartesian* cube. An n-cube is *strongly biCartesian* if it is both strongly Cartesian and strongly coCartesian.

Now, we can define the properties of interest:

Definition 5.7.10. Let $\mathbf{C}$ be an ∞-category with finite colimits and let $\mathbf{D}$ be an ∞-category with finite limits and let $n \geq 0$. Then a functor $F\colon \mathbf{C} \to \mathbf{D}$ is called *(weakly) n-excisive* if for every strongly coCartesian (strongly biCartesian) $(n+1)$-cube X in $\mathbf{C}$, the composition $F \circ X$ is Cartesian in $\mathbf{D}$. Dually, F is *(weakly) n-coexcisive* if for every strongly Cartesian (strongly biCartesian) $(n+1)$-cube X in $\mathbf{C}$, the composition $F \circ X$ is a coCartesian in $\mathbf{D}$.

Just as a linear function can be seen as a degenerate quadratic function, n-excisiveness also passes up to a higher degree:

Lemma 5.7.11 ([Lur17, Corollary 6.1.1.14])**.** *Let F be an n-excisive functor for $n \geq 0$. Then F is m-excisive for any $m \geq n$.* □

Note that the nerve N has a left adjoint, the homotopy functor ho: sSet $\to$ Cat and that taking the nerve is fully faithful, so the counit $\mathrm{ho} \circ N(\mathcal{C}) \to \mathcal{C}$ is an equivalence for every category $\mathcal{C}$. Next, note that the ∞-categorical constructions for limits and colimits are related with the respective (ordinary) categorical constructions in homotopy:

Lemma 5.7.12 ([Lur23, Example 02JD])**.** *Let $\mathcal{C}$ be an ordinary category. Then a natural transformation $\Delta_I Y \to F$ for an object Y of $\mathcal{C}$ is the limit of a diagram $F\colon I \to \mathcal{C}$ if and only if its nerve is the limit of the nerve of the diagram $N(F)\colon N(I) \to N(\mathcal{C})$. The dual holds for colimits.* □

5.7.13 Middle exact functors and weakly excisive functors

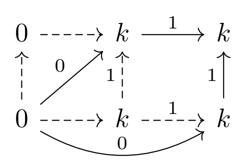
Next, we associate properly exact representations with these notions. Consider now a functor M in $\mathrm{Mod}\, kP$ for a field k and a poset P. This gives rise to an functor $N(M)\colon N(P) \to N(\mathrm{Mod}\, k)$. Now, for $n = 2, 3$, an n-cube X in $N(P)$ corresponds to a functor $\mathrm{ho}\, X\colon \mathcal{P}([n]) \to P$, that is a morphism of posets from $\mathcal{P}([n])$ to P. X is moreover strongly coCartesian if and only if $\mathrm{ho}\, X$ is a cube in P with the possible exception that the minimal element of $\mathrm{ho}\, X$ is smaller than the infimum of all vertices, by Lemma 5.7.12 and Lemma 5.7.6. Similarly, X is strongly biCartesian if and only if $\mathrm{ho}\, X$ is a biCartesian square or cube in P.

In view of other work, see an upcoming version of [BBF22], it suggests itself to ask, whether the class of representations we are studying is excisive in this sense, but this is not the case:

Example 5.7.14. Consider the following 2-short exact representation of the 3-step ladder:

$$\begin{array}{ccccc} 0 & \longrightarrow & k & \xrightarrow{1} & k \\ \uparrow & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} \\ 0 & \longrightarrow & k & \xrightarrow{1} & k \end{array}$$

All involved rectangles are 2-middle exact. Now compose the morphisms on the upper left corner and on the lower edge:



The underlying shape of this new deformed rectangle is still strongly coCartesian, as the upper right vertex is the supremum of both middle vertices. But the corresponding diagram is not 2-middle exact. This suggests that we must restrict to those squares in P which are biCartesian.

We observe:

Proposition 5.7.15. *Let $P = R \times S$ (or $P = R \times S \times T$) and M in* $\operatorname{Mod} kP$*. Then M is 2-short exact if and only if the nerve $N(M)$ is weakly 1-excisive. Moreover, M is 3-right (left) exact if and only if the nerve $N(M)$ is weakly 2-excisive (weakly 2-coexcisive).*

Proof. As discussed above, strongly biCartesian 2-cubes (3-cubes) X of the nerve $N(P)$ correspond to squares (cubes) $\operatorname{ho} X$ in the homotopy category P. Then the weak 1-excisiveness means exactly that the square is a pushout (the maximal/minimal point of the cube is a colimit/limit). □

Note that $N(M)$ is weakly 1-excisive if and only if it is weakly 1-coexcisive by the self-dual nature of 2-middle exactness. Also, in this context Lemma 5.2.10 implies a result similar to Lemma 5.7.11 for this kind of functors.

Weakly excisive functors and middle exactness

Let $P = R \times S$ be the product of two totally ordered sets and let $\mathcal{D}(k)$ denote the derived ∞-category of vector spaces, that is an enhancement of the classical (unbounded) derived category $D(k)$ of vector spaces as an ∞-category. Then a weakly excisive functor

$$\mathcal{M}\colon N(P) \to \mathcal{D}(k)$$

gives rise to a middle exact representation: By definition, every biCartesian square is mapped to a biCartesian square in $\mathcal{D}(k)$:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ C & \xrightarrow{\gamma} & D \end{array}$$

Then, applying the representable functor $\operatorname{Hom}_{\mathcal{D}(k)}(k, -)$, where k denotes the *stalk complex in degree 0*, which is the complex having k in degree 0 and vanishes everywhere else, yields a homotopy pullback in the simplicial category of Kan-complexes [Lur09, Section 4.4.2]:

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{D}(k)}(k, A) & \longrightarrow & \operatorname{Hom}_{\mathcal{D}(k)}(k, B) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathcal{D}(k)}(k, C) & \longrightarrow & \operatorname{Hom}_{\mathcal{D}(k)}(k, D), \end{array}$$

and therefore a homotopy fibre sequence, giving rise to the long exact sequence

$$\begin{aligned} \ldots \to \pi_0 \operatorname{Hom}_{\mathcal{D}(k)}(k, A) \to \pi_0 \operatorname{Hom}_{\mathcal{D}(k)}(k, B) \oplus \pi_0 \operatorname{Hom}_{\mathcal{D}(k)}(k, C) \\ \to \pi_0 \operatorname{Hom}_{\mathcal{D}(k)}(k, D) \to \ldots, \end{aligned}$$

see for instance [Lur23, Theorem 00WM]. Now, we have the isomorphism:

$$\pi_i \operatorname{Hom}_{\mathcal{D}(k)}(k, X) \cong \operatorname{Hom}_{D(k)}(k, X[i]) \cong H_i(X)$$

for all chain complexes X in $D(k)$, so we obtain the homology long exact sequence

$$\dots \to H_{i+1}D \to H_iA \to H_iB \oplus H_iC \to H_iD \to H_{i-1}A \to \dots$$

Restriction to a single degree yields a middle exact sequence as in Equation (5.3). Hence, we get:

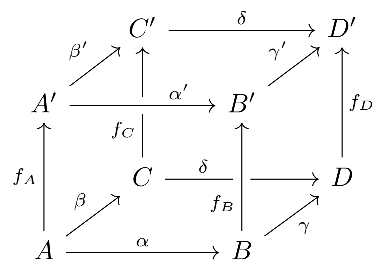
Corollary 5.7.16. *For $P = R \times S$ a product of totally ordered sets, consider a weakly excisive functor $\mathcal{M} \colon N(P) \to \mathcal{D}(k)$. Then $H_0 \circ \operatorname{ho} \mathcal{M}$ is a middle exact representation.* □

Pushing down to the homotopy category, we are studying functors $P \to D(k)$ such that for every biCartesian square in P the canonical sequence

$$A \to B \oplus C \to D$$

is a triangle in $D(k)$.

This raises the question if middle exact representations in three parameters are a manifestation of a similar phenomenon. But this relation seems to be more complicated. For this, consider a strongly biCartesian cube of chain complexes in $\mathrm{Ch}(\mathcal{A})$ as in diagram (5.4):



(5.10)

By assumption, all faces are biCartesian, in particular the top and the bottom square. Analogously as above we obtain two homology long exact sequences and a map between them:

$$\begin{array}{ccccccccc}
H_{i+1}(D') & \longrightarrow & H_i(A') & \longrightarrow & H_i(B') \oplus H_i(C') & \longrightarrow & H_i(D') & \longrightarrow & H_{i-1}(A') \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_{i+1}(D) & \longrightarrow & H_i(A) & \longrightarrow & H_i(B) \oplus H_i(C) & \longrightarrow & H_i(D) & \longrightarrow & H_{i-1}(A)
\end{array}$$

Similarly to Example 5.2.5, we can then take the mapping cone of this morphism, truncate it and obtain the exact sequence

$$\begin{aligned}
&H_{i+1}(D') \oplus H_i(A) \to H_i(B) \oplus H_i(C) \oplus H_i(A') \\
&\qquad \to H_i(D) \oplus H_i(B') \oplus H_i(C') \to H_i(D') \oplus H_{i-1}(A).
\end{aligned}$$

But $H_{i+1}(D')$ in the first term and $H_i(D')$ in the last term do not necessarily vanish and therefore we do not completely obtain the desired four-term middle exact sequence.

Conversely, it is not clear to us, yet, whether all middle exact representations come from a weakly excisive functor.

Some persistence modules are revealed to be excisive functors in an upcoming version of [BBF22]. There, they are interpreted as homological functors from a poset which carries a much richer ∞-structure.

Chapter 6

Outlook

The spectrum of persistence modules

In Chapter 4 we saw that for one parameter persistence modules the spectrum can easily described in terms of the ordered space of ideals. But multiparameter persistence modules are also of great interest. This raises the question if these results can also be generalised to such posets.

The structure of indecomposable injectives and finitely presented objects is more complicated in general and the lack of a barcode decomposition theorem prohibits a straightforward generalisation of our approach. A priori it is not even clear that for an arbitrary poset the representations are locally coherent. But for upper semi-lattices this was shown in [CJT21]. Another hitch is that more general posets may make case distinctions as in Section 4.5 more complicated. Building on this analysis, an interesting question is then to ask about the relation between the spectrum for each factor of the product and the spectrum of the product.

Middle exact representations

In Chapter 5 we proved a block decomposition theorem for three parameter persistence modules. An immediate follow-up question is if this generalises to more parameters. The foundation of the generalisations of both middle exactness and blocks to three parameters was the discovery of an inductive principal. Therefore there seems to be no problem to define these notions for more parameters. Though we have not yet attempted a further generalisation, many steps also seem to work in higher dimensions.

Nonetheless, one of the main challenges appears to be to get an intuition for cubes in dimension four and higher, as well as for their respective exactness conditions. The use of computer algebra systems for examples seems to be inevitable, here. Among all steps, the hardest to generalise seems to be Lemma 5.6.17, because it embodies the inclusion-exclusion principle and therefore hideous combinatorics.

Another question, brought up in a discussion with V. Lebovici, F. Petit and S. Oudot, was if it is possible to use the method of functorial filtrations for an alternative proof. We followed another approach here. Though the author raised concerns that this alternative approach possibly just transfers the complications in the proof to another point, this might be worth further investigations.

Then, in Section 5.7 we exposed some relations of middle exact representations with constructions from other fields of mathematics. Though we could not find a counterpart of Corollary 5.7.16 for three-parameter representations, this seems to be desirable to reduce the conditions of demanding 2- and 3-middle exactness to only one condition. Furthermore, to answer the question positively if every 2-middle exact representation comes from an excisive functor might boil down to the question if we can lift weak pushout-pullback squares of vector spaces to biCartesian squares of chain complexes, or to such weak squares in the derived category $D(k)$ of complexes over k. It is known that long exact sequences can be lifted to short exact sequences of chain complexes of abelian groups, see [Pre90].

Moreover, studying how middle exact representations in three or more parameters naturally appear in applied topology seems desirable.

Bibliography

[ASS06] Ibrahim Assem, Daniel Simson and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1.* Vol. 65. London Mathematical Society Student Texts. Techniques of representation theory. Cambridge University Press, Cambridge, 2006, pp. x+458. DOI: 10.1017/CBO9780511614309.

[Azu50] Gorô Azumaya. “Corrections and supplementaries to my paper concerning Krull-Remak-Schmidt’s theorem”. In: *Nagoya Math. J.* 1 (1950), pp. 117–124.

[BBF22] Ulrich Bauer, Magnus Bakke Botnan and Benedikt Fluhr. *Structure and Interleavings of Relative Interlevel Set Cohomology.* 2022. arXiv: 2108.09298 [math.AT].

[BCB20] Magnus Bakke Botnan and William Crawley-Boevey. “Decomposition of persistence modules”. In: *Proc. Amer. Math. Soc.* 148.11 (2020), pp. 4581–4596. DOI: 10.1090/proc/14790.

[BEMP13] Paul Bendich, Herbert Edelsbrunner, Dmitriy Morozov and Amit Patel. “Homology and robustness of level and interlevel sets”. In: *Homology Homotopy Appl.* 15.1 (2013), pp. 51–72. DOI: 10.4310/HHA.2013.v15.n1.a3.

[BT82] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology.* Vol. 82. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982, pp. xiv+331.

[Bor94a] Francis Borceux. *Handbook of Categorical Algebra. 1.* Vol. 50. Encyclopedia of Mathematics and its Applications. Basic category theory. Cambridge University Press, Cambridge, 1994, pp. xvi +345.

[Bor94b] Francis Borceux. *Handbook of Categorical Algebra. 2.* Vol. 51. Encyclopedia of Mathematics and its Applications. Categories and structures. Cambridge University Press, Cambridge, 1994, pp. xviii+443.

[Bru78] Hermann Brune. “Flache Darstellungen von geordneten Mengen”. In: *Manuscripta Math.* 26.1-2 (1978/79), pp. 141–154. DOI: 10.1007/BF01167970.

[CB94] William Crawley-Boevey. "Locally finitely presented additive categories". In: *commun. Algebra* 22.5 (1994), pp. 1641–1674. DOI: 10.1080/00927879408824927.

[CJT21] Wojciech Chacholski, Alvin Jin and Francesca Tombari. *Realisations of posets and tameness*. 2021. arXiv: 2112.12209 [math.AT].

[CO20] Jérémy Cochoy and Steve Oudot. "Decomposition of exact pfd persistence bimodules". In: *Discrete Comput. Geom.* 63.2 (2020), pp. 255–293. DOI: 10.1007/s00454-019-00165-z.

[CS10] Gunnar Carlsson and Vin de Silva. "Zigzag persistence". In: *Found. Comput. Math.* 10.4 (2010), pp. 367–405. DOI: 10.1007/s10208-010-9066-0.

[CSEH07] David Cohen-Steiner, Herbert Edelsbrunner and John Harer. "Stability of persistence diagrams". In: *Discrete Comput. Geom.* 37.1 (2007), pp. 103–120. DOI: 10.1007/s00454-006-1276-5.

[CSEH08] David Cohen-Steiner, Herbert Edelsbrunner and John Harer. "Extending Persistence Using Poincaré and Lefschetz Duality". In: *Foundations of Computational Mathematics* 9.1 (Apr. 2008), pp. 79–103. DOI: 10.1007/s10208-008-9027-z.

[CSM09] Gunnar Carlsson, Vin de Silva and Dmitriy Morozov. "Zigzag Persistent Homology and Real-Valued Functions". In: *Proceedings of the Twenty-Fifth Annual Symposium on Computational Geometry*. SCG '09. Aarhus, Denmark: Association for Computing Machinery, 2009, 247–256. DOI: 10.1145/1542362.1542408.

[CVJ22] Gunnar Carlsson and Mikael Vejdemo-Johansson. *Topological data analysis with applications*. Cambridge University Press, Cambridge, 2022, pp. xi+220. DOI: 10.1017/9781108975704.

[Car09] Gunnar Carlsson. "Topology and data". In: *Bull. Amer. Math. Soc. (N.S.)* 46.2 (2009), pp. 255–308. DOI: 10.1090/S0273-0979-09-01249-X.

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*. Vol. 180. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2019, pp. xviii+430. DOI: 10.1017/9781108588737.

[Cur14] Justin Curry. *Sheaves, Cosheaves and Applications*. 2014. arXiv: 1303.3255 [math.AT].

[Dor32] J. L. Dorroh. "Concerning adjunctions to algebras". In: *Bull. Amer. Math. Soc.* 38.2 (1932), pp. 85–88. DOI: 10.1090/S0002-9904-1932-05333-2.

[EH10] Herbert Edelsbrunner and John L. Harer. *Computational topology*. An introduction. American Mathematical Society, Providence, RI, 2010, pp. xii+241. DOI: 10.1090/mbk/069.

[EH16] Emerson G. Escolar and Yasuaki Hiraoka. “Persistence modules on commutative ladders of finite type”. In: *Discrete Comput. Geom.* 55.1 (2016), pp. 100–157. DOI: 10.1007/s00454-015-9746-2.

[EH18] Karin Erdmann and Thorsten Holm. *Algebras and representation theory.* Springer Undergraduate Mathematics Series. Springer, Cham, 2018, pp. ix+298. DOI: 10.1007/978-3-319-91998-0.

[ELZ02] Edelsbrunner, Letscher and Zomorodian. “Topological Persistence and Simplification”. In: *Discrete & Computational Geometry* 28.4 (Nov. 2002), pp. 511–533. DOI: 10.1007/s00454-002-2885-2.

[Eld16] Rosona Eldred. “Goodwillie calculus via adjunction and LS cocategory”. In: *Homology Homotopy Appl.* 18.2 (2016), pp. 31–58. DOI: 10.4310/HHA.2016.v18.n2.a2.

[Fre60] Peter J. Freyd. *Functor theory.* Thesis (Ph.D.)–Princeton University. ProQuest LLC, Ann Arbor, MI, 1960, p. 56.

[GJ73] Laurent Gruson and Christian U. Jensen. “Modules algébriquement compacts et foncteurs $\varprojlim^{(i)}$”. In: *C. R. Acad. Sci. Paris Sér. A-B* 276 (1973), A1651–A1653.

[GO66] Peter Gabriel and Ulrich Oberst. “Spektralkategorien und reguläre Ringe im von-Neumannschen Sinn”. In: *Math. Z.* 92 (1966), pp. 389–395. DOI: 10.1007/BF01112218.

[GR97] Pierre Gabriel and Andrei V. Roiter. *Representations of finite-dimensional algebras.* Translated from the Russian, With a chapter by B. Keller, Reprint of the 1992 English translation. Springer-Verlag, Berlin, 1997, pp. iv+177. DOI: 10.1007/978-3-642-58097-0.

[GS18] Dejan Govc and Primoz Skraba. “An approximate nerve theorem”. In: *Found. Comput. Math.* 18.5 (2018), pp. 1245–1297. DOI: 10.1007/s10208-017-9368-6.

[GUD23] The GUDHI Project. *GUDHI User and Reference Manual.* https://gudhi.inria.fr/doc/3.8.0/. 2023.

[Gab62] Pierre Gabriel. “Des catégories abéliennes”. In: *Bull. Soc. Math. France* 90 (1962), pp. 323–448.

[Gab72] Peter Gabriel. “Unzerlegbare Darstellungen. I”. In: *Manuscripta Math.* 6 (1972), 71–103; correction, ibid. 6 (1972), 309. DOI: 10.1007/BF01298413.

[Goo03] Thomas G. Goodwillie. “Calculus. III. Taylor series”. In: *Geom. Topol.* 7 (2003), pp. 645–711. DOI: 10.2140/gt.2003.7.645.

[Gro57] Alexander Grothendieck. “Sur quelques points d’algèbre homologique”. In: *Tohoku Math. J. (2)* 9 (1957), pp. 119–221. DOI: 10.2748/tmj/1178244839.

[HL81] Michael Höppner and Helmut Lenzing. "Projective diagrams over partially ordered sets are free". In: *J. Pure Appl. Algebra* 20.1 (1981), pp. 7–12. DOI: 10.1016/0022-4049(81)90045-1.

[HLŠV22] Lidia Angeleri Hügel, Rosanna Laking, Jan Šťovíček and Jorge Vitória. *Mutation and torsion pairs*. 2022. arXiv: 2201.02147 [math.RT].

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544.

[Her97] Ivo Herzog. "The Ziegler spectrum of a locally coherent Grothendieck category". In: *Proc. London Math. Soc. (3)* 74.3 (1997), pp. 503–558. DOI: 10.1112/S002461159700018X.

[Höp81] Michael Höppner. *Diagramme über geordneten Mengen und einige Analogien zur Theorie abelscher Gruppen*. Dissertation. Universität-Gesamthochschule-Paderborn, Fachbereich Mathematik-Informatik, Paderborn, 1981, pp. i+69.

[Höp83] Michael Höppner. "A note on the structure of injective diagrams". In: *Manuscripta Math.* 44.1-3 (1983), pp. 45–50. DOI: 10.1007/BF01166072.

[Hus95] A. A. Husainov. "On the Hochschild-Mitchell dimension of subsets of the reals". In: *Second International Conference on Algebra (Barnaul, 1991)*. Vol. 184. Contemp. Math. Amer. Math. Soc., Providence, RI, 1995, pp. 207–213. DOI: 10.1090/conm/184/02116.

[IRT23] Kiyoshi Igusa, Job D. Rock and Gordana Todorov. "Continuous quivers of type A (I) foundations". In: *Rend. Circ. Mat. Palermo (2)* 72.2 (2023), pp. 833–868. DOI: 10.1007/s12215-021-00691-x.

[JL89] Christian U. Jensen and Helmut Lenzing. *Model-theoretic algebra with particular emphasis on fields, rings, modules*. Vol. 2. Algebra, Logic and Applications. Gordon and Breach Science Publishers, New York, 1989, pp. xiv+443.

[Joh82] Peter T. Johnstone. *Stone spaces*. Vol. 3. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1982, pp. xxi+370.

[KOTMH18] M. Kimura, I. Obayashi, Y. Takeichi, R. Murao and Y. Hiraoka. "Non-Empirical Identification of Trigger Sites in Image Data using Persistent Homology: Crack Formation during Heterogeneous Reduction of Iron-Ore Sinters". In: *Microscopy and Microanalysis* 24.S2 (2018), pp. 540–541.

[Kap58] Irving Kaplansky. "Projective modules". In: *Ann. of Math (2)* 68 (1958), pp. 372–377. DOI: 10.2307/1970252.

[Kra22] Henning Krause. *Homological theory of representations*. Vol. 195. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2022, p. 482.

[Kra97] Henning Krause. "The spectrum of a locally coherent category". In: *J. Pure Appl. Algebra* 114.3 (1997), pp. 259–271. DOI: 10.1016/S0022-4049(95)00172-7.

[Kra98] Henning Krause. "Exactly definable categories". In: *J. Algebra* 201.2 (1998), pp. 456–492. DOI: 10.1006/jabr.1997.7252.

[LS00] Zbigniew Leszczyński and Andrzej Skowroński. "Tame triangular matrix algebras". In: *Colloq. Math.* 86.2 (2000), pp. 259–303. DOI: 10.4064/cm-86-2-259-303.

[Ler22] Jan-Paul Lerch. *The spectrum of the real line*. 2022. arXiv: 2209.01101 [math.RT].

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. DOI: 10.1515/9781400830558.

[Lur17] Jacob Lurie. *Higher Algebra*. https://people.math.harvard.edu/~lurie/papers/HA.pdf. 2017.

[Lur23] Jacob Lurie. *Kerodon*. https://kerodon.net. 2023.

[MHM20] Leland McInnes, John Healy and James Melville. *UMAP: Uniform Manifold Approximation and Projection for Dimension Reduction*. 2020. arXiv: 1802.03426 [stat.ML].

[ML63] Saunders Mac Lane. *Homology*. Die Grundlehren der mathematischen Wissenschaften, Band 114. Academic Press Inc. Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963, pp. x+422.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*. Second. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, pp. xii+314.

[Mat58] Eben Matlis. "Injective modules over Noetherian rings". In: *Pacific J. Math.* 8 (1958), pp. 511–528.

[Mit65] Barry Mitchell. *Theory of Categories*. Pure and Applied Mathematics, Vol. XVII. Academic Press, New York-London, 1965, pp. xi+273.

[Mit68] Barry Mitchell. "On the dimension of objects and categories. II. Finite ordered sets". In: *J. Algebra* 9 (1968), pp. 341–368. DOI: 10.1016/0021-8693(68)90029-X.

[Mit72] Barry Mitchell. "Rings with several objects". In: *Advances in Math.* 8 (1972), pp. 1–161. DOI: 10.1016/0001-8708(72)90002-3.

[Mit78] Barry Mitchell. "Some applications of module theory to functor categories". In: *Bull. Amer. Math. Soc.* 84.5 (1978), pp. 867–885. DOI: 10.1090/S0002-9904-1978-14530-3.

[NLC11] Monica Nicolau, Arnold J. Levine and Gunnar Carlsson. "Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival". In: *Proceedings of the National Academy of Sciences* 108.17 (2011), pp. 7265–7270. DOI: 10.1073/pnas.1102826108.

[Oud15] Steve Y. Oudot. *Persistence theory: from quiver representations to data analysis.* Vol. 209. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015, pp. viii+218. DOI: 10.1090/surv/209.

[PABEJ+19] Pratyush Pranav, Robert J. Adler, Thomas Buchert, Herbert Edelsbrunner, Bernard J. T. Jones, Armin Schwartzman, Hubert Wagner and Rien van de Weygaert. "Unexpected topology of the temperature fluctuations in the cosmic microwave background". en. In: *Astronomy & Astrophysics* 627 (July 2019), A163. DOI: 10.1051/0004-6361/201834916.

[PG64] Nicolae Popesco and Pierre Gabriel. "Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes". In: *C. R. Acad. Sci. Paris* 258 (1964), pp. 4188–4190.

[Pop73] N. Popescu. *Abelian categories with Applications to Rings and Modules.* London Mathematical Society Monographs, No. 3. Academic Press, London-New York, 1973, pp. xii+467.

[Pre09] Mike Prest. *Purity, spectra and localisation.* Vol. 121. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2009, pp. xxviii+769. DOI: 10.1017/CBO9781139644242.

[Pre90] Irwin S. Pressman. "Realization of long exact sequences of abelian groups". In: *Publ. Mat.* 34.1 (1990), pp. 67–76. DOI: 10.5565/PUBLMAT_34190_05.

[Pre93] Mike Prest. "Remarks on elementary duality". In: vol. 62. 2. Stability in model theory, III (Trento, 1991). 1993, pp. 185–205. DOI: 10.1016/0168-0072(93)90174-C.

[Qui96] Daniel Quillen. *Module theory over nonunital rings.* https://www.claymath.org/library/Quillen/Working_papers/quillen%201996/1996-2.pdf. 1996.

[Sga] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Vol. Vol. 269. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4)., Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Springer-Verlag, Berlin-New York, 1972, pp. xix+525.

[Sta23] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2023.

[Ste75] Bo Stenström. *Rings of quotients*. Die Grundlehren der mathematischen Wissenschaften, Band 217. Springer-Verlag, New York-Heidelberg, 1975, pp. viii+309.

[Vie27] L. Vietoris. "Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen". In: *Math. Ann.* 97.1 (1927), pp. 454–472.

[Wal20] Tashi Walde. "Higher Segal spaces via higher excision". In: *Theory Appl. Categ.* 35 (2020), Paper No. 28, 1048–1086.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. DOI: 10.1017/CBO9781139644136.

[ZH79] Birge Zimmermann-Huisgen. "Rings whose right modules are direct sums of indecomposable modules". In: *Proc. Amer. Math. Soc.* 77.2 (1979), pp. 191–197. DOI: 10.2307/2042637.

[Zie84] Martin Ziegler. "Model theory of modules". In: *Ann. Pure Appl. Logic* 26.2 (1984), pp. 149–213. DOI: 10.1016/0168-0072(84)90014-9.