Fragments of Existential Second-Order Logic and Logics with Team Semantics

Von der Fakultät für Mathematik, Informatik und Naturwissenschaften der
RWTH Aachen University zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften genehmigte Dissertation

vorgelegt von

Matthias Hoelzel, M.Sc.
aus Ibbenbüren

Berichter: Universitätsprofessor Dr. Erich Grädel
Professor Dr. Lauri Hella

Tag der mündlichen Prüfung: 10. Dezember 2019

Diese Dissertation ist auf den Internetseiten der Universitätsbibliothek online verfügbar.

Abstract

In this thesis different fragments of logics with team semantics and of existential second-order logic will be studied. The fragments we are interested in are the union closed fragments of these logics, inclusion logic of restricted arity and variants of logics with team semantics using dependency concepts which can distinguish elements only up to a given equivalence.

Logics with team semantics are extensions of first-order logic that allow to express concepts like (in)dependence in the form of atomic statements. To this end formulae are not evaluated against a single assignment but against so-called teams which are sets of such assignments. There is a strong connection between these logics and Σ_1^1, the existential fragment of second-order logic, which is reflected in the possibility to express formulae of logics with team semantics as equivalent Σ_1^1-sentences with an additional predicate for the team and vice versa. Dependence logic goes back to Väänänen and has the same expressive power as the fragment of Σ_1^1 in which the team predicate occurs only negatively. Independence logic, introduced by Grädel und Väänänen, has, as Galliani has proved, the full expressive power of Σ_1^1 and is equivalent to inclusion-exclusion logic, in whose formulae so-called in- resp. exclusion atoms can be used. If one allows only in- or only exclusion atoms, one speaks of the in- or exclusion logic. The latter corresponds exactly to dependence logic, while Galliani and Hella have shown that the inclusion logic corresponds to the greatest fixed-point logic GFP. Even though formulae of inclusion logic are closed under unions, not every union closed formulae is expressible in inclusion logic. This leads to the question how such formulae can be characterised.

In this thesis, it will be proved that union closed Σ_1^1-sentences can be characterised syntactically by myopic Σ_1^1-sentences. Towards this end, we will define and study novel inclusion-exclusion games that are precisely the model-checking games of Σ_1^1. Using these games it is also possible to identify a corresponding syntactical fragment of inclusion-exclusion logic. Furthermore, these games give rise to the definition of an atom that, when added to first-order logic, also precisely captures the union-closed fragment.

Another, so far open, problem that this thesis deals with is the question of Rönnholm, which fragment of GFP corresponds to the inclusion logic of restricted arity. In this thesis such a fragment is going to be introduced and effective translations between it and the restricted inclusion logic and vice versa are provided.

Finally, we study variants of logics with dependency concepts, which can distinguish elements only up to a given equivalence. We juxtapose these new logics with equivalent fragments of Σ_1^1 and study their expressive powers on different classes of structures.

Zusammenfassung

In dieser Arbeit werden verschiedene Fragmente von Logiken mit Teamsemantik und der existenziellen Logik zweiter Stufe untersucht. Die Fragmente, an denen wir interessiert sind, sind die unter Vereinigungen abgeschlossenen Fragmente dieser Logiken, Inklusionslogik mit eingeschränkter Stelligkeit sowie Varianten von Logiken mit Teamsemantik mit Abhängigkeitskonzepten, welche Elemente nur bis auf eine gegebene Äquivalenz unterscheiden können.

Logiken mit Teamsemantik sind Erweiterungen der Prädikatenlogik, welche es erlauben Konzepte wie (Un-)Abhängigkeiten in Form von atomaren Aussagen auszudrücken. Dazu werden Formeln nicht mithilfe einer einzigen Variablenbelegung, sondern mit sogenannten Teams, Mengen von solchen Belegungen, ausgewertet. Es gibt eine starke Verbindung zwischen diesen Logiken und Σ^1_1, dem existenziellen Fragment der Logik zweiter Stufe, was sich in der Möglichkeit widerspiegelt, Formeln aus Logiken mit Teamsemantik als äquivalente Σ^1_1-Sätze mit einem zusätzlichen Prädikat für das Team und umgekehrt auszudrücken. Die Abhängigkeitslogik geht zurück auf Väänänen und hat die gleiche Ausdrucksstärke wie das Fragment von Σ^1_1, in dem das Teamprädikat nur negativ verwendet wird. Die von Grädel und Väänänen eingeführte Unabhängigkeitslogik hat die volle Ausdrucksstärke von Σ^1_1 und ist, wie Galliani bewiesen hat, äquivalent zur Inklusion-Exklusionslogik, in deren Formeln sogenannte In- bzw. Exklusionsatome verwendet werden können. Erlaubt man nur In- bzw. nur Exklusionsatome, so spricht man von der In- bzw. Exklusionslogik. Letztere entspricht genau der Abhängigkeitslogik, während Galliani und Hella gezeigt haben, dass die Inklusionslogik der größten Fixpunktlogik GFP entspricht. Zwar sind Formeln der Inklusionslogik abgeschlossen unter Vereinigungen, aber nicht jede unter Vereinigungen abgeschlossene Formel ist in der Inklusionslogik ausdrückbar. Dies führt zu der Frage, wie man solche Formeln charakterisieren kann.

In dieser Arbeit wird bewiesen, dass unter Vereinigungen abgeschlossene Σ^1_1-Sätze syntaktisch durch die myopischen Σ^1_1-Sätze charakterisiert werden können. Dafür werden neuartige Inklusion-Exklusionsspiele definiert und untersucht, welche genau die Modellauswertungsspiele von Σ^1_1 sind. Mithilfe dieser Spiele ist es auch möglich, ein entsprechendes syntaktisches Fragment von der Inklusions-Exklusionslogik zu identifizieren. Darüber hinaus ermöglichen es diese Spiele ein Atom zu definieren, welches, hinzugefügt zur Prädikatenlogik, ebenfalls dieses Fragment beschreibt.

Ein weiteres bislang offenes Problem, mit dem sich diese Arbeit befasst, ist die Frage von Rönnholm, welches Fragment von GFP der Inklusionslogik mit eingeschränkter Stelligkeit gegenüber zu stellen ist. In dieser Arbeit wird ein solches Fragment vorgestellt und effektive Übersetzungen zu und von diesem Fragment werden angegeben.

Schließlich betrachten wir Varianten von teamsemantischen Atomen, in denen Elemente nur noch bis auf eine gegebene Äquivalenz betrachtet werden können. Diesen neuen Logiken stellen wir äquivalente Fragmente von Σ^1_1 gegenüber und untersuchen dessen Ausdrucksstärke auf unterschiedlichen Strukturklassen.

Contents

1 Introduction **1**
- 1.1 Team Semantics . . . 1
- 1.2 Previous Research . . . 2
- 1.3 Main Contributions of this Thesis . . . 3

2 Preliminaries **9**
- 2.1 Logics with Team Semantics . . . 10
  - 2.1.1 Witnesses . . . 11
  - 2.1.2 Closure Properties for Logics with Team Semantics . . . 13
- 2.2 The Second-Order Nature of Logics with Team Semantics . . . 13
  - 2.2.1 Closure Properties for Existential Second-Order Logic . . . 15
- 2.3 First-Order Interpretations . . . 16

3 Syntactic Normal Forms **19**
- 3.1 Inclusion-Exclusion Games . . . 20
  - 3.1.1 Second-Order Reachability Games . . . 24
  - 3.1.2 Model-Checking Games for Existential Second-Order Logic . . . 25
- 3.2 Union Closed Existential Second Order Logic Sentences . . . 27
- 3.3 Union Games . . . 32
- 3.4 Myopic Formulae of Inclusion-Exclusion Logic . . . 35
  - 3.4.1 Proving Corollary 3.27 using a Limited Number of Dependency Concepts . . . 40
  - 3.4.2 Proving Corollary 3.27 via Skolem Normal Form . . . 48
  - 3.4.3 Optimality of the Myopic Fragment . . . 49
- 3.5 An Atom capturing the Union Closed Fragment . . . 53
- 3.6 Other Closure Properties and More Games . . . 57
  - 3.6.1 Closure under Intersections . . . 57
  - 3.6.2 The Downwards/Upwards Closed Fragment . . . 58
  - 3.6.3 Games for Inclusion and Exclusion Logic . . . 58

4 Arity Fragments of Inclusion Logic **61**
- 4.1 Simultaneous Fixed-Point Logic . . . 62
  - 4.1.1 Avoiding Empty Fixed-Points . . . 66
- 4.2 Greatest Fixed-Points in Bounded Inclusion Logic . . . 70
  - 4.2.1 Dealing with Unary Fixed-Points . . . 73
  - 4.2.2 Myopic GFP-Sentences . . . 75

4.3 From Inclusion Logic to Bounded GFP 78

5 Dependency Concepts up to Equivalences 81
5.1 Logics with Concepts up to Equivalences 82
5.2 The Expressive Power of these new Logics 84
5.2.1 From ESO to Inclusion-Exclusion Logic up to Equivalences . 84
5.2.2 From Inclusion-Exclusion Logic to ESO up to Equivalences . 87
5.3 Inclusion Logic up to Equivalences vs. GFP 90
5.3.1 From Inclusion Logic up to Equivalences to GFP 91
5.3.2 The Translation in the Other Direction 92
5.4 ESO up to Equivalences on Restricted Classes of Structures 94

6 Concluding Remarks 99

Bibliography 101

1 Introduction

The main goal of this thesis is to explore certain fragments of existential second-order logic (Σ^1_1), which is known to exhibit a strong connection to logics with team semantics. The fragments we are interested in are the union closed fragment of Σ^1_1 and inclusion-exclusion logic, the arity hierarchy of inclusion logic and novel variants of logics for (in)dependence up to a given equivalence. In the following we outline the development of logics with team semantics, the current state of research and then we present our contributions.

1.1 Team Semantics

Logics with team semantics are extensions of first-order logic whose formulae are evaluated against *sets* of assignments, called teams, rather than single assignments, which are utilized in Tarski semantics. Special cases of these logics are logics of dependence and independence (sometimes called logics of imperfect information) that originally go back to the work of Henkin [Hen61], Enderton [End70], Walkoe [Wal70], Blass and Gurevich [BG86], and others on Henkin quantifiers, whose semantics can be naturally described in terms of games of imperfect information. Later, independence-friendly logics, which is first-order logic with annotations of independencies of quantifiers on each other, were introduced by Hintikka and Sandu [HS89]. Again, the semantics of independence-friendly logics was originally defined in terms of games of imperfect information. For more details about independence-friendly logics we refer to [MSS12].

The introduction of a model-theoretic semantics for independence-friendly logics by Hodges [Hod97b] in terms of what he called *trumps* was an important step towards the modern framework for logics of dependence and independence. Today, this semantics is called *team semantics*, where a *team* is a set of assignments $s\colon \mathcal{V} \longrightarrow A$, mapping a common finite domain of variables into the universe of a structure. Since such an assignment $s\colon \mathcal{V} \longrightarrow A$ is already described by a tuple $(s(v_1), \dots, s(v_n))$ (for a fixed enumeration $\{v_1, \dots, v_n\}$ of $\mathcal{V}$), teams correspond to relations or, more precisely, they admit relational encodings. This fact already foreshadows the tight connection between logics with team-semantics and existential second-order logic (abbreviated as ESO or Σ^1_1), which was studied by Galliani, Väänänen and Kontinen [Gal12, KV09, Vää07]. In 2007, Väänänen [Vää07] proposed *dependence logic* which is first-order logic together with dependence atoms $\mathrm{dep}(x_1, \dots, x_m, y)$ expressing that the value of y is functionally dependent on $x_1, \dots, x_m$. Besides functional dependence there are various other atomic dependency notions that give rise to interesting logics based on team semantics. In

[GV13] the notion of independence (which is more than just the absence of dependence) and *independence logics* have been introduced. Furthermore, Galliani [Gal12] and Engström [Eng12] have defined several other new logics with team properties based on notions originating in database dependency theory. The most important new logics are *inclusion logic* and *exclusion logic* which are both extensions of first-order logic by inclusion resp. exclusion statements. An inclusion atom like $\bar{x} \subseteq \bar{y}$ says that every values occuring for $\bar{x}$ also occurs for $\bar{y}$, whereas the exclusion statement $\bar{x} \mid \bar{y}$ expresses the disjointness of the possible values for $\bar{x}$ and $\bar{y}$.

1.2 Previous Research

Exclusion logic has turned out to be equivalent to dependence logic [Gal12], while inclusion logic has a strong connection to greatest fixed-point logics, which we will denote as GFP^+. Galliani and Hella have proven that inclusion logic and GFP^+ are equally expressive [GH13, Corollary 17] and, later, Grädel has shown by analysing the model-checking games for these logics that the formulae of inclusion logics correspond to myopic GFP^+-sentences, i.e. sentences of the form $\forall\bar{x}(X\bar{x} \longrightarrow \varphi(X,\bar{x}))$ where X is a relation symbol that occurs only positively in $\varphi(X,\bar{x}) \in \text{GFP}^+$ and is used to represent teams by their relational encoding. It is worth pointing out that GFP^+ can be considered to be a fragment of Σ^1_1, because $[\text{GFP}\, R\bar{x} : \varphi(R,\bar{x})](\bar{y})$ is, due to the Theorem of Knaster-Tarski, equivalent to $\exists R(\forall\bar{x}(R\bar{x} \longrightarrow \varphi(R,\bar{x})) \land R\bar{y})$. Galliani also discovered in [Gal12] that $\text{FO}(\subseteq, \mid)$, the logic that results by adding both inclusion and exclusion statements to first-order logic, captures precisely existential second-order logic, which we denote as Σ^1_1 or abbreviate as ESO. Again, in order to prove a correspondence between these logics, Σ^1_1-sentences $\varphi(X)$ with an additional relation symbol X are considered where X fulfils the task of representing teams in the form of their relational encodings.

Such a sentence $\varphi(X) \in \Sigma^1_1$ is called downwards closed, if $\mathfrak{A} \models \varphi(R)$ and $S \subseteq R$ already implies $\mathfrak{A} \models \varphi(S)$. It was observed by Kontinen and Vänäänen [KV09], that the downwards closed fragment can be characterized syntactically by demanding that the relation symbol X occurs only negatively in $\varphi(X) \in \Sigma^1_1$. Clearly such sentences are downwards closed and, conversely, every downwards closed sentence $\psi(X) \in \Sigma^1_1$ is equivalent to $\exists Y(X \subseteq Y \land \psi(Y))$ where Y is a new relation symbol and $X \subseteq Y$ is just an abbreviation for $\forall\bar{x}(X\bar{x} \longrightarrow Y\bar{x})$. Notice that X occurs now only negatively. Closure properties like downwards closure can be considered not only for existential second-order logic but also for logics with team semantics as well. A formula $\varphi(\bar{x})$ of, say, inclusion-exclusion logic is downwards-closed, if the satisfaction of $\varphi(\bar{x})$ by some team X implies that $\varphi(\bar{x})$ is also satisfied by all subteams (i.e. subsets) of X. It is well-known that exclusion logic and dependence logic are downwards closed and translations from the downwards closed fragment of Σ^1_1 to dependence logic and vice versa have been found by Kontinen and Väänänen [KV09]. So, the downwards closed fragment of Σ^1_1 is well understood. However, for union closure the situation is different.

Even though FO($\subseteq$)-formulae are known to be closed under unions, Galliani and Hella [GH13] have proven that not all union closed Σ_1^1-definable properties of relations are expressible in FO($\subseteq$) by presenting a concrete union closed atom $\mathcal{R}$ that allows to express even cardinality of finite structures, a property being inexpressible in GFP$^+$ and in FO($\subseteq$). So there is a gap between inclusion logic and the union closed fragment of Σ_1^1, which is why they asked in [GH13] what kind of logic with team semantics corresponds to the union closed fragment in Σ_1^1. Later, this question was presented again at the Dagstuhl seminar [GKKV19].

Since inclusion-exclusion logic is equally expressive as Σ_1^1, its expressive power is rather strong. Therefore it is reasonable to study certain fragments of it that are weaker and, thus, more manageable. One possible approach to do this is to study $\Sigma_1^1[k]$, the k-ary fragment of Σ_1^1, where the existential second-order quantifiers range only over relations of arity $\leq k$. In 1983, Ajtai [Ajt83] has proven that the arity hierarchy of Σ_1^1 is strict. On the level of logics with team semantics, one could define a similar restriction like FO($\subseteq, |$)[k], which is the fragment of FO($\subseteq, |$) where only in-/exclusion atoms using tuples of length $\leq k$ are allowed. Rönnholm [Rön18] established a strong connection between $\Sigma_1^1[k]$ and FO($\subseteq, |$)[k]: for given formulae $\varphi(\bar{x}) \in$ FO($\subseteq, |$)[k] he constructed equivalent sentences $\psi(X) \in \Sigma_1^1[k]$ with $\mathrm{ar}(X) = |\bar{x}|$ and, conversely, he demonstrated how sentences $\psi(X) \in \Sigma_1^1[k]$ can be turned into equivalent formulae $\varphi(\bar{x}) \in$ FO($\subseteq, |$)[k'] where $k' := \max\{k, \mathrm{ar}(X)\}$. In this context "*equivalent*" means that, for all suitable structures $\mathfrak{A}$ and teams X,

$$\mathfrak{A} \models_X \varphi(\bar{x}) \iff \mathfrak{A} \models \psi(X(\bar{x}))$$

where $X(\bar{x})$ is the relational encoding of X w.r.t. the free variables $\bar{x}$ of φ. In particular, when considering only sentences with a team predicate of arity $\leq k$ and only formulae with at most k free variables, then these translations are indeed in opposite directions and, hence, the k-ary fragment of inclusion-exclusion logic is well understood. However, the situation for inclusion logic is different. In 2015, Hannula [Han15] showed that the arity hierarchy of inclusion logic is strict, i.e. FO($\subseteq$)[1] < FO($\subseteq$)[2] < … (over the signature of graphs), but it is still an open question to what exact fragment of other logics the fragment FO($\subseteq$)[k] corresponds to. This is why Rönnholm has presented the quest for a characterisation of FO($\subseteq$)[k] in terms of a suitable fragment of GFP$^+$ during the Dagstuhl seminar 2019 [GKKV19]. This is the second open problem we are going to address.

1.3 Main Contributions of this Thesis

In this thesis, we are going to propose solutions for these open questions regarding the union closed fragment and the arity fragments of inclusion logic are analysed in Chapter 3 resp. Chapter 4. Furthermore, Chapter 5 is dedicated to the discussion of dependencies up to equivalences. Chapter 3 is based on the papers [HW19, HW20] which are joint work with Richard Wilke, while the results of Chapter 5 can mostly also be found in the paper [GH18] that is joint work with Erich Grädel.

For the union closed fragment we will provide syntactic characterisations on the level of Σ_1^1 and of $\mathrm{FO}(\subseteq, |)$. Such characterisation results are an important topic in model theory, because not only is a difficult and undecidable property turned into a syntactical one, which is easy to check, but they also enable a more in-depth analysis of these fragments. Prominent examples are van Benthem's Theorem characterising the bisimulation invariant fragment of first-order logic as the modal-logic [vB76] or preservation theorems like the Łoś-Tarski Theorem, which states that formulae preserved in substructures are equivalent to universal formulae [Hod97a].

A sentence $\varphi(X) \in \Sigma_1^1$ using an additional relation symbol X is closed under unions, if $\mathfrak{A} \models \varphi(X_i)$ for every $i \in I$ already implies $\mathfrak{A} \models \varphi(Y)$ where $Y := \bigcup_{i \in I} X_i$.[1] We shall characterise this semantic property as the myopic fragment of Σ_1^1 that consists of all those formulae that have the form $\forall \bar{x}(X\bar{x} \longrightarrow \psi(X, \bar{x}))$ where X occurs only positively in ψ by proving the following theorem.

Theorem. $\varphi(X) \in \Sigma_1^1$ *is union closed if and only if* $\varphi(X)$ *is equivalent to some myopic* Σ_1^1*-sentence.*

We will present two proofs for the direction from left to right. The proof that is more involved produces a myopic formula with a limited number of literals using X or quantified second-order symbols and it will rely on novel inclusion-exclusion games that turn out to be precisely the corresponding model-checking games of Σ_1^1-sentences $\varphi(X)$ with an additional free relation symbol X. More precisely, every tuple $\bar{a}$ that could be part of a relation X over $\mathfrak{A}$ with $(\mathfrak{A}, X) \models \varphi(X)$ will be a so-called *target* node, which is a special of kind node of the game $\mathcal{G}(\mathfrak{A}, \varphi)$ in which player 0 will have winning strategies S containing exactly those target nodes that form a relation satisfying $\varphi(X)$. It is possible to define restricted versions of these games that correspond to other closure properties. One particular version of these restricted games are called *union games* that are only able to express union closed properties of relations. These games are naturally obtained as the model-checking games for myopic Σ_1^1-sentences and because they consists of several components, it is possible to obtain new winning strategies for player 0 by a componentwise combination of other winning strategies — this is also the reason, why the sets described by these games are always union closed. This game-theoretic analysis can be transferred to a novel fragment of inclusion-exclusion logic that consists of so-called $\bar{x}$-myopic formulae. More precisely, we say that a formula $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq, |)$ is $\bar{x}$-myopic, if its free variables $\bar{x}$ are never quantified in φ and, more importantly, the in-/exclusion atoms occurring in $\varphi(\bar{x})$ obey the following two restrictions:

(i) Exclusion atoms are always of the form $\bar{x}\bar{y} \mid \bar{x}\bar{z}$.

[1] Here, I is an arbitrary index set. Please notice, that we do *not* exclude $I = \emptyset$ and, therefore, the empty relation must always satisfy a union closed formula, i.e. $\mathfrak{A} \models \varphi(\emptyset)$ is true for every suitable structure $\mathfrak{A}$ and every union closed sentence $\varphi(X)$.

(ii) Inclusion atoms are of the form $\bar{x}\bar{y} \subseteq \bar{x}\bar{z}$ or $\bar{v} \subseteq \bar{x}$, but $\bar{v} \subseteq \bar{x}$ is disallowed below disjunctions.

That this $\bar{x}$-myopic fragment of FO($\subseteq$, |) captures the essence of the union closed fragment can be stated as the following theorem.

Theorem. $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq, |)$ *is union closed if and only if it is equivalent to some $\bar{x}$-myopic formula.*

The game-theoretic proof produces an $\bar{x}$-myopic formula with a limited number of inclusion/exclusion atoms, but we also present two different proofs that are less intricate but still interesting. Furthermore, we will prove that the restrictions to the in-/exclusion atoms cannot be dropped and that all these atoms are indeed necessary to capture the union closed fragment.

Another interesting observation is that union games serve as a complete problem for union closed properties and, hence, this allows the formalization of a team-based atom $\cup$–game that, when added to first-order logic, also captures the union closed fragment of Σ^1_1 resp. FO($\subseteq$, |).

Theorem. FO($\cup$–game) *captures the union closed fragment.*

This result is motivated by an open question of Galliani and Hella, which can be found in [GH13] and was also presented at the Dagstuhl seminar 2019 [GKKV19]. These results and their details are in Chapter 3 and in [HW19, HW20].

In Chapter 4, we present a connection between the arity fragments of inclusion logic, for whose strictness Hannula has already furnished proof [Han15], and fragments of greatest fixed-points (in symbols: GFP⁺). This is motivated by an open question that was presented by Rönnholm at the Dagstuhl seminar 2019 [GKKV19]. He has asked whether or not there exists such a connection between the arity fragments of inclusion logic and some fragments of GFP⁺. We will demonstrate that $\mathrm{GFP}^+[k]$, the fragment of GFP⁺ where only fixed-points of arity $\leq k$ are allowed (and additional free first-order variables are disallowed in formulae defining fixed-point operators), corresponds to $\mathrm{FO}(\subseteq)[k]$, which is the fragment of FO($\subseteq$) where inclusion atoms only of the form $\bar{x} \subseteq \bar{y}$ with $|\bar{x}| = |\bar{y}| \leq k$ are allowed. To establish this connection, we provide effective translations in both directions, turning formulae from the one fragment into equivalent formulae from the other one.

Theorem. *For every formula $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq)[k]$ there exists some myopic sentence $\psi(X) \in \mathrm{GFP}^+[k]$ and, conversely, for every myopic sentence $\psi(X) \in \mathrm{GFP}^+[k]$ there is a $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq)[k']$ where $k' := \max\{k, \mathrm{ar}(X)\}$ such that*

$$\mathfrak{A} \models_X \varphi(\bar{x}) \iff \mathfrak{A} \models \psi(X(\bar{x}))$$

for all suitable structures $\mathfrak{A}$ and teams X.

When considering only formulae $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq)[k]$ with $|\bar{x}| \leq k$ and myopic sentences $\psi(X) \in \mathrm{GFP}^{+}[k]$ with $\mathrm{ar}(X) \leq k$, these translations are in opposite directions. Furthermore, there is a good reason for the choice of k' in this theorem, because, by using Hannula's results regarding the arity hierarchy of inclusion logic, we are able to prove that certain myopic first-order sentences $\varphi_{\ell}(X)$ with $\mathrm{ar}(X) = \ell$ *cannot* be turned into any equivalent formula from $\mathrm{FO}(\subseteq)[\ell - 1]$.

Instead of considering myopic $\mathrm{GFP}^{+}[k]$-sentences, we are also interested in $\mathrm{GFP}^{+}[k]$-*formulae* with free variables. By evaluating such a formula in a flat manner against a team we can compare them with (downwards-closed) inclusion logic formulae. The result [GH13, Theorem 16(b)] by Galliani and Hella does this for GFP^{+}-formulae of the form $[\mathrm{GFP}\, R\bar{x} \,:\, \eta(R,\bar{x})]\,(\bar{z})$ where $\eta(R,\bar{x})$ is a first-order formula. However, this result cannot be used for $\mathrm{GFP}^{+}[k]$, because it is unclear how one could transform a formula $\varphi(\bar{x}) \in \mathrm{GFP}^{+}[k]$ with several fixed-points into this normal form *without increasing the arity.* Utilising simultaneous fixed-point operators, we can circumvent these problems and obtain the following result for $\mathrm{GFP}^{+}[k]$-formulae:

Theorem. *For every* $\mathrm{GFP}^{+}[k]$*-formula* $\psi(\bar{x})$ *there exists a (downwards-closed) formula* $\gamma(\bar{x}) \in \mathrm{FO}(\subseteq)[k]$ *such that for all suitable structures* $\mathfrak{A}$ *and teams* X*,*

$$\mathfrak{A} \models_X \gamma(\bar{x}) \iff \mathfrak{A} \models_s \psi(\bar{x}) \text{ for every } s \in X.$$

In Chapter 5, logics with weaker versions of dependency concepts that can only distinguish elements up to a given equivalence relation $\approx$ are explored. The question arises whether and how known results for logics with team semantics carry over to these new logics. While this is not difficult for many known results, it turns out to be more challenging for the connection of inclusion-exclusion logic to Σ^1_1. In order to address this and to develop a better understanding of the expressive power of $\mathrm{FO}(\subseteq_{\approx}, |_{\approx})$, which is first-order logic extended by inclusion/exclusion atoms up to equivalence, we present the logic $\Sigma^1_1(\approx)$: a fragment of Σ^1_1 whose existential second-order quantifiers can only quantify over relations that are closed under the given equivalence. The connection between $\mathrm{FO}(\subseteq_{\approx}, |_{\approx})$ and $\Sigma^1_1(\approx)$ is made precise in our following result.

Theorem. $\mathrm{FO}(\subseteq_{\approx}, |_{\approx})$ *and* $\Sigma^1_1(\approx)$ *have the same expressive power on the level of sentences. Furthermore, for every* $\psi(X) \in \Sigma^1_1(\approx)$ *where* X *occurs only* $\approx$*-guarded, i.e. only in the form* $X_{\approx}\bar{v} := \exists\bar{w}(\bar{v} \approx \bar{w} \wedge X\bar{w})$*, there is some* $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq_{\approx}, |_{\approx})$ *that cannot distinguish between teams equivalent w.r.t.* $\approx$ *and, conversely, for every such* $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq_{\approx}, |_{\approx})$ *there exists such a* $\psi(X) \in \Sigma^1_1(\approx)$ *with*

$$\mathfrak{A} \models_X \varphi(\bar{x}) \iff (\mathfrak{A}, X(\bar{x})) \models \psi(X)$$

for all suitable structures $\mathfrak{A}$ *and teams* X*.*

The connection between inclusion logic and greatest fixed-point logics carries over to $\mathrm{FO}(\subseteq_{\approx})$. We define $\mathrm{GFP}^{+}_{\approx}$ as a variant of GFP^{+} where update operators build the $\approx$-closure after every computation step and we shall prove the following result.

Theorem. $\mathrm{FO}(\subseteq_\approx)$ *has the same expressive power as* $\mathrm{GFP}^+_\approx$ *on the level of sentences.*

Furthermore, the expressive power of $\Sigma^1_1(\approx)$- or, equivalently, $\mathrm{FO}(\subseteq_\approx, |_\approx)$-sentences can be investigated on different classes of structures, where either the number of equivalence classes and/or their sizes are bounded by given constant numbers. We will prove the following results:

- On any class of structures on which $\approx$ has only a bounded number of equivalence classes, $\Sigma^1_1(\approx)$, and hence all logics with dependencies up to equivalence as well, collapse to FO.
- On any class of structures in which all equivalence classes have bounded size, and only a bounded number of classes have more than one element, $\Sigma^1_1(\approx) \equiv \Sigma^1_1$.
- In general, and in particular on the classes of structures where all equivalence classes have size at most k (for $k > 1$), or that have only a bounded number of equivalence classes of size >1, the expressive power of $\Sigma^1_1(\approx)$, and all the considered logics of dependence up to equivalence, are strictly between FO and Σ^1_1.

All these results and their details are in the Chapters 3 to 5, which can be mostly read independent of each other. Chapter 2 is recommended for readers not familiar with team semantics, because logics with team semantics, frequently used notations and other concepts used in the following chapters are explained there.

Acknowledgements

I thank my supervisor Professor Dr. Erich Grädel for this guidance and support and I thank my co-examiner Professor Dr. Lauri Hella.

Some results of this thesis have been obtained in collaboration. Chapter 5 is based on joint work [GH18] with Erich Grädel, while Chapter 3 is based on an [HW19, HW19] which is joint work with Richard Wilke. I am especially thankful for all the insightful discussions we have had while working on these papers.

I am grateful for all the helpful comments that my colleagues Katrin Dannert, Benedikt Pago and Richard Wilke have provided. Further, I would like thank my colleagues at the research groups Mathematische Grundlagen der Informatik and Informatik 7 for creating such an enjoyable workplace.

Last, but not least, I want to thank my family and everyone who has accompanied me in recent years.

2 Preliminaries

This chapter is designated to explain frequently used notations, to recapitulate concepts like logical interpretations and to serve as an introduction to logics with team semantics.

A structure $\mathfrak{A}$ of signature τ is a tuple $\mathfrak{A} = (A, (S^{\mathfrak{A}})_{S\in\tau})$, where $A \neq \emptyset$ is the universe of $\mathfrak{A}$ and $S^{\mathfrak{A}}$ is the interpretation of the relation or function symbol $S \in \tau$. Structures are often denoted by letters like $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ while the corresponding Latin letters denote their universes. The arity of a $S \in \tau$ is $\operatorname{ar}(S)$. An expansion of $\mathfrak{A}$ is usually denoted as $(\mathfrak{A}, R \longmapsto X)$ in which the additional symbol R is interpreted by X. When R is clear from the context, we will just write $(\mathfrak{A}, X)$ instead. By a slight abuse of notation, something like $(\mathfrak{A}, \bar{R}) \vDash \varphi(\bar{R})$ indicates that certain relations $\bar{R}$ are interpreting the relation *symbols* $\bar{R}$ in φ. However, it will always be clear from the context whether $\bar{R}$ are just relation symbols or actual relations. For a formula $\varphi(\bar{S})$ using additional relation symbols $\bar{S} = (S_1, \ldots, S_n)$, a notation like $\mathfrak{A} \vDash \varphi(\bar{R})$ for a given tuple $\bar{R} = (R_1, \ldots, R_n)$ of relations (with $\operatorname{ar}(R_i) = \operatorname{ar}(S_i)$) is just a shorthand for $(\mathfrak{A}, \bar{R} \longmapsto \bar{S}) \vDash \varphi(\bar{R})$ where $(\mathfrak{A}, \bar{R} \longmapsto \bar{S}) := (\mathfrak{A}, R_1 \longmapsto S_1, \ldots, R_n \longmapsto S_n)$.

Graphs are $\{E\}$-structures and are often denoted as $G = (V, E)$, where the universe V is referred to as the vertex set of G while $E \subseteq V \times V$ is called the set of edges of G. Sometimes, we will nominate V or E as $V(G)$ resp. $E(G)$. A graph $H = (W, F)$ is a subgraph of G, if $W \subseteq V$ and $E \subseteq F$. For a set W we let $G{\restriction_W} := (V \cap W, E \cap (W \times W))$ be the subgraph of G which is induced by W. Similarly, for two graphs G and H we let $G{\restriction_H} := G{\restriction_{V(H)}}$. For $F \subseteq V \times V$, the extension of G by F is the graph $G + F := (V, E \cup F)$. The symmetric closure of E is $\operatorname{sym} E := E \cup \{(w, v) : (v, w) \in E\}$. The neighbourhood $\mathrm{N}_G(v)$ of a vertex v of G is the set $\{w \in V(G) : (v, w) \in E(G)\}$. Vertices $v \in V$ with $\mathrm{N}_G(v) = \emptyset$ are called terminal (or final) vertices.

Notations like $\bar{v}, \bar{w}$ always indicate that $\bar{v} = (v_1, \ldots, v_k)$ and $\bar{w} = (w_1, \ldots, w_\ell)$ are some (finite) tuples. Here $k = |\bar{v}|$ and $\ell = |\bar{w}|$, so $\bar{v}$ is a k-tuple while $\bar{w}$ is an ℓ-tuple. We write $\{\bar{v}\}$ or $\{\bar{v}, \bar{w}\}$ as abbreviations for $\{v_1, \ldots, v_k\}$ resp. $\{v_1, \ldots, v_k, w_1, \ldots, w_\ell\}$. A tuple $\bar{v}$ is called a *subtuple* of $\bar{w}$ (in symbols $\bar{v} \subseteq \bar{w}$), if $\{\bar{v}\} \subseteq \{\bar{w}\}$. To denote sets whose elements are tuples, we use a notation like $\{(\bar{v}), (\bar{w})\}$, which denotes the set containing exactly $\bar{v}$ and $\bar{w}$ (as elements). The *concatenation* of $\bar{v}$ and $\bar{w}$ is $(\bar{v}, \bar{w}) := (v_1, \ldots, v_k, w_1, \ldots, w_\ell)$ The powerset of a set A is denoted by $\mathcal{P}(A)$ and $\mathcal{P}^+(A) := \mathcal{P}(A) \setminus \{\emptyset\}$ is the powerset of A without the empty set. Following [Blu18], we write On to denote the class of ordinal numbers and assume familiarity with the concept of transfinite inductions.

We assume basic familiarity with first-order logic. For a given τ-structure $\mathfrak{A}$ and $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ we define $\varphi^{\mathfrak{A}} := \{\bar{a} : \mathfrak{A} \vDash \varphi(\bar{a})\}$, which is an $|\bar{x}|$-ary relation over A. Following the usual conventions, we often use greek letters to denote formulae. The set of free *first-order* variables of a formula φ is denoted as $\operatorname{free}(\varphi)$, while $\operatorname{subf}(\varphi)$ is the set of all subformulae of φ.

2.1 Logics with Team Semantics

In 2007, Väänänen introduced dependence logic [Vää07] based on the concepts of teams, which are sets of assignments. Using such teams it is possible to formalise dependency statements about variables. Formally, a *team* X over $\mathfrak{A}$ is a *set* of assignments mapping a common domain $\mathrm{dom}(X) = \{\bar{x}\}$ of variables into A. For a given subtuple $\bar{y} = (y_1, \dots, y_\ell) \subseteq \bar{x}$ and every $s \in X$ we define $s(\bar{y}) := (s(y_1), \dots, s(y_\ell))$. Furthermore, we frequently use $X(\bar{y}) := \{s(\bar{y}) : s \in X\}$ which is an ℓ-ary relation over $\mathfrak{A}$. For an assignment s, a variable x and $a \in A$ we use $s[x \mapsto a]$ to denote the assignment resulting from s by adding x to its domain (if it is not already contained) and declaring a as the image of x.

Definition 2.1. Let $\mathfrak{A}$ be a τ-structure, X a team of $\mathfrak{A}$. In the following, γ denotes an FO(τ)-literal and φ, ψ arbitrary formulae in negation normal form.

- $\mathfrak{A} \models_X \gamma :\Longleftrightarrow \mathfrak{A} \models_s \gamma$ for all $s \in X$
- $\mathfrak{A} \models_X \varphi \land \psi :\Longleftrightarrow \mathfrak{A} \models_X \varphi$ and $\mathfrak{A} \models_X \psi$
- $\mathfrak{A} \models_X \varphi \lor \psi :\Longleftrightarrow \mathfrak{A} \models_Y \varphi$ and $\mathfrak{A} \models_Z \psi$ for some Y, Z such that $Y \cup Z = X$
- $\mathfrak{A} \models_X \forall x \varphi :\Longleftrightarrow \mathfrak{A} \models_{X[x \mapsto A]} \varphi$
- $\mathfrak{A} \models_X \exists x \varphi :\Longleftrightarrow \mathfrak{A} \models_{X[x \mapsto F]} \varphi$ for some $F : X \longrightarrow \mathcal{P}^+(A)$

Here $X[x \mapsto A] := \{s[x \mapsto a] : s \in X,\ a \in A\}$ and $X[x \mapsto F] := \{s[x \mapsto a] : s \in X,\ a \in F(s)\}$.

Sometimes we call a team Y an $\{x\}$-extension of X, if $Y = X[x \mapsto F]$ for some function $F : X \longrightarrow \mathcal{P}^+(A)$. Furthermore, we sometimes write $X[x \mapsto B] := \{s[x \mapsto a] : s \in X,\ a \in B\}$ for subsets $B \subset A$. These notations generalize in the obvious way for tuples $\bar{x}$ instead of single variables x. We say that two formulae $\varphi(\bar{x})$ and $\psi(\bar{x})$ of some logic with team semantics (over the same signature τ) are *equivalent*, if $\mathfrak{A} \models_X \varphi(\bar{x}) \iff \mathfrak{A} \models_X \psi(\bar{x})$ holds for all τ-structures $\mathfrak{A}$ and teams X over $\mathfrak{A}$ with $\{\bar{x}\} \subseteq \mathrm{dom}(X)$.

One may wonder why it is appropriate to provide a non-empty set of values for an existentially quantified variable rather than just a single value as in standard Tarski semantics for first-order logic. Indeed a function $F : X \longrightarrow A$ rather than $F : X \longrightarrow \mathcal{P}^+(A)$ suffices, if the logic is *downwards closed*, i.e. when $\mathfrak{A} \models_X \psi$ implies that also $\mathfrak{A} \models_Y \psi$ for all subteams $Y \subseteq X$. Examples of downwards closed logics are dependence logic and exclusion logic, but not all logics with team semantics are downwards closed. For instance, inclusion logic and independence logic are not downwards closed. In these logics the so-called strict semantics requiring single values for existentially quantified variables leads to pathologies such as *non-locality*: the meaning of a formula might depend on the values of variables that do not even occur in it [Gal12]. This is why, we will always use the lax semantics that uses non-empty sets of values for existential quantifiers and splits that are not required to be disjoint for disjunctions.

All logics with team semantics that are considered in this thesis have the well-known *empty team property*: we always have $\mathfrak{A} \vDash_\emptyset \varphi(\bar{x})$ for all formulae $\varphi(\bar{x})$ and structures $\mathfrak{A}$. To evaluate sentences, i.e. formulae without free variables, we therefore do not use the empty team, but the team $\{\emptyset\}$ consisting just of the empty assignment. For a sentence ψ we write $\mathfrak{A} \vDash \psi$ if and only if $\mathfrak{A} \vDash_{\{\emptyset\}} \psi$. It is also worth mentioning that negation signs are only allowed in literals, but since it is not difficult to bring $\neg\varphi$ into negation normal form, denoted as $\mathrm{nnf}(\neg\varphi)$, we are able to simulate negation signs as long as φ is just a first-order formula. However, this is no longer possible if φ uses one of the dependency concepts that are defined below.

Team semantics for a first-order formula φ (without any dependency concepts) boils down to evaluating φ against every single assignment, i.e. more formally we have $\mathfrak{A} \vDash_X \varphi \iff \mathfrak{A} \vDash_s \varphi$ for *every* $s \in X$ (in usual Tarski semantics). This is also known as the so-called *flatness property* of FO. The reason for considering teams instead of single assignments is that they allow to formalise the meaning of dependency statements in the form of dependency atoms. Among the most important atoms are the following.

- $\mathfrak{A} \vDash_X \mathrm{dep}(\bar{x}, y) :\iff s(\bar{x}) = s'(\bar{x})$ implies $s(y) = s'(y)$ for all $s, s' \in X$
- $\mathfrak{A} \vDash_X \bar{x} \subseteq \bar{y} :\iff X(\bar{x}) \subseteq X(\bar{y})$
- $\mathfrak{A} \vDash_X \bar{x} \mid \bar{y} :\iff X(\bar{x}) \cap X(\bar{y}) = \emptyset$
- $\mathfrak{A} \vDash_X \bar{x} \perp \bar{y} :\iff X(\bar{x}, \bar{y}) = X(\bar{x}) \times X(\bar{y})$

These are called *dependence* [Vää07], *inclusion, exclusion* [Gal12] and (unconditional) *independence* [GV13] atoms, respectively. When we speak about a logic that may use certain atomic dependency notions, for example inclusion, we denote it by writing $\mathrm{FO}(\subseteq)$.

Let φ be a first-order formula and ψ be any formula of a logic with team semantics. For a given team X with $\mathrm{free}(\varphi) \subseteq \mathrm{dom}(X)$, the restriction of X to some first-order formula $\varphi(\bar{x})$ is $X\restriction_\varphi := \{s \in X : \mathfrak{A} \vDash_s \varphi\}$. We define $\varphi \longrightarrow \psi$ as $\mathrm{nnf}(\neg\varphi) \vee (\varphi \wedge \psi)$ where $\mathrm{nnf}(\neg\varphi)$ is the negation normal form of $\neg\varphi$. It is easy to see that $\mathfrak{A} \vDash_X \varphi \longrightarrow \psi \iff \mathfrak{A} \vDash_{X\restriction_\varphi} \psi$ for all teams X with $\mathrm{free}(\varphi) \cup \mathrm{free}(\psi) \subseteq \mathrm{dom}(X)$. It is worth pointing out, that the formula $\mathrm{nnf}(\neg\varphi) \vee \psi$ is in general *not* equivalent to $\varphi \longrightarrow \psi = \mathrm{nnf}(\neg\varphi) \vee (\varphi \wedge \psi)$, unless ψ is a first-order formula.

2.1.1 Witnesses

In order to prove a statement like $\mathfrak{A} \vDash_X \varphi$, one has to provide teams associated to all subformulae in a way that respects the conditions of Definition 2.1 and for the dependency concepts. A manageable way to do this is to use witnesses, which will be explained in this section. Since it is not difficult to manipulate these witnesses or to construct a new one from existing ones, this concept will be handy in various proofs.

For a formula φ let T_φ denote the syntax tree of φ in which every *occurrence* of a subformula ψ of φ corresponds to some node v_ψ of T_φ. It is very important that we use different nodes for different occurrences of (possibly identical) subformulae. Furthermore, there is an edge from v_ψ to v_ϑ, if ϑ is a *direct* subformula of ϑ.

A (team-)labelling of T_φ is a function λ mapping every node v_ψ to a team $\lambda(v_\psi)$ whose domain includes $\mathrm{free}(\psi)$.

Definition 2.2. A (team-)labelling λ is a *witness* for $\mathfrak{A} \models_X \varphi$, if $\lambda(v_\varphi) = X$ and for every $v_\psi \in V(T_\varphi)$ holds:

- If ψ is a literal, then $\mathfrak{A} \models_{\lambda(v_\psi)} \psi$.
- If $\psi = \vartheta_1 \vee \vartheta_2$, then $\lambda(v_\psi) = \lambda(v_{\vartheta_1}) \cup \lambda(v_{\vartheta_2})$.
- If $\psi = \vartheta_1 \wedge \vartheta_2$, then $\lambda(v_\psi) = \lambda(v_{\vartheta_1}) = \lambda(v_{\vartheta_2})$.
- If $\psi = \exists x \vartheta$, then $\lambda(v_\vartheta)$ is an $\{x\}$-extension of $\lambda(\vartheta)$.
- If $\psi = \forall x \vartheta$, then $\lambda(v_\vartheta) = \lambda(v_\vartheta)[x \mapsto A]$.

We often just write $\lambda(\psi)$ instead of $\lambda(v_\psi)$ if it is clear from the context which *occurrence* of the subformula ψ of φ is meant.

Since this definition basically captures Definition 2.1 and the condition for the dependency notions, we obtain the following lemma.

Lemma 2.3. *For every formula φ of some logic with team semantics, every structure $\mathfrak{A}$ and every team X over $\mathfrak{A}$ with* $\mathrm{dom}(X) \supseteq \mathrm{free}(\varphi)$*, $\mathfrak{A} \models_X \varphi$ if and only if there exists a witness λ for $\mathfrak{A} \models_X \varphi$.*

It is worth mentioning that if λ is a witness for $\mathfrak{A} \models_X \varphi$, then the restriction of λ to the subtree rooted at some $v_\psi \in V(T_\varphi)$ is a witness for $\mathfrak{A} \models_{\lambda(\psi)} \psi$. Furthermore, we like to point out that in order to prove the existence of witness λ for $\mathfrak{A} \models_X \varphi$, it suffices to give only a *partial* function λ', if one furnishes proof that for every $v_\zeta \in V$ where $\lambda'(v_\zeta)$ is undefined there exists an ancestor v_ϑ of v_ζ in T_φ with $\mathfrak{A} \models_{\lambda'(v_\vartheta)} \vartheta$ while the conditions of Definition 2.2 only have to be verified for those $v_\psi \in V(T_\varphi)$ where $\lambda'(v_\psi)$ and $\lambda'(w)$ are defined for every neighbour w of v. The reason for this is that $\mathfrak{A} \models_{\lambda'(v_\vartheta)} \vartheta$ gives rise to a witness λ'' for $\mathfrak{A} \models_{\lambda'(v_\vartheta)} \vartheta$, which can used to supplement λ' at the subtree rooted at v_ϑ. This observation is often useful in situations where a formula $\varphi^\star$ has been defined inductively from φ and one would like to turn witness for φ into a new ones for $\varphi^\star$.

2.1.2 Closure Properties for Logics with Team Semantics

A formula φ of a logic with team semantics may obey different closure properties with respect to the team, among which the following are the most important ones. We call a formula φ of any logic with team semantics

- *union closed* if $\mathfrak{A} \models_{X_i} \varphi$ for all $i \in I$ implies $\mathfrak{A} \models_X \varphi$, where $X = \bigcup_{i \in I} X_i$,
- *downwards closed* if $\mathfrak{A} \models_X \varphi$ implies $\mathfrak{A} \models_Y \varphi$ for all $Y \subseteq X$, and
- *upwards closed* on non-empty teams if $\mathfrak{A} \models_X \varphi$ and $X \neq \emptyset$ entail $\mathfrak{A} \models_Y \varphi$ for all teams Y with $X \subseteq Y$.

Since all logics we are interested in have the empty team property, i.e. we always have $\mathfrak{A} \models_\emptyset \varphi$, upwards closure is only an interesting concept as long as the empty team is excluded.

One might wonder why we skipped closure under intersection. The reason for this is that closure under intersection does not constitute a reasonable closure property, because, as we shall prove in Section 3.6.1, it is not preserved under conjunctions.

2.2 The Second-Order Nature of Logics with Team Semantics

In second-order logic, quantifiers may not only range over elements of the underlying structure but also quantifiers of the shape $\exists S$ or $\forall S$ are allowed where S is either a relation or function symbol. In the *existential* fragment of second-order logic (ESO), formally denoted as Σ^1_1, quantifiers of the form $\forall S$ are disallowed, that is, only existential second-order (but arbitrary first-order quantifers) are allowed. A notation like $\varphi(X, \bar{x})$ indicates that φ may contain an additional relation symbol X and free first-order variables $\bar{x}$. In such a case we may refer to X as a free second-order variable of φ, but we would like to point out that $\mathrm{free}(\varphi(X, \bar{x}))$ still only consists of the free *first-order* variables. It is worth mentioning that every Σ^1_1-sentence can be rewritten equivalently in the shape $\exists R_1 \dots \exists R_n \varphi(R_1, \dots, R_n)$ where $\varphi(R_1, \dots, R_n)$ is a first-order sentence. The key observation to prove this is that $\forall y \exists R \varphi(R, x)$ (where y is not quantified again in φ) can be rewritten as $\exists R' \forall y \varphi'(R', y)$ where φ' emerges from φ by replacing every $R\bar{x}$ by $R'y\bar{x}$.

In general, logical operations on teams have a second-order nature, and indeed, dependencies and team semantics may take the power of first-order logic FO up to existential second-order logic Σ^1_1. To make this precise we recall the *standard translation*, due to [Vää07, KV09], from formulae with team semantics into sentences of existential second-order logic using an additional relation symbol for (relational encodings of) the teams.

Due to the different nature of team semantics and classical Tarski semantics, one has to compare formulae $\varphi(\bar{x})$ of a logic with team semantics with sentences $\psi(X) \in \Sigma^1_1$

that use a special relation symbol for the team. In fact one could identify a team X of assignments $s : \{\bar{x}\} \longrightarrow A$ with the relation

$$X(\bar{x}) := \{s(\bar{x}) \in A^{|\bar{x}|} : s \in X\} \subseteq A^{|\bar{x}|}.$$

We say that $\varphi(\bar{x})$ and $\psi(X)$ are equivalent, if for every structure $\mathfrak{A}$ and every team X over $\mathfrak{A}$ with $\bar{x} \subseteq \mathrm{dom}(X)$ holds

$$\mathfrak{A} \vDash_X \varphi(\bar{x}) \iff (\mathfrak{A}, X(\bar{x})) \vDash \psi(X).$$

Please notice, that X occurs in $\psi(X)$ as a relation symbol while it is simultaneously used as a team to evaluate $\varphi(\bar{x})$. Furthermore, $(\mathfrak{A}, X(\bar{x}))$ is abbreviation for $(\mathfrak{A}, X \mapsto X(\bar{x}))$, because in order to evaluate the sentence $\psi(X)$ we have to interpret the relation symbol X by some relation and in this case the relation symbol X is mapped to the relation $X(\bar{x})$. When we use notations like this, it will always be clear from the context whether X refers to a team or to a relation symbol, because inside formulae X can only refer to a relation symbol while $X(\bar{x})$ only makes sense when X refers to a team. The reason we do not use different characters is to avoid an inflation of characters that all refer to almost the same concept.

To illustrate the second-order nature of logical operations in team semantics we recall the meaning of disjunctions and existential quantifications in team semantics, and their standard translation into Σ^1_1. Disjunctions split the team, i.e.

$$\mathfrak{A} \vDash_X \varphi_1 \vee \varphi_2 :\iff \text{there is a split } X = Y \cup Z \text{ such that } \mathfrak{A} \vDash_Y \varphi_1 \text{ and } \mathfrak{A} \vDash_Z \varphi_2.$$

Therefore, the standard translation of this disjunctions is

$$(\varphi_1 \vee \varphi_2)^\star(X) := \exists Y \exists Z(\underbrace{\forall \bar{x}(X\bar{x} \leftrightarrow Y\bar{x} \vee Z\bar{x})}_{X = Y \cup Z} \wedge \varphi_1^\star(Y) \wedge \varphi_2^\star(Z)).$$

Please recall that existential quantification requires the extension of the given team by providing for each of its assignments a *non-empty* set of values for quantified variables, that is

$$\mathfrak{A} \vDash_X \exists y \varphi :\iff \text{there exists a function } F : X \longrightarrow \mathcal{P}^+(A) \text{ such that } \mathfrak{A} \vDash_{X[y \mapsto F]} \varphi.$$

This leads to the following standard translation of existential quantifiers:

$$(\exists y \varphi)^\star(X) := \exists Y \forall \bar{x}((X\bar{x} \leftrightarrow \exists y Y\bar{x}y) \wedge \varphi^\star(Y))$$

In order to understand the expressive power of a first-order logic with dependencies, one is interested in identifying some fragment $\mathcal{F}$ of existential second-order logic which is equivalent in the sense just described. Here are some of the most important results that are already known:

- Dependence logic and exclusion logic are equivalent to the fragment of all Σ^1_1-sentences $\psi(X)$ in which the predicate X describing the team appears only negatively [KV09].
- Independence logic and inclusion-exclusion logic are equivalent to full Σ^1_1 (and thus can describe all NP-properties of teams) [Gal12].
- Inclusion logic FO(⊆) corresponds to GFP$^+$, the fragment of fixed-point logic that uses only (non-negated) greatest fixed-points. Since $[\mathrm{GFP} R\bar{x}\,.\,\psi(R,\bar{x})](\bar{y})$ readily translates into $\exists R(\forall \bar{x}(R\bar{x} \longrightarrow \psi(R,\bar{x})) \land R\bar{y})$, GFP$^+$ can be viewed as a fragment of Σ^1_1. Galliani and Hella [GH13] established the equivalence of inclusion logic and GFP$^+$ on the level of sentences [GH13, Corollary 17]. In [Grä16] it is shown that inclusion logics corresponds to *myopic* GFP$^+$-sentences, which are sentences of the form $\forall \bar{x}(X\bar{x} \longrightarrow \varphi(X,\bar{x}))$, where X occurs only positively in $\varphi \in \mathrm{GFP}^+$.[1]
- k-ary inclusion-exclusion logic correspond to k-ary existential second-order logic [Rön18]. For instance, the extension of FO by inclusion and exclusion atoms of single variables only (not tuples of variables) is equivalent to monadic Σ^1_1 [Rön18].
- First-order logic without any dependence atoms has the *flatness property*: $\mathfrak{A} \models_X \varphi \iff \mathfrak{A} \models_s \varphi$ for all $s \in X$. It thus corresponds to a very small fragment of Σ^1_1, namely FO-sentences of the form $\forall \bar{x}(X\bar{x} \longrightarrow \varphi(\bar{x}))$ where $\varphi(\bar{x})$ does not contain X.

2.2.1 Closure Properties for Existential Second-Order Logic

The closure properties for logics with team semantics naturally correspond to closure properties of existential second-order logic. A formula $\varphi(X) \in \Sigma^1_1$ using an additional free relation symbol X is called

- *union-closed*, if $(\mathfrak{A}, X_i) \models \varphi$ for all $i \in I$ implies $(\mathfrak{A}, X) \models \varphi$ for all $i \in I$ where $X := \bigcup_{i \in I} X_i$,
- *downwards closed*, if $(\mathfrak{A}, X) \models \varphi$ implies $(\mathfrak{A}, Y) \models \varphi$ for all relations $Y \subseteq X$,
- *upwards closed* (w.r.t. non-empty relations), if $(\mathfrak{A}, X) \models \varphi$ and $\emptyset \neq X \subseteq Y$ entail $(\mathfrak{A}, Y) \models \varphi$.

[1] Galliani and Hella also discovered a way to translate formulae of inclusion logic into myopic GFP$^+$-sentence (cf. [GH13, Theorem 15]), but their result for the converse direction always produces a downwards-closed formula, because their resulting formula had the shape $\exists \bar{x}(\bar{z} \subseteq \bar{x} \land \eta^+(\bar{x}))$ (cf. [GH13, Theorem 16(b)]). This is the reason why their converse direction could be used to rewrite sentences of the shape $\forall \bar{z}(X\bar{z} \longrightarrow [\mathrm{GFP}\, R\bar{x} \;:\; \eta(R,\bar{x})]\,(\bar{z}))$, where X does *not* occur in $\eta \in \mathrm{FO}$, as (downward-closed) inclusion logic formulae.

2.3 First-Order Interpretations

Logical interpretations are used to define structures inside another structure while translating formulae over the defined structure in the other direction. Let σ, τ be relational signatures. A first-order interpretation from σ to τ (of arity k) is a tuple $\mathcal{I} = (\delta, \varepsilon, (\psi_S)_{S\in\tau})$ of FO(σ)-formulae where

- $\delta = \delta(\bar{x})$ is the domain formula,
- $\varepsilon = \varepsilon(\bar{x}, \bar{y})$ is the equality formula and
- $\psi_S = \psi_S(\bar{x}_1, \dots, \bar{x}_{\mathrm{ar}(S)})$ are the relation formulae for each $S \in \tau$.

Here, the tuples $\bar{x}, \bar{y}, \bar{x}_1, \dots$ are of length k respectively.

For the remainder of this section, let $\mathfrak{A}$ be a σ-structure and $\mathfrak{B}$ some τ-structure. We say that $\mathcal{I}$ interprets $\mathfrak{B}$ in $\mathfrak{A}$ (and write $\mathfrak{B} \cong \mathcal{I}(\mathfrak{A})$) if and only if there exists a surjective function h, called the coordinate map, that maps $\delta^{\mathfrak{A}} = \{\bar{a} \in A^k : \mathfrak{A} \models \delta(\bar{a})\}$ to B such that

- for all $\bar{a}, \bar{b} \in \delta^{\mathfrak{A}}$ we have $\mathfrak{A} \models \varepsilon(\bar{a}, \bar{b}) \iff h(\bar{a}) = h(\bar{b})$, and
- for all $S \in \tau$ and $\bar{a}_1, \dots, \bar{a}_{\mathrm{ar}(S)} \in \delta^{\mathfrak{A}}$ holds

$$\mathfrak{A} \models \psi_S(\bar{a}_1, \dots, \bar{a}_{\mathrm{ar}(S)}) \iff (h(\bar{a}_1), \dots, h(\bar{a}_{\mathrm{ar}(S)})) \in S^{\mathfrak{B}}.$$

These conditions express that $\varepsilon^{\mathfrak{A}}$ is a congruence relation over

$$\mathfrak{C} := (\delta^{\mathfrak{A}}, ((\delta^{\mathfrak{A}})^{\mathrm{ar}(S)} \cap \psi_S^{\mathfrak{A}})_{S\in\tau})$$

and that $\mathfrak{B}$ is isomorphic to the quotient structure $\mathfrak{C}/\varepsilon^{\mathfrak{A}}$.

An interpretation $\mathcal{I}$ from σ to τ can also be used to translate a given τ-formula (of various logics) into a σ-formula. We will briefly describe how this works for Σ_1^1 and FO($\subseteq, \mid$). First, let $\varphi(S_1, \dots, S_n, x_1, \dots, x_m) \in \Sigma_1^1(\tau)$ be a formula with additional relation symbols $S_1, \dots, S_n, S_{n+1}, \dots, S_{n+n'}$ (here $S_1, \dots, S_n$ occur freely in φ while $S_{n+1}, \dots, S_{n+n'}$ are quantified in φ) and free variables $x_1, \dots, x_m$. In this process every variable v is replaced by a k-tuple $\bar{v}$ while every additional relation symbol S_i is replaced by a $(k \cdot \mathrm{ar}(S_i))$-ary relation symbol $S_i^\star$. We define a σ-formula $\varphi^{\mathcal{I}}(S_1^\star, \dots, S_n^\star, \bar{x}_1, \dots, \bar{x}_m) \in \Sigma_1^1$ by induction:

- $(S v_1 \cdots v_{\mathrm{ar}(S)})^{\mathcal{I}} := \psi_S(\bar{v}_1, \dots, \bar{v}_{\mathrm{ar}(S)})$ for $S \in \tau$,
- $(y = z)^{\mathcal{I}} := \varepsilon(\bar{y}, \bar{z})$,
- $(\neg\vartheta)^{\mathcal{I}} := \neg\vartheta^{\mathcal{I}}$,
- $(\exists y \vartheta)^{\mathcal{I}} := \exists \bar{y}(\delta(\bar{y}) \land \vartheta^{\mathcal{I}})$,
- $(\forall y \vartheta)^{\mathcal{I}} := \forall \bar{y}(\delta(\bar{y}) \to \vartheta^{\mathcal{I}})$,

- $(\vartheta_1 \circ \vartheta_2)^{\mathcal{I}} := \vartheta_1^{\mathcal{I}} \circ \vartheta_2^{\mathcal{I}}$ for $\circ \in \{\wedge, \vee\}$,
- $(S_i v_1 \cdots v_{\mathrm{ar}(S_i)})^{\mathcal{I}} := \exists \bar{w}_1 \cdots \bar{w}_{\mathrm{ar}(S_i)} \big(\bigwedge_{j=1}^{\mathrm{ar}(S_i)} (\delta(\bar{w}_j) \wedge \varepsilon(\bar{v}_j, \bar{w}_j)) \wedge S_i^\star \bar{w}_1 \cdots \bar{w}_{\mathrm{ar}(S_i)} \big)$,
- $(\exists S_j \vartheta)^{\mathcal{I}} := \exists S_j^\star \big(\forall \bar{x}_1 \cdots \bar{x}_{\mathrm{ar}(S_i)} \big(S_j^\star \bar{x}_1 \cdots \bar{x}_{\mathrm{ar}(S_i)} \longrightarrow \bigwedge_{j=1}^{\mathrm{ar}(S_j)} \delta(\bar{x}_j) \big) \wedge \vartheta^{\mathcal{I}} \big)$.

An assignment $s : \{\bar{x}_1, \ldots, \bar{x}_m\} \longrightarrow A$ is well-formed (w.r.t. $\mathcal{I}$), if $s(\bar{x}_i) \in \delta^{\mathcal{I}} (= \mathrm{dom}(h))$ for every $i = 1, \ldots, m$. Such an assignment encodes $h \circ s : \{x_1, \ldots, x_m\} \longrightarrow B$ with $(h \circ s)(x_i) := h(s(\bar{x}_i))$ which is an assignment over $\mathfrak{B}$. Similarly, a relation Q is well-formed (w.r.t. $\mathcal{I}$), if $Q \subseteq (\delta^{\mathfrak{A}})^\ell$ where $\ell = \frac{\mathrm{ar}(Q)}{k} \in \mathbb{N}$, and we define $h(Q) := \{(h(\bar{a}_1), \ldots, h(\bar{a}_\ell)) : (\bar{a}_1, \ldots, \bar{a}_\ell) \in Q\}$, which is the ℓ-ary relation over $\mathfrak{B}$ that was described by Q. The connection between $\varphi^{\mathcal{I}}$ and φ is made precise in the well-known interpretation lemma.

Lemma 2.4 (Interpretation Lemma for Σ_1^1). *Let $\varphi \in \Sigma_1^1$ and $\mathcal{I}$ be as above. Let $R_i^\star \subseteq A^{r \cdot \mathrm{ar}(S_i)}$ for $i = 1, \ldots, n$ and $s : \{\bar{x}_1, \ldots, \bar{x}_m\} \longrightarrow A$ be well-formed. Then:* $(\mathfrak{A}, R_1^\star, \ldots, R_n^\star) \models_s \varphi^{\mathcal{I}} \iff (\mathfrak{B}, h(R_1^\star), \ldots, h(R_n^\star)) \models_{h \circ s} \varphi$.

It is an easy consequence of this interpretation lemma, that, for all relations $R_1, \ldots, R_n$ over $\mathfrak{B}$ of arity $\mathrm{ar}(S_1), \ldots, \mathrm{ar}(S_n)$ respectively and every assignment $t : \{x_1, \ldots, x_m\} \longrightarrow B$, holds

$$(\mathfrak{B}, R_1, \ldots, R_n) \models_t \varphi \iff (\mathfrak{A}, h^{-1}(R_1), \ldots, h^{-1}(R_n)) \models_s \varphi^{\mathcal{I}} \text{ for some/all } s \in h^{-1}(t)$$

where $h^{-1}(s) := \{s : h \circ s = t\}$ and

$$h^{-1}(R_i) := \{(\bar{a}_1, \ldots, \bar{a}_{\mathrm{ar}(S_i)}) \in (\delta^{\mathfrak{A}})^{\mathrm{ar}(S_i)} : (h(\bar{a}_1), \ldots, h(\bar{a}_{\mathrm{ar}(S_i)})) \in R_i\},$$

because, since $h : \delta^{\mathfrak{A}} \longrightarrow A$ is surjective, it is easy to verify that the $h^{-1}(R_i)$ are well-formed and satisfy $h(h^{-1}(R_i)) = R_i$ while $h^{-1}(t)$ is a non-empty set with $h \circ s = t$ for every $s \in h^{-1}(t)$.

Now consider a formula $\psi(x_1, \ldots, x_m) \in \mathrm{FO}(\subseteq, |)$ with free variables $x_1, \ldots, x_m$. Here, the definition of $\psi^{\mathcal{I}} \in \mathrm{FO}(\subseteq, |)$ is similar to the one we used for Σ_1^1-formulae. The only differences occur when dealing with in-/exclusion atoms:

$$(v_1 \ldots v_\ell \subseteq w_1 \ldots w_\ell)^{\mathcal{I}} := \exists \bar{v}_1' \ldots \bar{v}_\ell' \Big(\bigwedge_{i=1}^{\ell} \delta(\bar{v}_i') \wedge \varepsilon(\bar{v}_i, \bar{v}_i') \wedge \bar{v}_1' \ldots \bar{v}_\ell' \subseteq \bar{w}_1 \ldots \bar{w}_\ell \Big)$$

$$(v_1 \ldots v_\ell \mid w_1 \ldots w_\ell)^{\mathcal{I}} := \forall \bar{v}_1' \ldots \bar{v}_\ell' \Big(\Big[\bigwedge_{i=1}^{\ell} \delta(\bar{v}_i') \wedge \varepsilon(\bar{v}_i, \bar{v}_i') \Big] \longrightarrow \bar{v}_1' \ldots \bar{v}_\ell' \mid \bar{w}_1 \ldots \bar{w}_\ell \Big)$$

Here, every $\bar{v}_i'$ is a $|\bar{v}_i|$-tuple of (pairwise different) new variables. A team X over $\mathfrak{A}$ with $\mathrm{dom}(X) = \{\bar{x}_1, \ldots, \bar{x}_m\}$ is said to be well-formed, if every $s \in X$ is well-formed (w.r.t. $\mathcal{I}$). For such a team, $h(X) := \{h \circ s : s \in X\}$ is a well-defined team over $\mathfrak{B}$ with $\mathrm{dom}(h(X)) = \{x_1, \ldots, x_m\}$.

Lemma 2.5 (Interpretation Lemma for FO($\subseteq$, $|$)). *Let $\psi \in$ FO($\subseteq$, $|$) and $\mathcal{I}$ be as above. For every well-formed team X over $\mathfrak{A}$ with* $\mathrm{dom}(X) = \{\bar{x}_1, \ldots, \bar{x}_m\}$, *holds* $\mathfrak{A} \models_X \psi^{\mathcal{I}} \iff \mathfrak{B} \models_{h(X)} \psi$.

Proof. A straightforward induction over ψ. □

Consider any team Y over $\mathfrak{B}$ with $\mathrm{dom}(Y) = \{x_1, \ldots, x_m\}$. Then it is an easy consequence of the interpretation lemma for FO($\subseteq$, $|$), that $\mathfrak{B} \models_Y \psi \iff \mathfrak{A} \models_{h^{-1}(Y)} \psi^{\mathcal{I}}$ where $h^{-1}(Y) := \bigcup_{t \in Y} h^{-1}(t) = \{s : h \circ s \in Y\}$ can be viewed as the "full" team describing Y. Of course, different tuples of the base structure $\mathfrak{A}$ may encode the same element of the target structure $\mathfrak{B}$, thus Y usually contains redundant assignments. For the same reason, two (well-formed) teams $X \neq X'$ (with the same domain and co-domain $\mathfrak{A}$) may describe the same team over $\mathfrak{B}$. We say that X and X' are *h-similar*, if $h(X) = h(X')$. Another easy consequence of the interpretation lemma is that h-similar teams satisfy the same formulae.

Lemma 2.6 (Similarity Lemma). *Let ψ and $\mathcal{I}$ be as above and X, X' be well-formed teams that are h-similar. Then:* $\mathfrak{A} \models_X \psi^{\mathcal{I}} \iff \mathfrak{A} \models_{X'} \psi^{\mathcal{I}}$.

Proof. Since X, X' are h-similar, we have $h(X) = h(X')$. By the interpretation lemma, $\mathfrak{A} \models_X \psi^{\mathcal{I}} \iff \mathfrak{B} \models_{h(X)} \psi \underset{h(X)=h(X')}{\iff} \mathfrak{B} \models_{h(X')} \psi \iff \mathfrak{A} \models_{X'} \psi^{\mathcal{I}}$. □

3 Syntactic Normal Forms for Union-Closed Formulae

In this chapter we analyse the semantical fragments of Σ_1^1 and FO($\subseteq$, $|$) consisting of formulae exhibiting certain closure properties, but the focus lies mostly upon union-closure, and we prove that these fragments can be characterised by syntactic restrictions imposed upon Σ_1^1 as well as FO($\subseteq$, $|$). In order to improve the translation from the semantical to the syntactical fragments w.r.t. some logical resources like number of literals with quantified symbols, we will use a new variant of second-order reachability games, which we will call inclusion-exclusion games. These games are also interesting in their own right because they turn out to be the model checking games for Σ_1^1-sentences with a free relation variable and it is even possible to define restricted variants of these games that are the model-checking games for union-closed formulae.

The most important results of this chapter are that a Σ_1^1-sentence $\psi(X)$ with a free relation variable X is union closed if and only if it is equivalent to a formula of the form $\forall\bar{x}(X\bar{x} \longrightarrow \exists\bar{R}\varphi(X, \bar{R}, \bar{x}))$ (where X occurs only positively in φ) and it will be shown that this corresponds to the fragment of inclusion-exclusion logic where only in-/exclusion atoms of the form $\bar{x}\bar{y} \mid \bar{x}\bar{z}$, $\bar{x}\bar{y} \subseteq \bar{x}\bar{z}$ and $\bar{v} \subseteq \bar{x}$ are allowed, but $\bar{v} \subseteq \bar{x}$ is disallowed in the scope of disjunctions. For FO($\subseteq$, $|$) this is somewhat the optimal solution, because we will also show that nullifying any of these restrictions or disallowing one of these three atoms results in a logic those expressive power is either too high or too low for the union-closed fragment. Last but not least, we will present a Σ_1^1-definable atom that when added to FO produces a logic, that also captures the union-closed fragment.

In Section 3.1, we will introduce the inclusion-exclusion games. They provide an alternative way to prove our characterisation results of union-closed formulae within Σ_1^1 which can be found in Section 3.2. In Section 3.3 we present the union games which are structurally restricted variants of the games from Section 3.1. The before mentioned fragment of FO($\subseteq$, $|$) is then defined and analysed in Section 3.4, while the same fragment is captured by the atom which is introduced in Section 3.5. Finally, in Section 3.6, we take a brief look at other closure properties and we will show that the inclusion/exclusion games can be adapted for fragments like inclusion resp. exclusion logic.

This chapter is based closely on the paper [HW20] that is joint work with my colleague Richard Wilke. Most results and proofs from this chapter can also be found in [HW19], which contains proofs omitted in [HW20]. However, the proofs for the characterisation results presented here are more direct and shorter in comparison to the proofs used in [HW19, HW20], while the original proofs are here analysed w.r.t. logical resources like number of in-/exclusion atoms or literals with quantified symbols.

Historically, I had been working on a solution of Rönnholm's problem, which is the main focus of Chapter 4, and, in this process, I have encountered the notions of myopic formulae that have been used in [GH13, Grä16] for GFP^+ and FO. This has sparked the idea to use an adaptation of this syntactic normal form to capture the union closed fragment of Σ^1_1 and, inspired by the approach of [Grä16], to rely on model-checking games for the proof. At first, these results were merely about Σ^1_1 and it is mostly Richard's merit to define the so-called union games as structurally restricted variants of these games in order to use them as an atom for a logic with team semantics. However, this results in a logic that may be viewed as being unnatural, because it is very cumbersome to write formulae in this new logic. Therefore, we began thinking about what kind of formulae are actually needed to express winning strategies of union games in $\mathrm{FO}(\subseteq, |)$. This line of thought eventually led to the $\bar{x}$-myopic fragment of $\mathrm{FO}(\subseteq, |)$. Only later, I have found different ways to prove the characterisation results without relying on the inclusion-exclusion games. However, inclusion-exclusion games still offer deep insights for various fragments of Σ^1_1 resp. $\mathrm{FO}(\subseteq, |)$. For example union games have inspired the definition of the myopic fragment of $\mathrm{FO}(\subseteq, |)$. In this regard, not having found the more direct proofs right away may have been a fortunate oversight. In this chapter, we will present the newer proofs alongside slightly modified versions of the original proofs, because it can be observed that they actually prove stronger statements, if one keeps track of certain logical resources. Since Richard was of the opinion that my very first proof, which was later refined, for the characterisation result with $\bar{x}$-myopic formulae was too technical and too long, he later came up with a more direct proof that relies on constructions first used by Väänänen, Kontinen and Galliani. These proofs have their (dis)advantages, which is why this thesis contains all of them. The results and concepts of this chapter have gone through numerous discussions with Richard and a countless number of changes before they eventually converged to the current refined form.

3.1 Inclusion-Exclusion Games

Classical model-checking games are designed to express satisfiability of sentences, i.e. formulae without free variables. Since we are interested in formulae in a free relational variable we are in need for a game that is able to not only express that a formula is satisfied, but moreover that it is satisfied by a certain relation. In the games we are about to describe a set of designated positions is present — called the *target set* — which corresponds to the full relation A^k (where the free relational variable has arity k). A winning strategy is said to be *adequate* for a subset X of the target positions, if the target vertices visited by it are X. On the level of logics this matches the relation satisfying the corresponding formula, i.e. there is a winning strategy adequate for X if and only if the formula is satisfied by X.

An inclusion-exclusion game $\mathcal{G} = (V, V_0, V_1, E, I, T, E_{\mathrm{ex}})$ is played by two players 0

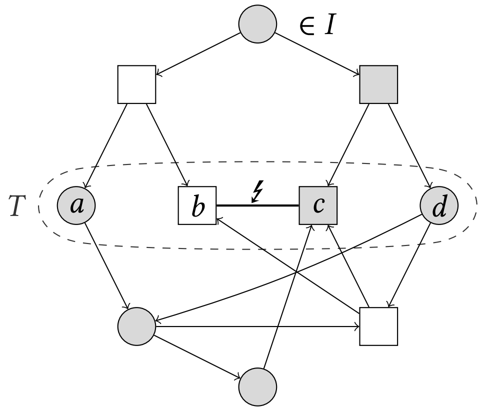

Figure 3.1: A drawing of an inclusion-exclusion game $\mathcal{G}$. Circular nodes belong to player 0, while the nodes of player 1 are rectangular. The set of target vertices is $T = \{a, b, c, d\}$, the possible moves are drawn as arrows and there is one exclusion edge between b and c, indicated by the lightning symbol. The grey vertices together with all edges between them form a winning strategy according to Definition 3.1. Another winning strategy can be obtained by dropping a. Since these are the only winning strategies, we have $\mathcal{T}(\mathcal{G}) = \{\{a, c, d\}, \{c, d\}\}$.

and 1 where

- V_σ is the set of vertices of player σ,
- $V = V_0 \uplus V_1$,
- $E \subseteq V \times V$ is set of possible moves,
- $I \subseteq V$ is the (possibly empty) set of initial positions,
- $T \subseteq V$ is the set of target vertices and
- $E_{\text{ex}} \subseteq V \times V$ is the exclusion condition, which defines the winning condition for player 0.[1]

The edges going into T, that is $E_{\text{in}} := E \cap (V \times T)$, are called *inclusion edges*, while E_{ex} is the set of *exclusion edges* (sometimes also called conflicting pairs). Figure 3.1 shows an example that illustrates these games.

Unlike *first-order* games where single plays[2] are considered and a certain winning condition is used to define whether or not such a play is winning for player 0 or 1, we do not have such a definition for inclusion-exclusion games, because these games are *second-order* games. This means that we do not consider single plays, but instead we

[1] E_{ex} can always be replaced by the symmetric closure of E_{ex} without altering its semantics.
[2] A play is a path through the game graph.

are looking at sets of plays or, to be more precise, the set of vertices visited by such a collection of plays. We will refer to the intersection of such a set of visited vertices with T as a so-called *target set* (which is not to be confused with T, the set of target *vertices*).

To put it in other words, for a subset $X \subseteq T$ the aim of player 0 is to provide a strategy (which induces a set of consistent plays) such that the vertices of T that are visited by this strategy is exactly X.

Definition 3.1. A *winning strategy* (for player 0) is a possibly empty subgraph $\mathcal{S} = (W, F)$ of $G = (V, E)$ ensuring the following four consistency conditions.

(i) For every $v \in W \cap V_0$ holds $\mathrm{N}_{\mathcal{S}}(v) \neq \emptyset$.[3]

(ii) For every $v \in W \cap V_1$ holds $\mathrm{N}_{\mathcal{S}}(v) = \mathrm{N}_G(v)$.

(iii) $I \subseteq W$.

(iv) $(W \times W) \cap E_{\mathrm{ex}} = \emptyset$.

Intuitively, the conditions (i) and (ii) state that the strategy must provide at least one move from each node of player 0 used by the strategy but does not make assumptions about the moves that player 1 may make whenever the strategy contains a node belonging to that player. In particular, the strategy must not play any with terminal vertices that are in V_0. Furthermore, (iii) enforces that at least the initial positions are contained. In a game with $I = \emptyset$, this condition becomes trivial. Finally, (iv) disallows playing with conflicting pairs $(v, w) \in E_{\mathrm{ex}}$, i.e. v and w must not coexist in any winning strategy for player 0. As an example, the grey vertices in Figure 3.1 induce a winning strategy. Another interesting observation is that (W, F) is a winning strategy for player 0, if and only if the subgraph of the game induced by W, which is $(W, (W \times W) \cap E)$, is a winning strategy. In this regard, the most relevant part of a winning strategy is its set of vertices. Please notice that we only define winning strategies for player 0 and the author is currently not aware of a possible definition for the other player.

We are mainly interested in the subset of target vertices that are visited by a winning strategy $\mathcal{S} = (W, F)$. More formally, a winning strategy $\mathcal{S}$ induces $\mathcal{T}(\mathcal{S}) := W \cap T$, which we also call the *target* of $\mathcal{S}$. This allows us to associate with every inclusion-exclusion game $\mathcal{G}$ the set of targets of winning strategies:

$$\mathcal{T}(\mathcal{G}) := \{\mathcal{T}(\mathcal{S}) : \mathcal{S} \text{ is a winning strategy for player 0 in } \mathcal{G}\}$$

Intuitively, as already pointed out, games of this kind will be the model-checking games for Σ_1^1-formulae $\varphi(X)$ that have a free relational variable X. Given a structure $\mathfrak{A}$ and such a formula, we are interested in the possible relations Y that satisfy the formula, in symbols $\mathfrak{A} \models \varphi(Y)$.[4] We will construct the game such that Y satisfies φ

[3] $\mathrm{N}_{\mathcal{S}}(v) = \{w \in W : (v, w) \in F\}$ is the set of neighbours of v in the graph $\mathcal{S}$.

[4] $\mathfrak{A} \models \varphi(Y)$ means the same as $(\mathfrak{A}, X \mapsto Y) \models \varphi(X)$ which is more precise but also more cumbersome.

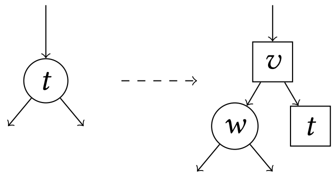

Figure 3.2: Gadget for making target vertices terminal.

if and only if there is a strategy of player 0 winning for the target set $Y \subseteq T$, thus $\mathcal{T}(\mathcal{G}) = \{Y : \mathfrak{A} \models \varphi(Y)\}$.

It will be more convenient for our purposes that the target vertices of an inclusion-exclusion game are not required to be terminal positions. As the gadget in Figure 3.2 shows (rectangle vertices belong to player 1), it is easy to transform any given game into one that agrees on the (possible) targets and in which all target vertices are terminal.

Of course, the winning condition of inclusion-exclusion games is first-order definable. For the sake of completeness we will provide the formula.

Proposition 3.2. *Let $\mathcal{G}$ be an inclusion-exclusion game. There are first-order formulae $\varphi_{\text{win}}(W, F)$ and $\varphi'_{\text{win}}(W)$ such that:*

- $\mathcal{G} \models \varphi_{\text{win}}(W, F) \iff (W, F)$ *is a winning strategy for player* 0 *in* $\mathcal{G}$.
- $\mathcal{G} \models \varphi'_{\text{win}}(W) \iff W$ *is the vertex set of a winning strategy for player* 0 *in* $\mathcal{G}$.

Proof. The formula

$$\begin{aligned}\varphi_{\text{win}}(W, F) := \forall v(Wv \longrightarrow [&(V_0 v \land \exists w(Evw \land Ww \land Fvw)) \lor \\ &(V_1 v \land \forall w(Evw \longrightarrow Ww \land Fvw))]) \land \\ \forall v(Iv \longrightarrow Wv) &\land \forall v \forall w((Wv \land Ww) \longrightarrow \neg E_{\text{ex}} vw) \land \\ \forall x \forall y(Fxy &\longrightarrow Exy \land Wx \land Wy)\end{aligned}$$

readably expresses the winning condition imposed on the graph (W, F). If (W, F) is a winning strategy, then $(W, (W \times W) \cap E)$ is a winning strategy as well. Therefore, the formula

$$\begin{aligned}\varphi'_{\text{win}}(W) := \forall v(Wv \longrightarrow [&(V_0 v \land \exists w(Evw \land Ww)) \lor \\ &(V_1 v \land \forall w(Evw \longrightarrow Ww))]) \land \\ \forall v(Iv \longrightarrow Wv) &\land \forall v \forall w((Wv \land Ww) \longrightarrow \neg E_{\text{ex}} vw).\end{aligned}$$

expresses that W is a vertex set of a winning strategy. □

Every in-/exclusion game can be associated with the following decision problem:
Given: $\mathcal{G} = (V, V_0, V_1, E, I, T, E_{\text{ex}})$ and $X \subseteq T$.
Decide: Does player 0 have a winning strategy $\mathcal{S}$ in $\mathcal{G}$ with $\mathcal{T}(\mathcal{S}) = X$?

Theorem 3.3. *The problem of deciding whether $X \in \mathcal{T}(\mathcal{G})$ for a finite inclusion-exclusion game $\mathcal{G}$ is NP-COMPLETE.*

Proof. Determining whether $X \in \mathcal{T}(\mathcal{G})$ holds is clearly in NP, as the winning strategy can be guessed and verified in polynomial time.

For the NP-hardness we present a reduction from the satisfiability problem of propositional logic. Let φ be formula in conjunctive normal form, i.e. $\varphi = \bigwedge_{j \leq m} C_j$ where $C_j = \bigvee_i L_i$ is a disjunction of literals (variables or negated variables). The game $\mathcal{G}_\varphi$ is constructed as follows. For every variable x we add two vertices (of player 1) x and $\neg x$ connected by an exclusion edge. Moreover, for every clause C_j we add a vertex, belonging to player 0, which has an outgoing edge into each literal L_i occurring in it. There are no initial vertices, i.e. $I := \emptyset$ and the target set T is the set of all clauses C_j of φ. Now, $S \in \mathcal{T}(\mathcal{G})$ if and only if $\bigwedge_{C_j \in S} C_j$ is satisfiable. In particular, $T \in \mathcal{T}(\mathcal{G}_\varphi)$ if and only if φ is satisfiable. □

3.1.1 Second-Order Reachability Games

We want to point out that inclusion-exclusion games can be seen as a certain kind of second-order reachability game, introduced by Grädel in [Grä13]. A second-order reachability game is a tuple $(V, V_0, V_1, E, I, F, \Omega)$ where V, V_0, V_1 and E are as usual, I is the set of initial positions and F the set of terminal (or final) positions (that do not belong to any player). Moreover, a winning condition on the terminal vertices $\Omega \subseteq \mathcal{P}(F)$ is given. In this context, a winning strategy for player 0 is a subgraph $\mathcal{S}$ of (V, E) that must satisfy the usual conditions for player 0 and 1 (see (i) and (ii) of Definition 3.1) and, furthermore, satisfies the winning condition imposed on the terminal vertices, that is $V(\mathcal{S}) \cap F \in \Omega$. A subset $X \subseteq I$ is said to be an *I-trap* if and only if there is a strategy[5] $\mathcal{S}$ for player 0 such that the initial positions visited by $\mathcal{S}$ are precisely X cf. [Grä16] — this corresponds to the target set of a strategy in inclusion-exclusion games. In this sense, I corresponds to the set T of target vertices. Thus, if we require that every vertex being part of an exclusion edge is a terminal position (analogously as we have seen earlier for the target vertices this is no restriction) we can formulate the winning condition of a second-order reachability game to correspond to the one of an inclusion-exclusion game: $\Omega = \{U \subseteq F : U \cap F_1 = \emptyset, (U \times U) \cap E_{\text{ex}} = \emptyset\}$, where F_1 are all final positions in which player 1 wins.

It comes as no surprise that inclusion-exclusion games can be embedded into second-order reachability games, since they are the model-checking games for existential second-order logic, the logic we are interested in in the present work. The notion we introduced here however generalises well to fragments of second-order logic that have certain closure properties such as union or downwards closure and is designed to work with formulae in a free relational variable.

[5]Contrary to our definition, Grädel did not require that $I \subseteq V(\mathcal{S})$ but he demanded that $\mathcal{S}$ contains only vertices reachable from $V(\mathcal{S}) \cap I$ via $E(\mathcal{S})$.

3.1.2 Model-Checking Games for Existential Second-Order Logic

In this section we define model-checking games for formulae $\varphi(X) \in \Sigma_1^1$ with a free relation variable. These games are inclusion-exclusion games whose target sets are precisely the sets of relations that satisfy $\varphi(X)$.

Definition 3.4. Let $\mathfrak{A}$ be a τ-structure and $\varphi(X) = \exists\bar{R}\varphi'(X,\bar{R}) \in \Sigma_1^1$ be in negation-normal form where $\varphi'(X,\bar{R}) \in \mathrm{FO}(\tau \cup \{X,\bar{R}\})$ using a free relation symbol X of arity $r := \mathrm{ar}(X)$. The game $\mathcal{G}_X(\mathfrak{A},\varphi) := (V, V_0, V_1, E, I, T, E_{\mathrm{ex}})$ consists of the following components:

- $V := \{(\vartheta, s) : \vartheta \in \mathrm{subf}(\varphi'), s : \ \mathrm{free}(\vartheta) \longrightarrow A\} \cup A^r$,
- $T := A^r$,
- $V_1 := \{(\vartheta, s) : \vartheta = \forall y\gamma \text{ or } \vartheta = \gamma_1 \wedge \gamma_2\} \cup$
  $\{(\gamma, s) : \gamma \text{ is a } \tau\text{-literal and } \mathfrak{A} \models_s \gamma\} \cup$
  $\{(\gamma, s) : \gamma \text{ is a } \{X,\bar{R}\}\text{-literal}\} \cup T$,
- $V_0 := V \setminus V_1$,
- $E := \{((\gamma\circ\vartheta, s), (\delta, s\restriction_{\mathrm{free}(\delta)})) : \circ \in \{\wedge,\vee\}, \delta \in \{\gamma,\vartheta\}\} \cup$
  $\{((X\bar{x}, s), s(\bar{x})) : X\bar{x} \in \mathrm{subf}(\varphi')\} \cup$
  $\{((Qx\gamma, s), (\gamma, s')) : Q \in \{\exists,\forall\}, s' = s[x \mapsto a], a \in A\}$,
- $I := \{(\varphi', \emptyset)\}$,
- $E_{\mathrm{ex}} := \{((R_i\bar{x}, s), (\neg R_i\bar{y}, s')) : s(\bar{x}) = s'(\bar{y})\} \cup \{((\neg X\bar{x}, s), \bar{a}) : s(\bar{x}) = \bar{a}\}$.

Figure 3.3 illustrates this definition.

These games capture the behaviour of existential second-order formulae which provides us with the following theorem.

Theorem 3.5. $(\mathfrak{A}, X) \models \varphi(X) \iff$ *Player 0 has a winning strategy* $\mathcal{S}$ *in* $\mathcal{G} := \mathcal{G}_X(\mathfrak{A},\varphi)$ *with* $\mathcal{T}(\mathcal{S}) = X$. *Or, in other words:* $\mathcal{T}(\mathcal{G}) = \{X \subseteq A^r : (\mathfrak{A}, X) \models \varphi(X)\}$.

Proof. "$\Longrightarrow$": First let $(\mathfrak{A}, X) \models \varphi = \exists\bar{R}\varphi'(X,\bar{R})$. Then there exist relations $\bar{R}$ such that $(\mathfrak{A}, X, \bar{R}) \models \varphi'(X,\bar{R})$. So player 0 wins the (first-order) model-checking game $\mathcal{G}' := \mathcal{G}((\mathfrak{A}, X, \bar{R}), \varphi'(X,\bar{R}))$. Let $\mathcal{S}' = (W', F')$ be a winning strategy for player 0 in $\mathcal{G}'$ and $\mathcal{S} := (W, F)$ where $W := W' \cup X$ and $F := F' \cup \{((X\bar{x}, s), \bar{a}) \in W' \times V : \bar{a} \in X \text{ and } s(\bar{x}) = \bar{a}\}$. Clearly we have that $\mathcal{T}(\mathcal{S}) = W \cap T = X$. To conclude this direction of the proof, we still need to prove that $\mathcal{S}$ is indeed a winning strategy. The required properties for player 0 and 1, that is (i) and (ii) of Definition 3.1, are inherited from $\mathcal{S}'$ for every node of the form $(\vartheta, s) \in V \setminus T$ with $\vartheta \neq X\bar{x}$. For nodes of the form $v = (X\bar{x}, s) \in W$ we have that $v \in V_1$ and, because of $v \in W'$ and the fact that $\mathcal{S}'$ is a winning strategy in $\mathcal{G}'$,

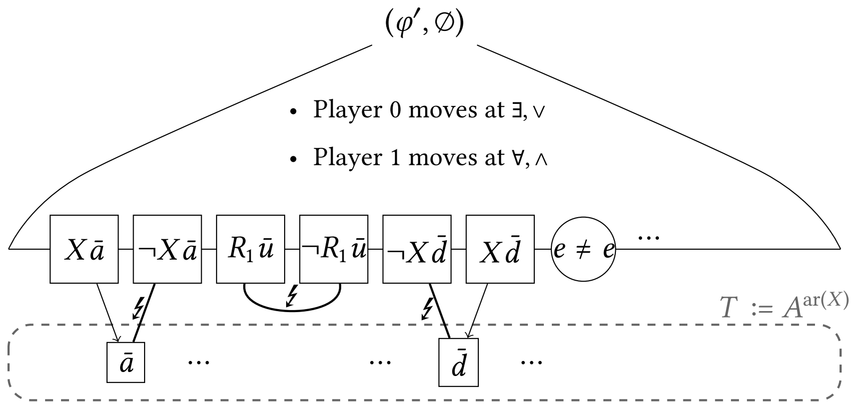

Figure 3.3: A rough sketch of a model-checking game $\mathcal{G}_X(\mathfrak{A}, \varphi(X))$ for a sentence $\varphi(X) = \exists R_1 \exists R_2 \dots \varphi'(X, R_1, R_2, \dots) \in \Sigma^1_1$. The game starts at $(\varphi', \emptyset)$ and proceeds as the classical model-checking game for FO until literals with X or R_i are reached. Please note that the assignments are only included implicitly at literals, i.e. we consider $X\bar{a}$ to be an abbreviation for $(X\bar{v}, \bar{v} \mapsto \bar{a})$ where $\bar{v}$ is the correct tuple of variables.

$s(\bar{x}) \in X$ must follow. As result, we have $s(\bar{x}) \in \mathrm{N}_{\mathcal{S}}(v)$. Since $T \subseteq V_1$ and target vertices are terminal positions in this particular game, condition (ii) is trivially satisfied for all $v \in W \cap T$.

Property (iii) is clearly satisfied, because $(\varphi', \emptyset)$ is the initial position of $\mathcal{G}'$ and, hence, $(\varphi', \emptyset) \in W' \subseteq W$. In order to prove that the last remaining condition, the exclusion condition (iv), is satisfied, consider any $(v, w) \in E_{\mathrm{ex}}$. Then there are two possible cases:

Case $(v, w) = ((R_i\bar{x}, s), (\neg R_i\bar{y}, s'))$ with $s(\bar{x}) = s'(\bar{y})$: Then either v or w is a losing position for player 0 in $\mathcal{G}'$. As a result, W' does not contain both v and w and, thus, neither does W.

Case $(v, w) = ((\neg X\bar{x}, s), \bar{a})$ and $s(\bar{x}) = \bar{a}$: If $v \in W$, then $v = (\neg X\bar{x}, s) \in W'$ and, since $\mathcal{S}'$ is a winning strategy for player 0 in $\mathcal{G}'$, it must be the case that $s(\bar{x}) \notin X$ which implies that $w = \bar{a} = s(\bar{x}) \notin W$. So, $v \in W$ and $w \in W$ cannot be true at the same time.

"$\Longleftarrow$": For the converse direction, let $\mathcal{S} = (W, F)$ now be a winning strategy for player 0 in $\mathcal{G}$ with $\mathcal{T}(\mathcal{S}) = X$. We have to show that $(\mathfrak{A}, X) \vDash \exists\bar{R}\varphi'(X, \bar{R})$. Let $R_i := \{s(\bar{x}) : (R_i\bar{x}, s) \in W\}$. Furthermore, we define $\mathcal{S}' := \mathcal{S}\restriction_{V \setminus A^{\mathrm{ar}(X)}}$, the restriction of the strategy $\mathcal{S}$ to $V(\mathcal{G}') = V \setminus A^{\mathrm{ar}(X)}$ which results from $\mathcal{S}$ by removing all nodes from $A^{\mathrm{ar}(X)}$ and by deleting all edges leading into $A^{\mathrm{ar}(X)}$.

We prove that $\mathcal{S}'$ is a winning strategy for player 0 in the first-order model-checking game $\mathcal{G}' := \mathcal{G}((\mathfrak{A}, X, \bar{R}), \varphi'(X, \bar{R}))$. First of all, the conditions for player 0 and 1 for non-terminal positions are inherited from $\mathcal{S}$. For the same reason we also have $(\varphi', \emptyset) \in V(\mathcal{S}')$. We still need to prove that $\mathcal{S}'$ contains only terminal positions that are winning for player 0. This in inherited for all terminal position that are not using any R_i nor

X. We will now investigate the other terminal positions, i.e. positions of the form $((\neg)R_i\bar{x}, s)$ or $((\neg)X\bar{x}, s)$. Clearly, if $\mathcal{S}'$ plays $(R_i\bar{x}, s)$, then $(R_i\bar{x}, s) \in W$ and $s(\bar{x}) \in R_i$ (by definition of R_i) implying that $(\mathfrak{A}, X, \bar{R}) \models_s R_i\bar{x}$ and, hence, $(R_i\bar{x}, s)$ is a winning position for player 0. In the case that $\mathcal{S}'$ visits $(\neg R_i\bar{x}, s)$, we know that $(\neg R_i\bar{x}, s) \in W$ and, because $\mathcal{S}$ respects the exclusion condition, a position of the form $(R_i\bar{y}, s')$ with $s'(\bar{y}) = s(\bar{x})$ cannot be in W. So, in this case, we have that $s(\bar{x}) \notin R_i$ and, hence, $(\neg R_i\bar{x}, s)$ is again a winning position for player 0. If $\mathcal{S}'$ contains $v := (X\bar{x}, s)$, then the edge $(v, s(\bar{x}))$ is played by $\mathcal{S}$ and, consequently, $s(\bar{x}) \in W \cap T = \mathcal{T}(\mathcal{S}) = X$ which shows that v is a winning position for player 0 in $\mathcal{G}'$. If, however, $(\neg X\bar{x}, s)$ is played by $\mathcal{S}'$, then $(\neg X\bar{x}, s) \in W$ and, due to exclusion condition, $s(\bar{x}) \notin W$ which proves that $s(\bar{x}) \notin W \cap T = X$ and, again, $(\neg X\bar{x}, s)$ is a winning for player 0 in $\mathcal{G}'$. As a result, we have that $(\mathfrak{A}, X) \models \varphi$. □

3.2 Union Closed Existential Second Order Logic Sentences

In this section we investigate formulae $\varphi(X)$ of existential second-order logic that are closed under unions with respect to their free relational variable X. Please recall that we call a formula $\varphi(X) \in \Sigma^1_1$ *union closed* if $\mathfrak{A} \models \varphi(X_i)$ for all $i \in I$ implies $\mathfrak{A} \models \varphi(X)$ for $X := \bigcup_{i \in I} X_i$.

Union closure, being a semantical property of formulae, is certainly undecidable. However, we present a *syntactical* characterisation of all such formulae via the following normal form.

Definition 3.6. A formula $\varphi(X) \in \Sigma^1_1$ is called *myopic*, if

$$\varphi(X) = \forall\bar{x}(X\bar{x} \longrightarrow \exists\bar{R}\varphi'(X, \bar{R}, \bar{x}))$$

where $\varphi' \in \mathrm{FO}$ and X occurs only positively[6] in φ'.

Variants of myopic formulae have already been considered for first-order logic [GH13, Definition 19] and for greatest fixed-point logics [Grä16, Theorem 24 and Theorem 26], but to our knowledge myopic Σ^1_1-formulae have not been studied so far.

Let $\mathcal{U}$ denote the set of all union closed Σ^1_1-formulae. To establish the claim that myopic formuale are a normal form of $\mathcal{U}$ we need to show that all myopic formulae are indeed closed under unions and, more importantly, that every union closed formula can be translated into an equivalent myopic formula. This translation is in particular constructive.

Theorem 3.7. *$\varphi(X) \in \Sigma^1_1$ is union closed if and only if $\varphi(X)$ is equivalent to some myopic Σ^1_1-sentence.*

[6] That is under an even number of negations.

We split the proof into two parts, the direction from right to left is handled in Proposition 3.8 and from left to right in Corollary 3.11.

Proposition 3.8. *Every myopic formula is union closed.*

Proof. Let $\varphi(X) = \forall\bar{x}(X\bar{x} \longrightarrow \exists\bar{R}\varphi'(X, \bar{R}, \bar{x}))$ be a myopic Σ_1^1-sentence and $(\mathfrak{A}, X_i) \vDash \varphi$ for all $i \in I$. We claim $(\mathfrak{A}, X) \vDash \varphi$ for $X = \bigcup_{i\in I} X_i$. Let $\bar{a} \in X_i \subseteq X$. By assumption $\mathfrak{A} \vDash_{\bar{x}\mapsto\bar{a}} \exists\bar{R}\varphi'(X_i, \bar{R}, \bar{x})$. A fortiori ($X$ occurs only positively in φ'), we obtain $(\mathfrak{A}, X) \vDash_{\bar{x}\mapsto\bar{a}} \exists\bar{R}\varphi'(X, \bar{R}, \bar{x})$. Since $\bar{a}$ was chosen arbitrarily, this property holds for all $\bar{a} \in X$, hence the claim follows. □

For a fixed formula $\varphi(X)$ the corresponding game $\mathcal{G}_X$ can be constructed by a first-order interpretation depending of course on the current structure.

Lemma 3.9. *Let* $\varphi(X) = \exists\bar{R}\varphi'(X, \bar{R}) \in \Sigma_1^1$ *where* $\varphi' \in \mathrm{FO}(\tau \cup \{X, \bar{R}\})$ *and* $r := \mathrm{ar}(X)$. *Then there exists a quantifier-free interpretation* $\mathcal{I}$ *such that* $\mathcal{G}_X(\mathfrak{A}, \varphi) \cong \mathcal{I}(\mathfrak{A})$ *for every structure* $\mathfrak{A}$ *(with at least two elements).*

Proof. The construction we use in this proof is similar to the one from [Grä16, Proposition 18]. An equality type $e(\bar{v})$ over a tuple $\bar{v} = (v_1, \dots, v_n)$ is a maximal consistent set of (in)equalities using only variables from $\bar{v}$. Since equality types over finitely many variables are finite, we can, by slight abuse of notation, identify $e(\bar{v})$ with the formula $\bigwedge e(\bar{v})$. Let n be chosen sufficiently large so that we can fix for every $\vartheta \in \mathrm{subf}(\varphi') \cup \{T\}$ a unique equality type $e_\vartheta(\bar{v})$.

Let $\bar{x} = (x_1, \dots, x_m)$ be a tuple of variables such that for every subformula $\vartheta \in \mathrm{subf}(\varphi')$ holds $\mathrm{free}(\vartheta) \subseteq \{\bar{x}\}$. For each variable $x_i \in \{\bar{x}\}$ let $\iota(x_i) := i$. A position (ϑ, s) of the game $\mathcal{G}_X(\mathfrak{A}, \varphi)$ will be encoded by an $(n+m)$-tuple of the form $(\bar{u}, \bar{a})$ where $\bar{u}$ has equality type e_ϑ and $s(x_i) = a_i$ for every $x_i \in \mathrm{free}(\vartheta)$, while a position of the form $\bar{a} \in T (= A^r)$ will be encoded by $(\bar{u}, \bar{a}\bar{b})$ such that $\bar{u}$ has equality type e_T whereas $\bar{b} \in A^{m-r}$ can be an arbitrary tuple. Now we are in the position to define the interpretation $\mathcal{I} = (\delta, \varepsilon, \psi_{V_0}, \psi_{V_1}, \psi_E, \psi_I, \psi_T, \psi_{E_{\mathrm{ex}}})$:

- $$\delta(\bar{v}, \bar{y}) := \bigvee_{\vartheta\in\mathrm{subf}(\varphi')\cup\{T\}} e_\vartheta(\bar{v})$$
- $$\varepsilon(\bar{v}, \bar{y}, \bar{w}, \bar{z}) := \bigvee_{\vartheta\in\mathrm{subf}(\varphi')} (e_\vartheta(\bar{v}) \wedge e_\vartheta(\bar{w}) \wedge \bigwedge_{x_i\in\mathrm{free}(\vartheta)} y_i = z_i) \vee (e_T(\bar{v}) \wedge e_T(\bar{w}) \wedge \bigwedge_{i=1}^{r} y_i = z_i)$$
- $\psi_{V_1}(\bar{v}, \bar{y}) := \bigvee_{(\vartheta,s)\in V_1(\mathcal{G}_X(\mathfrak{A},\varphi))} e_\vartheta(\bar{v}) \vee e_T(\bar{v})$ and $\psi_{V_0}(\bar{v}, \bar{y}) := \delta(\bar{v}, \bar{y}) \wedge \neg\psi_{V_1}(\bar{v}, \bar{y})$.
- Let $R := \{(\vartheta, \vartheta') : ((\vartheta, s), (\vartheta', s')) \in E(\mathcal{G}_X(\mathfrak{A}, \varphi))\}$. Then we define
$$\psi_E(\bar{v}, \bar{y}, \bar{w}, \bar{z}) := \bigvee_{(\vartheta,\vartheta')\in R} (e_\vartheta(\bar{v}) \wedge e_{\vartheta'}(\bar{w}) \wedge \bigwedge_{x_i\in\mathrm{free}(\vartheta)\cap\mathrm{free}(\vartheta')} y_i = z_i) \vee \bigvee_{X\bar{u}\in\mathrm{subf}(\varphi')} (e_{X\bar{u}}(\bar{v}) \wedge e_T(\bar{w}) \wedge \bigwedge_{i=1}^{r} y_{\iota(u_i)} = z_i).$$

- Let $S := \{(R_i\bar{u}, \neg R_i\bar{u}') : ((R_i\bar{u}, s), (\neg R_i\bar{u}', s')) \in E_{\text{ex}}(\mathcal{G}_X(\mathfrak{A}, \varphi))\}$[7]. Then we define

$$\psi_{E_{\text{ex}}}(\bar{v}, \bar{y}, \bar{w}, \bar{z}) := \bigvee_{(R_i\bar{u}, \neg R_i\bar{u}') \in S} (e_{R_i\bar{u}}(\bar{v}) \wedge e_{\neg R_i\bar{u}'}(\bar{w}) \wedge \bigwedge_{i=1}^{\text{ar}(R_i)} y_{\iota(u_i)} = z_{\iota(u_i')}) \vee$$
$$\bigvee_{\neg X\bar{u} \in \text{subf}(\varphi')} (e_{\neg X\bar{u}}(\bar{v}) \wedge e_T(\bar{w}) \wedge \bigwedge_{i=1}^{r} y_{\iota(u_i)} = z_i).$$

- $\psi_I(\bar{v}, \bar{y}) := e_{\varphi'}(\bar{v})$
- $\psi_T(\bar{v}, \bar{y}) := e_T(\bar{v})$

Now, for every $\mathfrak{A}$ (with at least two elements) we have that $\mathcal{I}(\mathfrak{A}) \cong \mathcal{G}_X(\mathfrak{A}, \varphi)$. □

Towards proving that union-closed Σ_1^1-sentences $\varphi(X)$ are equivalent to myopic formulae, we first prove the following slightly stronger result.

Theorem 3.10. *For every sentence $\varphi(X) \in \Sigma_1^1$ there is myopic sentence $\mu(X) \in \Sigma_1^1$ such that for every suitable structure $\mathfrak{A}$ and relation X over $\mathfrak{A}$ holds*

$$(\mathfrak{A}, X) \models \mu(X) \iff X \textit{ can be written as } X = \bigcup_{i \in I} X_i \textit{ where } \mathfrak{A} \models \varphi(X_i) \textit{ for every } i \in I.$$

Proof. Let $\mu(X) := \forall\bar{x}(X\bar{x} \longrightarrow \exists Y(Y \subseteq X \wedge Y\bar{x} \wedge \varphi(Y)))$ where $Y \subseteq X$ is a shorthand for the formula $\forall\bar{y}(Y\bar{y} \longrightarrow X\bar{y})$. Now, $\mu(X)$ is a myopic formula, since X occurs only positively after the implication. We still need to prove the two directions of the claim.

"$\Longrightarrow$": First assume that $(\mathfrak{A}, X) \models \mu(X)$. Then, for every $\bar{a} \in X$, there exists some $Y_{\bar{a}} \subseteq X$ with $(\mathfrak{A}, Y \mapsto Y_{\bar{a}}) \models Y\bar{a} \wedge \varphi(Y)$. Thus, we have $\mathfrak{A} \models \varphi(Y_{\bar{a}})$ and $\bar{a} \in Y_{\bar{a}} \subseteq X$ for every $\bar{a} \in X$. The last property entails that $X = \bigcup_{\bar{a} \in X} Y_{\bar{a}}$.

"$\Longleftarrow$": Now let $X = \bigcup_{i \in I} X_i$ where $\mathfrak{A} \models \varphi(X_i)$ for every $i \in I$. For every $\bar{a} \in X$, choose some index $i_{\bar{a}} \in I$ with $\bar{a} \in X_{i_{\bar{a}}}$. Then we have $(\mathfrak{A}, X, Y \mapsto X_{i_{\bar{a}}}) \models Y \subseteq X \wedge Y\bar{a} \wedge \varphi(Y)$ and, thus, $(\mathfrak{A}, X) \models \mu(X)$. □

Corollary 3.11. *For every union closed formula $\varphi(X) \in \Sigma_1^1$ there is an equivalent myopic formula $\mu(X) \in \Sigma_1^1$, that is $\varphi \equiv \mu$.*

Proof. Since $\varphi(X)$ is union closed, we have for every structure $\mathfrak{A}$ and relation X,

$$\mathfrak{A} \models \varphi(X) \iff X \text{ can be written as } X = \bigcup_{i \in I} X_i \text{ where } \mathfrak{A} \models \varphi(X_i) \text{ for every } i \in I.$$

Therefore, $\varphi(X)$ is indeed equivalent to the formula $\mu(X)$ that was constructed in the proof of Theorem 3.10. □

This was not the original proof that we have found and published in [HW19]. However, the original proof is still useful because it proves the following stronger statement where the usage of X and quantified second-order symbols is limited.

[7] Since the direction of exclusion edges does not matter we assume here that they are all of this form.

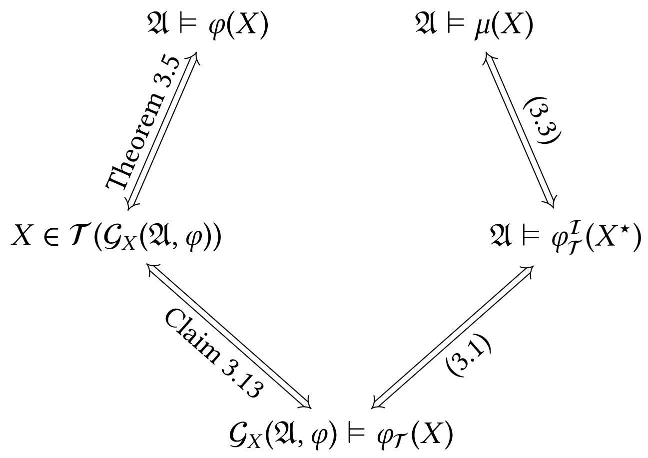

Figure 3.4: The most important steps of the proof of Theorem 3.12.

Theorem 3.12. *For every union closed formula $\varphi(X) \in \Sigma_1^1$ there is an equivalent myopic formula $\mu(X) \in \Sigma_1^1$ where exactly* 11 *literals use the symbol X or some quantified second-order symbol.*

Proof. Let $\varphi(X) = \exists \bar{R} \varphi'(X, \bar{R}) \in \Sigma_1^1(\tau)$ be closed under unions, $\mathfrak{A}$ be a τ-structure and $\mathcal{G} := \mathcal{G}_X(\mathfrak{A}, \varphi)$ be the corresponding game. W.l.o.g. $\mathfrak{A}$ has at least two elements. By Theorem 3.5, we have that $\mathcal{T}(\mathcal{G}) = \{X \subseteq A^r : (\mathfrak{A}, X) \models \varphi(X)\}$ where $r := \mathrm{ar}(X)$.

Since $\varphi(X)$ is union closed, it follows that $\mathcal{T}(\mathcal{G})$ is closed under unions as well. Now we observe that $\mathcal{T}(\mathcal{G})$ can be defined in the game $\mathcal{G}$ by the following myopic formula:

$$\varphi_{\mathcal{T}}(X) := \forall x (Xx \longrightarrow \psi_{\mathcal{T}}(X, x)) \text{ where}$$
$$\psi_{\mathcal{T}}(X, x) := \exists W (\varphi'_{\mathrm{win}}(W) \wedge Wx \wedge \forall y (Wy \wedge Ty \longrightarrow Xy))$$

Here, $\varphi'_{\mathrm{win}}(W)$ is the first-order formula from Proposition 3.2 that defines vertex sets of winning strategies. Furthermore, there are 6 W-atoms in φ_{win} and two additional W-atoms in $\psi_{\mathcal{T}}$, while X occurs twice in $\varphi_{\mathcal{T}}$. In total, X and W are used exactly 10 times. These 10 atoms will also occur in the final formula μ that are going to construct. This construction will use the interpretation lemma for Σ_1^1 (Lemma 2.4), which will introduce an additional $W^\star$-atom in order to simulate the quantifier $\exists W$ by a new quantifier $\exists W^\star$. Therefore, we will end up with exactly 11 literals using X or some quantified second-order symbol. Please note that $\varphi_{\mathcal{T}}$ is indeed a myopic formula, since X occurs only positively in $\psi_{\mathcal{T}}$.

Claim 3.13. For every $X \subseteq A^r$, $(\mathcal{G}, X) \models \varphi_{\mathcal{T}}(X) \iff X \in \mathcal{T}(\mathcal{G})$.

Proof of Claim 3.13. Assume that $(\mathcal{G}, X) \models \varphi_{\mathcal{T}}(X)$. By construction of $\varphi_{\mathcal{T}}$, for every $\bar{a} \in X$ there exists a winning strategy $\mathcal{S}_{\bar{a}} = (W_{\bar{a}}, F_{\bar{a}})$ with $\bar{a} \in W_{\bar{a}}$ and $\mathcal{T}(\mathcal{S}_{\bar{a}}) = W_{\bar{a}} \cap T \subseteq X$. It follows that $X = \bigcup_{\bar{a} \in X} \mathcal{T}(\mathcal{S}_{\bar{a}})$. Since $\mathcal{T}(\mathcal{G})$ is closed under unions, we also obtain that $X \in \mathcal{T}(\mathcal{G})$.

We want to remark that at this point the semantical property is translated into a syntactical one, as the formula only describes the correct winning strategy because the initial formula was closed under unions.

To conclude the proof of Claim 3.13, assume that $X \in \mathcal{T}(\mathcal{G})$. Then there exists a winning strategy $\mathcal{S} = (W, F)$ for player 0 with $\mathcal{T}(\mathcal{S}) = X$. Thus, for the quantifier $\exists W$ we can (for all $\bar{a} \in X$) choose the vertex set of $\mathcal{S}$, which, obviously, satisfies the formula. □

Let $\mathcal{I}$ be the interpretation from the proof of Lemma 3.9. We have $\mathcal{G} \cong \mathcal{I}(\mathfrak{A})$ with some coordinate map $h : \delta^{\mathfrak{A}} \longrightarrow V(\mathcal{G})$ and for every $\bar{a} \in T(\mathcal{G})$, $h^{-1}(\bar{a}) = \{(\bar{u}, \bar{a}, \bar{b}) : \mathfrak{A} \models e_T(\bar{u}), \bar{b} \in A^{m-r}\}$ where e_T is some equality type. By the interpretation lemma for Σ_1^1 (Lemma 2.4), we know that

$$(\mathfrak{A}, X^\star) \models \varphi_{\mathcal{T}}^{\mathcal{I}}(X^\star) \iff (\mathcal{G}, X) \models \varphi_{\mathcal{T}}(X) \tag{3.1}$$

where $X^\star := h^{-1}(X)$ is a relation of arity $(n + m) \cdot r = (n + m) \cdot \mathrm{ar}(X)$. The sentence $\psi_{\mathcal{T}}^{\mathcal{I}}(X^\star)$ "imports" the 9 atoms using X or W from $\psi_{\mathcal{T}}(X)$ and there is an additional occurrence of $W^\star$ used for the simulation of $\exists W$. Recall that every variable x occurring in $\varphi_{\mathcal{T}}$ is replaced by a tuple $\bar{x}$ of length $(n + m)$. Let $\bar{x} = (\bar{u}, \bar{v}, \bar{w})$ where $|\bar{u}| = n$, $|\bar{v}| = r$ and $|\bar{w}| = m - r$ and let

$$\mu(X) := \forall \bar{v}(X\bar{v} \longrightarrow \forall \bar{u} \forall \bar{w}(e_T(\bar{u}) \longrightarrow \psi^\star(X, \bar{u}, \bar{v}, \bar{w})))$$

where $\psi^\star$ is the formula that results from $\psi_{\mathcal{T}}^{\mathcal{I}}$ by replacing every occurrence of $X^\star \bar{u}' \bar{v}' \bar{w}'$ (where $|\bar{u}'| = n$, $|\bar{v}'| = r$ and $|\bar{w}'| = m - r$) by the formula $e_T(\bar{u}') \wedge X\bar{v}'$. By construction, μ is a myopic formula[8], because X occurred only positively in $\psi^{\mathcal{I}}$ and, hence, $X^\star$ (resp. X) occurs only positively in $\psi_{\mathcal{T}}^{\mathcal{I}}$ (resp. $\psi^\star$). Furthermore, $\mu(X)$ contains exactly 11 $\{X, W^\star\}$-literals.

We still need to verify the equivalence of $\varphi(X)$ and $\mu(X)$. Figure 3.4 shows the most important steps of this proof.

Recall that, in the game $\mathcal{G} \cong \mathcal{I}(\mathfrak{A})$, the set $X \subseteq T(\mathcal{G})$ is a unary relation over $\mathcal{G}$, while the elements of $T(\mathcal{G})$ themselves are r-tuples of elements of A. By the construction of the interpretation $\mathcal{I}$, we have that $h^{-1}(X) := \{(\bar{a}, \bar{b}, \bar{c}) \in A^n \times A^r \times A^{m-r} : \mathfrak{A} \models e_T(\bar{a}) \text{ and } \bar{b} \in X\}$. Because of this and $X^\star = h^{-1}(X)$, it follows that for every $s : \{\bar{u}', \bar{v}', \bar{w}'\} \longrightarrow A$ holds

$$\begin{aligned}(\mathfrak{A}, X^\star) \models_s X^\star \bar{u}' \bar{v}' \bar{w}' &\iff \mathfrak{A} \models e_T(s(\bar{u}')) \text{ and } s(\bar{v}') \in X \\ &\iff (\mathfrak{A}, X) \models_s e_T(\bar{u}') \wedge X\bar{v}'.\end{aligned} \tag{3.2}$$

By construction of $\psi^\star$, these are the only subformulae in which $\psi_{\mathcal{T}}^{\mathcal{I}}$ and $\psi^\star$ differ from each other. As a result, the following claim is true:

[8] Strictly speaking, the definition of myopic formulae (Definition 3.6) requires that the quantifier $\exists W^\star$ occurs right after the implication. This can be achieved by using the observation that $\forall v \exists S \gamma(S, v) \equiv \exists S' \forall v \gamma'(S', v)$ where γ' results from γ by replacing every $S\bar{x}$ by $S' v \bar{x}$.

Claim 3.14. For every $X \subseteq A^r$ and every assignment $s :\ \text{free}(\psi_{\mathcal{T}}^{\mathcal{I}}) \longrightarrow A$, holds

$$(\mathfrak{A}, X^\star) \models_s \psi_{\mathcal{T}}^{\mathcal{I}}(X^\star, \bar{x}) \iff (\mathfrak{A}, X) \models_s \psi^\star(X, \bar{x}).$$

Recall that $\bar{x} = (\bar{u}, \bar{v}, \bar{w})$ where $|\bar{u}| = n, |\bar{v}| = r$ and $|\bar{w}| = (m - r)$. Now we can see that

$$\begin{aligned}
& (\mathfrak{A}, X^\star) \models \varphi_{\mathcal{T}}^{\mathcal{I}}(X^\star) = \forall \bar{x}(X^\star \bar{x} \longrightarrow \psi_{\mathcal{T}}^{\mathcal{I}}(X^\star, \bar{x})) \\
\iff & (\mathfrak{A}, X^\star) \models_s \psi_{\mathcal{T}}^{\mathcal{I}}(X^\star, \bar{x}) \text{ for every } s \text{ with } s(\bar{x}) \in X^\star \\
\iff & (\mathfrak{A}, X) \ \ \models_s \psi^\star(X, \bar{x}) \text{ for every } s \text{ with } s(\bar{x}) \in X^\star \ \ \text{(Claim 3.14)} \\
\iff & (\mathfrak{A}, X) \ \ \models_s \psi^\star(X, \bar{x}) \text{ for every } s \text{ with } (\mathfrak{A}, X) \models_s e_T(\bar{u}) \wedge X\bar{v} \ \ \text{(due to (3.2))} \\
\iff & (\mathfrak{A}, X) \ \ \models \forall \bar{u} \forall \bar{v} \forall \bar{w}((e_T(\bar{u}) \wedge X\bar{v}) \longrightarrow \psi^\star(X, \bar{u}, \bar{v}, \bar{w}))) \equiv \mu.
\end{aligned}$$

As a result, we have that

$$(\mathfrak{A}, X) \models \mu(X) \iff (\mathfrak{A}, X^\star) \models \varphi_{\mathcal{T}}^{\mathcal{I}}(X^\star). \tag{3.3}$$

Furthermore, we also have:

$$(\mathfrak{A}, X^\star) \models \varphi_{\mathcal{T}}^{\mathcal{I}} \overset{(3.1)}{\iff} (\mathcal{G}, X) \models \varphi_{\mathcal{T}} \overset{\text{(Claim 3.13)}}{\iff} X \in \mathcal{T}(\mathcal{G}) \overset{\text{(Theorem 3.5)}}{\iff} (\mathfrak{A}, X) \models \varphi$$

Thus, the constructed myopic formula $\mu(X)$ is indeed equivalent to $\varphi(X)$. □

3.3 Union Games

In the previous section we have characterised the union closed fragment of Σ_1^1 by means of a syntactic normal form. Now we aim at a game theoretic description, which leads to the following restriction of inclusion-exclusion games that reveals *how* union closed properties are assembled.

Definition 3.15. A *union game* is an inclusion-exclusion game $\mathcal{G} = (V, V_0, V_1, E, I, T, E_{\text{ex}})$ obeying the following restrictions. For every $t \in T$ the subgraph reachable from t via the edges $E \setminus E_{\text{in}}$, that are the edges of E that do *not* go back into T, is denoted by $\mathcal{G}_t^\vartriangle$.[9] These components must be disjoint and form a partition of V, that is $V(\mathcal{G}_t^\vartriangle) \cap V(\mathcal{G}_{t'}^\vartriangle) = \emptyset$ for all $t \neq t' \in T$ and $V = \bigcup_{t \in T} V(\mathcal{G}_t^\vartriangle)$. Furthermore, there are no exclusion edges between *different* components of the target positions, that is $E_{\text{ex}} \subseteq \bigcup_{t \in T} V(\mathcal{G}_t^\vartriangle) \times V(\mathcal{G}_t^\vartriangle)$. The set of initial positions is empty, i.e. $I = \emptyset$.

See Figure 3.5 for a graphical representation of a union game. Since the exclusion edges are only inside a component we can in a way combine different strategies into one, which is the reason the target set of a union game is closed under unions.

Theorem 3.16. *Let $\mathcal{G}$ be a union game and $(\mathcal{S}_i)_{i \in J}$ be a family of winning strategies for player 0. Then there is a winning strategy $\mathcal{S}$ for player 0 such that $\mathcal{T}(\mathcal{S}) = \bigcup_{i \in J} \mathcal{T}(\mathcal{S}_i)$. In other words, the set $\mathcal{T}(\mathcal{G})$ is closed under unions.*

[9] Recall that $E_{\text{in}} := E \cap (V \times T)$.

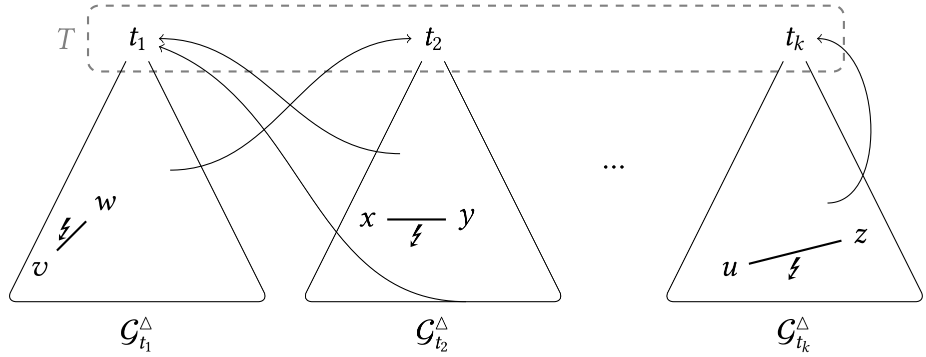

Figure 3.5: A drawing of a union game. The target positions $T = \{t_1, \dots, t_k\}$ are at the top of the components $\mathcal{G}^{\triangle}_t$ that are depicted by triangles. Recall that the inclusion edges, that are the edges going into the set of target vertices, do not account for the reachability of the components $\mathcal{G}^{\triangle}_t$. The exclusion edges E_{ex} are indicated by the symbol ↯ and, as seen here, are allowed only inside a component. In this example, a strategy for player 0 cannot contain $\{v, w\}, \{x, y\}$ or $\{u, z\}$ as a subset.

Proof. Let $\mathcal{S}_i = (W_i, F_i)$ for $i \in J$. We cannot directly combine the strategies as they might contain common target positions t, but differ on $\mathcal{G}^{\triangle}_t$. Thus the union of these strategies could contain two vertices that are connected via an edge of E_{ex}. Let $U := \bigcup_{i \in J} \mathcal{T}(\mathcal{S}_i)$ and $f : U \longrightarrow J$ be a function such that $t \in \mathcal{T}(\mathcal{S}_{f(t)})$ for all $t \in U$. Define $\mathcal{S} := \bigcup_{t \in U} (\mathcal{S}_{f(t)} \restriction_{\mathcal{G}^{\triangle}_t} + (E(\mathcal{S}_{f(t)}) \cap (V(\mathcal{G}^{\triangle}_t) \times T)))$. In words, $\mathcal{S}$ is defined on every component $\mathcal{G}^{\triangle}_t$ with $t \in U$ as an arbitrary strategy $\mathcal{S}_t$ that is defined on $\mathcal{G}^{\triangle}_t$, including the inclusion edges leaving this component. By definition $\mathcal{T}(\mathcal{S}) = U$, thus it remains to prove that indeed $\mathcal{S}$ is a winning strategy. Of course, $I = \emptyset \subseteq W$ and since $\mathcal{S}$ is defined on every component $\mathcal{G}^{\triangle}_t$ as the strategy $\mathcal{S}_{f(t)}$, it fulfils the requirements imposed on the neighbourhoods of the vertices of Definition 3.1 inside every $\mathcal{G}^{\triangle}_t$ while the edges leaving $\mathcal{G}^{\triangle}_t$ lead to vertices in $\mathcal{T}(\mathcal{S})$. Finally, since in an inclusion game there are no exclusion edges between components $\mathcal{G}^{\triangle}_t$ and $\mathcal{G}^{\triangle}_{t'}$ for $t \neq t'$, the strategy $\mathcal{S}$ cannot visit two vertices v and w that are connected by an exclusion edge (since this edge would be visible to $\mathcal{S}_t$ for some t which contradicts the assumption that it is indeed a winning strategy). □

Definition 3.17. Let $\mu(X) = \forall \bar{x}(X\bar{x} \longrightarrow \exists \bar{R}\varphi(X, \bar{R}, \bar{x}))$ be a myopic formula where φ is in negation-normal form and let $\mathfrak{A}$ be a τ-structure. The union game $\mathcal{G}(\mathfrak{A}, \mu) := (V, V_0, V_1, E, I, T, E_{\text{ex}})$ is defined similarly to Definition 3.4 with the difference being that for each $\bar{a} \in A^{\text{ar}(\bar{x})}$ we have to play on a copy of the game. Formally:

- $V := T \cup \{(\gamma, s, \bar{a}) : \gamma \in \text{subf}(\varphi), s : \text{free}(\gamma) \longrightarrow A, \bar{a} \in T\}$ where $T := A^{\text{ar}(X)}$
- $V_0 := \{(\gamma, s, \bar{a}) : \gamma = \delta \vee \vartheta, \exists x\delta, X\bar{x}$ or γ is a τ-literal with $\mathfrak{A} \not\models_s \gamma\}$

- $V_1 := V \setminus V_0$
- $E := \{(\bar{a}, (\varphi, \bar{x} \mapsto \bar{a}, \bar{a})) : \bar{a} \in T\} \cup$
  $\{((Qx\gamma, s, \bar{a}), (\gamma, s[x \mapsto a], \bar{a})) : Q \in \{\exists, \forall\}, a \in A\} \cup$
  $\{((\gamma \circ \delta, s, \bar{a}), (\vartheta, s, \bar{a})) : \vartheta \in \{\gamma, \delta\}, \circ \in \{\wedge, \vee\}, \bar{a} \in T\} \cup$
  $\{((X\bar{x}, s, \bar{a}), \bar{b}) : s(\bar{x}) = \bar{b}, \bar{a} \in T\}$
- $I := \emptyset$
- $E_{\text{ex}} := \{((R\bar{y}, s, \bar{a}), (\neg R\bar{z}, s', \bar{a})) : s(\bar{y}) = s'(\bar{z}) \text{ and } s(\bar{x}) = s'(\bar{x}), \bar{a} \in T\}$

Figure 3.6 illustrates this definition. Notice that there are still edges from $(X\bar{x}, s, \bar{a})$ to $s(\bar{x})$ — such edges are called inclusion edges. Recall that player 0 has to provide a strategy that has at least one outgoing edge from every vertex of V_0 on which the strategy is defined, thus positions with an unsatisfied literal belong to player 0. It is worth mentioning that the empty set is always included in $\mathcal{T}(\mathcal{G}(\mathfrak{A}, \mu))$ for all myopic μ because $(\emptyset, \emptyset)$ is a (trivial) winning strategy for player 0. This mimics the behaviour that in case $X = \emptyset$, the formula $\forall\bar{x}(X\bar{x} \longrightarrow \varphi)$ is satisfied regardless of everything else. The analogue of Theorem 3.5 holds for union games and myopic formulae.

Proposition 3.18. *Let $\mathfrak{A}$, μ and $\mathcal{G}(\mathfrak{A}, \mu)$ be as in Definition 3.17. Then* $(\mathfrak{A}, X) \models \mu \iff X \in \mathcal{T}(\mathcal{G}(\mathfrak{A}, \mu))$.

Proof. "$\Longrightarrow$": Assume $(\mathfrak{A}, X) \models \mu = \forall\bar{x}(X\bar{x} \longrightarrow \exists\bar{R}\varphi(X, \bar{R}, \bar{x}))$. Thus, for every $\bar{a} \in X$ there exist relations $\bar{R}_{\bar{a}}$ such that $\mathfrak{A} \models \varphi(X, \bar{R}_{\bar{a}}, \bar{a})$. Notice that every component $\mathcal{G}_{\bar{a}}^{\triangle}$ restricted to V, V_0, V_1, E is essentially isomorphic to the first-order model-checking game $\mathcal{G}((\mathfrak{A}, X, \bar{R}_{\bar{a}}), \varphi)$. Besides the additional node $\bar{a}$, the only differences are that in the first-order model-checking game the vertices of the form $(X\bar{y}, s)$ are terminal nodes where player 0 looses if and only if $s(\bar{y}) \notin X$, and that terminal positions of the form $(R_i\bar{v}, s)$ are evaluated similarly. For every $\bar{a} \in X$, let $\mathcal{S}_{\bar{a}}^{\text{FO}}$ be a winning strategy for player 0 in $\mathcal{G}((\mathfrak{A}, X, \bar{R}_{\bar{a}}), \varphi)$. Since either $\bar{b} \in R_{\bar{a}}$ or $\bar{b} \notin R_{\bar{a}}$ for all $\bar{b}$ and R, the vertex $(R\bar{x}, s)$ or $(\neg R\bar{y}, s')$ with $s(\bar{x}) = s'(\bar{y})$ is not visited by $\mathcal{S}_{\bar{a}}^{\text{FO}}$. Let $\mathcal{S}'$ be the subgraph of $\mathcal{G}(\mathfrak{A}, \mu)$ induced by $X \cup \bigcup_{\bar{a} \in X} V(\mathcal{S}_{\bar{a}}^{\text{FO}}) \times \{\bar{a}\}$ and $\mathcal{S} := \mathcal{S}' + E_{\text{in}} \cap (V(\mathcal{S}') \times V(\mathcal{S}')) + \{(\bar{a}, (\varphi, \bar{x} \mapsto \bar{a}, \bar{a})) : \bar{a} \in X\}$. In words, the strategy $\mathcal{S}$ combines all first-order strategies together, adds the reached inclusion edges and adds X and outgoing edges from X. By definition, $\mathcal{T}(\mathcal{S}) = X$. Whenever a node of the form $(X\bar{y}, s, \bar{a})$ is visited in $\mathcal{S}$ we have that $s(\bar{y}) \in X$ (because otherwise $\mathcal{S}_{\bar{a}}^{\text{FO}}$ would not be a winning strategy for player 0) and hence $((X\bar{y}, s, \bar{a}), s(\bar{y})) \in E_{\text{in}} \cap (V(\mathcal{S}') \times V(\mathcal{S}'))$ is a move that is available to player 0. That $\mathcal{S}'$ satisfies the conditions for a winning strategy on the other nodes is inherited from the fact that the individual strategies are winning strategies on the first-order part. As pointed out before, each strategy respects the exclusion condition.

"$\Longleftarrow$": For the other direction, let $\mathcal{S}$ be a winning strategy with $\mathcal{T}(\mathcal{S}) = X$. For every $\bar{a} \in X$ let $\bar{b} \in R_{\bar{a}}$ if and only if there is some $(R\bar{x}, s, \bar{a}) \in V(\mathcal{S})$ with $s(\bar{x}) = \bar{b}$. We have to show that $\mathfrak{A} \models \varphi(X, \bar{R}_{\bar{a}}, \bar{a})$ for all $\bar{a} \in X$. But there is nothing to do here

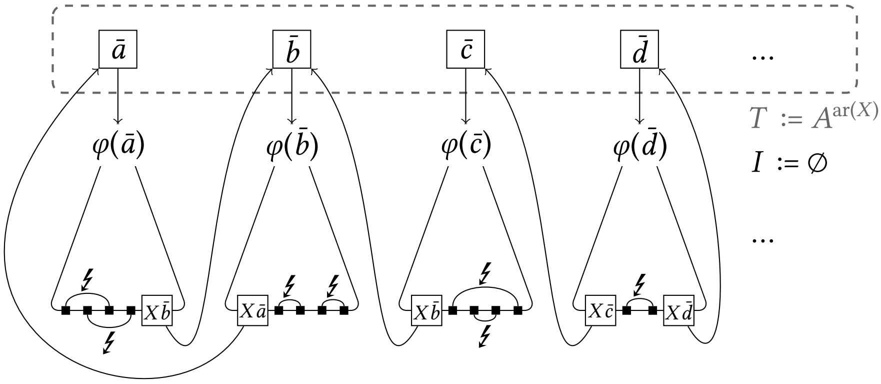

Figure 3.6: An example of how the game $\mathcal{G}(\mathfrak{A}, \mu(X))$ for a myopic sentence $\mu(X) = \forall \bar{x}(X\bar{x} \longrightarrow \varphi(X, \bar{x}))$ might look like. Starting at a tuple, the game proceeds to simulate a copy of the model-checking for φ and some assignment. These copies are depicted as triangles and they are similar to games defined in Definition 3.4 (see also Figure 3.1). Here inclusion edges can only originate at X-literals, which are always positive since $\mu(X)$ is a myopic sentence. As in Figure 3.3, the assignments are only included implicitly.

because $\mathcal{S}\restriction_{\mathcal{G}^{\triangle}_{\bar{a}}}$ induces a winning strategy for the first-order model-checking game for $\left\langle (\mathfrak{A}, X, \bar{R}_{\bar{a}}), \bar{x} \mapsto \bar{a}, \varphi \right\rangle$. □

It is worth mentioning that for other fragments with certain closure properties natural restrictions of inclusion-exclusion games exists. Especially, forbidding exclusion edges at all leads to model-checking games for inclusion logic, while forbidding inclusion edges results in games suited for exclusion logic. More details can be found in Section 3.6.

3.4 Myopic Formulae of Inclusion-Exclusion Logic

Similarly to the normal form of union closed Σ^1_1-formulae from Section 3.2 we present syntactic restrictions of inclusion-exclusion logic FO(⊆, |) that correspond precisely to the union closed fragment $\mathcal{U}$.[10] Analogously to myopic Σ^1_1-formulae we will also present a normal form for all union closed FO(⊆, |)-formulae.

Definition 3.19. A formula $\varphi(\bar{x}) \in$ FO(⊆, |) is $\bar{x}$-myopic, if the following conditions are satisfied:

(a) The variables from $\bar{x}$ are never quantified in φ.

[10] We have defined $\mathcal{U}$ to be the set of all union closed Σ^1_1-formulae, by slight abuse of notation we use the same symbol here to denote the set of all FO(⊆, |)-formulae that are closed under unions.

(b) Every exclusion atom occurring in φ is of the form $\bar{x}\bar{y} \mid \bar{x}\bar{z}$.

(c) Every inclusion atom occurring in φ is of the form $\bar{x}\bar{y} \subseteq \bar{x}\bar{z}$ or $\bar{v} \subseteq \bar{x}$, where the latter is not allowed to occur as a subformula of a disjunction.

Please note that $\varphi(\bar{x})$ must not have any additional free variables besides $\bar{x}$. We say that atoms of the form $\bar{x}\bar{y} \subseteq \bar{x}\bar{z}$ or $\bar{x}\bar{y} \mid \bar{x}\bar{z}$ are *$\bar{x}$-guarded* and $\bar{y} \subseteq \bar{z}$, respectively $\bar{y} \mid \bar{z}$, the corresponding *unguarded versions*. Analogously, we call a formula *$\bar{x}$-guarded*, if the variables from $\bar{x}$ are never quantified and every dependency atom occurring in the formula is $\bar{x}$-guarded. A formula ψ is called the *unguarded version* of an $\bar{x}$-guarded formula φ, if ψ emerges from φ by replacing every dependency atom by the respective unguarded version. In this situation we also call φ the *$\bar{x}$-guarded version* of ψ.

Our goal is to prove that $\bar{x}$-myopic formulae are (up to equivalence) precisely the union closed fragment of $\mathrm{FO}(\subseteq, \mid)$.

Theorem 3.20. *$\varphi(\bar{x}) \in \mathrm{FO}(\subseteq, \mid)$ is union closed if and only if it is equivalent to some $\bar{x}$-myopic formula.*

The direction "$\Longleftarrow$" is by Theorem 3.25 while "$\Longrightarrow$" is entailed by Corollary 3.27. Please recall the definition of witnesses from Section 2.1.1: λ is witness for $\mathfrak{A} \models_X \varphi$, if λ maps every (occurrence of a) subformula $\psi \in \mathrm{subf}(\varphi)$ to a team whose domain contains $\mathrm{free}(\psi)$ such that all literals γ occurring in φ are satisfied by $\lambda(\gamma)$, the conditions for logical operations are respected (e.g. $\lambda(\psi_1 \vee \psi_2) = \lambda(\psi_1) \cup \lambda(\psi_2)$) and $\lambda(\varphi) = X$ is true.

The intuition behind Definition 3.19 is that every $\bar{x}$-myopic formula can be evaluated componentwise on every team $X\restriction_{\bar{x}=\bar{a}} = \{s \in X : s(\bar{x}) = \bar{a}\}$ for all $\bar{a} \in X(\bar{x})$.

Definition 3.21. Let X be a team with $\{\bar{x}\} \subseteq \mathrm{dom}(X)$. The $\bar{x}$-components of X are the teams of the form $X\restriction_{\bar{x}=\bar{a}} = \{s \in X : s(x) = \bar{a}\}$.

It turns out that, if we do not have any inclusion atoms of the form $\bar{v} \subseteq \bar{x}$, then there is no mechanism that allows one $\bar{x}$-component to "look inside" other $\bar{x}$-components. This statement will be made precise in Lemma 3.23, which is a direct consequence of the following proposition, where the connections between a team and its $\bar{x}$-components is explored on the level of inclusion/exclusion atoms. Furthermore, the following proposition analyses the effect of the restrictions for inclusion atoms.

Proposition 3.22. *Let X be team over $\mathfrak{A}$ with $\mathrm{dom}(X) \supseteq \{\bar{x}, \bar{v}, \bar{w}\}$ and $\varphi(\bar{x})$ be $\bar{x}$-myopic.*

1. $\mathfrak{A} \models_X \bar{x}\bar{v} \subseteq \bar{x}\bar{w} \iff \mathfrak{A} \models_{X\restriction_{\bar{x}=\bar{a}}} \bar{v} \subseteq \bar{w}$ *for all* $\bar{a} \in X(\bar{x})$
2. $\mathfrak{A} \models_X \bar{x}\bar{v} \mid \bar{x}\bar{w} \iff \mathfrak{A} \models_{X\restriction_{\bar{x}=\bar{a}}} \bar{v} \mid \bar{w}$ *for all* $\bar{a} \in X(\bar{x})$
3. *For every subformula $\bar{v} \subseteq \bar{x}$ of φ and witness λ for $\mathfrak{A} \models_X \varphi$ we have $(\lambda(\bar{v} \subseteq \bar{x}))(\bar{x}) = X(\bar{x})$.*

Proof. We prove the first item. Let $\mathfrak{A} \models_X \bar{x}\bar{v} \subseteq \bar{x}\bar{w}$. That means for every assignment $s \in X$ there is another one, $s' \in X$, with $s(\bar{x}\bar{v}) = s'(\bar{x}\bar{w})$. Thus $s(\bar{x}) = s'(\bar{x})$ and therefore $s \in X\upharpoonright_{\bar{x}=\bar{a}} \iff s' \in X\upharpoonright_{\bar{x}=\bar{a}}$ from which $\mathfrak{A} \models_{X\upharpoonright_{\bar{x}=\bar{a}}} \bar{v} \subseteq \bar{w}$ follows for all $\bar{a} \in X(\bar{x})$.

Now assume $\mathfrak{A} \models_{X\upharpoonright_{\bar{x}=\bar{a}}} \bar{v} \subseteq \bar{w}$ for all $\bar{a} \in X(\bar{x})$ and let $s \in X$ be an arbitrary assignment. Since $\mathfrak{A} \models_{X\upharpoonright_{\bar{x}=s(\bar{x})}} \bar{v} \subseteq \bar{w}$ there is an assignment $s' \in X\upharpoonright_{\bar{x}=s(\bar{x})}$ with $s'(\bar{w}) = s(\bar{v})$. This means $s(\bar{x}\bar{v}) = s'(\bar{x}\bar{w})$, and because s was arbitrary, $\mathfrak{A} \models_X \bar{x}\bar{v} \subseteq \bar{x}\bar{w}$ follows.

Now, we prove the second item. $\mathfrak{A} \not\models_{X\upharpoonright_{\bar{x}=\bar{a}}} \bar{v} \mid \bar{w}$ holds for some $\bar{a} \in X(\bar{x})$ if and only if there are some $s, s' \in X$ with $s(\bar{x}) = \bar{a} = s'(\bar{x})$ and $s(\bar{v}) = s'(\bar{w})$, i.e. $s(\bar{x}\bar{v}) = s'(\bar{x}\bar{w})$ and hence $\mathfrak{A} \not\models_X \bar{x}\bar{v} \mid \bar{x}\bar{w}$. Conversely, if $s(\bar{x}\bar{v}) = s'(\bar{x}\bar{w})$ for some $s, s' \in X$, then $\mathfrak{A} \not\models_{X\upharpoonright_{\bar{x}=s(\bar{x})}} \bar{v} \mid \bar{w}$.

The third item follows from the simple fact that $\bar{x}$ is never quantified and that those atoms are not in the scope of a disjunction, hence the values of $\bar{x}$ are preserved. □

This proposition allows us to investigate the connection between a formula and its $\bar{x}$-guarded version.

Lemma 3.23. *Let $\varphi^\star(\bar{x}, \bar{y})$ be the $\bar{x}$-guarded version of $\varphi(\bar{y}) \in \mathrm{FO}(\subseteq, \mid)$. Then $\mathfrak{A} \models_X \varphi^\star(\bar{x}, \bar{y}) \iff \mathfrak{A} \models_{X\upharpoonright_{\bar{x}=\bar{a}}} \varphi(\bar{y})$ for every $\bar{a} \in X(\bar{x})$.*

Proof. By induction over φ where inclusion/exclusion atoms are handled by the items 1. and 2. of Proposition 3.22. □

Lemma 3.23 gives rise to the following lemma about myopic formula in the form $\exists \bar{x}'(\bar{x}' \subseteq \bar{x} \wedge \psi)$, which can be considered to be a normal-form for myopic formulae.

Lemma 3.24. *Let $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq, \mid)$ be an $\bar{x}$-myopic formula of the form $\exists \bar{x}'(\bar{x}' \subseteq \bar{x} \wedge \psi)$, where in ψ no inclusion atoms of the form $\bar{v} \subseteq \bar{x}$ occur. Then $\mathfrak{A} \models_X \varphi$ if and only if there exists $F : X \longrightarrow \mathcal{P}^+(A^{|\bar{x}|})$ such that $F(s) \subseteq X(\bar{x})$ for every $s \in X$ and $\mathfrak{A} \models_{X[\bar{x}' \mapsto F]\upharpoonright_{\bar{x}=\bar{a}}} \psi'$ for all $\bar{a} \in X(\bar{x})$, where ψ' is the unguarded version of ψ.*

The componentwise behaviour that can be observed in Lemma 3.23 is also the reason why $\bar{x}$-myopic formulae are union-closed.

Theorem 3.25. *Let $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq, \mid)$ be $\bar{x}$-myopic and $\mathfrak{A} \models_{X_i} \varphi(\bar{x})$ for all $i \in I$. Then $\mathfrak{A} \models_X \varphi(\bar{x})$ for $X = \bigcup_{i \in I} X_i$.*

Proof. Let λ_i be a witnesses for $\mathfrak{A} \models_{X_i} \varphi$ for $i \in I$. For every $\bar{a} \in X(\bar{x})$ choose $i_{\bar{a}} \in I$ such that $\bar{a} \in X_{i_{\bar{a}}}(\bar{x})$. Define $\lambda(\psi) := \bigcup_{\bar{a} \in X(\bar{x})} \lambda_{i_{\bar{a}}}(\psi)\upharpoonright_{\bar{x}=\bar{a}}$ for every $\psi \in \mathrm{subf}(\varphi)$. We show that λ is a witness for $\mathfrak{A} \models_X \varphi$. It is not difficult to see that the requirements on witnesses for composite formulae are satisfied. We prove that the requirements for the literals are fulfilled as well. By the flatness property, first-order literals are satisfied by λ.

We prove now that $\mathfrak{A} \models_{\lambda(\gamma)} \gamma$ for $\gamma = \bar{x}\bar{v} \subseteq \bar{x}\bar{w}$ or $\gamma = \bar{x}\bar{v} \mid \bar{x}\bar{w}$. Let γ' be the corresponding *unguarded* formula, that is the formula resulting from γ by removing $\bar{x}$, i.e. we have $\gamma' = \bar{v} \subseteq \bar{w}$ or $\gamma' = \bar{v} \mid \bar{w}$. Due to Proposition 3.22, it suffices to prove that $\mathfrak{A} \models_{\lambda(\gamma)\upharpoonright_{\bar{x}=\bar{a}}} \gamma'$ is true for every $\bar{a} \in (\lambda(\gamma))(\bar{x})$. Notice that $\lambda(\gamma)\upharpoonright_{\bar{x}=\bar{a}} = \lambda_{i_{\bar{a}}}(\gamma)\upharpoonright_{\bar{x}=\bar{a}}$. Since $\lambda_{i_{\bar{a}}}$ is a witness for $\mathfrak{A} \models_{X_{i_{\bar{a}}}} \varphi$, it must be the case that $\mathfrak{A} \models_{\lambda_{i_{\bar{a}}}(\gamma)} \gamma$. By Proposition 3.22, it follows that $\mathfrak{A} \models_{\lambda_{i_{\bar{a}}}(\gamma)\upharpoonright_{\bar{x}=\bar{b}}} \gamma'$ for every $\bar{b} \in (\lambda_{i_{\bar{a}}}(\gamma))(\bar{x})$. If $\bar{a} \in (\lambda_{i_{\bar{a}}}(\gamma))(\bar{x})$, then this implies that $\mathfrak{A} \models_{\lambda_{i_{\bar{a}}}(\gamma)\upharpoonright_{\bar{x}=\bar{a}}} \gamma'$. Otherwise we have that $\lambda_{i_{\bar{a}}}(\gamma)\upharpoonright_{\bar{x}=\bar{a}} = \emptyset$ and then $\mathfrak{A} \models_{\lambda_{i_{\bar{a}}}(\gamma)\upharpoonright_{\bar{x}=\bar{a}}} \gamma'$ follows from the empty team property of FO($\subseteq, \mid$). In both cases, $\mathfrak{A} \models_{\lambda_{i_{\bar{a}}}(\gamma)\upharpoonright_{\bar{x}=\bar{a}}} \gamma'$ holds as desired, which concludes the proof of $\mathfrak{A} \models_{\lambda(\gamma)} \gamma$.

We still need to prove that $\mathfrak{A} \models_{\lambda(\gamma)} \gamma$ for literals of the form $\gamma = \bar{v} \subseteq \bar{x} \in \mathrm{subf}(\varphi)$. Towards this end, let $s \in \lambda(\bar{v} \subseteq \bar{x})$ and $\bar{b} := s(\bar{v})$. By definition of λ, there is some $\bar{a} \in X(\bar{x})$ such that $s \in \lambda_{i_{\bar{a}}}(\bar{v} \subseteq \bar{x})\upharpoonright_{\bar{x}=\bar{a}}$. Since $\mathfrak{A} \models_{\lambda_{i_{\bar{a}}}(\bar{v}\subseteq\bar{x})} \bar{v} \subseteq \bar{x}$ and $\bar{b} = s(\bar{v}) \in \lambda_{i_{\bar{a}}}(\bar{v} \subseteq \bar{x})(\bar{v})$, it follows that $\bar{b} \in (\lambda_{i_{\bar{a}}}(\bar{v} \subseteq \bar{x}))(\bar{x})$. By Proposition 3.22 we have $(\lambda_{i_{\bar{a}}}(\bar{v} \subseteq \bar{x}))(\bar{x}) = X_{i_{\bar{a}}}(\bar{x})$, wherefore $\bar{b} \in X_{i_{\bar{a}}}(\bar{x}) \subseteq X(\bar{x})$ and, consequently, we have chosen some index $i_{\bar{b}} \in I$ with $\bar{b} \in X_{i_{\bar{b}}}(\bar{x})$. By Proposition 3.22 again, it follows that $X_{i_{\bar{b}}}(\bar{x}) = (\lambda_{i_{\bar{b}}}(\bar{v} \subseteq \bar{x}))(\bar{x})$. So there is some $s' \in \lambda_{i_{\bar{b}}}(\bar{v} \subseteq \bar{x})$ with $s'(\bar{x}) = \bar{b} = s(\bar{v})$ and thus $s' \in \lambda_{i_{\bar{b}}}(\bar{v} \subseteq \bar{x})\upharpoonright_{\bar{x}=\bar{b}} \subseteq \lambda(\bar{v} \subseteq \bar{x})$. This concludes the proof of $\mathfrak{A} \models_{\lambda(\bar{v}\subseteq\bar{x})} \bar{v} \subseteq \bar{x}$. □

We have thus shown that $\bar{x}$-myopic formulae are closed under unions. It remains to prove that indeed every union closed formula $\varphi(\bar{x})$ of FO($\subseteq, \mid$) is equivalent to some $\bar{x}$-myopic formula.

This is done by proving the following slightly stronger result. We will show how to transform any FO($\subseteq, \mid$)-formula $\varphi(\bar{x})$ into an $\bar{x}$-myopic one, that is satisfied by what can be considered to be the union-closure of $\varphi(\bar{x})$.

Theorem 3.26. *Let $\varphi(\bar{x}) \in$ FO($\subseteq, \mid$). There is an $\bar{x}$-myopic formula $\mu(\bar{x})$ such that for all suitable structures $\mathfrak{A}$ and teams X with* $\mathrm{dom}(X) = \{\bar{x}\}$ *holds*

$$\mathfrak{A} \models_X \mu(\bar{x}) \iff X \text{ can be written as } X = \bigcup_{i\in I} X_i \text{ where } \mathfrak{A} \models_{X_i} \varphi(\bar{x}) \text{ for every } i \in I.$$

Proof. Let $\bar{y}$ be a fresh tuple of variables. Let $\varphi^\star(\bar{x}, \bar{y})$ be the $\bar{x}$-guarded version of $\varphi(\bar{y})$, i.e. $\varphi^\star$ results from φ by first replacing the variables $\bar{x}$ by the new variables $\bar{y}$ and then by adding $\bar{x}$ on both sides of every inclusion or exclusion atoms occurring in $\varphi(\bar{y})$. We define

$$\mu(\bar{x}) := \exists\bar{y}(\bar{y} \subseteq \bar{x} \wedge \bar{x}\bar{x} \subseteq \bar{x}\bar{y} \wedge \varphi^\star(\bar{x}, \bar{y})),$$

which is $\bar{x}$-myopic. We still need to prove the two directions of the claim.

"$\Longrightarrow$": First assume $\mathfrak{A} \models_X \mu(\bar{x})$. Then there exists a team Y of the form $Y = X[\bar{y} \mapsto F]$ for some function $F : X \longrightarrow \mathcal{P}^+(A^{|\bar{x}|})$ such that

$$\mathfrak{A} \models_Y \bar{y} \subseteq \bar{x} \wedge \bar{x}\bar{x} \subseteq \bar{x}\bar{y} \wedge \varphi^\star(\bar{x}, \bar{y}).$$

Thus we have $Y(\bar{y}) \subseteq Y(\bar{x}) = X(\bar{x})$ and, due to Lemma 3.23, $\mathfrak{A} \models_{Y\upharpoonright_{\bar{x}=\bar{a}}} \bar{x} \subseteq \bar{y} \wedge \varphi(\bar{y})$ for every $\bar{a} \in Y(\bar{x})$, because $\bar{x}\bar{x} \subseteq \bar{x}\bar{y} \wedge \varphi^\star(\bar{x}, \bar{y})$ is the $\bar{x}$-guarded version of $\bar{x} \subseteq \bar{y} \wedge \varphi(\bar{y})$. We can deduce, for every $\bar{a} \in Y(\bar{x})$, that $\{\bar{a}\} = Y\upharpoonright_{\bar{x}=\bar{a}}(\bar{x}) \subseteq Y\upharpoonright_{\bar{x}=\bar{a}}(\bar{y}) \subseteq Y(\bar{y}) \subseteq Y(\bar{x})$. This implies that

$$\bar{a} \in Y\upharpoonright_{\bar{x}=\bar{a}}(\bar{y}) \subseteq Y(\bar{x}) \text{ for every } \bar{a} \in Y(\bar{x}). \tag{3.4}$$

For every $\bar{a} \in Y(\bar{x})$, let $X_{\bar{a}}$ be the team with $\mathrm{dom}(X_{\bar{a}}) = \{\bar{x}\}$ and $X_{\bar{a}}(\bar{x}) = Y\upharpoonright_{\bar{x}=\bar{a}}(\bar{y})$. Because of this construction and due to $\mathfrak{A} \models_{Y\upharpoonright_{\bar{x}=\bar{a}}} \varphi(\bar{y})$, it follows that $\mathfrak{A} \models_{X_{\bar{a}}} \varphi(\bar{x})$.

Furthermore, (3.4), $Y(\bar{x}) = X(\bar{x})$ and $Y\restriction_{\bar{x}=\bar{a}}(\bar{y}) = X_{\bar{a}}(\bar{x})$ for every $\bar{a} \in Y(\bar{x})$ lead to

$$\bar{a} \in X_{\bar{a}}(\bar{x}) \subseteq X(\bar{x}) \text{ for every } \bar{a} \in X(\bar{x})$$

and, consequently, to $\bigcup_{\bar{a}\in X(\bar{x})} X_{\bar{a}}(\bar{x}) = X(\bar{x})$. Because of $\mathrm{dom}(X) = \{\bar{x}\} = \mathrm{dom}(X_{\bar{a}})$, we indeed have $X = \bigcup_{\bar{a}\in X(\bar{x})} X_{\bar{a}}$ where $\mathfrak{A} \models_{X_{\bar{a}}} \varphi(\bar{x})$ for every $\bar{a} \in X(\bar{x})$.

"$\Longleftarrow$": For the converse direction, we assume that X can be written as $X = \bigcup_{i\in I} X_i$ where $\mathfrak{A} \models_{X_i} \varphi(\bar{x})$ for every $i \in I$. Our goal is to prove that $\mathfrak{A} \models_X \mu(\bar{x})$. Towards this end, for every $\bar{a} \in X(\bar{x})$, we choose some index $i_{\bar{a}} \in I$ with $\bar{a} \in X_{i_{\bar{a}}}(\bar{x})$. Let $F : X \longrightarrow \mathcal{P}^+(A^{|\bar{y}|})$ be defined by $F(s) := X_{i_{s(\bar{x})}}(\bar{x})$. F is well-defined, because $s(\bar{x}) \in X_{i_{s(\bar{x})}}(\bar{x})$ implies $F(s) \neq \emptyset$ for every $s \in X$. Moreover, we have

$$s(\bar{x}) \in F(s) \subseteq X(\bar{x}) \text{ for every } s \in X. \tag{3.5}$$

Let $Y := X[\bar{y} \mapsto F]$. By construction, it follows that

$$F(s) = Y\restriction_{\bar{x}=s(\bar{x})}(\bar{y}) \text{ for every } s \in X. \tag{3.6}$$

Furthermore, we also have

$$Y(\bar{y}) = \bigcup_{s\in X} F(s) \underset{(3.5)}{=} X(\bar{x}) = Y(\bar{x}).$$

In particular, this implies $Y(\bar{y}) \subseteq Y(\bar{x})$ and, hence, $\mathfrak{A} \models_Y \bar{y} \subseteq \bar{x}$. Now, in order to prove $\mathfrak{A} \models_Y \bar{x}\bar{x} \subseteq \bar{x}\bar{y} \wedge \varphi^\star(\bar{x},\bar{y})$ we will use Lemma 3.23. This means that we only need to prove that $\mathfrak{A} \models_{Y\restriction_{\bar{x}=\bar{a}}} \bar{x} \subseteq \bar{y} \wedge \varphi(\bar{y})$ for every $\bar{a} \in Y(\bar{x})$. Towards this end, pick any $\bar{a} \in Y(\bar{x})$. Because of $Y(\bar{x}) = X(\bar{x})$, there must be an assignment $s_{\bar{a}} \in X$ with $s_{\bar{a}}(\bar{x}) = \bar{a}$. We clearly have

$$Y\restriction_{\bar{x}=\bar{a}}(\bar{x}) = \{\bar{a}\} = \{s_{\bar{a}}(\bar{x})\} \underset{(3.5)}{\subseteq} F(s_{\bar{a}}) \underset{(3.6)}{=} Y\restriction_{\bar{x}=s_{\bar{a}}(\bar{x})}(\bar{y}) = Y\restriction_{\bar{x}=\bar{a}}(\bar{y})$$

and, thus, $\mathfrak{A} \models_{Y\restriction_{\bar{x}=\bar{a}}} \bar{x} \subseteq \bar{y}$. By assumption, we know that $\mathfrak{A} \models_{X_{i_{\bar{a}}}} \varphi(\bar{x})$. Since we also have $Y\restriction_{\bar{x}=\bar{a}}(\bar{y}) = F(s_{\bar{a}}) = X_{i_{s_{\bar{a}}(\bar{x})}}(\bar{x}) = X_{i_{\bar{a}}}(\bar{x})$, we can deduce that $\mathfrak{A} \models_{Y\restriction_{\bar{x}=\bar{a}}} \varphi(\bar{y})$. Therefore, we indeed have $\mathfrak{A} \models_{Y\restriction_{\bar{x}=\bar{a}}} \bar{x} \subseteq \bar{y} \wedge \varphi(\bar{y})$. As a result, we have $\mathfrak{A} \models_Y \bar{y} \subseteq \bar{x} \wedge \bar{x}\bar{x} \subseteq \bar{x}\bar{y} \wedge \varphi^\star(\bar{x},\bar{y})$ which leads to $\mathfrak{A} \models_X \mu(\bar{x})$. □

Corollary 3.27 (Normal form of myopic-FO($\subseteq$, |)). *Let $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq, |)$ be union closed formula. There is an equivalent $\bar{x}$-myopic formula $\mu(\bar{x}) = \exists\bar{y}(\bar{y} \subseteq \bar{x} \wedge \vartheta(\bar{x},\bar{y}))$ where $\vartheta(\bar{x},\bar{y}) \in \mathrm{FO}(\subseteq, |)$ is some $\bar{x}$-guarded formula*

Proof. Since $\varphi(\bar{x})$ is union closed, we have

$$\mathfrak{A} \models_X \varphi(\bar{x}) \iff X \text{ can be written as } X = \bigcup_{i\in I} X_i \text{ where } \mathfrak{A} \models_{X_i} \varphi(\bar{x}) \text{ for every } i \in I.$$

Therefore, $\varphi(\bar{x})$ is equivalent to the $\bar{x}$-myopic formula $\mu(\bar{x})$ from Theorem 3.26, which is in the desired normal form. □

Here is what union closure property amounts to on an intuitive level. Assume you have a formula $\varphi(X)$ or $\varphi(\bar{x})$ that is closed under unions (and Σ^1_1-definable). Now, if you ask whether $\mathfrak{A} \models \varphi(X)$, or $\mathfrak{A} \models_X \varphi(\bar{x})$, holds, you can do the following. For every assignment / tuple s in X test whether a certain Σ^1_1-definable property ψ holds for s (independent of the other assignments in X). Since myopic FO($\subseteq$, |)-formulae are allowed to freely use $\bar{x}$-guarded dependency atoms, this behaviour can be expressed in this logic. But, this alone does not suffice, as then the formula would be flat, which in general is not the case. Thus, moreover, for every assignment $s \in X$ a set Y_s is chosen (non-deterministically) and it is required that $Y_s \subseteq X$ for every $s \in X$. This behaviour is captured by the inclusion atoms of form $\bar{y} \subseteq \bar{x}$. In a way this enables the assignments s to request that the team X must include certain other assignments.

3.4.1 Proving Corollary 3.27 using a Limited Number of Dependency Concepts

The construction of μ presented in the proof of Theorem 3.26 resp. Corollary 3.27 turns all dependency concepts of φ into guarded ones and adds two additional inclusion atoms. One might ask whether or not so many dependency concepts are actually needed.

In this section we will show that a limited number of dependency concept suffices. This is achieved by using a different proof technique that similar to the proof of Theorem 3.12 exploits that winning strategies of union games are definable in the myopic fragment of FO($\subseteq$, |)-formulae and that the model-checking games of myopic formulae are first-order interpretable. The rest of this section is organised as follows. First, we define target sets of a union game by a myopic FO($\subseteq$, |)-formula and then we present the alternative proof of Corollary 3.27.

Example 3.28. Let us demonstrate that (vertex sets of) winning strategies of *general* inclusion-exclusion games can be defined in FO($\subseteq$, |):

$$\begin{aligned}
\psi_{\text{win}}(y) &:= \psi_{\text{init}}(y) \wedge \psi_{\text{move}}(y) \wedge \psi_{E_{\text{ex}}}(y) \text{ where} \\
\psi_{\text{init}}(y) &:= \forall z(Iz \longrightarrow z \subseteq y) \\
\psi_{\text{move}}(y) &:= \exists z([(V_0 y \wedge \exists z'(Eyz' \wedge z' \subseteq z)) \vee (V_1 y \wedge \forall z'(Eyz' \longrightarrow z' \subseteq z))] \wedge z \subseteq y) \\
\psi_{E_{\text{ex}}}(y) &:= \forall z((E_{\text{ex}} yz \vee E_{\text{ex}} zy) \longrightarrow y \mid z)
\end{aligned}$$

Please recall that $\varphi \longrightarrow \psi$ is defined as $\text{nnf}(\neg\varphi) \vee (\varphi \wedge \psi)$ for $\varphi \in \text{FO}$ (in negation-normal form) and that $\mathfrak{A} \models_X \varphi \longrightarrow \psi$ is equivalent to $\mathfrak{A} \models_{X\upharpoonright_\varphi} \psi$ where $X\upharpoonright_\varphi := \{s \in X : \mathfrak{A} \models_s \varphi\}$. It is not difficult to verify that ψ_{win} expresses the conditions for winning strategies (cf. Definition 3.1). More formally, we have the following claim:

Claim 3.29. Let $\mathcal{G}$ be an inclusion-exclusion game and Y be a non-empty team over $\mathcal{G}$ with $y \in \text{dom}(Y)$. Then $\mathcal{G} \models_Y \psi_{\text{win}}(y)$ if and only if $Y(y)$ is the vertex set of a winning strategy for player 0 in $\mathcal{G}$.

With these formulae at hand, it is easy to define the target sets in FO(⊆, |):

$$\psi_{\mathcal{T}}(z) := Tz \wedge \exists y(\psi_{\text{win}}(y) \wedge z \subseteq y \wedge (Ty \longrightarrow y \subseteq z)).$$

Claim 3.30. Let $\mathcal{G}$ be an inclusion-exclusion game and let X be a non-empty team over $\mathcal{G}$ with $z \in \text{dom}(X)$. Then $\mathcal{G} \models_X \psi_{\mathcal{T}}(z)$ if and only if $X(z) \in \mathcal{T}(\mathcal{G})$.

Proof. "$\Longrightarrow$": Let $\mathcal{G} \models_X \psi_{\mathcal{T}}(z)$. Then $X(z) \subseteq T(\mathcal{G})$ and $\mathcal{G} \models_Y \psi_{\text{win}}(y) \wedge z \subseteq y \wedge (Ty \longrightarrow y \subseteq z)$ where $Y := X[y \mapsto F]$ for some $F : X \longrightarrow \mathcal{P}^+(V(\mathcal{G}))$. From $\mathcal{G} \models_Y z \subseteq y \wedge (Ty \longrightarrow y \subseteq z)$ we obtain that $Y(z) \subseteq Y(y)$ and $Y(y) \cap T(\mathcal{G}) = (Y{\upharpoonright}_{Ty})(y) \subseteq (Y{\upharpoonright}_{Ty})(z)$. By Claim 3.29 and $\mathcal{G} \models_Y \psi_{\text{win}}(y)$, we have $Y(y) = V(\mathcal{S})$ for some winning strategy $\mathcal{S}$. Now we prove that $Y(z) = Y(y) \cap T(\mathcal{G})$. The direction "$\supseteq$" follows from $Y(y) \cap T(\mathcal{G}) \subseteq (Y{\upharpoonright}_{Ty})(z) \subseteq Y(z)$, while the direction "$\subseteq$" is entailed by $Y(z) \subseteq Y(y)$ and $Y(z) = X(z) \subseteq T(\mathcal{G})$. Thus, $X(z) = Y(z) = Y(y) \cap T(\mathcal{G}) = \mathcal{T}(\mathcal{S}) \in \mathcal{T}(\mathcal{G})$.

"$\Longleftarrow$": Now let $X(z) \in \mathcal{T}(\mathcal{G})$. Then there is some winning strategy $\mathcal{S}$ with $\mathcal{T}(\mathcal{S}) = X(z)$. So $\mathcal{G} \models_X Tz$. By letting $Y := X[y \mapsto V(\mathcal{S})]$. Since X is non-empty by assumption, we have that $V(\mathcal{S})$ and Y are non-empty as well. Therefore, we obtain $\mathcal{G} \models_Y \psi_{\text{win}}(y)$, because of Claim 3.29 and $Y(y) = V(\mathcal{S})$, and $\mathcal{G} \models_Y z \subseteq y$, since $Y(z) = X(z) = \mathcal{T}(\mathcal{S}) \subseteq V(\mathcal{S}) = Y(y)$. We still need to prove that $\mathcal{G} \models_Y Ty \longrightarrow y \subseteq z$. Towards this end, consider any $s \in Y{\upharpoonright}_{Ty}$. Then $s(y) \in Y(y) \cap T(\mathcal{G}) = \mathcal{T}(\mathcal{S}) = X(z)$ and, thus, there exists some $s' \in X$ with $s'(z) = s(y)$. Let $s'' := s'[y \mapsto s(y)]$. It follows that $s'' \in Y{\upharpoonright}_{Ty}$, because we have $s''(y) = s(y) \in \mathcal{T}(\mathcal{S}) = V(\mathcal{S}) \cap T(\mathcal{G})$. So we have $s''(z) = s'(z) = s(y)$ and $s'' \in Y{\upharpoonright}_{Ty}$, which concludes the proof of $\mathcal{G} \models_Y Ty \longrightarrow y \subseteq z$. All in all, this proves that $\mathcal{G} \models_X \psi_{\mathcal{T}}(z)$. ☐

In the last example, we have learned that target sets of general inclusion-exclusion games can be expressed in full FO(⊆, |). In particular, this formula also defines the target sets of union games. But is it possible to do the same in the *myopic fragment* of FO(⊆, |)?

Towards giving a positive answer to this question, let $\psi^G_{\mathcal{T}}(x, z)$ be the corresponding x-guarded version of $\psi_{\mathcal{T}}(z)$. Recall that this just means that we add x on both sides of every occurring in-/exclusion atom. For example, the inclusion atom $z \subseteq y$ occurring in the subformula ψ_{init} will be transformed into $xz \subseteq xy$ when constructing $\psi^G_{\mathcal{T}}(x, z)$.

Example 3.31. Consider the x-myopic formula $\vartheta_{\mathcal{T}}(x)$ given by

$$\vartheta_{\mathcal{T}}(x) := \exists x'(x' \subseteq x \wedge \zeta(x, x')) \text{ where } \zeta(x, x') := xx \subseteq xx' \wedge \psi^G_{\mathcal{T}}(x, x').$$

We claim (and prove) that $\vartheta_{\mathcal{T}}(x)$ defines the target sets in *union* games which, more formally, means that for every team X over some union game $\mathcal{G}$ with $x \in \text{dom}(X)$ holds $\mathcal{G} \models_X \vartheta_{\mathcal{T}}(x) \iff X(x) \in \mathcal{T}(\mathcal{G})$.[11] To see this, let ζ' be the corresponding unguarded version of ζ, which turns out to be the following formula:

$$\zeta'(x, x') = x \subseteq x' \wedge \psi_{\mathcal{T}}(x')$$

[11] Since union games do not have any initial vertices, the formula $\vartheta_{\mathcal{T}}$ could be simplified by removing the subformula ψ^G_{init} which is in a union game just trivially satisfied.

Claim 3.32. Let $\mathcal{G}$ be an inclusion-exclusion game and let X be a non-empty team over $\mathcal{G}$ with $x, x' \in \mathrm{dom}(X)$. Then $\mathcal{G} \vDash_X \zeta'(x, x')$ if and only if $X(x) \subseteq X(x') \in \mathcal{T}(\mathcal{G})$.

Proof. Follows immediately from Claim 3.30. □

With this claim at hand, we can prove that $\vartheta_{\mathcal{T}}$ really defines what we have promised:

Claim 3.33. Let $\mathcal{G} = (V, V_0, V_1, E, I, T, E_{\mathrm{ex}})$ be a *union* game and let X be a team with $x \in \mathrm{dom}(X)$. Then $\mathcal{G} \vDash_X \vartheta_{\mathcal{T}}(x) \iff X(x) \in \mathcal{T}(\mathcal{G})$.

Proof. For $X = \emptyset$ the claim is true, because $\emptyset \in \mathcal{T}(\mathcal{G})$ is true for every union game and $\mathcal{G} \vDash_\emptyset \vartheta_{\mathcal{T}}$ is due to the empty team property. Let $X \neq \emptyset$. Applying Lemma 3.24 immediately yields the equivalence of the following statements:

(a) $\mathcal{G} \vDash_X \vartheta_{\mathcal{T}}(x) = \exists x'(x' \subseteq x \wedge \zeta(x, x'))$

(b) There is a function $F : X \longrightarrow \mathcal{P}^+(V)$ such that
   (i) $F(s) \subseteq X(x)$ for every $s \in X$, and
   (ii) $\mathcal{G} \vDash_{X_t} \zeta'(x, x')$ where $X_t := X[x' \mapsto F]\restriction_{x=t}$ for every $t \in X(x)$.

Next we will prove that the following propositions are also equivalent:

(c) For every $t \in X(x)$ exists a winning strategy $\mathcal{S}_t$ with $t \in \mathcal{T}(\mathcal{S}_t) \subseteq X(x)$.

(d) $X(x) \in \mathcal{T}(\mathcal{G})$.

"$(d) \Longrightarrow (c)$": If $X(x) \in \mathcal{T}(\mathcal{G})$, then there is a strategy $\mathcal{S}$ with $\mathcal{T}(\mathcal{S}) = V(\mathcal{S}) \cap T = X(x)$ which in particular implies that $t \in \mathcal{T}(\mathcal{S}) \subseteq X(x)$ for every $t \in X(x)$.

"$(c) \Longrightarrow (d)$": For the converse direction assume (c). Since $\mathcal{G}$ is a union game, we are allowed to use Theorem 3.16 to combine the family $(\mathcal{S}_t)_{t \in X(x)}$ into a single winning strategy $\mathcal{S}$ with $\mathcal{T}(\mathcal{S}) = \bigcup_{t \in X(x)} \mathcal{T}(\mathcal{S}_t)$ which is, due to assumptions about $\mathcal{T}(\mathcal{S}_t)$, equal to $X(x)$.

So we have already established that $(a) \iff (b)$ and $(c) \iff (d)$, but our goal was to show that $(a) \iff (d)$ which is exactly what Claim 3.33 states. Thus, in order to complete our proof of Claim 3.33, we just have to verify the missing link $(b) \iff (c)$.

"$(b) \Longrightarrow (c)$": Suppose that there is some function $F : X \longrightarrow \mathcal{P}^+(V)$ such that (i) and (ii) are true. So $\mathcal{G} \vDash_{X_t} \zeta'(x, x')$ for every $t \in X(x)$ which, by Claim 3.32, yields that $X_t(x) \subseteq X_t(x') \in \mathcal{T}(\mathcal{G})$. By definition of X_t in (ii), we have $X_t(x) = (X[x' \mapsto F]\restriction_{x=t})(x) = \{t\}$ and, consequently, $t \in X_t(x') \in \mathcal{T}(\mathcal{G})$ which, by definition of $\mathcal{T}(\mathcal{G})$, leads to the existence of winning strategies $\mathcal{S}_t$ with $X_t(x') = \mathcal{T}(\mathcal{S}_t)$ for every $t \in X(x)$. Because of (i) we can also conclude that $X_t(x') = \bigcup_{s \in X\restriction_{x=t}} F(s) \subseteq X(x)$. As a result, we obtain $t \in \mathcal{T}(\mathcal{S}_t) \subseteq X(x)$ for every $t \in X(x)$ as desired.

"$(b) \Longleftarrow (c)$": We assume now that for every $t \in X(x)$ there exists some winning strategy $\mathcal{S}_t$ with $t \in \mathcal{T}(\mathcal{S}_t) \subseteq X(x)$. Define $F : X \longrightarrow \mathcal{P}^+(V)$ as $F(s) := \mathcal{T}(\mathcal{S}_{s(x)})$ — notice that $s(x) \in \mathcal{T}(\mathcal{S}_{s(x)})$ holds by assumption, so $\mathcal{T}(\mathcal{S}_{s(x)}) \neq \emptyset$ and, hence, F is indeed well-defined. Then (i) is true, because for every $s \in X$ we have also assumed that

$\mathcal{T}(\mathcal{S}_{s(x)}) \subseteq X(x)$. Because $F(s)$ depends only on $s(x)$ and we have $X_t(x) = \{t\}$, it follows that $X_t(x') = \mathcal{T}(\mathcal{S}_t)$. As a result, we have $X_t(x) = \{t\} \subseteq \mathcal{T}(\mathcal{S}_t) = X_t(x')$ from which immediately follows that $X_t(x) \subseteq X_t(x') \in \mathcal{T}(\mathcal{G})$ for every $t \in X(x)$. Thus, by Claim 3.32, we obtain $\mathcal{G} \models_{X_t} \zeta'(x, x')$ for every $t \in X(x)$, which is exactly (ii). □

With the myopic formula $\vartheta_{\mathcal{T}}(x)$ defining target sets in union games at hand, we are now ready to present the variant of the game-theoretic proof of Theorem 3.12.

It is well known that every FO(⊆, |)-formula can be translated into an equivalent Σ_1^1-formula [Gal12]. As we have already seen in Theorem 3.7 and Theorem 3.12, every union closed formula of existential second-order logic is equivalent to some myopic Σ_1^1-formula. This is why, the following theorem starts w.l.o.g. with a myopic second-order-formula.

Theorem 3.34. *For every myopic second-order-formula $\varphi(X)$ there exists an equivalent myopic formula $\mu(\bar{x}) \in$ FO(⊆, |) using eight inclusion atoms and one exclusion atom.*

Proof. Let $\varphi(X)$ be a myopic Σ_1^1-formula. We are going to find a $\bar{y}$-myopic formula $\mu(\bar{y}) \in$ FO(⊆, |) with $(\mathfrak{A}, X(\bar{y})) \models \varphi(X) \iff \mathfrak{A} \models_X \mu(\bar{y})$ for every τ-structure $\mathfrak{A}$ and every team X with $\bar{y} \subseteq \mathrm{dom}(X)$. The most important steps of this proof are illustrated in Figure 3.7.

Let $\mathfrak{A}$ be a τ-structure with at least 2 elements.[12] In Definition 3.17 we have defined the model-checking game $\mathcal{G} := \mathcal{G}(\mathfrak{A}, \varphi)$ for $\mathfrak{A}$ and the myopic sentence $\varphi(X)$. By Proposition 3.18, we know for every *relation* $X \subseteq A^r$ that:

$$(\mathfrak{A}, X) \models \varphi(X) \iff X \in \mathcal{T}(\mathcal{G}) \tag{3.7}$$

Due to Claim 3.33 we may conclude for every *team* X with $x \in \mathrm{dom}(X)$:

$$\mathcal{G} \models_X \vartheta_{\mathcal{T}}(x) \iff X(x) \in \mathcal{T}(\mathcal{G}) \tag{3.8}$$

By combining (3.7) and (3.8), we obtain that for every *team* X with $x \in \mathrm{dom}(X)$ over $\mathcal{G}$:

$$\mathcal{G} \models_X \vartheta_{\mathcal{T}}(x) \iff (\mathfrak{A}, X(x)) \models \varphi(X) \tag{3.9}$$

Notice that $\mathcal{G} \models_X \vartheta_{\mathcal{T}}(x)$ implies that $X(x) \subseteq T(\mathcal{G}) = A^r$ and, hence, $X(x)$ is then actually a relation of the correct arity for the formula $\varphi(X)$. By counting, it is easy to verify that exactly eight inclusion atoms and one exclusion atom occurs in $\vartheta_{\mathcal{T}}$. The $\bar{y}$-myopic formula $\mu(\bar{y})$ that we are going to construct will result from $\vartheta_{\mathcal{T}}(x)$ and will use the same number of inclusion/exclusion atoms.

Using the technique of Lemma 3.9, it is possible to devise a (quantifier-free) interpretation $\mathcal{I} = (\delta, \varepsilon, \psi_{V_0}, \psi_{V_1}, \psi_E, \psi_I, \psi_T, \psi_{E_{ex}})$ such that $\mathcal{G} \cong \mathcal{I}(\mathfrak{A})$ with coordinate map $h : \delta^{\mathfrak{A}} \longrightarrow V(\mathcal{G})$. This interpretation encodes a position of the game $\mathcal{G}$ as a tuple

[12] This assumption is without loss of generality, since the resulting formulae $\mu(\bar{y})$ can be modified to be equivalent to $\varphi(X)$ even on structures with only one element. This modification does not require additional inclusion or exclusion atoms.

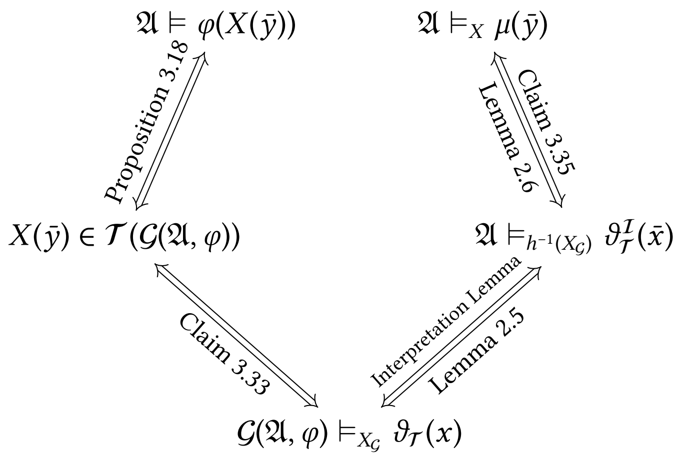

Figure 3.7: Overview of the proof of Theorem 3.34. Here, X is a team over $\mathfrak{A}$ with $\mathrm{dom}(X) = \{\bar{y}\}$, while $X_{\mathcal{G}}$ is the "game version" of X with $\mathrm{dom}(X_{\mathcal{G}}) = \{x\}$ and $X_{\mathcal{G}}(x) = X(\bar{y})$.

$(\bar{u}, \bar{v}) \in A^{n+m}$ where the n-tuple $\bar{u}$ has a certain equality type (indicating at which type of position we are encoding, e.g. at which formula we are) while the m-tuple $\bar{v}$ stores certain values (e.g. values of free variables and to which component the node belongs). More importantly, a position $\bar{a} \in T(\mathcal{G}) = A^r$ is described by the tuple $(\bar{u}, \bar{a}, \bar{b}) \in A^{n+m}$ where $\bar{u}$ has equality type e_T while $\bar{b} \in A^{m-r}$ can be chosen arbitrarily. Also recall that every variable v is replaced by an $(n+m)$-tuple $\bar{v}$ of pairwise different variables. In particular, let $\bar{x} = (\bar{u}, \bar{y}, \bar{z})$ where $\bar{u}$ is a n-tuple, $\bar{y}$ some r-tuple and $\bar{z}$ an $(m-r)$-tuple.

Every inclusion/exclusion atoms occurring in $\vartheta_{\mathcal{T}}(x)$ has one of the following three possible forms where v, w are some variables:

- $\beta_1(x, v, w) := xv \mid xw$
- $\beta_2(x, v, w) := xv \subseteq xw$
- $\beta_3(x, v) := v \subseteq x$

Notice that the only inclusion atom of the form of β_3 is in $\vartheta_{\mathcal{T}}(x)$ not within the scope of a disjunction (it is $x' \subseteq x$ right after the existential quantifier). In $\vartheta^{\mathcal{I}}_{\mathcal{T}}(\bar{x})$, these formulae are replaced by:

- $\beta^{\mathcal{I}}_1(\bar{x}, \bar{v}, \bar{w}) := \forall \bar{x}' \forall \bar{v}'([\delta(\bar{x}') \land \delta(\bar{v}') \land \varepsilon(\bar{x}, \bar{x}') \land \varepsilon(\bar{v}, \bar{v}')] \longrightarrow \bar{x}'\bar{v}' \mid \bar{x}\bar{w})$
- $\beta^{\mathcal{I}}_2(\bar{x}, \bar{v}, \bar{w}) := \exists \bar{x}' \exists \bar{v}'(\delta(\bar{x}') \land \delta(\bar{v}') \land \varepsilon(\bar{x}, \bar{x}') \land \varepsilon(\bar{v}, \bar{v}') \land \bar{x}'\bar{v}' \subseteq \bar{x}\bar{w})$
- $\beta^{\mathcal{I}}_3(\bar{x}, \bar{v}) := \exists \bar{v}'(\delta(\bar{v}') \land \varepsilon(\bar{v}, \bar{v}') \land \bar{v}' \subseteq \bar{x})$

Clearly, these formulas are not allowed in $\bar{y}$-myopic formulae, because the occurring inclusion/exclusion atoms are not $\bar{y}$-guarded. However, we know that the tuple $\bar{x}$ =

$(\bar{u}, \bar{y}, \bar{z})$ is used in $\vartheta^{\mathcal{I}}_{\mathcal{T}}(\bar{x})$ to store elements from $h^{-1}(T(\mathcal{G})) = \{(\bar{v}, \bar{a}, \bar{b}) \in A^{n+m} : \mathfrak{A} \models e_T(\bar{u}), \bar{a} \in A^r, \bar{b} \in A^{m-r}\}$, because whenever a team Y interprets $\bar{x}$ by values encoding different game positions, it follows that $h(Y)(x)$ cannot be a target set of a winning strategy, so Claim 3.33 leads to $\mathcal{G} \not\models_{h(Y)} \vartheta_{\mathcal{T}}(x)$ and then, by the Interpretation Lemma (Lemma 2.5), $\mathfrak{A} \not\models_Y \vartheta^{\mathcal{I}}_{\mathcal{T}}$. So for every team X which is well-formed (w.r.t. $\mathcal{I}$) and satisfies $\mathfrak{A} \models_X \vartheta^{\mathcal{I}}_{\mathcal{T}}$ we must have that $\mathfrak{A} \models_X \psi_T(\bar{x}) = e_T(\bar{u})$. This observation enables us to define versions of $\beta^{\mathcal{I}}_i$ that are allowed in $\bar{y}$-myopic formulae:

- $\beta^{\star}_1(\bar{y}, \bar{v}, \bar{w}) := \forall \bar{v}'([\delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}')] \longrightarrow \bar{y}\bar{v}' \mid \bar{y}\bar{w})$
- $\beta^{\star}_2(\bar{y}, \bar{v}, \bar{w}) := \exists \bar{v}'(\delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}') \wedge \bar{y}\bar{v}' \subseteq \bar{y}\bar{w})$
- $\beta^{\star}_3(\bar{y}, \bar{v}) := e_T(\bar{v}_{(1)}) \wedge \bar{v}_{(2)} \subseteq \bar{y}$ where $\bar{v} = \bar{v}_{(1)}\bar{v}_{(2)}\bar{v}_{(3)}$ and $|\bar{v}_{(1)}| = n$, $|\bar{v}_{(2)}| = r$ and $|\bar{v}_{(3)}| = m - r$.

Claim 3.35. For every team X with $\mathrm{dom}(X) = \{\bar{x}, \bar{u}, \bar{w}\}$ with $X(\bar{x}) \subseteq h^{-1}(T)$ and every $i = 1, 2, 3$ holds $\mathfrak{A} \models_X \beta^{\mathcal{I}}_i \iff \mathfrak{A} \models_X \beta^{\star}_i$.

Proof of Claim 3.35. $i = 1$: Let $X' := X[\bar{x}' \mapsto \delta^{\mathfrak{A}}, \bar{v}' \mapsto \delta^{\mathfrak{A}}]\upharpoonright_{\varepsilon(\bar{x},\bar{x}') \wedge \varepsilon(\bar{v},\bar{v}')}$. Then we can observe that

$$\mathfrak{A} \models_X \beta^{\mathcal{I}}_1 \iff \mathfrak{A} \models_{X'} \bar{x}'\bar{v}' \mid \bar{x}\bar{w} \overset{(!)}{\iff} \mathfrak{A} \models_{X'} \bar{y}\bar{v}' \mid \bar{y}\bar{w} \iff \mathfrak{A} \models_X \beta^{\star}_1,$$

but the equivalence of $\mathfrak{A} \models_{X'} \bar{x}'\bar{v}' \mid \bar{x}\bar{w}$ and $\mathfrak{A} \models_{X'} \bar{y}\bar{v}' \mid \bar{y}\bar{w}$ requires proof. Let $\bar{x}' := (\bar{u}', \bar{y}', \bar{z}')$ where $|\bar{u}'| = |\bar{u}|, |\bar{y}'| = |\bar{y}|$ and $|\bar{z}'| = |\bar{z}|$. Because of $\mathfrak{A} \models_{X'} \varepsilon(\bar{x}, \bar{x}')$ and $X'(\bar{x}) \subseteq h^{-1}(T)$, we have that $s(\bar{y}) = s(\bar{y}')$ for every $s \in X'$. Now, we prove the two directions of $\mathfrak{A} \models_{X'} \bar{x}'\bar{v}' \mid \bar{x}\bar{w} \iff \mathfrak{A} \models_{X'} \bar{y}\bar{v}' \mid \bar{y}\bar{w}$ separately:

"$\Longleftarrow$": First, assume that $\mathfrak{A} \models_{X'} \bar{y}\bar{v}' \mid \bar{y}\bar{w}$. It follows that $s_1(\bar{y}'\bar{v}') = s_1(\bar{y}\bar{v}') \neq s_2(\bar{y}\bar{w})$ for every $s_1, s_2 \in X'$. Because $\bar{y}$ and $\bar{y}'$ are subtuples of $\bar{x} = (\bar{u}, \bar{y}, \bar{z})$ resp. $\bar{x}' = (\bar{u}', \bar{y}', \bar{z}')$, this implies that $s_1(\bar{x}'\bar{v}') \neq s_2(\bar{x}\bar{w})$ for every $s_1, s_2 \in X'$. Hence, $\mathfrak{A} \models_{X'} \bar{x}'\bar{v}' \mid \bar{x}\bar{w}$.

"$\Longrightarrow$": Now let $\mathfrak{A} \models_{X'} \bar{x}'\bar{v}' \mid \bar{x}\bar{w}$. Towards a contradiction assume that $\mathfrak{A} \not\models_{X'} \bar{y}\bar{v}' \mid \bar{y}\bar{w}$. Then there are assignments $s_1, s_2 \in X'$ with $s_1(\bar{y}\bar{v}') = s_2(\bar{y}\bar{w})$. Since $X(\bar{x}) \subseteq h^{-1}(T)$ it follows that $\mathfrak{A} \models e_T(s_1(\bar{u})) \wedge e_T(s_2(\bar{u}))$. Because we also have $\mathfrak{A} \models_{X'} \varepsilon(\bar{x}, \bar{x}')$, it must be the case that $\mathfrak{A} \models_{s_1} e_T(\bar{u}) \wedge e_T(\bar{u}') \wedge \bar{y} = \bar{y}'$.

Consider $s'_1 := s_1[\bar{u}' \mapsto s_2(\bar{u}), \bar{z}' \mapsto s_2(\bar{z})]$. s_1 and s'_1 only differ on $\bar{u}'$ and $\bar{z}'$, but both still encode the equality type e_T in $\bar{u}'$ while the values of $\bar{z}'$ are irrelevant. So we still have $\mathfrak{A} \models_{s'_1} \delta(\bar{x}') \wedge \varepsilon(\bar{x}, \bar{x}')$, implying that $s'_1 \in X'$.

By definition of s'_1, holds

$$s'_1(\bar{x}') = s'_1(\bar{u}', \bar{y}', \bar{z}') = (s'_1(\bar{u}'), s'_1(\bar{y}'), s'_1(\bar{z}')) = (s_2(\bar{u}), s'_1(\bar{y}'), s_2(\bar{z}))$$

and, because of $s'_1(\bar{y}') = s'_1(\bar{y}) = s_1(\bar{y}) = s_2(\bar{y})$, we even have $(s_2(\bar{u}), s'_1(\bar{y}'), s_2(\bar{z})) = s_2(\bar{x})$. So $s'_1(\bar{x}') = s_2(\bar{x})$. Because we also have $s'_1(\bar{v}') = s_2(\bar{w})$, this leads to $s'_1(\bar{x}'\bar{v}') = s_2(\bar{x}\bar{w})$, which is impossible due to $\mathfrak{A} \models_{X'} \bar{x}'\bar{v}' \mid \bar{x}\bar{w}$. Contradiction! Therefore, $\mathfrak{A} \models_{X'} \bar{y}\bar{v}' \mid \bar{y}\bar{w}$ must be true.

$i = 2$: Again, we prove both directions separately. "$\Longleftarrow$": First let $\mathfrak{A} \models_X \beta_2^\star = \exists \bar{v}'(\delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}') \wedge \bar{y}\bar{v}' \subseteq \bar{y}\bar{w})$. Then there is a function $F : X \longrightarrow \mathcal{P}^+(A^{n+m})$ such that $\mathfrak{A} \models_{X'} \delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}') \wedge \bar{y}\bar{v}' \subseteq \bar{y}\bar{w}$ where $X' := X[\bar{v}' \mapsto F]$.

Due to $\mathfrak{A} \models_{X'} \bar{y}\bar{v}' \subseteq \bar{y}\bar{w}$, for every assignment $s \in X'$ there exists some $\tilde{s} \in X'$ with $s(\bar{y}\bar{v}') = \tilde{s}(\bar{y}\bar{w})$. Now consider $Y := \{s[\bar{x}' \mapsto \tilde{s}(\bar{x})] : s \in X'\}$. Since $X'(\bar{x}) = X(\bar{x}) \subseteq h^{-1}(T)$ and $s(\bar{y}) = \tilde{s}(\bar{y})$ for every $s \in X'$, we can conclude that $s(\bar{x}) = s[\bar{x}' \mapsto \tilde{s}(\bar{x})](\bar{x})$ and $\tilde{s}(\bar{x}) = s[\bar{x}' \mapsto \tilde{s}(\bar{x})](\bar{x}')$ encode the same target vertex $s(\bar{y})$ for every $s \in X'$ and, thus, $\mathfrak{A} \models_Y \delta(\bar{x}') \wedge \varepsilon(\bar{x}, \bar{x}')$. Because of $\mathfrak{A} \models_{X'} \delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}')$, we also have $\mathfrak{A} \models_Y \delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}')$. So, $\mathfrak{A} \models_Y \delta(\bar{x}') \wedge \delta(\bar{v}') \wedge \varepsilon(\bar{x}, \bar{x}') \wedge \varepsilon(\bar{v}, \bar{v}')$.

Towards proving $\mathfrak{A} \models_Y \bar{x}'\bar{v}' \subseteq \bar{x}\bar{w}$, let $t \in Y$. Then $t = s[\bar{x}' \mapsto \tilde{s}(\bar{x})]$ for some $s \in X'$ and, hence, $t(\bar{x}'\bar{v}') = (\tilde{s}(\bar{x}), s(\bar{v}'))$. Since $\tilde{s} \in X'$ has the property $s(\bar{y}\bar{v}') = \tilde{s}(\bar{y}\bar{w})$, we have $s(\bar{v}') = \tilde{s}(\bar{w})$ in particular. Thus, $t(\bar{x}'\bar{v}') = (\tilde{s}(\bar{x}), s(\bar{v}')) = (\tilde{s}(\bar{x}), \tilde{s}(\bar{w})) \in X'(\bar{x}, \bar{w}) = Y(\bar{x}, \bar{w})$.

Therefore, $\mathfrak{A} \models_Y \delta(\bar{x}') \wedge \delta(\bar{v}') \wedge \varepsilon(\bar{x}, \bar{x}') \wedge \varepsilon(\bar{v}, \bar{v}') \wedge \bar{x}'\bar{v}' \subseteq \bar{x}\bar{w}$ which proves that $\mathfrak{A} \models_X \beta_2^{\mathcal{I}}$, because $Y\upharpoonright_{\mathrm{dom}(X)} = X$.

"$\Longrightarrow$": Now let $\mathfrak{A} \models_X \beta_2^{\mathcal{I}}$. Then there is a function $F : X \longrightarrow \mathcal{P}^+(A^{2(n+m)})$ such that $\mathfrak{A} \models_{X'} \delta(\bar{x}') \wedge \delta(\bar{v}') \wedge \varepsilon(\bar{x}, \bar{x}') \wedge \varepsilon(\bar{v}, \bar{v}') \wedge \bar{x}'\bar{v}' \subseteq \bar{x}\bar{w}$ where $X' := X[\bar{x}'\bar{v}' \mapsto F]$.

Since $\mathfrak{A} \models_{X'} \bar{x}'\bar{v}' \subseteq \bar{x}\bar{w}$ it follows that $\mathfrak{A} \models_{X'} \bar{y}'\bar{v}' \subseteq \bar{y}\bar{w}$, because $\bar{y}, \bar{y}'$ are subtuples of $\bar{x}, \bar{x}'$. Due to $X'(\bar{x}) = X(\bar{x}) \subseteq h^{-1}(T)$, $\mathfrak{A} \models_{X'} \delta(\bar{x}') \wedge \varepsilon(\bar{x}, \bar{x}')$ implies that $\mathfrak{A} \models_{X'} \bar{y} = \bar{y}'$. So, $\mathfrak{A} \models_{X'} \bar{y}'\bar{v}' \subseteq \bar{y}\bar{w}$ is equivalent to $\mathfrak{A} \models_{X'} \bar{y}\bar{v}' \subseteq \bar{y}\bar{w}$. Hence, we have that $\mathfrak{A} \models_{X'} \delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}') \wedge \bar{y}\bar{v}' \subseteq \bar{y}\bar{w}$ which implies that $\mathfrak{A} \models_X \beta_2^\star$.

$i = 3$: "$\Longleftarrow$": First let $\mathfrak{A} \models_X \beta_3^\star = e_T(\bar{v}_{(1)}) \wedge \bar{v}_{(2)} \subseteq \bar{y}$ where $\bar{v}_{(1)}\bar{v}_{(2)}\bar{v}_{(3)} = \bar{v}$ and $|\bar{v}_{(1)}| = n, |\bar{v}_{(2)}| = r$ and $|\bar{v}_{(3)}| = m - r$. Then $X(\bar{v}_{(1)}) \subseteq e_T^{\mathfrak{A}}$ and $X(\bar{v}_{(2)}) \subseteq X(\bar{y})$. So, for every $s \in X$ there exists some $\tilde{s} \in X$ with $s(\bar{v}_{(2)}) = \tilde{s}(\bar{y})$. Let $X' := \{s[\bar{v}' \mapsto \tilde{s}(\bar{x})] : s \in X\}$. By construction, $\mathfrak{A} \models_{X'} \bar{v}' \subseteq \bar{x}$.

Let $t \in X'$ be chosen arbitrarily. By construction of X', we have that $t = s[\bar{v}' \mapsto \tilde{s}(\bar{x})]$ for some $s \in X$. Because $X(\bar{x}) \subseteq h^{-1}(T)$ and $X(\bar{v}_{(1)}) \subseteq e_T^{\mathfrak{A}}$, it is the case that $t(\bar{v}) = s(\bar{v})$ and $t(\bar{v}') = \tilde{s}(\bar{x})$ are both encoding the same target vertex $s(\bar{v}_{(2)}) = \tilde{s}(\bar{y})$. Thus, $\mathfrak{A} \models_t \delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}')$.

As a result, we obtain $\mathfrak{A} \models_{X'} \delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}') \wedge \bar{v}' \subseteq \bar{x}$ which proves that $\mathfrak{A} \models_X \beta_3^{\mathcal{I}}$, because $X'\upharpoonright_{\mathrm{dom}(X)} = X$.

"$\Longrightarrow$": Let $\mathfrak{A} \models_X \beta_3^{\mathcal{I}}$. Then there is a function $F : X \longrightarrow \mathcal{P}^+(A^{n+m})$ such that $\mathfrak{A} \models_{X'} \delta(\bar{v}') \wedge \varepsilon(\bar{v}, \bar{v}') \wedge \bar{v}' \subseteq \bar{x}$ where $X' := X[\bar{v}' \mapsto F]$. Let $\bar{v}' = \bar{v}'_{(1)}\bar{v}'_{(2)}\bar{v}'_{(3)}$ where $|\bar{v}'_{(1)}| = n, |\bar{v}'_{(2)}| = r$ and $|\bar{v}'_{(3)}| = m - r$.

Since $X'(\bar{u}\bar{y}\bar{z}) = X'(\bar{x}) = X(\bar{x}) \subseteq h^{-1}(T)$, we have $X'(\bar{u}) \subseteq e_T^{\mathfrak{A}}$ and, due to $\mathfrak{A} \models_{X'} \bar{v}' \subseteq \bar{x}$, it follows that $X'(\bar{v}'_{(1)}) \subseteq e_T^{\mathfrak{A}}$ as well as $\mathfrak{A} \models_{X'} \bar{v}'_{(2)} \subseteq \bar{y}$. Because of $\mathfrak{A} \models_{X'} \varepsilon(\bar{v}, \bar{v}')$ and $X'(\bar{v}'_{(1)}) \subseteq e_T^{\mathfrak{A}}$, we also have $X'(\bar{v}_{(1)}) \subseteq e_T^{\mathfrak{A}}$ and $\mathfrak{A} \models_{X'} \bar{v}_{(2)} = \bar{v}'_{(2)}$. This is why, we have that $\mathfrak{A} \models_{X'} e_T(\bar{v}_{(1)}) \wedge \bar{v}_{(2)} \subseteq \bar{y} = \beta_3^\star(\bar{y}, \bar{v})$. This concludes the proof of Claim 3.35. □

Let $\vartheta_{\mathcal{T}}^\star(\bar{x})$ be the formula that results from $\vartheta_{\mathcal{T}}^{\mathcal{I}}$ by replacing every subformulae $\beta_i^{\mathcal{I}}(\bar{x}, \dots)$ by $\beta_i^\star(\bar{y}, \dots)$. Now all the inclusion/exclusion atoms occurring in $\vartheta_{\mathcal{T}}^\star(\bar{x})$ go conform with the conditions of Definition 3.19 but it is still not quite a $\bar{y}$-myopic formula, since $\vartheta_{\mathcal{T}}^\star(\bar{x}) = \vartheta_{\mathcal{T}}^\star(\bar{u}, \bar{y}, \bar{z})$ has too many free variables. In order to get rid of the superfluous variables $\bar{u}, \bar{z}$ we simply define

$$\mu(\bar{y}) := \exists\bar{u}\exists\bar{z}(e_T(\bar{u}) \wedge \vartheta^\star_{\mathcal{T}}(\bar{u}, \bar{y}, \bar{z}))$$

which is now a $\bar{y}$-myopic formula.[13] For a team X with $\mathrm{dom}(X) = \{\bar{y}\}$ we call the team Y a T-expansion of X, if $\mathrm{dom}(Y) = \{\bar{u}, \bar{y}, \bar{z}\}$, $Y(\bar{y}) = X(\bar{y})$ and $Y(\bar{u}) \subseteq e^{\mathfrak{A}}_T$. Clearly, we have that $\mathfrak{A} \models_X \mu(\bar{y})$ if and only if $\mathfrak{A} \models_Y \vartheta^\star_{\mathcal{T}}(\bar{x})$ for some T-expansion of X and two T-expansions of Y, Y' are always h-similar to each other, because $h(Y(\bar{u}, \bar{y}, \bar{z})) = X(\bar{y}) = h(Y'(\bar{u}, \bar{y}, \bar{z}))$ already implies that $h(Y) = h(Y')$ which, by the Similarity Lemma (Lemma 2.6), leads to $\mathfrak{A} \models_Y \vartheta^{\mathcal{I}}_{\mathcal{T}} \iff \mathfrak{A} \models_{Y'} \vartheta^{\mathcal{I}}_{\mathcal{T}}$.

Furthermore, it follows from the construction of $\vartheta^\star_{\mathcal{T}}$ and Claim 3.35 that $\mathfrak{A} \models_Y \vartheta^{\mathcal{I}}_{\mathcal{T}} \iff \mathfrak{A} \models_Y \vartheta^\star_{\mathcal{T}}$ for every T-expansion Y of X. The reason for this is that T-expansions satisfy the property $Y(\bar{x}) \subseteq h^{-1}(T)$ which will be maintained at the subformulae, because, due to $\vartheta_{\mathcal{T}}$ being a x-myopic formula, variables from $\bar{x}$ are never quantified in $\vartheta^{\mathcal{I}}_{\mathcal{T}}$ or $\vartheta^\star_{\mathcal{T}}$.

The full T-expansion X_T of X is $X_T := \{s[\bar{u} \mapsto \bar{a}, \bar{z} \mapsto \bar{c}] : s \in X, \bar{a} \in e^{\mathfrak{A}}_T, \bar{c} \in A^{m-r}\}$ while the "game version" $X_{\mathcal{G}}$ of X_T is a team with $\mathrm{dom}(X_{\mathcal{G}}) = \{x\}$ defined by $X_{\mathcal{G}} := \{s_{\bar{b}} : \bar{b} \in X(\bar{y})\}$ where $s_{\bar{b}} : \{x\} \longrightarrow T(\mathcal{G}),\ x \mapsto \bar{b}$. It is not difficult to verify that $X_{\mathcal{G}}(x) = X(\bar{y})$ and $h(X_T) = X_{\mathcal{G}}$.

Putting everything together, we obtain:

$$\begin{aligned}
&\mathfrak{A} \models_X \mu(\bar{y})\\
\iff\ &\mathfrak{A} \models_Y \vartheta^\star_{\mathcal{T}}(\bar{x}) \text{ for some } T\text{-expansion } Y \text{ of } X \text{ (by construction of } \mu(\bar{y}))\\
\iff\ &\mathfrak{A} \models_Y \vartheta^{\mathcal{I}}_{\mathcal{T}}(\bar{x}) \text{ for some } T\text{-expansion } Y \text{ of } X \text{ (follows from Claim 3.35)}\\
\iff\ &\mathfrak{A} \models_{X_T} \vartheta^{\mathcal{I}}_{\mathcal{T}}(\bar{x}) \ (X_T \text{ is } h\text{-similar to every } T\text{-expansion of } X\text{, Lemma 2.6})\\
\iff\ &\mathcal{G} \models_{X_{\mathcal{G}}} \vartheta_{\mathcal{T}}(x) \text{ (Interpretation Lemma (Lemma 2.5))}\\
\iff\ &(\mathfrak{A}, X_{\mathcal{G}}(x)) \models \varphi(X) \text{ (due to (3.9))}\\
\iff\ &(\mathfrak{A}, X(\bar{y})) \models \varphi(X) \text{ (because } X_{\mathcal{G}}(x) = X(\bar{y}))
\end{aligned}$$

Thus, $\mathfrak{A} \models_X \mu(\bar{y}) \iff (\mathfrak{A}, X(\bar{y})) \models \varphi(X)$ follows and our proof is completed. □

Corollary 3.36. *Every $\bar{x}$-myopic formula $\mu(\bar{x})$ is equivalent to a $\bar{x}$-myopic formula that uses exactly seven inclusion atoms and one guarded exclusion atom.*

Proof. Every $\bar{x}$-myopic formula $\mu(\bar{x})$ can be transformed into a equivalent myopic Σ^1_1-sentence $\mu'(X)$ and the construction used in the proof of Theorem 3.34 produces exactly as many atoms as specified, if one replaces the unneeded subformula ψ_{init} by any tautology (this does not harm, because union games do not have initial vertices, so ψ_{init} was trivially satisfied). □

[13] Notice that the interpretation does *not* introduce any disjunction above $\beta^\star_3(\bar{x}, \bar{z})$, because $x \subseteq z$ is only in the scope of existential quantifiers and we only a need a conjunction in order to guard the translated existential quantifier in $\vartheta^{\mathcal{I}}_{\mathcal{T}}$.

3.4.2 Proving Corollary 3.27 via Skolem Normal Form

In this section, we present a third proof for Corollary 3.27 that uses a special Skolem normal form. Historically, this was the second proof for this result and it has been found by Richard Wilke as a shorter and direct version of the proof we have present in the previous section. It also appeared in [HW19, HW20] and it is based on methods of Galliani, Kontinen and Väänänen [Gal12, KV09].

Theorem 3.37. *Let $\varphi(X)$ be a myopic Σ^1_1-formula. There is an equivalent $\bar{x}$-myopic formula of* $\mathrm{FO}(\subseteq, \mid)$ *where* $|\bar{x}| = \mathrm{ar}(X)$.

Proof. First of all let us introduce a normal form of myopic Σ^1_1-formulae. Since in myopic formulae the symbol X may occur only positively after the implication, we can transform every $\forall\bar{x}(X\bar{x} \longrightarrow \exists\bar{R}\varphi'(\bar{R}, X, \bar{x}))$ into the equivalent formula

$$\forall\bar{x}(X\bar{x} \longrightarrow \exists S(S \subseteq X \land \exists\bar{R}\varphi'(\bar{R}, S, \bar{x})))$$

where $S \subseteq X$ is a shorthand for $\forall\bar{y}(S\bar{y} \longrightarrow X\bar{y})$. We now apply the Skolem normal form of Σ^1_1-formulae to $\exists\bar{R}\varphi'(\bar{R}, S, \bar{x})$, which yields the formula

$$\sigma(S, \bar{x}) := \exists\bar{f}\forall\bar{y}((f_1(\bar{w}) = f_2(\bar{w}) \leftrightarrow S\bar{w}) \land \psi(\bar{f}, \bar{y}, \bar{x}))$$

where ψ is a quantifier-free first-order formula and $\bar{w}$ is a subtuple of $\bar{y}$ and, moreover, every f_i occurs in ψ only with a unique tuple $\bar{w}_i$ (consisting of pairwise different variables) as argument, that is $f_i(\bar{w}_i)$ (see [KV09, Theorem 4.9] where an analogous construction is made). The original formula can thus be transformed into

$$\forall\bar{x}(X\bar{x} \longrightarrow \exists S(S \subseteq X \land \sigma(S, \bar{x}))).$$

Similarly to [Gal12] we embed $\sigma(S, \bar{x})$ into inclusion-exclusion logic as

$$\vartheta(\bar{s}, \bar{x}) := \forall\bar{y}\exists\bar{z}\Big(\bigwedge_i \mathrm{dep}(\bar{x}\bar{w}_i, z_i) \land ((\bar{x}\bar{w} \subseteq \bar{x}\bar{s} \land z_1 = z_2) \lor (\bar{x}\bar{w} \mid \bar{x}\bar{s} \land z_1 \neq z_2)) \land \psi'(\bar{x}, \bar{y}, \bar{z})\Big).$$

Here ψ' is obtained from ψ by simply replacing every occurrence of $f_i(\bar{w}_i)$ by z_i. The only difference in our case is that every dependency atom is $\bar{x}$-guarded due to the fact that the subformula at hand is inside the scope of the universally quantified variables $\bar{x}$ in $\forall\bar{x}(X\bar{x} \longrightarrow \dots)$. Notice that dependence atoms of the form $\mathrm{dep}(\bar{x}\bar{w}_i, z_i)$ can also be regarded as $\bar{x}$-myopic. Formally, we can embed such an atom into exclusion logic via the formula $\forall v(\bar{x}\bar{w}_i v \mid \bar{x}\bar{w}_i z_i \lor z_i = v)$, which has the intended shape [Gal12]. The whole formula $\varphi(X)$ thus translates into

$$\mu(\bar{x}) := \exists\bar{s}(\bar{s} \subseteq \bar{x} \land \vartheta(\bar{s}, \bar{x})).$$

Let $\vartheta'(\bar{s}, \bar{x})$ be the unguarded version of $\vartheta(\bar{s}, \bar{x})$. Analogously to the argumentation of Galliani [Gal12] by additionally making use of Proposition 3.22, $(\mathfrak{A}, Y\restriction_{\bar{x}=\bar{a}}(\bar{s})) \models_{\bar{x}\mapsto\bar{a}} \sigma(S, \bar{x})$ if and only if $\mathfrak{A} \models_{Y\restriction_{\bar{x}=\bar{a}}} \vartheta'(\bar{s}, \bar{x})$ for $\bar{a} \in Y(\bar{s})$, where Y is a team with domain

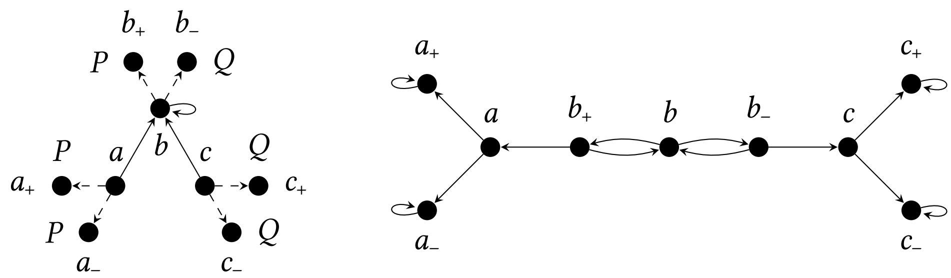

Figure 3.8: The structures $\mathfrak{A}$ and $\mathfrak{B}$. The structure $\mathfrak{A} = (V, E^{\mathfrak{A}}, F^{\mathfrak{A}}, P^{\mathfrak{A}}, Q^{\mathfrak{A}})$ on the left side uses two different kinds of edges: the dashed edges belong to F, while the other are E-edges. Furthermore, $\mathfrak{A}$ exhibits two predicates P, Q. The structure $\mathfrak{B} = (V, E^{\mathfrak{B}})$ depicted on the right is just a directed graph. Please notice that both structures are using the same universe V.

$\{\bar{s}, \bar{x}\}$ (here the variable S takes the role of the team). Using Lemma 3.24 we have $\mathfrak{A} \models_X \mu(\bar{x})$ if and only if there is a function $F : X \longrightarrow \mathcal{P}^+(A^{\mathrm{ar}(\bar{s})})$ such that $F(s) \subseteq X(\bar{x})$ for every $s \in X$ and $\mathfrak{A} \models_{X[\bar{s} \mapsto F]\upharpoonright_{\bar{x}=\bar{a}}} \vartheta'(\bar{s}, \bar{x})$ for all $\bar{a} \in X(\bar{x})$, which again holds if and only if there exists such an F with $(\mathfrak{A}, F(s)) \models_s \sigma(S, \bar{x})$ for all $s \in X$, but this just means $(\mathfrak{A}, X(\bar{x})) \models \forall \bar{x}(X\bar{x} \longrightarrow \exists S(S \subseteq X \wedge \sigma(S, \bar{x})))$. □

3.4.3 Optimality of the Myopic Fragment

One might ask whether the restrictions of Definition 3.19 are actually imperative to capture the union closed fragment w.r.t. its expressive power. In this section, we will show that neither condition can be dropped and that every single atom of Definition 3.19 is required to express all union closed properties.

We start by showing that neither condition can be dropped. First of all, it is pretty clear that exclusion atoms have to be $\bar{x}$-guarded, because $x_1 \mid x_2$ is not guarded and obviously not closed under unions. Furthermore, it is very clear that the variables among $\bar{x}$ must not be quantified to ensure the effectiveness of the above restrictions. This points out the necessity of conditions (a) and (b) of Definition 3.19. In the next example we demonstrate that neither restriction of condition (c) can be dropped.

Example 3.38. Consider the structures $\mathfrak{A} = (V, E^{\mathfrak{A}}, F^{\mathfrak{A}}, P^{\mathfrak{A}}, Q^{\mathfrak{A}})$ and $\mathfrak{B} = (V, E^{\mathfrak{B}})$ drawn in Figure 3.8 and the following formulae:

$$\varphi(x) := \exists y \exists z(Fxy \wedge Fxz \wedge xy \mid xz \wedge [(Py \wedge \vartheta(x)) \vee (Qy \wedge \vartheta(x))])$$
$$\text{where } \vartheta(x) := \exists v(Exv \wedge v \subseteq x)$$
$$\psi(x) := \exists y \exists z(Exy \wedge Exz \wedge xy \mid xz \wedge \exists w(Eyw \wedge x \subseteq w))$$

Neither $\varphi(x)$ nor $\psi(x)$ is x-myopic, because the inclusion atom $v \subseteq x$ from ϑ occurs inside the scope of a disjunction (and it is not x-guarded), while the atom $x \subseteq w$ is neither x-guarded nor of the form that is allowed outside the scope of disjunctions, because x appears on the wrong side of the inclusion atom.

For every $v \in V$ let $s_v : \{x\} \longrightarrow V$ be the assignment with $s_v(x) := v$. We define the teams $X_1 := \{s_a, s_b\}$, $X_2 := \{s_b, s_c\}$ and $X := X_1 \cup X_2 = \{s_a, s_b, s_c\}$. In order to shows that the restrictions of Definition 3.19 are indeed necessary, we will now demonstrate that neither $\varphi(x)$ nor $\psi(x)$ are closed under unions:

Claim 3.39. $\mathfrak{A} \models_{X_i} \varphi(x)$ and $\mathfrak{B} \models_{X_i} \psi(x)$ for $i = 1, 2$ but $\mathfrak{A} \not\models_X \varphi(x)$ and $\mathfrak{B} \not\models_X \psi(x)$.

Proof. First, we prove the statements for $\mathfrak{A}$ and φ. We will prove $\mathfrak{A} \models_{X_1} \varphi(x)$ and $\mathfrak{A} \not\models_X \varphi(x)$ but skip the proof for $\mathfrak{A} \models_{X_2} \varphi(x)$, because this can be proven by using very similar (in fact almost symmetric) arguments.

We will now prove that $\mathfrak{A} \models_{X_1} \varphi(x)$. Let $Y_1 := \{t_a, t_b\}$ be the team consisting of $t_a := s_a[y \mapsto a_+, z \mapsto a_-]$ and $t_b := s_b[y \mapsto b_+, z \mapsto b_-]$ or, slightly more readable, $t_a : xyz \mapsto aa_+a_-$ and $t_b : xyz \mapsto bb_+b_-$. Clearly, Y_1 is an extension of X_1 that now also interprets y, z and we have that $\mathfrak{A} \models_{Y_1} Fxy \wedge Fxz$. Furthermore, we also have $\mathfrak{A} \models_{Y_1} xy \mid xz$, because $t_a(x) \neq t_b(x)$, $t_a(y) \neq t_a(z)$ and $t_b(y) \neq t_b(z)$. We still need to prove that $\mathfrak{A} \models_{Y_1} [(Py \wedge \vartheta) \vee (Qy \wedge \vartheta)]$. By the empty team property, it suffices to prove that $\mathfrak{A} \models_{Y_1} Py \wedge \vartheta$. Since $t_a(y) = a_+ \in P$ and $t_b(y) = b_+ \in P$ we have $\mathfrak{A} \models_{Y_1} Py$.

In order to show $\mathfrak{A} \models_{Y_1} \vartheta(x) = \exists v(Exv \wedge v \subseteq x)$, we define $Z_1 := \{r_a, r_b\}$ where $r_a := t_a[v \mapsto b]$ and $r_b := t_b[v \mapsto b]$. Since $r_a(x, v) = (a, b) \in E$ and $r_b(x, y) = (b, b) \in E$, it follows that $\mathfrak{A} \models_{Z_1} Exv$. Furthermore, we have $Z_1(v) = \{b\} \subseteq \{a, b\} = Z_1(x)$ and hence also $\mathfrak{A} \models_{Z_1} v \subseteq x$. This concludes the proof of $\mathfrak{A} \models_{Y_1} \vartheta$ and, thus, also of $\mathfrak{A} \models_{X_1} \varphi(x)$.

Now we will prove that $\mathfrak{A} \not\models_X \varphi$. Towards a contradiction, we assume that $\mathfrak{A} \models_X \varphi$ would be true. Then there exists an extension Y of X that additionally interprets the variables y, z such that $\mathfrak{A} \models_Y Fxy \wedge Fxz \wedge xy \mid xz \wedge [(Py \wedge \vartheta) \vee (Qy \wedge \vartheta)]$.

We analyse how assignments from X are manipulated in this process. Let $t \in Y$. Since $\mathfrak{A} \models_Y Fxy \wedge Fxz$, we can deduce $t(y), t(z) \in \{t(x)_+, t(x)_-\} \in \{\{a_+, a_-\}, \{b_+, b_-\}, \{c_+, c_-\}\}$. However, due to $\mathfrak{A} \models_Y xy \mid xz$, it must be the case that $t(y) \neq t(z)$. This implies that $\{t(y), t(z)\} = \{t(x)_+, t(x)_-\}$. Please note, that this leaves only two possibilities for each $t \in X$: either $t(y) = t(x)_+$ and $t(z) = t(x)_-$ or $t(y) = t(x)_-$ and $t(z) = t(x)_+$. We call $t \in Y$ *positive* if $t(y) = t(x)_+$ and *negative* if we have $t(y) = t(x)_-$ instead. It is not possible that a positive $t \in Y$ *coexists* with a negative $t' \in Y$ with $t(x) = t'(x)$, because then we would have $t(x, y) = (t(x), t(x)_+) = (t'(x), t'(x)_+) = t'(x, z)$ which contradicts $\mathfrak{A} \models_Y xy \mid xz$. As a result, every $s \in X$ was extended to an assignment of Y by adding either $s(x)_+$ to y and $s(x)_-$ to z or $s(x)_-$ to y and $s(x)_+$ to z. For every $s \in X$ we define $s^+ := s[y \mapsto s(x)_+, z \mapsto s(x)_-]$ and $s^- := s[y \mapsto s(x)_-, z \mapsto s(x)_+]$ which we sometimes also call the positive resp. negative extension of s.

We have already seen that for each $s \in X$ either $s^+ \in Y$ or $s^- \in Y$ happens. This means Y is of the form $Y = \{t_a, t_b, t_c\}$ where $t_v \in \{s_v^+, s_v^-\}$ for $v \in \{a, b, c\}$. In particular we either have $t_b = s_b^+$ or $t_b = s_b^-$.

We only deal with the case $t_b = s_b^+$, because the other cases is very analogous. So $Y = \{t_a, s_b^+, t_c\}$. By construction of $\mathfrak{A}$, we have that $t_a(y), s_b^+(y) \in P \setminus Q$ while $t_c(y) \in Q \setminus P$.

Therefore, the only possible split of Y witnessing the satisfaction of $[(Py \wedge \vartheta) \vee (Qy \wedge \vartheta)]$ is $Y = Y_P \cup Y_Q$ with $Y_P := \{t_a, s_b^+\}$ and $Y_Q = \{t_c\}$.

But then $\mathfrak{A} \models_{Y_Q} \vartheta(x) = \exists v(Exv \wedge v \subseteq x)$, which implies the existence of a function $H : \{t_c\} \longrightarrow \mathcal{P}^+(A)$ such that $\mathfrak{A} \models_Z Exv \wedge v \subseteq x$ for $Z := Y_Q[v \mapsto H]$. Due to $Z(x) = Y_Q(x) = \{t_c(x)\} = \{c\}$ and $\mathfrak{A} \models_Z v \subseteq x$, it follows that $H(t_c) = \{c\}$. So, we have $Z = \{r\}$ where $r := t_c[v \mapsto c]$. But then $r(x, v) = (c, c) \notin E$ in contradiction to $\mathfrak{A} \models_Z Exv$.

Now we prove the statements for $\mathfrak{B}$ and ψ. Again we define the positive resp. negative extension as follows:

$$s^+ := s[y \mapsto s(x)_+, y \mapsto s(x)_-] \text{ and } s^- := s[y \mapsto s(x)_-, y \mapsto s(x)_+]$$

In order prove $\mathfrak{B} \models_{X_1} \psi(x)$, it suffices to verify that $\mathfrak{B} \models_{Y_1} Exy \wedge Exz \wedge xy \mid xz \wedge \exists w(Eyw \wedge x \subseteq w)$ is true for $Y_1 := \{s_a^+, s_b^+\}$. Since $s_a^+(x, y) = (a, a_+)$, $s_a^+(x, z) = (a, a_-)$, $s_b^+(x, y) = (b, b_+)$ and $s_b^+(x, z) = (b, b_-)$ are edges in $\mathfrak{B}$, it follows that $\mathfrak{B} \models_{Y_1} Exy \wedge Exz$. We also have $\mathfrak{B} \models_{Y_1} xy \mid xz$, because $s_a^+(x) \neq s_b^+(x)$, $s_a^+(y) \neq s_a^+(z)$ and $s_b^+(y) \neq s_b^+(z)$. We still need to check that $\mathfrak{B} \models_{Y_1} \exists w(Eyw \wedge x \subseteq w)$. Towards this end, let $Z_1 = \{r_1, r_2, r_3\}$ where

$$\begin{aligned} r_1 &:= s_a^+[w \mapsto a_+] : \ xyzw \mapsto aa_+a_-a_+ \\ r_2 &:= s_b^+[w \mapsto a] : \ xyzw \mapsto bb_+b_-a \\ r_2 &:= s_b^+[w \mapsto b] : \ xyzw \mapsto bb_+b_-b \end{aligned}$$

Clearly, $\mathfrak{B} \models_{Z_1} Eyw \wedge x \subseteq w$ and Z_1 is an extension of Y_1. This concludes the proof for $\mathfrak{B} \models_{X_1} \psi(x)$. The proof for $\mathfrak{B} \models_{X_2} \psi(x)$ is analogous.

Now we will prove that $\mathfrak{B} \not\models_X \psi(x)$. Towards a contradiction, we assume that $\mathfrak{B} \models_X \psi(x)$. It follows that there is an extension Y of X that additionally interprets the variables y, z such that $\mathfrak{B} \models_Y Exy \wedge Exz \wedge xy \mid xz \wedge \exists w(Eyw \wedge x \subseteq w)$.

An analysis that is very similar to the analysis for $\mathfrak{A}$ and φ yields that for every $s \in X$ either the *positive* or the *negative* extension of s, which are s^+ resp. s^-, is an element of Y. Moreover, s^+ and s^- cannot coexist in Y for every $s \in X$. It follows again that $Y = \{t_a, t_b, t_c\}$ where $t_v \in \{s_v^+, s_v^-\}$ for $v \in \{a, b, c\}$. In particular, either $t_b = s_b^+$ or $t_b = s_b^-$.

We only deal with the case that $t_b = s_b^+$; the argumentation for the other case is analogous. $\mathfrak{B} \models_Y \exists w(Eyw \wedge x \subseteq w)$ states that $\mathfrak{B} \models_Z Eyw \wedge x \subseteq w$ where $Z := Y[w \mapsto F]$ for some function $F : Y \longrightarrow \mathcal{P}^+(B)$. It is not difficult to verify that $Z(x) = \{a, b, c\}$ while $Z(w)$ cannot contain c, because due to $t_b = s_b^+ \in Y$ we have that $s_b^- \notin Y$ and as a result $b_- \notin Y(y)$ but b_- is the only vertex with an edge to c. So $\mathfrak{B} \not\models_Z x \subseteq w$, which contradicts $\mathfrak{B} \models_Z Eyw \wedge x \subseteq w$.

Since the assumption of $\mathfrak{B} \models_X \psi(x)$ has led to a contradiction, it must be the case that $\mathfrak{B} \not\models_X \psi$. □

Thus the atoms allowed by Definition 3.19 suffice to capture the union closed fragment of FO($\subseteq$, |). On the contrary, one may ask whether the set of atoms given in Definition 3.19 is necessary. Let us argue for all rules of Definition 3.19.

Assume that all exclusion atoms are forbidden then every formula is already in inclusion logic in which one cannot define every union closed property as was shown by Galliani and Hella [GH13, p. 16].

If inclusion atoms were only allowed in the form $\bar{x}\bar{y} \subseteq \bar{x}\bar{z}$, that means the atoms $\bar{v} \subseteq \bar{x}$ are forbidden, the formulae become flat, as can be seen by considering Proposition 3.22, but not all union closed properties are flat.

The case where inclusion atoms of form $\bar{x}\bar{y} \subseteq \bar{x}\bar{z}$ are forbidden is a bit more delicate. To prove that such a formula cannot express every union closed property consider the formula

$$\mu(x) := \exists z(z \subseteq x \land \forall y(Exy \rightarrow xy \subseteq xz)),$$

where $\tau = \{E\}$ for a binary predicate symbol E. This formula axiomatises the set of all teams X over a graph $G = (V, E)$ such that whenever $v \in X(x)$ and $(v, w) \in E$, then already $w \in X(x)$. The formula obviously describes a union closed property. Consider the graph G: $b \longleftarrow a \longrightarrow c$. Here, $G \models_X \mu(x)$ for precisely those teams X that satisfy "$a \in X(x)$ implies $b, c \in X(x)$". For every $v \in V(G)$ let s_v be the assignment $x \mapsto v$ and let $X_v := \{s_v\}$. Furthermore, we define $X_{abc} := \{s_a, s_b, s_c\}$.

Let $\psi(x)$ be an x-myopic formula in which the construct $x\bar{y} \subseteq x\bar{z}$ does not appear. So the only inclusion atoms occurring in $\psi(x)$ are of the form $z \subseteq x$ which are not allowed in the scope of disjunctions. Notice that z cannot be universally quantified, as the team $X_b = \{s_b\}$ satisfies the described property, but not $\forall z(z \subseteq x)$. Thus we may assume without loss of generality that $\psi(x)$ has the form

$$\psi(x) = \exists z(z \subseteq x \land \psi'(x, z)),$$

where in $\psi'(x, z)$ no inclusion atom occurs (we postpone the argumentation for the case that more than one atom of the form $z \subseteq x$ occurs). Let $\eta(x, z)$ be the unguarded version of $\psi'(x, z)$. By Lemma 3.24, there is a function $F\colon X_{abc} \longrightarrow \mathcal{P}^+(V(G))$ such that $F(s) \subseteq X_{abc}(x) = V(G)$ for $s \in X_{abc}$ and $G \models_{X_{abc}[z \mapsto F]\restriction_{x=v}} \eta$ for every $v \in X_{abc}(x)$. Please notice that $X_{abc}[z \mapsto F]\restriction_{x=v} = X_v[z \mapsto F\restriction_{X_v}]$. Moreover, because in $\eta(x, z)$ no inclusion atom occurs it is downwards closed. Assume $a \in F(s_a)$. By downwards closure of $\eta(x, z)$ we obtain $G \models_{X_a[z \mapsto a]} \eta$, which (by Lemma 3.24) implies that $G \models_{X_a} \psi$ contradicting our assumption that ψ describes the desired property. Otherwise, because of symmetry, b is in $F(s_a)$, and hence $G \models_{X_a[z \mapsto b]} \eta$. Additionally, since $G \models_{X_b} \psi$ we know (by Lemma 3.24) that $G \models_{X_b[z \mapsto b]} \eta$. Together this implies $G \models_{X_{ab}[z \mapsto b]\restriction_{x=v}} \eta$ for $v = a, b$ and, due to Lemma 3.24, we get $G \models_{X_{ab}} \psi$ which is again in conflict with our assumption about ψ describing the desired property.

We want to remark that generally the formula ψ could have the form $\exists \bar{z}(z_1 \subseteq x \land \cdots \land z_n \subseteq x \land \psi')$. In this case we would have to consider a graph G similar to the one above, where instead of two successors b and c the vertex a has $n + 1$ successors b_i, and argue that there are functions F_1 to F_n such that G satisfies η under $X[z_1 \mapsto F_1] \cdots [z_n \mapsto F_n]$. Inside this team there would be an assignment $s\colon x \mapsto a, z_1 \mapsto b_1, \ldots, z_n \mapsto b_n$ with possibly $b_i \neq b_j$ for all i, j. Again, every team X_{b_i} satisfies ψ, hence $s\colon x \mapsto b_i, z_i \mapsto b_i$ for every i is an assignment satisfying η. Together the team $\{x \mapsto a\} \cup \{x \mapsto b_i : 1 \leq i \leq n\}$ satisfies ψ but misses b_{n+1} hence we obtain a contradiction.

3.5 An Atom capturing the Union Closed Fragment

The present work was motivated by a question of Galliani and Hella in 2013 [GH13]. Galliani and Hella asked whether there is a union closed atomic dependency notion α that is definable in existential second-order logic such that $\mathrm{FO}(\alpha)$ corresponds precisely to all union closed properties of $\mathrm{FO}(\subseteq, |)$. In [GH13] they have already shown that $\mathrm{FO}(\subseteq)$ does not suffice, as there are union closed properties not definable in it. Moreover, they have established a theorem stating that every union-closed atomic property that is definable in first-order logic (where the formula has access to the team via a predicate) is expressible in inclusion logic. Thus, whatever atom characterises all union closed properties of $\mathrm{FO}(\subseteq, |)$ must axiomatise an inherently non-first-order property.

Intuitively speaking, as we have seen in Section 3.3, solving union games is a complete problem for $\mathcal{U}$, the class of all union closed Σ^1_1-definable properties. Therefore, a canonical solution to this question is to propose an atomic formula that defines the winning regions in a union game. Towards this we must describe how a game can be encoded into a team. For instance, this is not as straightforward as one might think, because there is a technical pitfall we need to avoid. The union of two teams describing union games, each won by player 0, might encode a game won by player 1, but by union closure it must satisfy the atomic formula.

We encode union games in teams by using variable tuples for the respective components, where we also encode the complementary relations in order to ensure that the union of two different games cannot form a different game. For $k \in \mathbb{N}$ let $\mathcal{V}_k$ be the set of distinct k-tuples of variables $\{\bar{u}, \bar{v}_0, \bar{v}_1, \bar{v}, \bar{w}, \bar{t}, \bar{v}_{\mathrm{ex}}, \bar{w}_{\mathrm{ex}}, \bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{u}^{\complement}, \bar{v}^{\complement}, \bar{w}^{\complement}, \bar{t}^{\complement}, \bar{v}^{\complement}_{\mathrm{ex}}, \bar{w}^{\complement}_{\mathrm{ex}}, \bar{\varepsilon}^{\complement}_1, \bar{\varepsilon}^{\complement}_2\}$.

Definition 3.40. Let X be a team with $\mathcal{V}_k \subseteq \mathrm{dom}(X)$ and codomain A. We define $\sim := X(\bar{\varepsilon}_1, \bar{\varepsilon}_2)$ and $\mathfrak{A}^X := (V, V_0, V_1, E, I, T, E_{\mathrm{ex}})$ where the components are as follows.

- $V := X(\bar{u})$
- $V_0 := X(\bar{v}_0)$
- $V_1 := X(\bar{v}_1)$
- $I := \emptyset$
- $E := X(\bar{v}, \bar{w})$
- $T := X(\bar{t})$
- $E_{\mathrm{ex}} := X(\bar{v}_{\mathrm{ex}}, \bar{w}_{\mathrm{ex}})$

If the following consistency requirements are satisfied, then we define $\mathcal{G}^A_X := \mathfrak{A}^X_{/\sim}$.

(i) $X(\bar{u}^{\complement}) = A^k \setminus V$

(ii) $X(\bar{v}^{\complement}, \bar{w}^{\complement}) = (A^k \times A^k) \setminus E$

(iii) $X(\bar{t}^{\complement}) = A^k \setminus T$

(iv) $X(\bar{v}^{\complement}_{\mathrm{ex}}, \bar{w}^{\complement}_{\mathrm{ex}}) = (A^k \times A^k) \setminus E_{\mathrm{ex}}$

(v) $X(\bar{\varepsilon}^{\complement}_1, \bar{\varepsilon}^{\complement}_2) = (A^k \times A^k) \setminus \sim$

(vi) $V_0 = V \setminus V_1$

[14]This ensures that $V_0 \subseteq V$ and so forth.

(vii) $\mathfrak{A}^X$ is a structure.[14]

(viii) $\sim$ is a congruence on $\mathfrak{A}^X$.

(ix) $\mathfrak{A}^X_{/\sim}$ is a union game.

Otherwise, if any of these requirements is not fulfilled, we let $\mathcal{G}^A_X$ be undefined.

We call X complete (w.r.t. A), if $X(\bar{y}) \cup X(\bar{y}^\complement)$ is A^k or $A^k \times A^k$ for every $\bar{y} \in \{(\bar{u}), (\bar{v}, \bar{w}), (\bar{t}), (\bar{v}_\cap, \bar{w}_\cap), (\bar{\varepsilon}_1, \bar{\varepsilon}_2)\}$ and $V = V_0 \cup V_1$, and incomplete otherwise. It is easy to observe that $\mathcal{G}^A_X$ is undefined for every incomplete team X. Furthermore complete subteams of teams describing a game actually describe the same game and the same congruence relation.

Lemma 3.41. *Let X, Y be teams with codomain A and $\mathcal{V}_k \subseteq \mathrm{dom}(X) = \mathrm{dom}(Y)$. If X is complete, $X \subseteq Y$ and $\mathcal{G}^A_Y$ is defined, then $\mathcal{G}^A_X = \mathcal{G}^A_Y$ and $\sim_X := X(\bar{\varepsilon}_1, \bar{\varepsilon}_2) = Y(\bar{\varepsilon}_1, \bar{\varepsilon}_2) =: \sim_Y$.*

Proof. Suppose that X is complete, $X \subseteq Y$ and $\mathcal{G}^A_Y$ is defined. First, we prove that $\mathcal{G}^A_X$ is defined. Towards this end, we prove that X satisfies the consistency requirements of Definition 3.40. By completeness of X, we know already that $X(\bar{u}) \cup X(\bar{u}^\complement) = A^k$. Since $X \subseteq Y$ and $\mathcal{G}^A_Y$ is defined, we have $X(\bar{u}) \cap X(\bar{u}^\complement) \subseteq Y(\bar{u}) \cap Y(\bar{u}^\complement) = \emptyset$. Thus, we have $X(\bar{u}) \cup X(\bar{u}^\complement) = A^k$ and $X(\bar{u}) \cap X(\bar{u}^\complement) = \emptyset$, which implies that $X(\bar{u}^\complement) = A^k \setminus X(\bar{u})$. This proves condition (i) of Definition 3.40. The proof for (ii)-(vi) is very analogous.

Towards proving the remaining conditions (vii)-(ix), it suffices to show that $\mathfrak{A}^X = \mathfrak{A}^Y$ and $\sim_X = \sim_Y$, because $\mathcal{G}^A_Y$ is defined and thus the conditions (vii)-(ix) must be true for $\mathfrak{A}^Y$ and $\sim_Y$.

Thus, we need to prove that $X(\bar{y}) = Y(\bar{y})$ for every tuple $\bar{y} \in \{(\bar{u}), (\bar{v}, \bar{w}), (\bar{t}), (\bar{v}_\cap, \bar{w}_\cap), (\bar{\varepsilon}_1, \bar{\varepsilon}_2)\}$. Since the argumentation is very analogous for these different tuples, we prove this only for $\bar{y} = \bar{u}$. Towards a contradiction, assume that $X(\bar{u}) \neq Y(\bar{u})$. Since X and Y are complete, we can conclude that $X(\bar{u}^\complement) = A^k \setminus X(\bar{u})$ and $Y(\bar{u}^\complement) = A^k \setminus Y(\bar{u})$. Since $X(\bar{u}) \neq Y(\bar{u})$, we have that $Y(\bar{u}) \setminus X(\bar{u}) \neq \emptyset$ or $X(\bar{u}) \setminus Y(\bar{u}) \neq \emptyset$. In the first case, follows $\emptyset \neq Y(\bar{u}) \setminus X(\bar{u}) = Y(\bar{u}) \cap (A^k \setminus X(\bar{u})) = Y(\bar{u}) \cap X(\bar{u}^\complement) \subseteq Y(\bar{u}) \cap Y(\bar{u}^\complement)$. In the second case, a similar line of thought leads to $\emptyset \neq X(\bar{u}) \setminus Y(\bar{u}) \subseteq Y(\bar{y}) \cap Y(\bar{u}^\complement)$. In both cases we have $Y(\bar{u}) \cap Y(\bar{u}^\complement) \neq \emptyset$ which contradicts $Y(\bar{u}^\complement) = A^k \setminus Y(\bar{u})$.

Therefore, all conditions of Definition 3.40 are fulfilled. Because of $\mathfrak{A}^X = \mathfrak{A}^Y$ and $\sim_X = \sim_Y$, it is even the case that $\mathcal{G}^A_X = \mathfrak{A}^X_{/\sim_X} = \mathfrak{A}^Y_{/\sim_Y} = \mathcal{G}^A_Y$. □

Note that in this way we cannot encode every union game that is formally possible. The reason for this is that in a non-empty team at least one value must be present for every variable (of its domain). For us this results in at least one tuple $\bar{a} \in X(\bar{u}^\complement)$, hence there must be one tuple that does not take part in the game. While this is no restriction when it comes to modelling inclusion-exclusion games (just increase the arity by one and not use an arbitrary value), for the other variables it makes a small difference. That is, there must be at least one vertex in V_0, at least one edge must be present and so forth. Of course, it is easy to see that this is no actual restriction, as one can transform every inclusion-exclusion game into this form without changing its size drastically or altering the set $\mathcal{T}(\mathcal{G})$.

Now let us show that inclusion-exclusion games in the sense of Definition 3.40 are definable in plain first-order team semantics.

Lemma 3.42. *Let Ω be a set of dependency concepts. Let $\varphi(X) = \forall\bar{x}(X\bar{x} \longrightarrow \exists\bar{R}\varphi'(X, \bar{R}, \bar{x}))$ be a myopic Σ^1_1-formula and $\psi(\mathcal{V}_k, \bar{x}) \in \mathrm{FO}(\Omega)$ (where k is large enough such that the game $\mathcal{G}(\mathfrak{A}, \varphi)$ can be encoded). There is a $\mathrm{FO}(\Omega)$-formula $\vartheta^{\psi}_{\varphi}(\bar{x})$ such that $\mathfrak{A} \models_X \vartheta^{\psi}_{\varphi} \iff \mathfrak{A} \models_Y \psi$, where Y is a (uniquely determined) team that results as an extension of X with $\mathcal{G}^A_Y \cong \mathcal{G}(\mathfrak{A}, \varphi)$ and $X(\bar{x}) = Y(\bar{x})$.*

Proof. Using the same technique as in the proof of Lemma 3.9, it is easy to construct a (quantifier-free) first-order interpretation $\mathcal{I} := (\delta, \varepsilon, \psi_V, \psi_{V_0}, \psi_{V_1}, \psi_E, \psi_I, \psi_T, \psi_{E_{\mathrm{ex}}})$ with $\mathcal{I}(\mathfrak{A}) \cong \mathcal{G}(\mathfrak{A}, \varphi)$. Now let $\vartheta^{\psi}_{\varphi}(\bar{x}) := \forall\mathcal{V}_k(\gamma(\mathcal{V}_k) \longrightarrow \psi(\mathcal{V}_k, \bar{x}))$ where the formula

$$\begin{aligned}\gamma(\mathcal{V}_k) := {} & \delta(\bar{u}) \land \psi_{V_0}(\bar{v}_0) \land \psi_{V_1}(\bar{v}_1) \land \psi_E(\bar{v}, \bar{w}) \land \psi_T(\bar{t}) \land \psi_{E_{\mathrm{ex}}}(\bar{v}_{\mathrm{ex}}, \bar{w}_{\mathrm{ex}}) \land \varepsilon(\bar{\varepsilon}_1, \bar{\varepsilon}_2) \land \\ & \neg\delta(\bar{u}^{\complement}) \land \neg\psi_E(\bar{v}^{\complement}, \bar{w}^{\complement}) \land \neg\psi_T(\bar{t}^{\complement}) \land \neg\psi_{E_{\mathrm{ex}}}(\bar{v}^{\complement}_{\mathrm{ex}}, \bar{w}^{\complement}_{\mathrm{ex}}) \land \neg\varepsilon(\bar{\varepsilon}^{\complement}_1, \bar{\varepsilon}^{\complement}_2)\end{aligned}$$

enforces that the game $\mathcal{G}(\mathfrak{A}, \varphi)$ will be “loaded” into the team. As long as none of these conjuncts are unsatisfiable this construction is correct. This is safe to assume because one can easily transform a union game into an equivalent one w.r.t. the target set such that none of its components are empty. □

This knowledge enables us to finally define the atomic formula we sought after. For this we need to show that the atom is definable in Σ^1_1, is union closed and its first-order closure can express all of $\mathcal{U}$.

Definition 3.43. The atomic team formula $\cup\text{-}\mathsf{game}(\mathcal{V}_k, \bar{x})$ for the respective tuples of variables has the following semantics. For non-empty teams X with $\mathcal{V}_k, \bar{x} \subseteq \mathrm{dom}(X)$ we define

$$\begin{aligned}\mathfrak{A} \models_X \cup\text{-}\mathsf{game}(\mathcal{V}_k, \bar{x}) :\iff {} & X \text{ is complete and} \\ & \text{if } \mathcal{G}^A_X \text{ is defined, then } X(\bar{x})_{/X(\bar{\varepsilon}_1, \bar{\varepsilon}_2)} \in \mathcal{T}(\mathcal{G}^A_X)\end{aligned}$$

and we set $\mathfrak{A} \models_\emptyset \cup\text{-}\mathsf{game}(\mathcal{V}_k, \bar{x})$ to be always true (to ensure the empty team property).

It is easy to see that this atom is definable in existential second-order logic, as on the one hand we know that winning regions of union games can be described in Σ^1_1 and on the other hand the syntax check is of course definable in Σ^1_1.

Proposition 3.44. *The atomic formula $\cup\text{-}\mathsf{game}$ is union closed.*

Proof. Assume that $\mathfrak{A} \models_{X_i} \cup\text{-}\mathsf{game}(\mathcal{V}_k, \bar{x})$ for $i \in I$. We prove: $\mathfrak{A} \models_X \cup\text{-}\mathsf{game}(\mathcal{V}_k, \bar{x})$ holds for the union $X := \bigcup_{i\in I} X_i$. If $X = \emptyset$, there is nothing to prove. Otherwise at least one X_j is non-empty and, since $\mathfrak{A} \models_{X_j} \cup\text{-}\mathsf{game}(\mathcal{V}_k, \bar{x})$, X_j must be complete implying that X is also complete (because $X \supseteq X_j$). For the remainder of this proof, we assume w.l.o.g. that all involved teams X_i (and X) are non-empty. If $\mathcal{G}^A_X$ is undefined, then $\mathfrak{A} \models_X \cup\text{-}\mathsf{game}(\mathcal{V}_k, \bar{x})$

follows from the definition of $\cup\text{-game}$. Otherwise, if $\mathcal{G}^A_X$ is defined, then we can use Lemma 3.41 to obtain that $\mathcal{G}^A_X = \mathcal{G}^A_{X_i}$ and $\sim := X(\bar{\varepsilon}_1, \bar{\varepsilon}_2) = X_i(\bar{\varepsilon}_1, \bar{\varepsilon}_2)$ for every $i \in I$. Since $\mathfrak{A} \models_{X_i} \cup\text{-game}(\mathcal{V}_k, \bar{x})$, we can conclude that $X_i(\bar{x})_{/\sim} \in \mathcal{T}(\mathcal{G}^A_{X_i}) = \mathcal{T}(\mathcal{G}^A_X)$ for each $i \in I$. By Theorem 3.16, $X(\bar{x})_{/\sim} = \bigcup_{i \in I} X_i(\bar{x})_{/\sim} \in \mathcal{T}(\mathcal{G}^A_X)$ and, hence, $\mathfrak{A} \models_X \cup\text{-game}(\mathcal{V}_k, \bar{x})$. □

Theorem 3.45. *Let $\varphi \in$ FO($\subseteq$, |) be a union closed formula. There is a logically equivalent formula $\zeta \in$ FO($\cup$-game). In other words,* FO($\cup$-game) *captures precisely the union closed fragment of* FO($\subseteq$, |).

Proof. Let $\mathfrak{A}$ be an arbitrary structure. Due to [Gal12, Theorem 6.1] there exists a formula $\varphi'(X) \in \Sigma^1_1$ which is logically equivalent to $\varphi(\bar{x})$ in the sense that $\mathfrak{A} \models_X \varphi(\bar{x}) \iff (\mathfrak{A}, X(\bar{x})) \models \varphi'(X)$ for every team X with $\bar{x} \subseteq \mathrm{dom}(X)$. By Theorem 3.7, there is a myopic formula $\mu \equiv \varphi'$. So, we have $(\mathfrak{A}, X(\bar{x})) \models \mu(X) \iff \mathfrak{A} \models_X \varphi(\bar{x})$.

The game $\mathcal{G}(\mathfrak{A}, \mu)$ from Definition 3.17 is a union game and Lemma 3.42 allows us to load this game into a team. Please notice, that Lemma 3.42 is using a similar first-order interpretation $\mathcal{I}$ as Lemma 3.9, which encodes a target vertex $\bar{a} \in T(\mathcal{G}(\mathfrak{A}, \mu))$ by tuples of the form $(\bar{u}, \bar{a}, \bar{w})$ of length $k = n + m$ where the n-tuple $\bar{u}$ has the equality type e_T while $\bar{w}$ is an arbitrary tuple of length $m - |\bar{a}|$. Let $\psi(\mathcal{V}_k, \bar{x}) := \forall \bar{u} \forall \bar{w}(e_T(\bar{u}) \longrightarrow \cup\text{-game}(\mathcal{V}_k, \bar{u}\bar{x}\bar{w}))$ and $\zeta(\bar{x}) := \vartheta^{\psi}_{\mu}$ be as in Lemma 3.42, that is $\forall \mathcal{V}_k(\gamma(\mathcal{V}_k) \longrightarrow \psi(\mathcal{V}_k, \bar{x}))$. So $\mathfrak{A} \models_X \zeta(\bar{x}) \iff \mathfrak{A} \models_Y \psi(\mathcal{V}_k, \bar{x})$ where $Y = X[\mathcal{V}_k \mapsto A]\restriction_\gamma$. As in Lemma 3.42, we have $\mathcal{G}^A_Y \cong \mathcal{I}(\mathfrak{A}) \cong \mathcal{G}(\mathfrak{A}, \mu)$ and $X(\bar{x}) = Y(\bar{x})$. Furthermore, we have defined $\mathcal{G}^A_Y = \mathfrak{A}^Y_{/\sim}$ where $\sim := Y(\bar{\varepsilon}_1, \bar{\varepsilon}_2)$.

Because of the construction of ψ, holds $\mathfrak{A} \models_Y \psi(\mathcal{V}_k, \bar{x}) \iff \mathfrak{A} \models_Z \cup\text{-game}(\mathcal{V}_k, \bar{u}\bar{x}\bar{w})$ where $Z := Y[\bar{u} \mapsto e^{\mathfrak{A}}_T, \bar{w} \mapsto A^{m-|\bar{x}|}]$. Since $\mathcal{G}^A_Z = \mathcal{G}^A_Y \cong \mathcal{G}(\mathfrak{A}, \mu)$ is a well-defined union game, this is equivalent to $Z(\bar{u}\bar{x}\bar{w})_{/\sim} \in \mathcal{T}(\mathcal{G}^A_Y)$. Let $h : \delta^{\mathfrak{A}}_{\mathcal{I}} \longrightarrow V(\mathcal{G}(\mathfrak{A}, \mu))$ be the coordinate map for $\mathcal{G}(\mathfrak{A}, \mu) \cong \mathcal{I}(\mathfrak{A})$. By construction, h induces an isomorphism between $\mathfrak{A}^Y_{/\sim}$ and $\mathcal{G}(\mathfrak{A}, \mu)$. In particular each element of any equivalence class $[(\bar{u}', \bar{a}, \bar{w}')]_\sim \in Z(\bar{u}\bar{x}\bar{w})_{/\sim}$ is mapped by h to $\bar{a}$. Therefore, $Z(\bar{u}\bar{x}\bar{w})_{/\sim} \in \mathcal{T}(\mathcal{G}^A_Y) \iff Z(\bar{x}) = X(\bar{x}) \in \mathcal{T}(\mathcal{G}(\mathfrak{A}, \mu))$. Thus we have $\mathfrak{A} \models_X \zeta(\bar{x}) \iff X(\bar{x}) \in \mathcal{T}(\mathcal{G}(\mathfrak{A}, \mu))$. Putting everything together, we have $\mathfrak{A} \models_X \zeta(\bar{x}) \iff X(\bar{x}) \in \mathcal{T}(\mathcal{G}(\mathfrak{A}, \mu)) \iff (\mathfrak{A}, X(\bar{x})) \models \mu \iff \mathfrak{A} \models_X \varphi(\bar{x})$ as desired. □

We want to end this section with a remark on the “naturalness” of the atom $\cup\text{-game}$. Certainly inclusion, exclusion and the notions alike can be regarded as natural atomic dependency formulae, whereas the just introduced atom has to be classified differently. Nevertheless, it is a canonical candidate since it solves a complete problem of the desired class. Of course, a more natural — and more usable — atom might be found, but it will not be as simplistic as e.g. inclusion for Galliani and Hella have shown that every first-order definable union closed property is already expressible in inclusion logic. Hence, whatever formula one proposes, it must in a non-trivial way existentially quantify over the team at hand.

3.6 Other Closure Properties and More Games

In this final section we want to remark on the other closure properties of logics with team semantics. Particularly we present adaptations of inclusion-exclusion games suitable for inclusion, respectively exclusion, logic to demonstrate the generality of these games.

3.6.1 Closure under Intersections

To our knowledge there has been no study of formulae that are closed under intersections. It turns out that closure under intersection is not a well behaved closure property for logics with team semantics, since it is not preserved under conjunctions. The following definition is an attempt to define closure under intersections.

Definition 3.46. A formula $\psi \in \mathrm{FO}(\subseteq, |)$ is *closed under (finite) intersections* if for all teams X, Y with $\mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_Y \psi$ already $\mathfrak{A} \models_Z \psi$ follows, where $Z := (X\upharpoonright_{\mathrm{free}(\psi)}) \cap (Y\upharpoonright_{\mathrm{free}(\psi)})$.

This definition appears to be overcautious when it comes to the free variables of ψ. This is not by accident, because the naive variant of Definition 3.46 with $Z := X \cap Y$ would be equivalent to downward closure. To see this, consider any team T with $\mathfrak{A} \models_T \psi$, $\mathrm{dom}(T) = \mathrm{free}(\psi)$ and let $S \subseteq T$ be any subteam of T. Then for a variable $z \notin \mathrm{free}(\psi)$ and two different elements $a, b \in A$ we could define $X := T[z \mapsto a]$ and $Y := S[z \mapsto a] \cup (T \setminus S)[z \mapsto b]$. Since $\mathfrak{A} \models_T \psi$, it would follow by the locality principle that $\mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_Y \psi$. So, the naive version of Definition 3.46 would yield $\mathfrak{A} \models_{X\cap Y} \psi$. Because of $X \cap Y = S[z \mapsto a]$ and the locality principle, this would imply that $\mathfrak{A} \models_S \psi$. Hence, the naive definition with $Z := X \cap Y$ implies downwards closure and, because downwards closure always entails intersection closure, this naive variant of Definition 3.46 would be equivalent to downwards closure.

However, we will now see that our definition still does not describe a meaningful closure property for logics with team semantics, because it is not closed under conjunctions. To see this let P, Q be unary relation symbols and $\psi(x) \in \mathrm{FO}(\subseteq, |)$ be a formula with $(A, P, Q) \models_X \psi(x) \iff X(x) \in \{P, Q, P \cap Q, \emptyset\}$ and $\varphi(z) := z = z$. Then it is easy to verify that both $\psi(x)$ and $\varphi(z)$ are closed under intersections w.r.t. Definition 3.46.

Now, consider $\mathfrak{A} := (\{1, 2, 3, 4\}, P^{\mathfrak{A}}, Q^{\mathfrak{A}})$ with $P^{\mathfrak{A}} = \{1, 2, 3\}$ and $Q^{\mathfrak{A}} := \{2, 3, 4\}$ and the teams S, T with $\mathrm{dom}(S) = \mathrm{dom}(T) = \{x\}$ and $S(x) = \{2\} \subseteq \{2, 3\} = T(x)$. Then $\mathfrak{A} \models_T \psi(x)$ but $\mathfrak{A} \not\models_S \psi(x)$, since $T(x) = \{2, 3\} = P^{\mathfrak{A}} \cap Q^{\mathfrak{A}}$ but $S(x) = \{2\} \notin \{P^{\mathfrak{A}}, Q^{\mathfrak{A}}, P^{\mathfrak{A}} \cap Q^{\mathfrak{A}}, \emptyset\}$. This shows that $\psi(x)$ is not downwards closed. We will prove now that $\psi(x) \wedge \varphi(z)$ is no longer closed under intersections despite the fact that $\psi(x)$ and $\varphi(z)$ are both closed under intersections. Towards this end, let $T' := T[z \mapsto 1]$ and $S' := S[z \mapsto 1] \cup (T \setminus S)[z \mapsto 2]$. Then $\mathfrak{A} \models_{T'} \psi(x) \wedge \varphi(z)$ and $\mathfrak{A} \models_{S'} \psi(x) \wedge \varphi(z)$, because $T'\upharpoonright_x = T = S'\upharpoonright_x$ and $\mathfrak{A} \models_T \psi(x)$ while $\varphi(z) = (z = z)$ is a tautology. However, $\mathfrak{A} \not\models_Z \psi(x) \wedge (z = z)$ for $Z := (T'\upharpoonright_{\{x,z\}}) \cap (S'\upharpoonright_{\{x,z\}}) = T' \cap S'$, since $Z = T' \cap S' = S[z \mapsto 1]$ while $\mathfrak{A} \not\models_{S[z\mapsto 1]} \psi(x)$ follows from $\mathfrak{A} \not\models_S \psi(x)$ and the locality principle.

3.6.2 The Downwards/Upwards Closed Fragment

In a similar fashion to the normal form presented in Section 3.4 we can give syntactical normal forms of downwards, respectively upwards, closed formulae of FO(⊆, |). Notice that the upcoming formulae are similar in nature to the form given in Corollary 3.27.

Proposition 3.47. *Let $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq, |)$ and let $\bar{y}$ be a fresh tuple of variables with $|\bar{y}| = |\bar{y}|$. Then:*

(a) $\varphi(\bar{x})$ is downwards closed if and only if $\varphi(\bar{x})$ is equivalent to $\exists\bar{y}(\bar{x} \subseteq \bar{y} \wedge \varphi(\bar{y}))$.

(b) $\varphi(\bar{x})$ is upwards closed on non-empty teams if and only if $\varphi(\bar{x})$ is equivalent to $\exists\bar{y}(\bar{y} \subseteq \bar{x} \wedge \varphi(\bar{y}))$.

3.6.3 Games for Inclusion and Exclusion Logic

Let us show that inclusion-exclusion games can easily be adapted to fit inclusion or exclusion logic by either forbidding exclusion or inclusion edges.

Exclusion Games

Definition 3.48. An *exclusion game $\mathcal{G}$* is an inclusion-exclusion game $\mathcal{G} = (V, V_0, V_1, E, I, T, E_{\mathrm{ex}})$, if there are no inclusion edges, i.e. $E_{\mathrm{in}} := E \cap (V \times T) = \emptyset$.

Proposition 3.49. *Let $\mathcal{G}$ be an exclusion game. The set $\mathcal{T}(\mathcal{G})$ is downwards closed. In other words, for every $L \subseteq U \in \mathcal{T}(\mathcal{G})$ we already have $L \in \mathcal{T}(\mathcal{G})$.*

Proof. Let $\mathcal{S}$ be a winning strategy for player 0 witnessing $U \subseteq T$, that is $\mathcal{T}_{\mathcal{G}}(\mathcal{S}) = U$. Let $L \subseteq U$ be chosen arbitrarily and let $\mathcal{G}_L$ be the part of the game that is reachable from L. We define $\mathcal{S}' := \mathcal{S}\restriction_{\mathcal{G}_L}$ plus the subgraph of $\mathcal{S}$ that is reachable from the initial position (that is disjoint from T since there are no inclusion edges). Because there are no inclusion edges $\mathcal{S}'$ is a winning strategy for player 0 with $\mathcal{T}_{\mathcal{G}}(\mathcal{S}') = L$. □

Now we consider a Σ_1^1-formula $\varphi(X) = \exists\bar{R}\varphi'(X, \bar{R})$ in which X occurs only negatively. We have already defined the model-checking game $\mathcal{G}_X(\mathfrak{A}, \varphi(X))$ in Definition 3.4. The only difference is that this game no longer employs edges of the form $((X\bar{x}, s), \bar{a})$ with $s(\bar{x}) = \bar{a}$, because X occurs only negatively in φ. Thus, player 0 can put any subset of the target vertices $T = A^r$ into her winning strategy as long she respects the exclusion condition.

This construction can be adapted for exclusion logic. For $\psi(\bar{x}) \in \mathrm{FO}(\,|\,)$, the game $\mathcal{G}(\mathfrak{A}, \psi)$ is defined very similarly but with the following differences. Tuples of the form (ϑ, s) where ϑ is an (occurrence of) a subformula from ψ and assignments $s\colon \mathrm{free}(\psi) \to A$ are the target positions of this game. Such a target position $s\colon \mathrm{free}(\psi) \to A$ exhibits exactly one outgoing edge to (ψ, s), from where the games is played as usual. Again, (ϑ, s) belongs to player 0, if ϑ starts with a disjunction or some existential quantifier or is an unsatisfied literal, and (ϑ, s) has an edge to (ϑ', s'), if ϑ' is the direct subformula of the non-literal ϑ and $s(v) = s'(v)$ for all $v \in \text{free}(\vartheta) \cap \text{free}(\vartheta')$. The exclusion condition, E_{ex}, consists of all conflicting pairs $((\bar{x} \mid \bar{y}, s), (\bar{x} \mid \bar{y}, s'))$ where $s(\bar{x}) = s'(\bar{y})$ and $\bar{x} \mid \bar{y}$ refers to the same occurrence.

Theorem 3.50. *Let $\psi(\bar{x}) \in \text{FO}(\mid)$ and $\varphi(X) \in \Sigma_1^1$ be a formula in which X occurs only negatively. Then for every team resp. relation X over some structure $\mathfrak{A}$, we have:*

(a) $\mathfrak{A} \models_X \psi(\bar{x}) \iff X \in \mathcal{T}(\mathcal{G}(\mathfrak{A}, \psi))$

(b) $(\mathfrak{A}, X) \models \varphi(X) \iff X \in \mathcal{T}(\mathcal{G}_X(\mathfrak{A}, \varphi(X)))$

Proof. Item (b) is a direct consequence of Theorem 3.5. Therefore, we only have to prove item (a).

"$\Longrightarrow$": If $\mathfrak{A} \models_X \psi(\bar{x})$, then there exists a witness λ for $\mathfrak{A} \models_X \psi(\bar{x})$. It is not difficult to prove that defining $\mathcal{S}_\lambda$ as the subgraph of $\mathcal{G}(\mathfrak{A}, \psi)$ induced by $X \cup \{(\vartheta, s) : \vartheta \in \text{subf}(\psi), s \in \lambda(\vartheta)\}$ yields a winning strategy with $\mathcal{T}(\mathcal{S}_\lambda) = X$.

"$\Longleftarrow$": Conversely, if we have $X \in \mathcal{T}(\mathcal{G}(\mathfrak{A}, \psi))$, then player 0 has a winning strategy $\mathcal{S} = (W, F)$ with $\mathcal{T}(\mathcal{S}) = X$. Let $W' := \{w \in W : w \text{ is reachable in } \mathcal{S} \text{ from some } s \in X\}$. For every occurrence of a subformula $\vartheta \in \text{subf}(\psi)$ let $\lambda_{\mathcal{S}}(\vartheta) := \{s : (\vartheta, s) \in W'\}$. It is straightforward to show that $\lambda_{\mathcal{S}}$ is a witness for $\mathfrak{A} \models_X \psi$. □

Observation 3.51. Let $\varphi(\bar{x}) \in \text{FO}(\mid)$. Then $\varphi_{\mathcal{T}}(X) := \exists W \exists F(\varphi_{\text{win}}(W, F) \wedge \forall v(Xv \longrightarrow Wv \wedge Tv))$ defines $\mathcal{T}(\mathcal{G}(\mathfrak{A}, \varphi))$ and X occurs only negatively in $\varphi_{\mathcal{T}}(X)$.

Inclusion Games

Definition 3.52. An *inclusion game* $\mathcal{G}$ is an inclusion-exclusion game $\mathcal{G} = (V, V_0, V_1, E, I, T, E_{\text{ex}})$ with $E_\cap = \emptyset$ and $I = \emptyset$.

The next definition is adaptation of definition of safety games from [Grä16].

Definition 3.53. A *safety game* $\mathcal{G}$ is a game $\mathcal{G} = (V, V_0, V_1, E, V_{\text{safe}}, T)$ without initial vertices where the goal of player 0 is to ensure that the play remains inside $V_{\text{safe}} \subseteq V$.

Theorem 3.54. *Inclusion games and safety games are equivalent to each other. More formally, this means that for every inclusion game $\mathcal{G}$ there exists a safety game $\mathcal{G}_{safe}$ and vice versa such that $\mathcal{T}(\mathcal{G}) = \mathcal{T}(\mathcal{G}_{safe})$.*

Proof. Let $\mathcal{G} = (V, V_0, V_1, E, \emptyset, T, \emptyset)$ be an inclusion game. We define $V_{\text{safe}} := V \setminus \{v \in V_0 : \text{N}_{\mathcal{G}}(v) = \emptyset\}$ and let $\mathcal{G}_{\text{safe}} := (V, V_0, V_1, E, V_{\text{safe}}, T)$. Then $\mathcal{S} = (W, F)$ is a winning strategy for player 0 in $\mathcal{G}$ if and only if it is a winning strategy in $\mathcal{G}_{\text{safe}}$. So, $\mathcal{G}$ and $\mathcal{G}_{\text{safe}}$ are equivalent in this sense.

Conversely, let $\mathcal{G}_{\text{safe}} = (V, V_0, V_1, E, V_{\text{safe}}, T)$ be a safety game. Let $V_{\text{unsafe}} := V \setminus V_{\text{safe}}$. Now, $\mathcal{G}_{\text{safe}}$ can be converted into the following equivalent inclusion game $(V, V_0 \cup V_{\text{unsafe}}, V_1 \setminus (V_{\text{unsafe}}), E', \emptyset, T, \emptyset)$ where $E' := E \setminus (V_{\text{unsafe}} \times V)$. □

Proposition 3.55. *In inclusion or safety games, the union of winning strategies is again a winning strategy (for player 0).*

Corollary 3.56. *Let $\mathcal{G}$ be an inclusion game. The set $\mathcal{T}(\mathcal{G})$ is union closed. In other words, if $X_j \in \mathcal{T}(\mathcal{G})$ for all $j \in J$ we already have $X \in \mathcal{T}(\mathcal{G})$, where $X = \bigcup_{j\in J} X_j$.*

It is known that the safety games are precisely the model-checking games for inclusion logic and for GFP^+ [Grä16].

This concludes this chapter about the syntactic normal forms for the union-closed fragment and for other closure properties. Besides these normal forms, we have defined inclusion-exclusion games, demonstrated their adaptability to various fragments of Σ_1^1 and showed that they give rise to an atom that captures the union-closed fragment.

4 Arity Fragments of Inclusion Logic

In this chapter we compare $\mathrm{GFP}^+[k]$, which is greatest fixed-point logic using only fixed-point operators of arities at most k, with $\mathrm{FO}(\subseteq)[k]$, the restricted version of inclusion logic where only tuples of length at most k are allowed in inclusion atoms.

It is well-known that there is a tight connection between inclusion logic and greatest fixed-point logic. Galliani and Hella have shown the equivalence of these logics on the level of sentences [GH13, Corollary 17] and later Grädel has re-examined this connection in [Grä16] and, with the aid of the model-checking games for these logics, he showed that inclusion logic corresponds to the class of *myopic* GFP^+-formulae, i.e. formulae of the shape $\forall\bar{x}(X\bar{x} \longrightarrow \psi(X,\bar{x}))$ where the relation symbol X, which is used to represent teams, occurs only positively in $\psi(X,\bar{x})$. Furthermore, it was shown by Hannula that the arity hierarchy of inclusion logic is strict, i.e. $\mathrm{FO}(\subseteq)[1] < \mathrm{FO}(\subseteq)[2] < \dots$ on the level of sentences (over the signature of graphs) [Han15].

We take a closer look at what happens with the arities of the involved inclusion atoms, the fixed-points and the team resp. relation symbol that encodes the team when translating $\mathrm{FO}(\subseteq)$-formulae into GFP-formulae and vice versa. This is motivated by an open question, which has been presented by Rönnholm at the Dagstuhl seminar 2019 [GKKV19]. Rönnholm has asked whether inclusion logic using only inclusion atoms of bounded arities corresponds to some fragment of greatest fixed-point logics. We will show that such correspondence exists between $\mathrm{FO}(\subseteq)[k]$ and a fragment $\mathrm{GFP}^+[k]$ of greatest fixed-point logic. $\mathrm{GFP}^+[k]$ is defined by allowing only greatest fixed-point formulae of the shape $[\mathrm{GFP}\, R\bar{x} \;:\; \psi(R,\bar{x})]$ where $\mathrm{ar}(R) \le k$ and, moreover, $\mathrm{free}(\psi) \subseteq \{\bar{x}\}$. Furthermore, all $\mathrm{GFP}^+[k]$-formulae are in negation normal form and every such greatest fixed-point formula is only allowed to occur positively, i.e. not in the scope of negation signs.

We are going to provide effective translations between these fragments. More precisely, here are our results:

(i) For every $\mathrm{FO}(\subseteq)[k]$-formula $\varphi(\bar{x})$ there exists a myopic $\mathrm{GFP}^+[k]$-sentence $\psi(X)$ where $\mathrm{ar}(X) = |\bar{x}|$ such that for every suitable structure $\mathfrak{A}$ and team X holds

$$\mathfrak{A} \models_X \varphi(\bar{x}) \iff (\mathfrak{A}, X(\bar{x})) \models \psi(X).^1 \tag{4.1}$$

(ii) Conversely, for every myopic $\mathrm{GFP}^+[k]$-sentence $\psi(X)$ there exists a formula $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq)[k']$ equivalent to $\psi(X)$ in the sense of (4.1) where $|\bar{x}| = \mathrm{ar}(X)$ and

[1] $\psi(X)$ indicates that X is an additional relation symbol occurring in ψ, which is interpreted by the relation $X(\bar{x})$ and not to be confused with the team X.

$k' := \max\{k, \mathrm{ar}(X)\}$. We will also show that k' cannot be chosen smaller than $\mathrm{ar}(X)$.

(iii) For every $\mathrm{GFP}^+[k]$-formula $\psi(\bar{x})$ there exists a (downwards-closed) $\mathrm{FO}(\subseteq)[k]$-formula $\gamma(\bar{x})$ such that for all suitable structures $\mathfrak{A}$ and teams X,

$$\mathfrak{A} \models_X \gamma(\bar{x}) \iff \mathfrak{A} \models_s \psi(\bar{x}) \text{ for every } s \in X.$$

Please notice that $k' = k$, if $\mathrm{ar}(X) \leq k$ resp. $|\bar{x}| \leq k$. So, if we would also impose a bound on the number of free variables and on the arity of the free relation variable used in myopic formulae to represent the team, then (i) and (ii) are translations in opposite directions. Using Hannula's results about the expressive power of $\mathrm{FO}(\subseteq)[k']$ we will prove that k' can in fact not be chosen smaller than $\mathrm{ar}(X)$.

The third result translates $\mathrm{GFP}^+[k]$-*formulae* rather than sentences with an additional team-predicate into a $\mathrm{FO}(\subseteq)[k]$-formulae having the same free variables. It is worth pointing out, that for $\psi(\bar{x}) \in \mathrm{GFP}^+[k]$ we have that $\mathfrak{A} \models_s \psi(\bar{x})$ for every $s \in X \iff (\mathfrak{A}, X(\bar{x})) \models \forall\bar{x}(X\bar{x} \longrightarrow \psi(\bar{x}))$, which is a special myopic formula where X does not occur at all in $\psi(\bar{x})$.

4.1 Simultaneous Fixed-Point Logic

In this section we will define a third logic, called simultaneous greatest fixed-point logic (denoted as sGFP^+_k), which, as we will see, has the same expressive power as $\mathrm{GFP}^+[k]$ and will aid us in the translations from $\mathrm{GFP}^+[k]$ to $\mathrm{FO}(\subseteq)[k]$ and vice versa.

Definition 4.1. sGFP^+_k is the extension of first-order logic (in negation normal form) by the following formula formation rule: Let $R_1, \dots, R_n$ be relation symbols and $\bar{x}_1, \dots, \bar{x}_n$ be tuples of variables such that $r_i := \mathrm{ar}(R_i) = |\bar{x}_i| \leq k$ and let S be the following system

$$S := \begin{cases} R_1\bar{x}_1 & : \eta_1(R_1, \dots, R_n, \bar{x}_1) \\ R_2\bar{x}_2 & : \eta_2(R_1, \dots, R_n, \bar{x}_2) \\ & \vdots \\ R_n\bar{x}_n & : \eta_n(R_1, \dots, R_n, \bar{x}_n) \end{cases}$$

where $\eta_1, \dots, \eta_n$ are formulae in which the relation symbols $R_1, \dots, R_n$ occur only positively. For $j \in \{1, \dots, n\}$ and a tuple $\bar{z}$ of variables with $|\bar{z}| = \mathrm{ar}(R_j)$, the expression

$$\varphi(\bar{z}) := [\mathrm{sGFP}\ S]_j\,(\bar{z}) \tag{4.2}$$

is a sGFP^+_k-formula. Then the system S defines, for a given structure $\mathfrak{A}$, a simultaneous update operator $\Gamma : \mathcal{P}(A^{r_1}) \times \mathcal{P}(A^{r_2}) \times \dots \times \mathcal{P}(A^{r_n}) \longrightarrow \mathcal{P}(A^{r_1}) \times \mathcal{P}(A^{r_2}) \times \dots \times \mathcal{P}(A^{r_n})$ via $\Gamma(\bar{S}) := (\Gamma_1(\bar{S}), \dots, \Gamma_n(\bar{S}))$ where $\Gamma_i(\bar{S}) := [\![\eta_i(\bar{S}, \bar{x}_i)]\!]^{\mathfrak{A}} = \{\bar{a} \in A^{r_i} : \mathfrak{A} \models \eta_i(\bar{S}, \bar{a})\}$.

For tuples $\bar{S} = (S_1, \dots, S_n), \bar{S}' = (S'_1, \dots, S'_n)$ of relations we write $\bar{S} \subseteq \bar{S}'$, if $S_i \subseteq S'_i$ for $i = 1, \dots, n$. Since the relation symbols $R_1, \dots, R_n$ occur only positively in $\eta_1, \dots, \eta_n$, it follows that Γ is monotone, i.e. $\bar{S} \subseteq \bar{S}'$ implies $\Gamma(\bar{S}) \subseteq \Gamma(\bar{S}')$. It is well-known that monotone operators have a unique greatest fixed-point $\bar{G} = (G_1, \dots, G_n)$, i.e. $\Gamma(\bar{G}) = \bar{G}$ and $\bar{S} \subseteq \bar{G}$ for every fixed-point $\bar{S}$ of Γ. The semantics of $\varphi(\bar{z})$ from (4.2) is given by $\mathfrak{A} \models_s \varphi(\bar{z}) :\iff s(\bar{z}) \in G_j$.

The greatest fixed-point can also be calculated in an inductive fashion. Let $(\bar{S}_\alpha)_{\alpha \in \mathrm{On}}$ where $\bar{S}_0$ is the tuple consisting of the full relations, $\bar{S}_{\alpha+1} = \Gamma(\bar{S}_\alpha)$ and $\bar{S}_\lambda = \bigcap_{\beta<\lambda} \bar{S}_\beta$, where this intersection is defined component-wise, i.e. $(\bar{S}_\lambda)_j = \bigcap_{\beta<\lambda} (\bar{S}_\beta)_j$.[2] A different characterization is provided by the following well-known Knaster-Tarski Theorem.

Theorem 4.2 (Theorem of Knaster-Tarski for Simultaneous Operators)**.** *Let* Γ *be as above. Then the greatest fixed-point of the operator* Γ*, denoted as* $\mathrm{GFP}(\Gamma)$*, is* $\bigcup\{\bar{S} : \bar{S} \subseteq \Gamma(\bar{S})\}$ *where this union is defined componentwise, i.e.* $(\mathrm{GFP}(\Gamma))_j = \bigcup\{S_j : \bar{S} = (S_1, \dots, S_n) \subseteq \Gamma(\bar{S})\}$.

Remark 4.3. $[\mathrm{GFP}\, R\bar{x} : \eta(R, \bar{x})]\,(\bar{y})$ is equivalent to $[\mathrm{sGFP}\, R_1\bar{x} : \eta(R_1, \bar{x})]_1\,(\bar{y})$.

The following lemma relies on a technique that is also known as the Bekic principle [GKL+07].

Lemma 4.4. *Every* sGFP^+_k*-formula of the form* $[\mathrm{sGFP}\, S]_i\,(\bar{y}_i)$ *can be expressed in* $\mathrm{GFP}^+[k]$.

The following proof is a straightforward adaptation of [GKL+07, Lemma 3.3.41] for sGFP^+_k. The construction that is used in this proof will be useful for the proof of Lemma 4.7.

Proof of Lemma 4.4. W.l.o.g. we assume that $i = 1$. Let S be a system of the following form:

$$S := \begin{cases} R_1\bar{x}_1 & : \eta_1(R_1, \dots, R_n, \bar{x}_1) \\ R_2\bar{x}_2 & : \eta_2(R_1, \dots, R_n, \bar{x}_2) \\ & \vdots \\ R_n\bar{x}_n & : \eta_n(R_1, \dots, R_n, \bar{x}_n) \end{cases}$$

If $n = 1$, then we can use Remark 4.3. Now we assume that $n > 1$. Let S' be the following system:

$$S' := \begin{cases} R_1\bar{x}_1 & : \eta'_1(R_1, \dots, R_{n-1}, \bar{x}_1) \\ R_2\bar{x}_2 & : \eta'_2(R_1, \dots, R_{n-1}, \bar{x}_2) \\ & \vdots \\ R_{n-1}\bar{x}_{n-1} & : \eta'_{n-1}(R_1, \dots, R_{n-1}, \bar{x}_{n-1}) \end{cases}$$

[2] On is the class of all ordinal numbers. For more information about ordinal numbers and transfinite inductions, we refer to [Blu18].

where η_i' results from η_i by replacing every occurrence of $R_n\bar{y}$ by

$$[\mathrm{GFP}\, R_n\bar{x}_n \;:\; \eta_n(R_1,\dots,R_n,\bar{x}_n)]\,(\bar{y}).$$

In the following, we will prove that $[\mathrm{sGFP}\, S]_1\,(\bar{y}_1)$ is equivalent to $[\mathrm{sGFP}\, S']_1\,(\bar{y}_1)$.

Let $\mathfrak{A}$ be any structure. Let $r_i := \mathrm{ar}(R_i)$ for $i = 1,\dots,n$, $Y := \mathcal{P}(A^{r_1}) \times \cdots \times \mathcal{P}(A^{r_{n-1}})$ and $X := Y \times \mathcal{P}(A^{r_n})$ and let $\Gamma : X \longrightarrow X$ and $\Gamma' : Y \longrightarrow Y$ be the update operators defined by S and S' (respectively).

Claim 4.5. $\mathrm{GFP}(\Gamma)_i = \mathrm{GFP}(\Gamma')_i$ holds for every $i \in \{1,\dots,n-1\}$.

As soon as this claim has been proven we are done, because then

$$[\![[\mathrm{sGFP}\, S]_1\,(\bar{y}_1)]\!]^{\mathfrak{A}} = \mathrm{GFP}(\Gamma)_1 = \mathrm{GFP}(\Gamma')_1 = [\![\left[\mathrm{sGFP}\, S'\right]_1\,(\bar{y}_1)]\!]^{\mathfrak{A}}$$

follows and, by applying this construction repeatedly, we end up with an equivalent $\mathrm{GFP}^+[k]$-formula.

For every $\bar{S} \in Y$ let $\Gamma_{\bar{S}} : \mathcal{P}(A^{r_n}) \longrightarrow \mathcal{P}(A^{r_n})$ be defined as $\Gamma_{\bar{S}}(R) := \Gamma_n(\bar{S},R)$. By construction of S', we have that $\Gamma'(\bar{S}) = \Gamma(\bar{S},\mathrm{GFP}(\Gamma_{\bar{S}}))$, because R_n was replaced by $[\mathrm{GFP}\, R_n\bar{x}_n \;:\; \eta_n(R_1,\dots,R_n,\bar{x}_n)]$ and it holds that

$$[\![\left[\mathrm{GFP}\, R_n\bar{x}_n \;:\; \eta_n(\bar{S},R_n,\bar{x}_n)\right]]\!]^{\mathfrak{A}} = \mathrm{GFP}(\Gamma_{\bar{S}}).$$

Claim 4.6. For every $\bar{S},\bar{S}' \in Y$ with $\bar{S} \subseteq \bar{S}'$ holds $\mathrm{GFP}(\Gamma_{\bar{S}}) \subseteq \mathrm{GFP}(\Gamma_{\bar{S}'})$.

Proof of Claim 4.6. Let $(R_\alpha)_{\alpha\in\mathrm{On}}$ and $(R'_\alpha)_{\alpha\in\mathrm{On}}$ be the inductive calculation of the greatest fixed-points $\mathrm{GFP}(\Gamma_{\bar{S}})$ and $\mathrm{GFP}(\Gamma_{\bar{S}'})$. So we have that $R_0 := A^{r_n} =: R'_0$, $R_{\alpha+1} := \Gamma_{\bar{S}}(R_\alpha)$, $R'_{\alpha+1} := \Gamma_{\bar{S}'}(R'_\alpha)$ and, for limit ordinals λ, $R_\lambda := \bigcap_{\beta<\lambda} R_\beta$ and $R'_\lambda := \bigcap_{\beta<\lambda} R'_\beta$. We will prove that $R_\alpha \subseteq R'_\alpha$ is true for every $\alpha \in \mathrm{On}$ by induction over α. For $\alpha = 0$, there is nothing to prove. If $R_\alpha \subseteq R'_\alpha$ is true for some ordinal α, then it immediately follows that

$$R_{\alpha+1} = \Gamma_{\bar{S}}(R_\alpha) = \Gamma_n(\bar{S},R_\alpha) \subseteq \Gamma_n(\bar{S}',R'_\alpha) = \Gamma_{\bar{S}'}(R'_\alpha) = R'_{\alpha+1}.$$

If λ is some limit ordinal and $R_\beta \subseteq R'_\beta$ is true for all $\beta < \lambda$, then we clearly have $R_\lambda = \bigcap_{\beta<\lambda} R_\beta \subseteq \bigcap_{\beta<\lambda} R'_\beta = R'_\lambda$. As a result, we obtain $\mathrm{GFP}(\Gamma_{\bar{S}}) = \bigcap_{\alpha\in\mathrm{On}} R_\alpha \subseteq \bigcap_{\alpha\in\mathrm{On}} R'_\alpha = \mathrm{GFP}(\Gamma_{\bar{S}'})$ as desired. This concludes the proof of Claim 4.6. □

Let $G_i := \mathrm{GFP}(\Gamma)_i$ for $i = 1,\dots,n$ and $G'_i := \mathrm{GFP}(\Gamma')_i$ for $i = 1,\dots,n-1$. Please recall that we want to prove $G_i = G'_i$ for $i = 1,\dots,n-1$. We will prove "$\subseteq$" and "$\supseteq$" separately and start with the direction "$\subseteq$". Towards this end, we observe that $\Gamma_{G_1,\dots,G_{n-1}}(G_n) = \Gamma_n(G_1,\dots,G_n) = G_n$ is true, because the first equation is due to the definition of $\Gamma_{G_1,\dots,G_{n-1}}$ while the second equation is entailed by the fact that $G_1,\dots,G_n$ is a fixed-point of Γ. So G_n is a fixed-point of $\Gamma_{G_1,\dots,G_{n-1}}$, which implies that $G_n \subseteq \mathrm{GFP}(\Gamma_{G_1,\dots,G_{n-1}})$. Using this, the definition of Γ'_i and the monotonicity of Γ_i, we can derive that

$$\begin{aligned}\Gamma'_i(G_1,\dots,G_{n-1}) \;&=\; \Gamma_i(G_1,\dots,G_{n-1},\mathrm{GFP}(\Gamma_{G_1,\dots,G_{n-1}}))\\ &\supseteq\; \Gamma_i(G_1,\dots,G_{n-1},G_n) \;=\; G_i.\end{aligned}$$

As a result, we obtain that $(G_1, \dots, G_{n-1}) \subseteq \Gamma'(G_1, \dots, G_{n-1})$ and, consequently, we can derive that

$$(G_1, \dots, G_{n-1}) \subseteq \bigcup \{\bar{S} \in Y \ : \ \bar{S} \subseteq \Gamma'(\bar{S})\} \underset{\text{(Theorem 4.2)}}{=} \mathrm{GFP}(\Gamma') = (G'_1, \dots, G'_{n-1}).$$

Towards proving the direction "$\supseteq$", let $(G_1^\alpha, \dots, G_n^\alpha)_{\alpha \in \mathrm{On}}$ be the inductive calculation of $\mathrm{GFP}(\Gamma)$. We will prove that $(G'_1, \dots, G'_{n-1}) \subseteq (G_1^\alpha, \dots, G_{n-1}^\alpha)$ and that $\mathrm{GFP}(\Gamma_{G'_1, \dots, G'_{n-1}}) \subseteq G_n^\alpha$ for every $\alpha \in \mathrm{On}$. By induction over α: For $\alpha = 0$, there is nothing to prove since $G_i^\alpha = A^{r_i}$. Now assume that $(G'_1, \dots, G'_{n-1}) \subseteq (G_1^\alpha, \dots, G_{n-1}^\alpha)$ and $\mathrm{GFP}(\Gamma_{G'_1, \dots, G'_{n-1}}) \subseteq G_n^\alpha$ is true for some $\alpha \in \mathrm{On}$. We will show that the statement also holds for $\alpha + 1$: For every $i \in \{1, \dots, n-1\}$, we have that

$$G_i^{\alpha+1} = \Gamma_i(G_1^\alpha, \dots, G_n^\alpha) \supseteq \Gamma_i(G'_1, \dots, G'_{n-1}, \mathrm{GFP}(\Gamma_{G'_1, \dots, G'_{n-1}})) = \Gamma'_i(G'_1, \dots, G'_{n-1}) = G'_i.$$

Furthermore, we have

$$\begin{aligned} G_n^{\alpha+1} &= \Gamma_n(G_1^\alpha, \dots, G_n^\alpha) \supseteq \Gamma_n(G'_1, \dots, G'_{n-1}, \mathrm{GFP}(\Gamma_{G'_1, \dots, G'_{n-1}})) \\ &= \Gamma_{G'_1, \dots, G'_{n-1}}(\mathrm{GFP}(\Gamma_{G'_1, \dots, G'_{n-1}})) = \mathrm{GFP}(\Gamma_{G'_1, \dots, G'_{n-1}}). \end{aligned}$$

The step for limit ordinals λ is trivial. This concludes the proof of Claim 4.5 and of Lemma 4.4. □

The last lemma allowed us to express simultaneous fixed-points in non-simultaneous fixed-points. The next lemma shows that the converse is also true, even if we additionally demand that only first-order formulae are used inside simultaneous fixed-point.

Lemma 4.7. *Let $\psi_1(\bar{y}_1) = [\mathrm{GFP}\, R_1 \bar{x}_1 \ : \ \eta_1(\bar{x}_1)]\,(\bar{y}_1) \in \mathrm{GFP}^+[k]$. Then $\psi_1(\bar{y}_1)$ is equivalent to a formula of the form $\varphi(\bar{y}_1) = [\mathrm{sGFP}\, S]_1\,(\bar{y}_1) \in \mathrm{sGFP}_k^+$ where S consists only of first-order formulae.*

Proof. Let $\psi_2, \dots, \psi_n$ be an enumeration of all subformulae of η_1 of the form

$$\psi_i(\bar{y}_i) = [\mathrm{GFP}\, R_i \bar{x}_i \ : \ \eta_i(\bar{x}_i)]\,(\bar{y}_i).$$

By renumbering the indices of these formulae, it is possible to achieve that, if $\psi_j(\bar{y}_j)$ happens to be a proper subformula of $\psi_i(\bar{y}_i)$, then $i < j$. Let $\eta'_1, \dots, \eta'_n$ be the formulae that result from $\eta_1, \dots, \eta_n$ by replacing $\psi_i(\bar{y}_i)$ by $R_i \bar{y}_i$. Because of the renumbering, we have $\eta'_n = \eta_n$. By construction, $\eta'_1, \dots, \eta'_n$ are first-order formulae. Now we define

$$\varphi(\bar{y}_1) := [\mathrm{sGFP}\, S]_1\,(\bar{y}_1)$$

where

$$S := \begin{cases} R_1 \bar{x}_1 & : \ \eta'_1(\bar{x}_1) \\ R_2 \bar{x}_2 & : \ \eta'_2(\bar{x}_2) \\ & \vdots \\ R_n \bar{x}_n & : \ \eta'_n(\bar{x}_n) \end{cases}$$

$\psi_1(\bar{y}_1)$ is equivalent to $\varphi(\bar{y}_1)$, because unfolding the repeated construction from the proof of Lemma 4.4 would yield the formula ψ_1 again. For example, the first step that was used in the proof of Lemma 4.4, would plug in $\eta'_n(\bar{x}_n) = \eta_n(\bar{x}_n)$ into the $\eta'_1, \dots, \eta'_{n-1}$ by switching $R_n\bar{y}_n$ back to $[\text{GFP}\, R_n\bar{x}_n \;:\; \eta'_n(\bar{x}_n)]\,(\bar{y}_n) = [\text{GFP}\, R_n\bar{x}_n \;:\; \eta_n(\bar{x}_n)]\,(\bar{y}_n) = \psi_n(\bar{y}_n)$. After this replacement η'_{n-1} would coincide with η_{n-1} and after having done $n-1$ such replacement steps we obtain the formula $\psi_1(\bar{y}_1)$ again. □

4.1.1 Avoiding Empty Fixed-Points

Sometimes it is desirable to ensure the non-emptiness of every component of GFP(Γ) where Γ is associated to some system S of first-order formulae. In this section we will demonstrate that this is not difficult to achieve for non-unary relations. Towards this end, we consider a system S of the form

$$S := \begin{cases} R_1\bar{x}_1 & : \eta_1(R_1, \dots, R_n, \bar{x}_1) \\ R_2\bar{x}_2 & : \eta_2(R_1, \dots, R_n, \bar{x}_2) \\ & \vdots \\ R_n\bar{x}_n & : \eta_n(R_1, \dots, R_n, \bar{x}_n) \end{cases}$$

where $\text{ar}(R_i) \geq 2$ for $i = 1, \dots, n$ and every η_i is a first-order formula. The idea is to replace S by a new system S' in which two non-empty variants R_i^+, R_i^- of R_i with the property $R_i^+ \cap R_i^- = R_i$ are calculated. Towards this end, let $\bar{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,|\bar{x}_i|})$ and for every $i \in \{1, \dots, n\}$ we let R_i^+ and R_i^- be new relation symbols of the same arity as R_i. In the following we will write $\bar{R}^+ = (R_1^+, \dots, R_n^+)$ and $\bar{R}^- = (R_1^-, \dots, R_n^-)$. Now consider the system

$$S' := \begin{cases} R_1^+\bar{x}_1 & : \eta_1^+(\bar{R}^+, \bar{R}^-, \bar{x}_1) := x_{1,1} = x_{1,2} \lor \eta'_1(\bar{R}^+, \bar{R}^-, \bar{x}_1) \\ R_1^-\bar{x}_1 & : \eta_1^-(\bar{R}^+, \bar{R}^-, \bar{x}_1) := x_{1,1} \neq x_{1,2} \lor \eta'_1(\bar{R}^+, \bar{R}^-, \bar{x}_1) \\ & \vdots \\ R_n^+\bar{x}_n & : \eta_n^+(\bar{R}^+, \bar{R}^-, \bar{x}_n) := x_{n,1} = x_{n,2} \lor \eta'_n(\bar{R}^+, \bar{R}^-, \bar{x}_n) \\ R_n^-\bar{x}_n & : \eta_n^-(\bar{R}^+, \bar{R}^-, \bar{x}_n) := x_{n,1} \neq x_{n,2} \lor \eta'_n(\bar{R}^+, \bar{R}^-, \bar{x}_n) \end{cases}$$

where $\eta'_i(\bar{R}^+, \bar{R}^-, \bar{x}_i)$ emerges from $\eta_i(\bar{R}, \bar{x}_i)$ by replacing every $R_\ell\bar{v}$ by $R_\ell^+\bar{v} \land R_\ell^-\bar{v}$. For any structure $\mathfrak{A}$ with at least two elements, let Γ, Γ' be the update operators associated with S, S' respectively. Furthermore, let $(G_1, \dots, G_n) = \text{GFP}(\Gamma)$ and $(G_1^+, G_1^-, \dots, G_n^+, G_n^+) = \text{GFP}(\Gamma')$.

Lemma 4.8. *For every i, $G_i^+ = G_i \cup D^+_{\text{ar}(R_i)}$ and $G_i^+ = G_i \cup D^-_{\text{ar}(R_i)}$ where $D_k^+ := \{\bar{a} \in A^k \;:\; a_1 = a_2\}$ and $D_k^- := \{\bar{a} \in A^k \;:\; a_1 \neq a_2\}$. Moreover, we have $G_i^+ \cap G_i^- = G_i$ and $G_i^+ \neq \emptyset \neq G_i^-$.*

Proof. Clearly, $D_k^+ \cap D_k^- = \emptyset$. So $G_i^+ \cap G_i^- = G_i$ is entailed by $G_i^- = G_i \cup D^+_{\text{ar}(R_i)}$ and $G_i^+ = G_i \cup D^-_{\text{ar}(R_i)}$.

Let $(\bar{S}_\alpha)_{\alpha\in\mathrm{On}}$ and $(\bar{S}^+_\alpha, \bar{S}^-_\alpha)_{\alpha\in\mathrm{On}}$ be the inductive calculation of the fixed-points $\bar{G}$ and $(\bar{G}^+, \bar{G}^-)$. In order to prove $G^+_i = G_i \cup D^+_{\mathrm{ar}(R_i)}$ and $G^-_i = G_i \cup D^-_{\mathrm{ar}(R_i)}$, we will now show that $S^+_{\alpha,i} = S_{\alpha,i} \cup D^+_{\mathrm{ar}(R_i)}$ and $S^-_{\alpha,i} = S_{\alpha,i} \cup D^-_{\mathrm{ar}(R_i)}$ by transfinite induction over $\alpha \in \mathrm{On}$.

Clearly, this is true for $\alpha = 0$ since at that stage we are only having full relations. Now assume that the claim is true for some α. Then, as above, $S_{\alpha,i} = S^+_{\alpha,i} \cap S^-_{\alpha,i}$ and, by construction of η'_i, we have $[\![\eta'_i(\bar{S}^+_\alpha, \bar{S}^-_\alpha, \bar{x}_i)]\!]^{\mathfrak{A}} = [\![\eta_i(\bar{S}_\alpha, \bar{x}_i)]\!]^{\mathfrak{A}} = S^{\alpha+1}_i$. Now, the part $x_{i,1} = x_{i,2} \vee \ldots$ of η^+_i adds $D^+_{\mathrm{ar}(R_i)}$ to $S_{\alpha+1,i}$, which is why we end up with $S^+_{\alpha+1,i} = S^+_{\alpha+1,i} \cup D^+_{\mathrm{ar}(R_i)}$. The proof for $S^-_{\alpha+1,i} = S^-_{\alpha+1,i} \cup D^-_{\mathrm{ar}(R_i)}$ is analogous.
For the transfinite step, let $\alpha = \lambda$ for some limit ordinal λ and let the claim be true for all $\beta < \lambda$. Then

$$S^+_{\lambda,i} = \bigcap_{\beta<\lambda} S^+_{\beta,i} = \bigcap_{\beta<\lambda} \left(S_{\beta,i} \cup D^+_{\mathrm{ar}(R_i)}\right) = \left(\bigcap_{\beta<\lambda} S_{\beta,i}\right) \cup D^+_{\mathrm{ar}(R_i)} = S_{\lambda,i} \cup D^+_{\mathrm{ar}(R_i)}$$

follows as desired. $S^-_{\lambda,i} = S_{\lambda,i} \cup D^-_{\mathrm{ar}(R_i)}$ is analogous. □

Corollary 4.9. *On the class of structures with at least two elements,* $[\mathrm{sGFP}\, S]_j\,(\bar{x})$ *is equivalent to* $[\mathrm{sGFP}\, S']_{j^+}\,(\bar{x}) \wedge [\mathrm{sGFP}\, S']_{j^-}\,(\bar{x})$ *where* $j^+ := 2j - 1$ *and* $j^- := 2j$.

In this remainder of this chapter, we will always assume that structures have at least two elements. Since the set of symbols occurring in a formula is always finite and there are (up to isomorphisms) only finitely many structures $\mathfrak{A}$ over a finite signature, this assumption does not cause a loss of generality.

Avoiding Empty Unary Relations

Corollary 4.9 offers a way to eliminate empty fixed-point relations without increasing their arities. However, the construction is only applicable when the relation symbols are of arity ≥ 2. This is not a problem when we just want to translate a formula of sGFP^+_k for $k \geq 2$, since then we could just replace an unary relation P by its binary variant $\{(v, v) : v \in P\}$. Of course, this replacement is not possible when we are in the special case $k = 1$. Therefore, we describe an alternative approach that also works for $k = 1$, but has the disadvantage of creating exponentially long formulae. Readers who are not interested in the special case $k = 1$ can safely skip this subsection and jump directly to Section 4.2.

For the remainder of this section, let $\mathfrak{A}$ be some arbitrary structure. Let $\psi(\bar{x}) = [\mathrm{sGFP}\, S]_j\,(\bar{x}) \in \mathrm{sGFP}^+_k$ where

$$S := \begin{cases} R_1\bar{x}_1 & : \eta_1(R_1, \ldots, R_n, \bar{x}_1) \\ R_2\bar{x}_2 & : \eta_2(R_1, \ldots, R_n, \bar{x}_2) \\ & \vdots \\ R_n\bar{x}_n & : \eta_n(R_1, \ldots, R_n, \bar{x}_n) \end{cases}$$

The idea is to employ tuples $\bar{b} \in \{0, 1\}^n$ that "indicate" which components of the greatest fixed-points w.r.t. S and some structure $\mathfrak{A}$ become empty. Let $\varphi^{\bar{b}}_i$ be the formula that results from η_i by replacing every $R_\ell \bar{v}$ with $b_\ell = 0$ by $\bot$, a formula that is always false. Furthermore, let $S_{\bar{b}}$ be defined as

$$S_{\bar{b}} := \begin{cases} R_1\bar{y}_1 & : \ \eta_1^{\bar{b}}(\bar{R}, \bar{y}_1) \\ R_2\bar{y}_2 & : \ \eta_2^{\bar{b}}(\bar{R}, \bar{y}_2) \\ & \vdots \\ R_n\bar{y}_n & : \ \eta_n^{\bar{b}}(\bar{R}, \bar{y}_n) \end{cases} \text{ where } \eta_i^{\bar{b}}(\bar{R}, \bar{x}_i) := \begin{cases} \varphi_i^{\bar{b}}(\bar{R}, \bar{x}_i), & \text{if } b_i = 1 \\ \bot, & \text{if } b_i = 0. \end{cases} \tag{4.3}$$

A variant of $S_{\bar{b}}$ is the system $S_{\bar{b}}^+$ which results from $S_{\bar{b}}$ by changing the update formula $\eta_\ell^{\bar{b}}(\bar{R}, \bar{x}_\ell)$ for R_ℓ with $\bar{b}_\ell = 0$ from $\bot$ to $\top$, a formula which is always true. Formally, we have

$$S_{\bar{b}}^+ := \begin{cases} R_1\bar{y}_1 & : \ \tilde{\eta}_1^{\bar{b}}(\bar{R}, \bar{y}_1) \\ R_2\bar{y}_2 & : \ \tilde{\eta}_2^{\bar{b}}(\bar{R}, \bar{y}_2) \\ & \vdots \\ R_n\bar{y}_n & : \ \tilde{\eta}_n^{\bar{b}}(\bar{R}, \bar{y}_n) \end{cases} \text{ where } \tilde{\eta}_i^{\bar{b}}(\bar{R}, \bar{x}_i) := \begin{cases} \varphi_i^{\bar{b}}(\bar{R}, \bar{x}_i), & \text{if } b_i = 1 \\ \top, & \text{if } b_i = 0. \end{cases} \tag{4.4}$$

We will use Γ, $\Gamma_{\bar{b}}$ and $\Gamma_{\bar{b}}^+$ to denote the update operators w.r.t. S,$S_{\bar{b}}$ resp. $S_{\bar{b}}^+$ (for some structure $\mathfrak{A}$). Furthermore, for any n-tuple of relations $\bar{R} = (R_1, \dots, R_n)$ we let $\bar{R}^{\bar{b}\downarrow}$ and $\bar{R}^{\bar{b}\uparrow}$ be given by:

$$\bar{R}_i^{\bar{b}\downarrow} = \begin{cases} R_i, & \text{if } b_i = 1 \\ \emptyset, & \text{if } b_i = 0 \end{cases} \text{ and } \bar{R}_i^{\bar{b}\uparrow} = \begin{cases} R_i, & \text{if } b_i = 1 \\ A^{\mathrm{ar}(R_i)}, & \text{if } b_i = 0 \end{cases} \tag{4.5}$$

By construction of $S_{\bar{b}}$ (see (4.3)), we have that

$$\Gamma_{\bar{b}}(\bar{R}) = (\Gamma(\bar{R}^{\bar{b}\downarrow}))^{\bar{b}\downarrow} \tag{4.6}$$

for every $\bar{R}$. Furthermore, since $(\Gamma_{\bar{b}}(\bar{R}))_i = \emptyset$ for every i with $b_i = 0$, we can derive that

$$\mathrm{GFP}(\Gamma_{\bar{b}})^{\bar{b}\downarrow} = \mathrm{GFP}(\Gamma_{\bar{b}}). \tag{4.7}$$

The construction of $S_{\bar{b}}^+$ (see (4.4)) leads to $(\Gamma_{\bar{b}}^+(\bar{R}))_i = A^{\mathrm{ar}(R_i)}$ for every i with $b_i = 0$. We can also observe that

$$\Gamma_{\bar{b}}^+(\bar{R}) = \Gamma_{\bar{b}}(\bar{R})^{\bar{b}\uparrow} \tag{4.8}$$

and, consequently, we have

$$\mathrm{GFP}(\Gamma_{\bar{b}}^+) = \mathrm{GFP}(\Gamma_{\bar{b}})^{\bar{b}\uparrow}. \tag{4.9}$$

This equation explains the connection between $\mathrm{GFP}(\Gamma_{\bar{b}}^+)$ and $\mathrm{GFP}(\Gamma_{\bar{b}})$, but what is the relationship between $\mathrm{GFP}(\Gamma_{\bar{b}})$ and $\mathrm{GFP}(\Gamma)$? The following lemma answers this question.

Lemma 4.10. *Let* $\bar{b} \in \{0, 1\}^n$.

(a) $\mathrm{GFP}(\Gamma_{\bar{b}}) \subseteq \mathrm{GFP}(\Gamma)$.

(b) For every $j \in \{1, \dots, n\}$ with $\bar{b}_j = 1$ holds $\mathrm{GFP}(\Gamma^+_{\bar{b}})_j = \mathrm{GFP}(\Gamma_{\bar{b}})_j$.

Proof. First, we prove (a). We have

$$\mathrm{GFP}(\Gamma_{\bar{b}}) = \Gamma_{\bar{b}}(\mathrm{GFP}(\Gamma_{\bar{b}})) \overset{(4.6)}{=} \Gamma(\mathrm{GFP}(\Gamma_{\bar{b}})^{\bar{b}\downarrow})^{\bar{b}\downarrow} \overset{(4.7)}{=} \Gamma(\mathrm{GFP}(\Gamma_{\bar{b}}))^{\bar{b}\downarrow} \subseteq \Gamma(\mathrm{GFP}(\Gamma_{\bar{b}}))$$

and, consequently, $\mathrm{GFP}(\Gamma_{\bar{b}}) \subseteq \Gamma(\mathrm{GFP}(\Gamma_{\bar{b}}))$. By the Theorem of Knaster-Tarski (Theorem 4.2), the claim follows.

Now we prove (b): $\mathrm{GFP}(\Gamma^+_{\bar{b}})_j \overset{(4.9)}{=} (\mathrm{GFP}(\Gamma_{\bar{b}})^{\bar{b}\uparrow})_j = \mathrm{GFP}(\Gamma_{\bar{b}})_j$ — the last equation is due to (4.5) and $\bar{b}_j = 1$. □

Definition 4.11. For a given tuple $\bar{R} = (R_1, \dots, R_n)$ of relations, we define $\chi(\bar{R}) = (\chi_1(\bar{R}), \dots, \chi_n(\bar{R})) \in \{0,1\}^n$ by $\chi_i(\bar{R}) = 0 :\Longleftrightarrow R_i = \emptyset$ and $\bar{R}^{\uparrow} := \bar{R}^{\chi(\bar{R})\uparrow}$.

In other words, $\bar{R}^{\uparrow}$ results from $\bar{R}$ by replacing its empty components by the full relations. As a result, $\bar{R}^{\uparrow}$ is a tuple consisting of non-empty relations. Furthermore, it is easy to see that

$$\bar{R}^{\chi(\bar{R})\downarrow} = \bar{R}. \tag{4.10}$$

Lemma 4.12. *Let $\bar{G} := \mathrm{GFP}(\Gamma)$. Then $\bar{G}^{\uparrow} = \Gamma^+_{\chi(\bar{G})}(\bar{G}^{\uparrow})$.*

Proof. For every tuple $\bar{R}$ of relations and every $\bar{b} \in \{0,1\}^n$ holds

$$(\bar{R}^{\bar{b}\downarrow})^{\bar{b}\uparrow} = \bar{R}^{\bar{b}\uparrow} \text{ and } (\bar{R}^{\bar{b}\uparrow})^{\bar{b}\downarrow} = \bar{R}^{\bar{b}\downarrow}. \tag{4.11}$$

By Definition 4.11, we have $\bar{G}^{\uparrow} = \bar{G}^{\chi(\bar{G})\uparrow}$. Now we can observe that

$$\begin{aligned}\Gamma^+_{\chi(\bar{G})}(\bar{G}^{\uparrow}) &\overset{(4.8)}{=} [\Gamma_{\chi(\bar{G})}(\bar{G}^{\uparrow})]^{\chi(\bar{G})\uparrow} \overset{(4.6)}{=} [\Gamma([\bar{G}^{\uparrow}]^{\chi(\bar{G})\downarrow})^{\chi(\bar{G})\downarrow}]^{\chi(\bar{G})\uparrow} \\ &\overset{(4.11)}{=} [\Gamma(\bar{G}^{\chi(\bar{G})\downarrow})]^{\chi(\bar{G})\uparrow} \overset{(4.10)}{=} \Gamma(\bar{G})^{\chi(\bar{G})\uparrow} = \bar{G}^{\chi(\bar{G})\uparrow} = \bar{G}^{\uparrow}.\end{aligned}$$ □

Lemma 4.12 shows that the greatest fixed-point $\bar{G}$ turns into a fixed-point $\bar{G}^{\uparrow}$ of $\Gamma^+_{\chi(\bar{G})}$, which does not have empty components.

Since we do not know $\chi(\bar{G})$ before computing $\bar{G}$, we will just try all possible combinations $\bar{b} \in \{0,1\}^n$ instead. The following proof of Theorem 4.13 demonstrates that we can indeed simulate the formula $\psi(\bar{x}) = [\mathrm{sGFP}\, S]_j(\bar{x})$ from above by a big disjunction of some $\psi_{\bar{b}}(\bar{x}) := \left[\mathrm{sGFP}\, S^+_{\bar{b}}\right]_j(\bar{x})$ where $\bar{b} \in \{0,1\}^n$.

Theorem 4.13. *Let $\psi(\bar{x})$ and $\psi_{\bar{b}}(\bar{x})$ be as above. For every structure $\mathfrak{A}$ and assignment $s\colon \{\bar{x}\} \longrightarrow A$,*

$$\mathfrak{A} \models_s \psi(\bar{x}) \Longleftrightarrow \mathfrak{A} \models_s \bigvee_{\substack{\bar{b}\in\{0,1\}^n \\ \bar{b}_j=1}} \psi_{\bar{b}}(\bar{x}).$$

Proof. Let Γ, $\Gamma_{\bar{b}}$ and $\Gamma^+_{\bar{b}}$ be the update operators for S, $S_{\bar{b}}$ resp. $S^+_{\bar{b}}$. We prove the two directions separately.

"$\Longleftarrow$": If $\mathfrak{A} \vDash_s \psi_{\bar{b}}(\bar{x})$ for some $\bar{b} \in \{0,1\}^n$ with $\bar{b}_j = 1$, then

$$s(\bar{x}) \in \mathrm{GFP}(\Gamma^+_{\bar{b}})_j \overset{(4.9)}{=} (\mathrm{GFP}(\Gamma_{\bar{b}})^{\bar{b}\uparrow})_j \overset{(4.5)}{=} \mathrm{GFP}(\Gamma_{\bar{b}})_j.$$

By Lemma 4.10 (a), it follows $s(\bar{x}) \in \mathrm{GFP}(\Gamma)_j$ and, thus, $\mathfrak{A} \vDash_s \psi(\bar{x})$.

"$\Longrightarrow$": Now assume $\mathfrak{A} \vDash_s \psi(\bar{x})$. Let $\bar{G} = (G_1, \dots, G_n) := \mathrm{GFP}(\Gamma)$. By Lemma 4.12, we know that $\bar{G}^\uparrow = \Gamma^+_{\chi(\bar{G})}(\bar{G}^\uparrow)$. In particular, we have $\bar{G}^\uparrow \subseteq \Gamma^+_{\chi(\bar{G})}(\bar{G}^\uparrow)$ and, due to Theorem 4.2, $\bar{G}^\uparrow \subseteq \mathrm{GFP}(\Gamma^+_{\chi(\bar{G})})$. Since $\mathfrak{A} \vDash_s \psi(\bar{x}) = [\mathrm{sGFP}\, S]_j\,(\bar{x})$, we have $s(\bar{x}) \in G_j$ and, hence, $\chi_j(\bar{G}) = 1$. Because of $\bar{G}^\uparrow = \bar{G}^{\chi(\bar{G})\uparrow\,\prime}$ and (4.5), this leads to $G_j = \bar{G}^\uparrow_j$. Thus $s(\bar{x}) \in \bar{G}^\uparrow_j \subseteq \mathrm{GFP}(\Gamma^+_{\chi(\bar{G})})_j$ and, consequently, $\mathfrak{A} \vDash_s \left[\mathrm{sGFP}\, S^+_{\chi(\bar{G})}\right]_j (\bar{x}) = \psi_{\chi(\bar{G})}(\bar{x})$. Because of $\chi_j(\bar{G}) = 1$, this concludes this proof. □

4.2 Greatest Fixed-Points in Bounded Inclusion Logic

In this section we will translate given GFP$^+[k]$-formulae into FO($\subseteq$)$[k]$-formulae. This will also enable the translation of myopic GFP$^+[k]$-sentences using a free relational variable X to represent the team into inclusion logic of bounded arity. We describe the translation for a τ-formula of the form $\psi(\bar{x}) = [\mathrm{sGFP}\, S]_j\,(\bar{x})$, which will be examined in the remainder of this section, where the system

$$S := \begin{cases} R_1\bar{y}_1 & : \varphi_1(\bar{R}, \bar{y}_1) \\ R_2\bar{y}_2 & : \varphi_2(\bar{R}, \bar{y}_2) \\ & \vdots \\ R_n\bar{y}_n & : \varphi_n(\bar{R}, \bar{y}_n) \end{cases}$$

consists of formulae $\varphi_1, \dots, \varphi_n \in \mathrm{FO}(\tau \cup \{\bar{R}\})$ in negation normal form and in which the relations symbols $\bar{R} = (R_1, \dots, R_n)$ occur only positively. For the remainder of this section, let $\mathfrak{A}$ be some arbitrary structure with at least 2 elements and Γ_S be the update operator that is defined by $\varphi_1, \dots, \varphi_n$ w.r.t. $\mathfrak{A}$.

Let $\bar{r}_1, \dots, \bar{r}_n, \bar{r}'_1, \dots, \bar{r}'_n$ be tuples of yet unused, pairwise different variables with $\mathrm{ar}(R_i) = |\bar{r}_i| = |\bar{r}'_i|$ for $i = 1, \dots, n$. We define

$$\eta_S := \exists \bar{r}'_1 \dots \exists \bar{r}'_n \Big(\bigwedge_{i=1}^{n} \bar{r}'_i \subseteq \bar{r}_i \wedge \varphi'_i(\bar{r}'_1, \dots, \bar{r}'_n, \bar{r}_i) \Big) \tag{4.12}$$

where $\varphi'_i(\bar{r}'_1, \dots, \bar{r}'_n, \bar{r}_i)$ results from $\varphi_i(\bar{R}, \bar{r}_i)$ by replacing every subformula of the form $R_\ell \bar{v}$ by $\bar{v} \subseteq \bar{r}'_\ell$.

Lemma 4.14. *Let Y be a team over $\mathfrak{A}$ with $\bar{r}_1, \dots, \bar{r}_n \subseteq \mathrm{dom}(Y)$ and $\bar{Y} := (Y(\bar{r}_1), \dots, Y(\bar{r}_n))$. Then $\mathfrak{A} \models_Y \eta_S \iff \bar{Y} \subseteq \Gamma_S(\bar{Y})$.*

Proof. If $Y = \emptyset$, then $\mathfrak{A} \models_Y \eta_S$ follows from the empty team property and $\bar{Y} \subseteq \Gamma_S(\bar{Y})$ is trivially satisfied. Therefore, we assume $Y \neq \emptyset$ for the rest of the proof, which handles the two directions separately.

"$\Longleftarrow$": Let $\bar{Y} \subseteq \Gamma_S(\bar{Y})$. Then $Y(\bar{r}_i) \subseteq (\Gamma_S(\bar{Y}))_i$ for $i = 1, \dots, n$ and, consequently, for every $i \in \{1, \dots, n\}$ and $\bar{a} \in Y(\bar{r}_i)$, there exists a witness $\lambda_{i,\bar{a}}$ for $(\mathfrak{A}, \bar{Y}) \models_{\{\bar{r}_i \mapsto \bar{a}\}} \varphi_i(\bar{Y}, \bar{r}_i)$. Let $Z := Y[\bar{r}'_1 \mapsto Y(\bar{r}_1), \dots, \bar{r}'_n \mapsto Y(\bar{r}_n)]$. It is the case that $Y(\bar{r}_i) \neq \emptyset$, because $Y \neq \emptyset$. Clearly, $\mathfrak{A} \models_Z \bigwedge_{i=1}^n \bar{r}'_i \subseteq \bar{r}_i$. We still need to prove that $\mathfrak{A} \models_Z \varphi'_i(\bar{r}'_1, \dots, \bar{r}'_n, \bar{r}_i)$ for $i = 1, \dots, n$. For every subformula ϑ of some φ_i, we denote by ϑ' the corresponding subformula of φ'_i, which can be obtained by replacing occurrences of $R_\ell \bar{v}$ by $\bar{v} \subseteq \bar{r}'_\ell$. Now, we can combine all the witnesses $\lambda_{i,\bar{a}}$ for $\bar{a} \in Y(\bar{r}_i)$ into a witness λ_i for $\mathfrak{A} \models_Z \varphi'_i(\bar{r}'_1, \dots, \bar{r}'_n, \bar{r}_i)$ by setting

$$\lambda_i(\vartheta') := \bigcup_{\bar{a} \in Y(\bar{r}_i)} \lambda_{i,\bar{a}}(\vartheta)[\bar{r}'_1 \mapsto Y(\bar{r}_1), \dots, \bar{r}'_n \mapsto Y(\bar{r}_n)] \text{ for } \vartheta \in \mathrm{subf}(\varphi).$$

Then $\lambda_i(\varphi') = \bigcup_{\bar{a} \in Y(\bar{r}_i)} \lambda_{i,\bar{a}}(\varphi)[\bar{r}'_1 \mapsto Y(\bar{r}_1), \dots, \bar{r}'_n \mapsto Y(\bar{r}_n)] = Z\restriction_{\mathrm{free}(\varphi')}$, because $\lambda_{i,\bar{a}}(\varphi) = \{\bar{r}_i \mapsto \bar{a}\}$. Furthermore, it is not difficult to verify that λ_i satisfies the requirements for all composite formulae ϑ' and all first-order literals. However, it still requires proof that the same is true for subformulae of the form $\vartheta' = \bar{v} \subseteq \bar{r}'_\ell$, which are the replacements of $\vartheta = R_\ell \bar{v} \in \mathrm{subf}(\varphi_i)$.

If $\lambda_i(\vartheta') = \emptyset$, then $\mathfrak{A} \models_{\lambda_i(\vartheta')} \vartheta'$ follows from the empty team property.

Now we deal with the case $\lambda_i(\vartheta') \neq \emptyset$. This implies that $\lambda_{i,\bar{a}}(\vartheta) \neq \emptyset$ for some $\bar{a} \in Y(\bar{r}_i)$ and, therefore, $(\lambda_i(\vartheta'))(\bar{r}'_\ell) = Y(\bar{r}_\ell)$. Since the $\lambda_{i,\bar{a}}$ are witnesses for $(\mathfrak{A}, \bar{Y}) \models_{\{\bar{r}_i \mapsto \bar{a}\}} \varphi_i(\bar{Y}, \bar{r}_i)$, we have that $(\mathfrak{A}, \bar{Y}) \models_{\lambda_{i,\bar{a}}(\vartheta)} \vartheta = R_\ell \bar{v}$, i.e. $(\lambda_{i,\bar{a}}(\vartheta))(\bar{v}) \subseteq Y(\bar{r}_\ell)$ for every $\bar{a} \in Y(\bar{r}_i)$. Because of that and by the definition of λ_i, it follows that $(\lambda_i(\vartheta'))(\bar{v}) \subseteq Y(\bar{r}_\ell) = (\lambda_i(\vartheta'))(\bar{r}'_\ell)$ and thus $\mathfrak{A} \models_{\lambda_i(\vartheta')} \bar{v} \subseteq \bar{r}'_\ell = \vartheta'$ follows as desired. So, λ_i is indeed a witness for $\mathfrak{A} \models_Z \varphi'_i$, which concludes the proof of $\mathfrak{A} \models_Y \eta_S$.

"$\Longrightarrow$": For the converse direction, we assume that $\mathfrak{A} \models_Y \eta_S$. Then there exists a team Z extending Y by $\bar{r}'_1, \dots, \bar{r}'_n$ such that $\mathfrak{A} \models_Z \bar{r}'_i \subseteq \bar{r}_i \wedge \varphi'_i(\bar{r}'_1, \dots, \bar{r}'_n, \bar{r}_i)$ for every $i = 1, \dots, n$. So $Z(\bar{r}'_i) \subseteq Z(\bar{r}_i)$ and there are witnesses λ_i for $\mathfrak{A} \models_Z \varphi'_i$ with $\lambda_i(\varphi'_i) = Z$. In order to prove $\bar{Y} \subseteq \Gamma_S(\bar{Y})$, we have to verify that $(\mathfrak{A}, \bar{Y}) \models_{\{\bar{r}_i \mapsto \bar{a}\}} \varphi_i(\bar{Y}, \bar{r}_i)$ holds for every $\bar{a} \in Y(\bar{r}_i)$, $i \in \{1, \dots, n\}$. Let $i \in \{1, \dots, n\}$ and $\bar{a} \in Y(\bar{r}_i)$ be chosen arbitrarily. Now, we construct witnesses $\lambda_{i,\bar{a}}$ for $(\mathfrak{A}, \bar{Y}) \models_{\{\bar{r}_i \mapsto \bar{a}\}} \varphi_i(\bar{Y}, \bar{r}_i)$ by setting

$$\lambda_{i,\bar{a}}(\vartheta) := \lambda_i(\vartheta')\restriction_{\bar{r}_i = \bar{a}} \text{ for every } \vartheta \in \mathrm{subf}(\varphi_i)$$

where ϑ' is the corresponding subformula emerging from ϑ by replacing every $R_\ell \bar{v}$ by $\bar{v} \subseteq \bar{r}'_\ell$. Then $(\lambda_{i,\bar{a}}(\varphi_i))\restriction_{\bar{r}_i} = (\lambda_i(\varphi'_i)\restriction_{\bar{r}_i = \bar{a}})\restriction_{\bar{r}_i} = \{\bar{r}_i \mapsto \bar{a}\}$ follows. Furthermore, the requirements for witnesses are inherited for all composite subformulae and all literals *not* of the form $R_\ell \bar{v}$. Consider some subformula $\vartheta = R_\ell \bar{v} \in \mathrm{subf}(\varphi_i)$. Then $\vartheta' = \bar{v} \subseteq \bar{r}'_\ell$ and, since λ_i is a witness for $\mathfrak{A} \models_Z \varphi'_i$, it follows that $(\lambda_i(\vartheta'))(\bar{v}) \subseteq (\lambda_i(\vartheta'))(\bar{r}'_\ell)$. Since $\bar{r}'_\ell$ is never quantified in φ'_i, it follows that $(\lambda_i(\vartheta'))(\bar{r}'_\ell) \subseteq (\lambda_i(\varphi'_i))(\bar{r}'_\ell)$.[3] As a result, we

[3]Here we only have $\subseteq$ in general, because we could loose values for $\bar{r}'_\ell$ at disjunctions.

obtain $(\lambda_{i,\bar{a}}(\vartheta))(\bar{v}) \subseteq (\lambda_i(\vartheta'))(\bar{v}) \subseteq (\lambda_i(\varphi_i'))(\bar{r}_\ell') = Z(\bar{r}_\ell') \subseteq Z(\bar{r}_\ell) = Y(\bar{r}_\ell)$, which proves that $(\mathfrak{A}, \bar{Y}) \vDash_{\lambda_{i,\bar{a}}(\vartheta)} R_\ell \bar{v} = \vartheta$. So, $\lambda_{i,\bar{a}}$ is indeed a witness for $(\mathfrak{A}, \bar{Y}) \vDash_{\{\bar{r}_i \mapsto \bar{a}\}} \varphi_i(\bar{Y}, \bar{r}_i)$. This shows that $\bar{Y} \subseteq \Gamma_S(\bar{Y})$. □

Now we have understood the formula η_S for a given system S of first-order update rules. We will now point out how this formula helps us to translate a given sGFP^+_k-formula into inclusion logic.

For the remainder of section, we make the additional assumption that $\mathrm{ar}(R_i) \geq 2$ for every $i = 1, \dots, n$. If $k \geq 2$, it is always possible to simulate a unary relation P by its binary variant $\{(u, u) : u \in P\}$. We will deal with the special case $k = 1$ in Section 4.2.1. This additional assumption allows us to use Corollary 4.9 to transform $\psi(\bar{x}) = [\mathrm{sGFP}\ S]_j(\bar{x})$ into the formula $[\mathrm{sGFP}\ S']_{j^+}(\bar{x}) \wedge [\mathrm{sGFP}\ S']_{j^-}(\bar{x})$, which is equivalent to $\psi(\bar{x})$ on the class of all structure with at least two elements.

Now consider the $\mathrm{FO}(\subseteq)[k]$-formula

$$\gamma(\bar{x}) := \exists \bar{r}_{1^+} \exists \bar{r}_{1^-} \dots \exists \bar{r}_{n^+} \exists \bar{r}_{n^-} (\bar{x} \subseteq \bar{r}_{j^+} \wedge \bar{x} \subseteq \bar{r}_{j^-} \wedge \eta_{S'})$$

where $\eta_{S'}$ is the analog of η_S from above for S' instead of S.[4]

Theorem 4.15. *Let $k \geq 2$, $\psi(\bar{x}) \in \mathrm{sGFP}^+_k$ and $\gamma(\bar{x}) \in \mathrm{FO}(\subseteq)[k]$ be as above. For every structure $\mathfrak{A}$ with at least two elements and every team X with $\{\bar{x}\} \subseteq \mathrm{dom}(X)$ holds*

$$\mathfrak{A} \vDash_X \gamma(\bar{x}) \iff \mathfrak{A} \vDash_s \psi(\bar{x}) \textit{ for every } s \in X.$$

Proof. Let Γ_S and $\Gamma_{S'}$ be the update operator associated with S resp. S'.

If $X = \emptyset$, then there is nothing to prove. Therefore, we now assume that $X \neq \emptyset$ and prove the two directions separately.

"$\Longrightarrow$": Let $\mathfrak{A} \vDash_X \gamma(\bar{x})$. Then $\mathfrak{A} \vDash_Y \bar{x} \subseteq \bar{r}_{j^+} \wedge \bar{x} \subseteq \bar{r}_{j^-} \wedge \eta_{S'}$ for some $\{\bar{r}_1, \dots, \bar{r}_{2n}\}$-extension Y of X. Because to Lemma 4.14, this corresponds to $Y(\bar{x}) \subseteq Y(\bar{r}_{j^+}) \cap Y(\bar{r}_{j^-})$ and

$$(Y(\bar{r}_{1^+}), Y(\bar{r}_{1^-}), \dots, Y(\bar{r}_{n^+}), Y(\bar{r}_{n^-})) \subseteq \Gamma_{S'}(Y(\bar{r}_{1^+}), Y(\bar{r}_{1^-}), \dots, Y(\bar{r}_{n^+}), Y(\bar{r}_{n^-})),$$

which, because of the Theorem of Knaster-Tarski (Theorem 4.2), implies that $X(\bar{x}) = Y(\bar{x}) \subseteq G_j^+ \cap G_j^-$ where $(G_1^+, G_1^-, \dots, G_n^+, G_n^-) := \mathrm{GFP}(\Gamma_{S'})$. By Lemma 4.8, we have $G_j^+ \cap G_j^- = G_j$ where $(G_1, \dots, G_n) := \mathrm{GFP}(\Gamma_S)$. Together this leads to $X(\bar{x}) \subseteq G_j$ or, in other words, $\mathfrak{A} \vDash_s [\mathrm{sGFP}\ S]_j(\bar{x}) = \psi(\bar{x})$ is true for every $s \in X$.

"$\Longleftarrow$": Now assume that $\mathfrak{A} \vDash_s \psi(\bar{x})$ for every $s \in X$. This implies $X(\bar{x}) \subseteq G_j = G_j^+ \cap G_j^-$ where G_i^+, G_i^-, G_i are as above. By Lemma 4.8, the relations G_i^+ and G_i^- are non-empty. Therefore, $Y := X[\bar{r}_{1^+} \mapsto G_1^+, \bar{r}_{1^-} \mapsto G_1^-, \dots, \bar{r}_{n^+} \mapsto G_n^+, \bar{r}_{n^-} \mapsto G_n^-]$ is indeed an extension of X and, hence, it suffices to prove that $\mathfrak{A} \vDash_Y \bar{x} \subseteq \bar{r}_{j^+} \wedge \bar{x} \subseteq \bar{r}_{j^-} \wedge \eta_{S'}$. Since $Y(\bar{x}) = X(\bar{x}) \subseteq G_j = G_j^+ \cap G_j^-$, we have $\mathfrak{A} \vDash_Y \bar{x} \subseteq \bar{r}_{j^+} \wedge \bar{x} \subseteq \bar{r}_{j^-}$. Furthermore, since the $G_i^+ = Y(\bar{r}_{i^+})$ and $G_i^- = Y(\bar{r}_i^-)$ are the components of the greatest fixed-point of $\Gamma_{S'}$, using Lemma 4.14 leads to $\mathfrak{A} \vDash_Y \eta_{S'}$. This completes the proof of $\mathfrak{A} \vDash_X \gamma(\bar{x})$. □

[4] Please recall that we defined $i^+ := 2i - 1$ and $i^- := 2i$ in Corollary 4.9.

4.2.1 Dealing with Unary Fixed-Points

Since Theorem 4.15 works only for $k \geq 2$, we describe an alternative construction for the special case $k = 1$. Readers who are not interested in this special case can jump directly to Corollary 4.18.

The goal of this section is to translate a given formula $\psi(\bar{x}) = [\mathrm{sGFP}\ S]_j\,(\bar{x}) \in \mathrm{sGFP}^+_k$ into some $\mathrm{FO}(\subseteq)[k]$-formula $\gamma(\bar{x})$ without using Lemma 4.8.

Towards this end, please recall the definition of $S^+_{\bar{b}}$ in (4.4) and the formulae

$$\psi_{\bar{b}}(\bar{x}) = \left[\mathrm{sGFP}\ S^+_{\bar{b}}\right]_j(\bar{x})$$

for $\bar{b} \in \{0,1\}^n$. In Theorem 4.13, we have seen the equivalence of $\psi(\bar{x})$ and the formula $\bigvee_{\bar{b}\in\{0,1\}^n, \bar{b}_j=1} \psi_{\bar{b}}(\bar{x})$. Now we define

$$\psi'_{\bar{b}}(\bar{x}) := \exists\bar{r}_1 \dots \exists\bar{r}_n(\bar{x} \subseteq \bar{r}_j \wedge \eta_{S^+_{\bar{b}}})$$

where $\eta_{S^+_{\bar{b}}}$ is the analogue of η_S, which was defined in (4.12).

Lemma 4.16. *Let $\bar{b} \in \{0,1\}^n$ and X be a team over some structure $\mathfrak{A}$ with $\bar{x} \subseteq \mathrm{dom}(X)$. If $\mathfrak{A} \models_X \psi'_{\bar{b}}(\bar{x})$, then $\mathfrak{A} \models_s \psi_{\bar{b}}(\bar{x})$ for every $s \in X$.*

Proof. If $X = \emptyset$, then there is nothing to prove. Therefore, we now assume that $X \neq \emptyset$. Let $\Gamma^+_{\bar{b}}$ be the update operator w.r.t. $S^+_{\bar{b}}$ and $\mathfrak{A}$.

Suppose we have $\mathfrak{A} \models_X \psi'_{\bar{b}}(\bar{x})$. Then $\mathfrak{A} \models_Y \bar{x} \subseteq \bar{r}_j \wedge \eta_{S^+_{\bar{b}}}$ for some $\{\bar{r}_1, \dots, \bar{r}_n\}$-extension Y of X. Because of Lemma 4.14, this corresponds to $Y(\bar{x}) \subseteq Y(\bar{r}_j)$ and $(Y(\bar{r}_1), \dots, Y(\bar{r}_n)) \subseteq \Gamma^+_{\bar{b}}(Y(\bar{r}_1), \dots, Y(\bar{r}_n))$, which, because of the Theorem of Knaster-Tarski (Theorem 4.2), implies that $X(\bar{x}) = Y(\bar{x}) \subseteq \mathrm{GFP}(\Gamma^+_{\bar{b}})_j$ or, in other words, $\mathfrak{A} \models_s \psi_{\bar{b}}(\bar{x})$ for every $s \in X$. □

Now consider the following formula:

$$\gamma(\bar{x}) := \bigvee_{\substack{\bar{b}\in\{0,1\}^n \\ \bar{b}_j=1}} \psi'_{\bar{b}}(\bar{x})$$

Theorem 4.17. *Let $\psi(\bar{x})$ and $\gamma(\bar{x})$ be as above. For every structure $\mathfrak{A}$ and every team X with $\{\bar{x}\} \subseteq \mathrm{dom}(X)$ holds*

$$\mathfrak{A} \models_X \gamma(\bar{x}) \iff \mathfrak{A} \models_s \psi(\bar{x}) \text{ for every } s \in X.$$

Proof. Let Γ, $\Gamma^+_{\bar{b}}$ and $\Gamma_{\bar{b}}$ be the update operators w.r.t. S, $S^+_{\bar{b}}$ resp. $S_{\bar{b}}$. We prove the two directions separately:

"$\Longrightarrow$": First let $\mathfrak{A} \models_X \gamma(\bar{x})$. Then there exists a family $(X_{\bar{b}})_{\bar{b}\in\{0,1\}^n, \bar{b}_j=1}$ of teams with $X = \bigcup_{\bar{b}\in\{0,1\}^n, \bar{b}_j=1} X_{\bar{b}}$ and $\mathfrak{A} \models_{X_{\bar{b}}} \psi'_{\bar{b}}(\bar{x})$ for every $\bar{b} \in \{0,1\}^n$ with $\bar{b}_j = 1$. Let $\bar{b} \in \{0,1\}$ with $\bar{b}_j = 1$ be chosen arbitrarily. By using Lemma 4.16 for $\psi_{\bar{b}}$ and $\psi'_{\bar{b}}$, we obtain $\mathfrak{A} \models_s \psi_{\bar{b}}(\bar{x}) = \left[\mathrm{sGFP}\ S^+_{\bar{b}}\right]_j(\bar{x})$ for every $s \in X_{\bar{b}}$. So $X_{\bar{b}}(\bar{x}) \subseteq \mathrm{GFP}(\Gamma^+_{\bar{b}})_j$. Because Lemma 4.10 entails $\mathrm{GFP}(\Gamma^+_{\bar{b}})_j \subseteq \mathrm{GFP}(\Gamma)_j$, we have $X_{\bar{b}}(\bar{x}) \subseteq \mathrm{GFP}(\Gamma)_j$. Since $\bar{b}$ was chosen arbitrarily, it follows that $X(\bar{x}) \subseteq \mathrm{GFP}(\Gamma)_j$ or, in other words, $\mathfrak{A} \models_s [\mathrm{sGFP}\, S]_j\,(\bar{x}) = \psi(\bar{x})$ for every $s \in X$.

"$\Longleftarrow$": For the converse direction, suppose that $\mathfrak{A} \models_s \psi(\bar{x})$ for every $s \in X$. If $X = \emptyset$, then $\mathfrak{A} \models_X \gamma(\bar{x})$ follows from the empty team property and there is nothing to prove. Thus, we now assume $X \neq \emptyset$. Let $\bar{G} := (G_1, \dots, G_n) = \mathrm{GFP}(\Gamma)$. In Definition 4.11, we have defined $\chi(\bar{G}) = (\chi_1(\bar{G}), \dots, \chi_n(\bar{G})) \in \{0,1\}^n$ such that $\chi_i(\bar{G}) = 0 \iff G_i = \emptyset$ and $\bar{G}^{\uparrow} := \bar{G}^{\chi(\bar{G})\uparrow}$ (see also (4.5)). This means that $\bar{G}^{\uparrow}$ results from $\bar{G}$ by replacing its empty components with full relations. In particular, every single component of $\bar{G}^{\uparrow} = (\bar{G}_1^{\uparrow}, \dots, \bar{G}_n^{\uparrow})$ is non-empty.

Let $Y := X[\bar{r}_1 \mapsto G_1^{\uparrow}, \dots, \bar{r}_n \mapsto G_n^{\uparrow}]$. Since the $G_i^{\uparrow}$ are all non-empty, Y is indeed an extension of X. Clearly, we have $Y(\bar{x}) = X(\bar{x})$ and, due to $\mathfrak{A} \models_s \psi(\bar{x}) = [\mathrm{sGFP}\, S]_j\,(\bar{x})$ for every $s \in X$, we also have $X(\bar{x}) \subseteq G_j$. So $Y(\bar{x}) \subseteq G_j$. Because of $X \neq \emptyset$, holds $X(\bar{x}) \neq \emptyset$ and, thus, $G_j \neq \emptyset$. This implies $\chi_j(\bar{G}) = 1$ and $G_j^{\uparrow} = G_j$. Thus $Y(\bar{x}) \subseteq G_j = G_j^{\uparrow} = Y(\bar{r}_j)$ which leads to $\mathfrak{A} \models_Y \bar{x} \subseteq \bar{r}_j$.

By Lemma 4.12, we know $\bar{G}^{\uparrow} = \Gamma^{+}_{\chi(\bar{G})}(\bar{G}^{\uparrow})$ which implies $\bar{G}^{\uparrow} \subseteq \Gamma^{+}_{\chi(\bar{G})}(\bar{G}^{\uparrow})$. Therefore, by using Lemma 4.14, we obtain $\mathfrak{A} \models_Y \eta_{S^{+}_{\chi(\bar{G})}}$. As a result, we have $\mathfrak{A} \models_Y \bar{x} \subseteq \bar{r}_j \wedge \eta_{S^{+}_{\chi(\bar{G})}}$, which proves $\mathfrak{A} \models_X \psi'_{\chi(\bar{G})}(\bar{x})$ and, thus, $\mathfrak{A} \models_X \gamma(\bar{x})$. □

Because of the exponential size of the resulting formula γ, we will use Theorem 4.17 only for the special case $k = 1$.

Corollary 4.18. *For every $\varphi(\bar{x}) \in \mathrm{GFP}^{+}[k]$ there exists a $\gamma(\bar{x}) \in \mathrm{FO}(\subseteq)[k]$ such that for every structure $\mathfrak{A}$ and every team X over $\mathfrak{A}$ with $\bar{x} \subseteq \mathrm{dom}(X)$, holds*

$$\mathfrak{A} \models_X \gamma(\bar{x}) \iff \mathfrak{A} \models_s \varphi(\bar{x}) \text{ for every } s \in X. \tag{4.13}$$

Proof. Let $\varphi_1(\bar{x}_1), \dots, \varphi_m(\bar{x}_m)$ be a complete enumeration of the outermost fixed-point formulae of $\varphi(\bar{x})$. So, we have $\varphi_i(\bar{x}_i) = [\mathrm{GFP}\, R_i \bar{y}_i \,:\, \varphi_i'(R_i, \bar{y}_i)]\,(\bar{x}_i)$ and therefore we can use Lemma 4.7 to obtain an equivalent formula $\psi_i(\bar{x}_i) = [\mathrm{sGFP}\, S_i]_1\,(\bar{x}_i) \in \mathrm{sGFP}^{+}_k$ for every $i = 1, \dots, m$ where S_i consists only of first-order formulae. Now Theorem 4.15 for $k \geq 2$ resp. Theorem 4.17 for $k = 1$ are applicable. Thus, there are formulae $\gamma_1(\bar{x}_1), \dots, \gamma_m(\bar{x}_m) \in \mathrm{FO}(\subseteq)[k]$ with the property

$$\mathfrak{A} \models_X \gamma_i(\bar{x}_i) \iff \mathfrak{A} \models_s \psi_i(\bar{x}_i) \text{ for every } s \in X$$

for every structure $\mathfrak{A}$ and every team X over $\mathfrak{A}$ with $\bar{x}_i \subseteq \mathrm{dom}(X)$. Because of the equivalence of φ_i and ψ_i, we this leads to

$$\mathfrak{A} \models_X \gamma_i(\bar{x}_i) \iff \mathfrak{A} \models_s \varphi_i(\bar{x}_i) \text{ for every } s \in X. \tag{4.14}$$

Now, let $\gamma(\bar{x})$ be the formula that results from $\varphi(\bar{x})$ by replacing its outermost fixed-points formulae $\varphi_i(\bar{x}_i)$ by $\gamma_i(\bar{x}_i) \in \mathrm{FO}(\subseteq)[k]$. Then we have $\gamma(\bar{x}) \in \mathrm{FO}(\subseteq)[k]$ and the desired property (4.13) is inherited from (4.14). □

4.2.2 Myopic GFP-Sentences

Corollary 4.18 allows us to translate a $\mathrm{GFP}^+[k]$-formula into a $\mathrm{FO}(\subseteq)[k]$-formula, but how does that help us to translate a $\mathrm{GFP}^+[k]$-*sentence* $\varphi(X)$ using a team predicate X into inclusion logic? We will answer this question in this section for sentences that are myopic in X.

Definition 4.19. A $\mathrm{GFP}^+[k]$-sentence is called *myopic* in X, if it is of the shape $\forall\bar{x}(X\bar{x} \longrightarrow \psi(X,\bar{x}))$ where X occurs only positively in $\psi(X,\bar{x})$.

Myopicity implies union closure, which is crucial for our translation into inclusion logic. As an example for the significance of myopicity, consider the non-myopic first-order sentence $\forall x(Xx \longrightarrow Px) \vee \forall x(Xx \longrightarrow \neg Px)$ which defines a property that is obviously not closed under unions and, hence, it *cannot* be expressed in inclusion logic.

Now consider a myopic sentence $\forall\bar{x}(X\bar{x} \longrightarrow \psi(X,\bar{x})) \in \mathrm{GFP}^+[k]$. Using Corollary 4.18 for $\psi(X,\bar{x})$, we obtain a formula $\gamma(X,\bar{x}) \in \mathrm{FO}(\subseteq)[k]$ such that for every structure $\mathfrak{A}$ and every team X over $\mathfrak{A}$ with $\mathrm{dom}(X) = \{\bar{x}\}$ holds

$$(\mathfrak{A}, X(\bar{x})) \vDash_X \gamma(X,\bar{x}) \iff (\mathfrak{A}, X(\bar{x})) \vDash_s \psi(X,\bar{x}) \text{ for every } s \in X. \tag{4.15}$$

Since the signature of $\psi(X,\bar{x})$ contains a relation symbol X, we apply Corollary 4.18 to the structure $(\mathfrak{A}, X(\bar{x}))$ and the team X. This is legitimate, because X, being a team over $\mathfrak{A}$, is also a team over the structure $(\mathfrak{A}, X(\bar{x}))$.

In (4.15), the team X is used simultaneously as a relation $X(\bar{x})$ interpreting the relation symbol X, which occurs in γ and ψ, while it is also used as a team X to evaluate γ and to provide assignments suitable for ψ. Since the construction of γ (before Theorem 4.15 resp. Theorem 4.17) does not touch the X-atoms, because they are viewed as first-order literals during the construction of γ, X occurs only positively in $\gamma(X,\bar{x})$.

To get rid of the simultaneous uses of X, we use a fresh tuple $\bar{x}'$ of variables with $|\bar{x}'| = |\bar{x}|$ and define

$$\zeta(\bar{x}) := \exists\bar{x}'(\bar{x}' \subseteq \bar{x} \wedge \gamma'(\bar{x}',\bar{x}))$$

where $\gamma'(\bar{x}',\bar{x})$ results from $\gamma(X,\bar{x})$ by replacing every atom of the form $X\bar{v}$ by $\bar{v} \subseteq \bar{x}'$. Since $\gamma(X,\bar{x}) \in \mathrm{FO}(\subseteq)[k]$, we have $\zeta(\bar{x}) \in \mathrm{FO}(\subseteq)[k']$ where $k' = \max\{k, \mathrm{ar}(X)\}$.

Claim 4.20. For every structure $\mathfrak{A}$ and every team X over $\mathfrak{A}$ with $\mathrm{dom}(X) = \{\bar{x}\}$ holds $(\mathfrak{A}, X(\bar{x})) \vDash \forall\bar{x}(X\bar{x} \longrightarrow \psi(X,\bar{x})) \iff \mathfrak{A} \vDash_X \zeta(\bar{x})$.

Proof. Clearly, $(\mathfrak{A}, X(\bar{x})) \vDash \forall\bar{x}(X\bar{x} \longrightarrow \psi(X,\bar{x}))$ is equivalent to $(\mathfrak{A}, X(\bar{x})) \vDash_s \psi(X,\bar{x})$ for every $s \in X$. By (4.15), the latter is equivalent to $(\mathfrak{A}, X(\bar{x})) \vDash_X \gamma(X,\bar{x})$. Thus, it suffices to prove that $(\mathfrak{A}, X(\bar{x})) \vDash_X \gamma(X,\bar{x}) \iff \mathfrak{A} \vDash_X \zeta(\bar{x})$. Since both sides are true for $X = \emptyset$ due to the empty team property, we assume $X \neq \emptyset$ for the remainder of this proof. We prove both directions separately.

"$\Longrightarrow$": First, assume that $(\mathfrak{A}, X(\bar{x})) \vDash_X \gamma(X,\bar{x})$. So, there is a witness λ for this assumption. For every subformula ϑ of γ let ϑ' be the corresponding subformula of γ', which results by replacing every $X\bar{v}$ by $\bar{v} \subseteq \bar{x}'$. Let $Y := X[\bar{x}' \mapsto X(\bar{x})]$. Since $X \neq \emptyset$, we have $X(\bar{x}) \neq \emptyset$. Therefore, proving $\mathfrak{A} \models_Y \bar{x}' \subseteq \bar{x} \wedge \gamma'(\bar{x}', \bar{x})$ suffices to prove that $\mathfrak{A} \models_X \zeta(\bar{x})$. Clearly, $\mathfrak{A} \models_Y \bar{x}' \subseteq \bar{x}$ is true.

We are still obligated to prove $\mathfrak{A} \models_Y \gamma'(\bar{x}', \bar{x})$. Towards this end, we define $\lambda'(\vartheta') := \lambda(\vartheta)[\bar{x}' \mapsto X(\bar{x})]$ and show that this is a witness for $\mathfrak{A} \models_Y \gamma'(\bar{x}', \bar{x})$. It is immediately clear that $\lambda'(\gamma') = Y$ and that λ' satisfies the required conditions for all subformulae ϑ' *not* of the form $\bar{v} \subseteq \bar{x}'$. For subformulae of the form $\vartheta' = \bar{v} \subseteq \bar{x}'$, we prove $\mathfrak{A} \models_{\lambda'(\vartheta')} \vartheta'$ as follows. If $\lambda'(\vartheta') = \emptyset$, then we can just use the empty team property. Otherwise we have $\lambda'(\vartheta') \neq \emptyset$, which, due to $\lambda'(\vartheta') = \lambda(\vartheta)[\bar{x}' \mapsto X(\bar{x})]$ and $X \neq \emptyset$, leads to $\lambda(\vartheta) \neq \emptyset$ and, moreover, $(\lambda'(\vartheta'))(\bar{x}') = X(\bar{x})$. Because λ is a witness for $(\mathfrak{A}, X(\bar{x})) \models_X \gamma(X, \bar{x})$, we have $(\mathfrak{A}, X(\bar{x})) \models_{\lambda(\vartheta)} \vartheta = X\bar{v}$, which implies that $(\lambda'(\vartheta'))(\bar{v}) = (\lambda(\vartheta))(\bar{v}) \subseteq X(\bar{x}) = (\lambda'(\vartheta'))(\bar{x}')$. Thus, $\mathfrak{A} \models_{\lambda'(\vartheta')} \bar{v} \subseteq \bar{x}' = \vartheta'$. As a result, λ' is indeed a witness for $\mathfrak{A} \models_Y \gamma'(\bar{x}', \bar{x})$. This concludes the proof of $\mathfrak{A} \models_X \zeta(\bar{x})$.

"$\Longleftarrow$": For the converse direction, suppose that $\mathfrak{A} \models_X \zeta(\bar{x})$. Then there exists a team $Y = X[\bar{x}' \mapsto F]$ for some $F : X \longrightarrow \mathcal{P}^+(A^{|\bar{x}'|})$ such that $\mathfrak{A} \models_Y \bar{x}' \subseteq \bar{x} \wedge \gamma'(\bar{x}', \bar{x})$. Thus, $Y(\bar{x}') \subseteq Y(\bar{x})$ and there is a witness λ' for $\mathfrak{A} \models_Y \gamma'(\bar{x}', \bar{x})$ with $\lambda'(\gamma') = Y$.

Towards proving $(\mathfrak{A}, X(\bar{x})) \models_X \gamma(X, \bar{x})$, let $\lambda(\vartheta) := \lambda'(\vartheta')$ for every subformula $\vartheta \in \mathrm{subf}(\gamma)$ where ϑ' is the corresponding subformula of γ', which results by replacing every $X\bar{v}$ by $\bar{v} \subseteq \bar{x}'$.

We show that λ is indeed a witness for $(\mathfrak{A}, X(\bar{x})) \models_X \gamma(X, \bar{x})$. First, we can observe that $\lambda(\gamma)\restriction_{\mathrm{free}(\gamma)} = \lambda'(\gamma')\restriction_{\mathrm{free}(\gamma)} = X[\bar{x}' \mapsto F]\restriction_{\mathrm{free}(\gamma)} = X$. Again, λ inherits the satisfaction of all conditions for formulae ϑ that are *not* of the form $X\bar{v}$. Now consider a subformula ϑ of γ with $\vartheta = X\bar{v}$. Then $\vartheta' = \bar{v} \subseteq \bar{x}'$ is the corresponding subformula of γ' and, because of $\mathfrak{A} \models_{\lambda'(\vartheta')} \vartheta'$, it follows that $(\lambda'(\vartheta'))(\bar{v}) \subseteq (\lambda'(\vartheta'))(\bar{x}')$. Since we can only "loose" values for $\bar{x}'$ at disjunctions, it is the case that $(\lambda'(\vartheta'))(\bar{x}') \subseteq (\lambda'(\gamma'))(\bar{x}') = Y(\bar{x}') \subseteq Y(\bar{x}) = X(\bar{x})$. Thus, $(\lambda(\vartheta))(\bar{v}) = (\lambda'(\vartheta'))(\bar{v}) \subseteq X(\bar{x})$ which proves that $(\mathfrak{A}, X(\bar{x})) \models_{\lambda(\vartheta)} X\bar{v}$. This concludes the proof of λ being a witness for $(\mathfrak{A}, X(\bar{x})) \models_X \gamma(X, \bar{x})$. □

This shows that a given myopic GFP$^+[k]$-sentence can be translated into inclusion logic where the arities of occurring inclusion atoms can be bounded by $\max\{k, \mathrm{ar}(X)\}$.

Theorem 4.21. *Let $\varphi(X) = \forall \bar{x}(X\bar{x} \longrightarrow \psi(X, \bar{x}))$ be myopic* GFP$^+[k]$*-sentence. Then $\varphi(X)$ is equivalent to some $\zeta(\bar{x}) \in \mathrm{FO}(\subseteq)[k']$ where $k' = \max\{k, \mathrm{ar}(X)\}$.*

It is no surprise that we need k' instead of k, because positive occurrences of X in a myopic sentence correspond to an inclusion atom. However, this, despite being intuitive, requires a more rigorous proof, which can be found in the next section.

Optimality of k'

We will now prove that k' cannot be chosen smaller in general. Towards this end, we use the result of Hannula [Han15] that for every $\ell \geq 1$ the TC-sentence[5]

$$\gamma_\ell := \neg[\mathrm{TC}_{\bar{x},\bar{y}}\, \mathrm{EDGE}_\ell(\bar{x}, \bar{y})](\bar{b}, \bar{c})$$

[5] TC is first-order logic extended by an operator for transitive closures.

over the signature $\tau_\ell := \{E, \bar{b}, \bar{c}\}$ is *not* expressible in FO(⊆)[$\ell - 1$]. Here $\bar{b}$ and $\bar{c}$ are ℓ-tuples of constant symbols while $\mathrm{EDGE}_\ell(\bar{x}, \bar{y})$ is a first-order formula expressing that $\bar{x}, \bar{y}$ form a 2ℓ-clique. The sentence γ_ℓ expresses the non-existence of an EDGE_ℓ-path from $\bar{b}$ to $\bar{c}$. Hannula also provided the following FO(⊆)[ℓ]-sentence

$$\zeta_\ell := \exists \bar{z} \zeta'_\ell(\bar{z}) \text{ where } \zeta'_\ell(\bar{z}) := \bar{b} \subseteq \bar{z} \wedge \bar{z} \neq \bar{c} \wedge \forall \bar{w}(\mathrm{nnf}(\neg \mathrm{EDGE}_\ell(\bar{z}, \bar{w})) \vee \bar{w} \subseteq \bar{z}),^{6}$$

which is equivalent to γ_ℓ, i.e. $\mathfrak{A} \models \gamma_\ell \iff \mathfrak{A} \models \zeta_\ell$ for τ_ℓ-structures $\mathfrak{A}$ [Han15, Theorem 7].

Theorem 4.22 ([Han15]). *The sentence $\zeta_\ell \in$ FO(⊆)[ℓ] is* not *equivalent to any* FO(⊆)[ℓ − 1]*-sentence.*

Now consider the myopic first-order sentence

$$\varphi_\ell(X) := \forall \bar{z}(X\bar{z} \longrightarrow (X\bar{b} \wedge \bar{z} \neq \bar{c} \wedge \forall \bar{w}(\mathrm{EDGE}_\ell(\bar{z}, \bar{w}) \longrightarrow X\bar{w}))).$$

Proposition 4.23. $(\mathfrak{A}, X(\bar{z})) \models \varphi_\ell(X) \iff \mathfrak{A} \models_X \zeta'_\ell(\bar{z})$ *for τ_ℓ-structures $\mathfrak{A}$ and teams X with $\bar{z} \subseteq \mathrm{dom}(X)$.*

Proof. We prove both directions separately.

"$\Longrightarrow$": First, assume that $(\mathfrak{A}, X(\bar{z})) \models \varphi_\ell(X)$. If $X = \emptyset$, then $\mathfrak{A} \models_X \zeta'_\ell(\bar{z})$ follows from the empty team property. Otherwise, we have $X(\bar{z}) \neq \emptyset$ and, due to $(\mathfrak{A}, X(\bar{z})) \models \varphi_\ell(X) = \forall \bar{z}(X\bar{z} \longrightarrow (X\bar{b} \wedge \bar{z} \neq \bar{c} \wedge \dots))$, it follows that $\mathfrak{A} \models_X \bar{b} \subseteq \bar{z} \wedge \bar{z} \neq \bar{c}$. Because of $(\mathfrak{A}, X(\bar{z})) \models \varphi_\ell = \forall \bar{z}(X\bar{z} \longrightarrow \dots \wedge \forall \bar{w}(\mathrm{EDGE}_\ell(\bar{z}, \bar{w}) \longrightarrow X\bar{w}))$, it is the case that $X(\bar{z})$ is closed under $[\![\mathrm{EDGE}_\ell]\!]^{\mathfrak{A}}$, which means that $\bar{a} \in X(\bar{z})$ and $\mathfrak{A} \models \mathrm{EDGE}_\ell(\bar{a}, \bar{d})$ already entails $\bar{d} \in X(\bar{z})$. We can use this to prove that $\mathfrak{A} \models_X \forall \bar{w}(\mathrm{nnf}(\neg \mathrm{EDGE}_\ell(\bar{z}, \bar{w})) \vee \bar{w} \subseteq \bar{z})$. Towards this end, let $Y := X[\bar{w} \mapsto A^\ell]$ and $Z := \{s \in Y : s(\bar{w}) \in X(\bar{z})\}$. We will prove that $\mathfrak{A} \models_{Y \setminus Z} \mathrm{nnf}(\neg \mathrm{EDGE}_\ell(\bar{z}, \bar{w}))$ and $\mathfrak{A} \models_Z \bar{w} \subseteq \bar{z}$.

Clearly $Z(\bar{z}) \subseteq X(\bar{z})$ and, since for every $s \in X$ holds $s[\bar{w} \mapsto s(\bar{z})] \in Z$ and $s(\bar{z}) = s[\bar{w} \mapsto s(\bar{z})](\bar{z})$, we also have $X(\bar{z}) \subseteq Z(\bar{z})$. So $X(\bar{z}) = Z(\bar{z})$. By definition of Z, holds $Z(\bar{w}) \subseteq X(\bar{z}) = Z(\bar{z})$ and, thus, it is indeed the case that $\mathfrak{A} \models_Z \bar{w} \subseteq \bar{z}$.

We still need to prove that $\mathfrak{A} \models_{Y \setminus Z} \mathrm{nnf}(\neg \mathrm{EDGE}_\ell(\bar{z}, \bar{w}))$. Towards this end, let $s \in Y \setminus Z$. So, we have that $s(\bar{w}) \notin X(\bar{z})$ and, hence, it must be the case that $\mathfrak{A} \not\models_s \mathrm{EDGE}_\ell(\bar{z}, \bar{w})$, because $s(\bar{z}) \in X(\bar{z})$ and $X(\bar{z})$ is closed under $[\![\mathrm{EDGE}_\ell]\!]^{\mathfrak{A}}$. This concludes the proof of $\mathfrak{A} \models_{Y \setminus Z} \mathrm{nnf}(\neg \mathrm{EDGE}_\ell(\bar{z}, \bar{w}))$ and of $\mathfrak{A} \models_X \zeta'_\ell(\bar{z})$.

"$\Longleftarrow$": For the converse direction, let $\mathfrak{A} \models_X \zeta'_\ell(\bar{z})$. Towards proving $(\mathfrak{A}, X(\bar{z})) \models \varphi_\ell(X) = \forall \bar{z}(X\bar{z} \longrightarrow \dots)$, let $\bar{a} \in X(z)$ be chosen arbitrarily. So, there exists some $s \in X$ with $s(\bar{z}) = \bar{a}$. We need to prove that $(\mathfrak{A}, X(\bar{z})) \models X\bar{b} \wedge \bar{a} \neq \bar{c} \wedge \forall \bar{w}(\mathrm{EDGE}_\ell(\bar{a}, \bar{w}) \longrightarrow X\bar{w})$ holds.

[6] Technically, the subformula $\bar{b} \subseteq \bar{z}$ is a shorthand for $\exists \bar{v}(\bar{v} = \bar{b} \wedge \bar{v} \subseteq \bar{z})$, because we only allow tuples of variables to occur in inclusion atoms. Furthermore, we use $\mathrm{nnf}(\neg \mathrm{EDGE}_\ell(\bar{z}, \bar{w}))$ to denote the negation normal form of $\neg \mathrm{EDGE}_\ell(\bar{z}, \bar{w})$, because negation signs are only allowed in first-order literals.

Since $\mathfrak{A} \models_X \zeta'_\ell(\bar{z}) = \bar{b} \subseteq \bar{z} \wedge \bar{z} \neq \bar{c} \wedge \ldots$ and $s \in X$, we clearly have that $(\mathfrak{A}, X(\bar{z})) \models_s X\bar{b} \wedge \bar{z} \neq \bar{c}$, which leads to $(\mathfrak{A}, X(\bar{z})) \models X\bar{b} \wedge \bar{a} \neq \bar{c}$.

We still need to prove that $(\mathfrak{A}, X(\bar{z})) \models \forall\bar{w}(\mathrm{EDGE}_\ell(\bar{a}, \bar{w}) \longrightarrow X\bar{w})$. Towards this end, it suffices to prove that for every $\bar{d} \in A^\ell$ with $\mathfrak{A} \models \mathrm{EDGE}_\ell(\bar{a}, \bar{d})$ holds $\bar{d} \in X(\bar{z})$. Let such a $\bar{d}$ be chosen arbitrarily. Because of $\mathfrak{A} \models_X \forall\bar{w}(\mathrm{nnf}(\neg\mathrm{EDGE}_\ell(\bar{z}, \bar{w})) \vee \bar{w} \subseteq \bar{z})$, we have $\mathfrak{A} \models_Y \mathrm{nnf}(\neg\mathrm{EDGE}_\ell(\bar{z}, \bar{w})) \vee \bar{w} \subseteq \bar{z}$ for $Y := X[\bar{w} \mapsto A^\ell]$ and, consequently, there are teams Y_0, Y_1 with $Y = Y_0 \cup Y_1$ such that $\mathfrak{A} \models_{Y_0} \mathrm{nnf}(\neg\mathrm{EDGE}_\ell(\bar{z}, \bar{w}))$ and $\mathfrak{A} \models_{Y_1} \bar{w} \subseteq \bar{z}$. Clearly, $t := s[\bar{w} \mapsto \bar{d}] \in Y$ but $t \notin Y_0$, because $t(\bar{z}, \bar{w}) = (\bar{a}, \bar{d}) \in [\![\mathrm{EDGE}_\ell]\!]^{\mathfrak{A}}$. Therefore, $t \in Y_1$ and, due to $\mathfrak{A} \models_{Y_1} \bar{w} \subseteq \bar{z}$, it must be the case that $\bar{d} = t(\bar{w}) \in Y_1(\bar{z}) \subseteq X(\bar{z})$ as desired. This concludes the proof of $(\mathfrak{A}, X(\bar{z})) \models \varphi_\ell(X)$. □

Using Theorem 4.21, we can translate the first-order sentence $\varphi_\ell(X)$ into an equivalent inclusion logic formula $\zeta''_\ell(\bar{z}) \in \mathrm{FO}(\subseteq)[k']$ where $k' = \mathrm{ar}(X) = \ell$. We can now prove that $\varphi_\ell(X)$ is not equivalent to any $\mathrm{FO}(\subseteq)[\ell - 1]$-formula, which proves that k' has in fact been chosen optimally in Theorem 4.21.

Corollary 4.24. *Let $\ell \geq 2$. Then $\varphi_\ell(X)$ is* not *equivalent to some* $\mathrm{FO}(\subseteq)[\ell - 1]$*-formula.*

Proof. Towards a contradiction, assume that $\varphi_\ell(X)$ would be equivalent to some $\vartheta(\bar{z}) \in \mathrm{FO}(\subseteq)[\ell - 1]$, i.e.

$$(\mathfrak{A}, X(\bar{z})) \models \varphi_\ell(X) \iff \mathfrak{A} \models_X \vartheta(\bar{z})$$

for all τ_k-structures $\mathfrak{A}$ and teams X. Because of Proposition 4.23, it would follow that

$$\mathfrak{A} \models_X \zeta'_\ell(\bar{z}) \iff \mathfrak{A} \models_X \vartheta(\bar{z})$$

for all $\mathfrak{A}, X$. Therefore, $\zeta_\ell = \exists\bar{z}\zeta'_\ell(\bar{z})$ is equivalent to $\exists\bar{z}\vartheta(\bar{z}) \in \mathrm{FO}(\subseteq)[\ell - 1]$ in contradiction to Hanulla's result (Theorem 4.22). □

4.3 From Inclusion Logic to Bounded GFP

Now we provide the translation in the opposide direction. This relies on the following construction that has been found by Rönnholm and was utilized in the proof of [Rön18, Theorem 4.2].

Theorem 4.25 (Rönnholm's Construction)**.** *Let $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq)[k]$. Then there exist first-order formulae $\varphi^\star(\bar{R}, \bar{x}), \varphi^{(1)}(\bar{R}, \bar{z}_1, \bar{x}), \ldots, \varphi^{(n)}(\bar{R}, \bar{z}_n, \bar{x})$ where $\bar{R} = (R_1, \ldots, R_n)$ is a tuple of new relation symbols of arity at most k, which occur only positively in $\varphi^\star, \varphi^{(1)}, \ldots, \varphi^{(n)}$, such that the following are equivalent:*

- $\mathfrak{A} \models_X \varphi$
- *There are relations $\bar{R} = (R_1, \ldots, R_n)$ over $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{R}) \models_X \varphi^\star$ and for every $i \in \{1, \ldots, n\}$, $\bar{a} \in R_i$ there exists some $s \in X$ such that $(\mathfrak{A}, \bar{R}) \models_{s[\bar{z}_i \mapsto \bar{a}]} \varphi^{(i)}$.*

For more details on Rönnholm's construction we refer to [Rön18] and to Section 5.2.2 where we adapt it for dependency concepts up to equivalence.

Let φ and $\varphi^\star, \varphi^{(1)}, \dots, \varphi^{(n)}$ be as in Theorem 4.25. For every $i \in \{1, \dots, n\}$ let

$$\eta_i(X, \bar{R}, \bar{z}_i) := \exists \bar{x}(X\bar{x} \wedge \varphi^{(i)}(\bar{R}, \bar{z}_i, \bar{x})).$$

Furthermore, let

$$\eta(X, \bar{R}) := \forall \bar{x}(X\bar{x} \longrightarrow \varphi^\star(\bar{R}, \bar{x})).$$

Because $\varphi^\star, \varphi^{(1)}, \dots, \varphi^{(n)}$ are first-order formulas, we obtain the following corollary.

Corollary 4.26. *Let φ and $\eta, \eta_1, \dots, \eta_n$ be as above. The following are equivalent:*

- $\mathfrak{A} \vDash_X \varphi$
- *There are relations $\bar{R} = (R_1, \dots, R_n)$ over $\mathfrak{A}$ such that $(\mathfrak{A}, X(\bar{x}), \bar{R}) \vDash \eta$ and, for every $i = 1, \dots, n$, $(\mathfrak{A}, X(\bar{x}), \bar{R}) \vDash \forall \bar{z}_i(R_i\bar{z}_i \longrightarrow \eta_i(X, \bar{R}, \bar{z}_i))$.*

Furthermore, the relation symbols $R_1, \dots, R_n$ occur only positively in $\eta, \eta_1, \dots, \eta_n$.

Theorem 4.27. *For every $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq)[k]$ there exists a myopic $\mathrm{GFP}^+[k]$-sentence $\psi(X)$ such that for every suitable structure $\mathfrak{A}$ and team X holds*

$$\mathfrak{A} \vDash_X \varphi(\bar{x}) \iff (\mathfrak{A}, X(\bar{x})) \vDash \psi(X). \tag{4.16}$$

Proof. Because of Lemma 4.4, it suffices to translate $\varphi(\bar{x})$ into an equivalent sGFP^+_k-sentence $\psi(X) := \forall \bar{x}(X\bar{x} \longrightarrow \psi'(X, \bar{x}))$ where X occurs only positively in the sGFP^+_k-formula ψ'. Towards this end, let $\psi'(X, \bar{x})$ be the formula that results from $\varphi^\star(\bar{R}, \bar{x})$ by replacing every occurrence of $R_i\bar{v}$ by $\gamma_i(\bar{v}) := [\mathrm{sGFP}\ S]_i\,(\bar{v})$ where

$$S := \begin{cases} R_1\bar{z}_1 \ :\ \eta_1(X, \bar{R}, \bar{z}_1) \\ \quad\vdots \\ R_n\bar{z}_n \ :\ \eta_n(X, \bar{R}, \bar{z}_n). \end{cases}$$

Please notice that X occurs only positively in $\eta_1, \dots, \eta_n$ and the construction used in the proof of Lemma 4.4 does not introduce negation signs above X. Let Γ be the simultaneous update operator defined w.r.t. S and $(\mathfrak{A}, X(\bar{x}))$. We prove the two directions of (4.16) separately.

"$\Longrightarrow$": First, assume that $\mathfrak{A} \vDash_X \varphi(\bar{x})$. By Corollary 4.26, it follows that there are some relations $\bar{R} = (R_1, \dots, R_n)$ such that $(\mathfrak{A}, X(\bar{x}), \bar{R}) \vDash \eta$ and $(\mathfrak{A}, X(\bar{x}), \bar{R}) \vDash \forall \bar{z}_i(R_i\bar{z}_i \longrightarrow \eta_i)$ for $i = 1, \dots, n$. This implies that $R_i \subseteq [\![\eta_i(X, \bar{R}, \bar{z}_i)]\!]^{(\mathfrak{A}, X(\bar{x}))} = \Gamma_i(\bar{R})$. By Theorem 4.2, we obtain $\bar{R} \subseteq \bigcup\{\bar{S} : \bar{S} \subseteq \Gamma(\bar{S})\} = \mathrm{GFP}(\Gamma)$. Since $(\mathfrak{A}, X(\bar{x}), \bar{R}) \vDash \eta$ and the relation symbols $R_1, \dots, R_n$ occur only positively in $\eta(X, \bar{R}) = \forall \bar{x}(X\bar{x} \longrightarrow \varphi^\star(\bar{R}, \bar{x}))$, we can conclude that $(\mathfrak{A}, X(\bar{x}), \mathrm{GFP}(\Gamma)) \vDash \forall \bar{x}(X\bar{x} \longrightarrow \varphi^\star(\mathrm{GFP}(\Gamma), \bar{x}))$, which implies that $(\mathfrak{A}, X(\bar{x})) \vDash \psi(X)$.

"$\Longleftarrow$": Now, suppose that $(\mathfrak{A}, X(\bar{x})) \vDash \psi(X) = \forall \bar{x}(X\bar{x} \longrightarrow \psi'(X, \bar{x}))$. Let $\bar{G} = (G_1, \dots, G_n)$ be the greatest fixed-point of Γ. By construction of ψ', it follows that

$(\mathfrak{A}, X(\bar{x}), \bar{R} \mapsto \bar{G}) \models \eta = \forall \bar{x}(X\bar{x} \longrightarrow \varphi^\star(\bar{R}, \bar{x}))$. Since $\bar{G}$ is the greatest fixed-point of Γ, we can deduce that $G_i = \Gamma_i(G_1, \dots, G_n) = [\![\eta_i(X, \bar{G}, \bar{z}_i)]\!]^{(\mathfrak{A}, X(\bar{x}))}$. This implies that $(\mathfrak{A}, X(\bar{x}), \bar{G}) \models \forall \bar{z}_i(G_i\bar{z}_i \longrightarrow \eta_i(X, \bar{G}, \bar{z}_i))$. By Corollary 4.26, we obtain $\mathfrak{A} \models_X \varphi(\bar{x})$ as desired. □

This result completes our analysis of the connections between the arity fragments of bounded inclusion logic and of bounded greatest-fixed point logic.

5 Dependency Concepts up to Equivalences

In this chapter we explore logics that are based on weaker variants of dependencies. We consider atomic dependence statements that do not distinguish elements up to equality, but only up to a coarser equivalence relation. This is motivated by the possible situation that elements, such as for instance states in a computation or values obtained in experiments, are subject to observational indistinguishabilities, which we model here via an equivalence relation $\approx$ on the set of possible values. This whole chapter is closely based on the paper [GH18] which is joint work with my supervisor Erich Grädel, who gave me this topic to work on and introduced me to his former conjectures about the possible connections of these new logics to (possible fragments of) Σ_1^1.

An example for the new atoms we will investigate is the new dependence atom $\text{dep}_\approx(\bar{x}, y)$ that says: whenever the values of $\bar{x}$ are indistinguishable for certain assignments in a team, then so are the values of y. Similarly an exclusion statement between x and y, up to an equivalence relation $\approx$, says that no value for x in the team is equivalent to a value of y, and an inclusion statement like $x \subseteq_\approx y$ means that every value for x is equivalent to some value for y. The most powerful of such notions, independence of x and y up to equivalence, means that additional information about the equivalence class of the value of one variable does not help to learn anything new about the equivalence class of the value of the other, or to put it differently, whenever a value a for x and a value b for y occur in the team, then there is an assignment in the team whose value for x is equivalent to a and whose value for y is equivalent to b. More general definitions of these dependencies, extended to tuples of variables, will be given in the next section. The main goal of this chapter is to understand the expressive power of the logics with dependencies up to equivalence.

The question arises how the results known so far (see Section 2.2) change when the standard dependency notions are replaced by dependencies up to equivalence. There is a natural conjecture: *one has to restrict existential second-order quantification to relations that are closed under the given equivalence relation, i.e. to relations that can be written as unions of equivalence classes* (where equivalence is extended to tuples component-wise). We denote the resulting variant of existential second-order logic by $\Sigma_1^1(\approx)$.

Notice however, that to decide this conjecture is far from being trivial, because the restriction of the standard translation to quantification over $\approx$-closed relations does not work as a proof. Even for simple disjunctions, the existential second-order expression given above describing the split of the team will not work anymore once we restrict quantification to $\approx$-closed relations because we cannot assume that the relevant subteams are ≈-closed. Here is a simple example, not even involving any dependencies: consider the formula $x = y \lor x \neq y$ which is trivially satisfied by any team X, by the split $X = Y \cup Z$ where Y contains the assignments s which $s(x) = s(y)$ and $Z = X \setminus Y$ (and this is the only split that works). However if there are elements $a \neq b$ with $a \approx b$ then in general neither Y nor Z are ≈-closed, even if X is.

Nevertheless we shall prove that the conjecture is true, and that we can characterize the expressive power of dependence logics up to equivalence by appropriate fragments of $\Sigma_1^1(\approx)$. This is based on a much more sophisticated translation from logics with team semantics into existential second-order logic that adapts ideas from [Rön18]. We shall also present a fragment of GFP^+ that has the same expressive power as inclusion logic up to equivalence.

Our next question is then how the expressive power of $\Sigma_1^1(\approx)$, and hence logics of dependence up to equivalence, compare to first-order logic and to full Σ_1^1. Of course this depends on the properties of the underlying equivalence relation, notably on the number and sizes of its equivalence classes.

1. On any class of structures on which ≈ has only a bounded number of equivalence classes, $\Sigma_1^1(\approx)$, and hence all logics with dependencies up to equivalence as well, collapse to FO.
2. On any class of structures in which all equivalence classes have bounded size, and only a bounded number of classes have more than one element, $\Sigma_1^1(\approx) \equiv \Sigma_1^1$.
3. In general, and in particular on the classes of structures where all equivalence classes have size at most k (for $k > 1$), or that have only a bounded number of equivalence classes of size >1, the expressive power of $\Sigma_1^1(\approx)$, and all the considered logics of dependence up to equivalence, are strictly between FO and Σ_1^1.

To prove this we shall use appropriate variants of Ehrenfeucht-Fraïssé games for these logics.

The sections below are taken from [GH18] with minor changes. However, the translation from $\text{FO}(\subseteq_\approx, |_\approx)$ to $\Sigma_1^1(\approx)$ in Section 5.2.2 has been improved in comparison to [GH18], because it now also works for certain formulae of $\text{FO}(\subseteq_\approx, |_\approx)$ teams rather than just for sentences.

5.1 Logics with Concepts up to Equivalences

Let τ be a signature containing a binary relation symbol ≈ and let $(\tau, \approx)$ denote the class of τ-structures $\mathfrak{A}$ in which ≈ is interpreted by an equivalence relation on the universe A of $\mathfrak{A}$. For every $\mathfrak{A} \in (\tau, \approx)$ and every $\bar{a}, \bar{b} \in A^n$ we write $\bar{a} \approx \bar{b}$, if $a_i \approx b_i$ for every $i \in \{1, \dots, n\}$. Given two relations $R, S \subseteq A^k$ of the same arity we write $R \subseteq_\approx S$ if for every $\bar{a} \in R$, there exists some $\bar{b} \in S$ with $\bar{a} \approx \bar{b}$. We further write $R \approx S$ if $R \subseteq_\approx S$ and $S \subseteq_\approx R$. Furthermore, we define the ≈-closure of R as $R_\approx := \{\bar{a} : \bar{a} \approx \bar{b} \text{ for some } \bar{b} \in R\}$

and say that R is $\approx$-closed if, and only if, $R = R_{\approx}$. The semantics of (in)dependence, inclusion and exclusion atoms up to $\approx$ is given as follows:

Definition 5.1. Let X be a team over $\mathfrak{A}$. Then we define

$$\begin{aligned}
\mathfrak{A} \models_X \mathrm{dep}_{\approx}(\bar{x}, y) \quad &:\Longleftrightarrow \quad \text{for all } s, s' \in X, \text{ if } s(\bar{x}) \approx s'(\bar{x}) \text{ then also } s(y) \approx s'(y),\\
\mathfrak{A} \models_X \bar{x} \perp_{\approx} \bar{y} \quad &:\Longleftrightarrow \quad \text{for all } s, s' \in X \text{ there exists some } s'' \in X \text{ such that}\\
& \qquad\quad s''(\bar{x}) \approx s(\bar{x}) \text{ and } s''(\bar{y}) \approx s'(\bar{y}),\\
\mathfrak{A} \models_X \bar{x} \subseteq_{\approx} \bar{y} \quad &:\Longleftrightarrow \quad X(\bar{x}) := \{s(\bar{x}) : s \in X\} \subseteq_{\approx} X(\bar{y}),\\
\mathfrak{A} \models_X \bar{x} \mid_{\approx} \bar{y} \quad &:\Longleftrightarrow \quad s(\bar{x}) \not\approx s'(\bar{y}) \text{ for all } s, s' \in X.
\end{aligned}$$

For $\Omega_{\approx} \subseteq \{\mathrm{dep}_{\approx}, \perp_{\approx}, \subseteq_{\approx}, |_{\approx}\}$ we denote by $\mathrm{FO}(\Omega_{\approx})$ the set of all first-order formulas in negation normal form where we additionally allow positive occurrences of $\Omega_{\approx}$-atoms. The semantics of first-order literals and of the logical operators are as in Definition 2.1. Many standard results concerning the closure properties and relationships between different logics of dependence and independence (see e.g. [Gal12]) carry over to this new setting with equivalences, by easy and straightforward adaptations of proofs (which are therefore omitted here). In particular, this includes the following observations:

- For all formulae in these logics the *locality principle* holds: $\mathfrak{A} \models_X \varphi$ if, and only if, $\mathfrak{A} \models_{X\restriction_{\mathrm{free}(\varphi)}} \varphi$ (where $X\restriction_{\mathrm{free}(\varphi)} := \{s\restriction_{\mathrm{free}(\varphi)} : s \in X\}$ is the restriction of X to the free variables of φ).
- The logics $\mathrm{FO}(\mathrm{dep}_{\approx})$ and $\mathrm{FO}(|_{\approx})$ are equivalent and downwards closed.
- The logic $\mathrm{FO}(\subseteq_{\approx})$ is closed under unions of teams, and incomparable with $\mathrm{FO}(\mathrm{dep}_{\approx})$ and $\mathrm{FO}(|_{\approx})$.
- Independence logic with equivalences, $\mathrm{FO}(\perp_{\approx})$, has the same expressive power as inclusion-exclusion logic with equivalences, $\mathrm{FO}(\subseteq_{\approx}, |_{\approx})$.

A much more difficult problem is to understand the expressive power of these logics in connection with existential second-order logic Σ^1_1. As mentioned above, formulae of independence logic or, equivalently, inclusion-exclusion logic (without equivalences) have the same expressive power as existential second-order sentences, and weaker logics such as dependence logic, exclusion logic, or inclusion logic correspond to fragments of Σ^1_1. To describe the expressive power of dependence logics with equivalences we introduce the $\approx$-closed fragment $\Sigma^1_1(\approx)$ of Σ^1_1 and show that it captures the expressiveness of $\mathrm{FO}(\subseteq_{\approx}, |_{\approx})$.

Definition 5.2. The logic $\Sigma^1_1(\approx)$ consists of sentences of the form

$$\psi := \exists_{\approx} R_1 \dots \exists_{\approx} R_k \varphi(R_1, \dots, R_k)$$

where $\varphi \in \mathrm{FO}(\tau \cup \{R_1, \dots, R_k\})$. The semantics of ψ is given in terms of $\approx$-closed relations:

$$\mathfrak{A} \models \psi \;:\Longleftrightarrow\; \text{there are } \approx\text{-closed relations } R_1, \dots, R_k \text{ such that } (\mathfrak{A}, R_1, \dots, R_k) \models \varphi.$$

5.2 The Expressive Power of these new Logics

In this section we establish that $\mathrm{FO}(\subseteq_\approx, |_\approx)$ has exactly the same expressive power as $\Sigma^1_1(\approx)$. This means that every formula $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq_\approx, |_\approx)$ that cannot distinguish between teams X, X' with $X(\bar{x}) \approx X'(\bar{x})$ can be translated into an equivalent sentence $\varphi' \in \Sigma^1_1(\approx)$ using an additional predicate X that occurs only $\approx$-guarded, i.e. only in the shape $X_\approx \bar{v} := \exists \bar{w}(\bar{v} \approx \bar{w} \wedge X\bar{w})$, such that

$$\mathfrak{A} \models_X \varphi(\bar{x}) \iff (\mathfrak{A}, X(\bar{x})) \models \varphi'(X) \tag{5.1}$$

and, vice versa, for such a $\varphi'(X) \in \Sigma^1_1(\approx)$ we can find such a $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq_\approx, |_\approx)$ that is equivalent in the sense of (5.1).

5.2.1 From ESO to Inclusion-Exclusion Logic up to Equivalences

Consider a sentence of the form $\exists_\approx R_1 \dots \exists_\approx R_k \varphi(X, \bar{R}) \in \Sigma^1_1(\approx)$ in $\mathrm{FO}(\subseteq_\approx, |_\approx)$ where the relation symbol X occurs only in the form $X_\approx \bar{v} := \exists \bar{w}(\bar{v} \approx \bar{w} \wedge X\bar{w})$, which we also call a $\approx$-guarded occurrence of X. In order to capture the semantics of such a sentence we adapt ideas by Rönnholm [Rön18] and use tuples of variables $\bar{x}, \bar{v}_1, \dots, \bar{v}_k$ of length $|\bar{x}| = \mathrm{ar}(X)$ and $|\bar{v}_i| = \mathrm{ar}(R_i)$ in order to simulate the ($\approx$-closed) relations $R_1, \dots, R_k$. The reason why this is possible lies in the fact that we are using team semantics: in a given team Y with $\{\bar{x}, \bar{v}_1, \dots, \bar{v}_k\} \subseteq \mathrm{dom}(Y)$ we naturally have that $Y(\bar{v}_i)$ corresponds to a (not necessarily $\approx$-closed) relation. The most important step is to find a formula $\varphi^\star(\bar{x}, \bar{v}_1, \dots, \bar{v}_k) \in \mathrm{FO}(\subseteq_\approx, |_\approx)$ such that for all teams X with $\bar{x} \in \mathrm{dom}(X)$ and all $\approx$-closed relations $R_1, \dots, R_k$,

$$(\mathfrak{A}, X(\bar{x}), \bar{R}) \models \varphi \iff \mathfrak{A} \models_Y \varphi^\star(\bar{x}, \bar{v}_1, \dots, \bar{v}_k)$$

where $Y = X[\bar{v}_1 \mapsto R_1, \dots, \bar{v}_k \mapsto R_k]$. Towards this end, $\varphi^\star$ is constructed (inductively) while using inclusion/exclusion atoms to express (non)membership in $X(\bar{x}), R_1, \dots, R_k$. For example, $\bar{w} \subseteq_\approx \bar{v}_i$ means that $s(\bar{w}) \in Y(\bar{v}_i)_\approx = R_i$ for every $s \in Y$, while $\bar{w} |_\approx \bar{v}_i$ expresses that $s(\bar{w}) \notin Y(\bar{v}_i)_\approx = R_i$ for every $s \in Y$. Therefore, the semantics of $R_i\bar{w}$ resp. $\neg R_i \bar{w}$ is captured by $\bar{w} \subseteq_\approx \bar{v}_i$ resp. $\bar{w} |_\approx \bar{v}_i$. Similarly, $\bar{w} \subseteq_\approx \bar{x}$ and $\bar{w} |_\approx \bar{x}$ are expressing membership in $(X(\bar{x}))_\approx$. But of course, it could be the case that φ is a much more complicated formula made up of quantifiers, conjunctions or disjunctions. It turns out that quantifiers and conjunctions can be handled with ease by simply setting

$$\begin{aligned}(Qu\vartheta)^\star &:= Qu(\vartheta^\star) \text{ for both quantifiers } Q \in \{\exists, \forall\}, \text{ and}\\ (\vartheta_1 \wedge \vartheta_2)^\star &:= \vartheta_1^\star \wedge \vartheta_2^\star,\end{aligned}$$

because when evaluating conjunctions in team semantics, the team is not modified and in the process of evaluating quantifiers there are just more columns[1] added to the team (w.l.o.g. we assume that every variable in the formula occurs either freely or is

[1] One may view a relation as a table and, conversely, teams have relational encodings.

quantified exactly once). However, for disjunctions the situation is much more delicate because it is *not* possible to define $(\vartheta_1 \vee \vartheta_2)^\star$ as $\vartheta_1^\star \vee \vartheta_2^\star$. The reason for this is that after splitting the team X into X_1, X_2 with $X = X_1 \cup X_2$ and $\mathfrak{A} \models_{X_j} \vartheta_j^\star$ it cannot be guaranteed that $X_j(\bar{v}_i)$ still describes the original R_i (up to equivalence). To make sure that we do not loose information about $R_1, \ldots, R_k$, we use instead an adaptation of the *value preserving disjunction* that was introduced by Rönnholm [Rön18].

Lemma 5.3. *Let* $\psi_1, \psi_2 \in \mathrm{FO}(\subseteq_\approx, |_\approx)$ *and* $\bar{w}_1, \ldots, \bar{w}_n$ *be some tuples of variables. Then there exists a formula* $\psi_1 \underset{\bar{w}_1,\ldots,\bar{w}_n}{\vee} \psi_2 \in \mathrm{FO}(\subseteq_\approx, |_\approx)$ *such that the following are equivalent:*

1. $\mathfrak{A} \models_X \psi_1 \underset{\bar{w}_1,\ldots,\bar{w}_n}{\vee} \psi_2$
2. $X = X_1 \cup X_2$ *for some teams* X_1, X_2 *such that for both* $j = 1$ *and* $j = 2$*:*
   - $\mathfrak{A} \models_{X_j} \psi_j$*, and*
   - *if* $X_j \neq \emptyset$*, then* $X_j(\bar{w}_i) \approx X(\bar{w}_i)$ *for all* $i \in \{1, \ldots, k\}$*.*

Proof. The construction of $\psi_1 \underset{\bar{w}_1,\ldots,\bar{w}_n}{\vee} \psi_2$ relies on the *intuitionistic disjunction* $\psi_1 \sqcup \psi_2$ with

$$\mathfrak{A} \models_X \psi_1 \sqcup \psi_2 \Longleftrightarrow \mathfrak{A} \models_X \psi_1 \text{ or } \mathfrak{A} \models_X \psi_2.$$

On structures $\mathfrak{A} \in (\tau, \approx)$ with $\approx^{\mathfrak{A}} \neq A^2$ this is definable in $\mathrm{FO}(\subseteq_\approx, |_\approx)$ since

$$\psi_1 \sqcup \psi_2 \equiv \exists c_\ell \exists c_r (\mathrm{dep}_\approx(c_\ell) \wedge \mathrm{dep}_\approx(c_r) \wedge [(c_\ell \approx c_r \wedge \psi_1) \vee (\neg c_\ell \approx c_r \wedge \psi_2)])$$

where c_ℓ and c_r are some variables not occurring in ψ_1 or ψ_2. Note that $\mathrm{dep}_\approx(c)$ expresses that c only assumes values from a single equivalence class. Now consider the following formula, which is a modification of a construction by Rönnholm [Rön18]. $\psi_1 \underset{\bar{w}_1,\ldots,\bar{w}_n}{\vee'} \psi_2$ is defined as

$$(\psi_1 \sqcup \psi_2) \sqcup \exists c_\ell \exists c_r \Big(\mathrm{dep}_\approx(c_\ell) \wedge \mathrm{dep}_\approx(c_r) \wedge c_\ell \not\approx c_r \wedge \\ \exists y \Big([(y \approx c_\ell \wedge \psi_1) \vee (y \approx c_r \wedge \psi_2)] \wedge \bigwedge_{i=1}^{k} \Theta_i \wedge \Theta'_i \Big) \Big)$$

Θ_i and Θ'_i are given by

$$\begin{aligned} \Theta_i &:= \exists \bar{z}_1 \exists \bar{z}_2 ([(y \approx c_\ell \wedge \bar{z}_1 = \bar{w}_i \wedge \bar{z}_2 = \bar{c}_\ell) \vee (y \approx c_r \wedge \bar{z}_1 = \bar{c}_\ell \wedge \bar{z}_2 = \bar{w}_i)] \wedge \\ &\qquad \bar{w}_i \subseteq_\approx \bar{z}_1 \wedge \bar{w}_i \subseteq_\approx \bar{z}_2), \\ \Theta'_i &:= \exists \bar{z}_1 \exists \bar{z}_2 ([(y \approx c_\ell \wedge \bar{z}_1 = \bar{w}_i \wedge \bar{z}_2 = \bar{c}_r) \vee (y \approx c_r \wedge \bar{z}_1 = \bar{c}_r \wedge \bar{z}_2 = \bar{w}_i)] \wedge \\ &\qquad \bar{w}_i \subseteq_\approx \bar{z}_1 \wedge \bar{w}_i \subseteq_\approx \bar{z}_2). \end{aligned}$$

where $\bar{c}_\ell = (c_\ell, c_\ell, \ldots, c_\ell)$ and $\bar{c}_r = (c_r, c_r, \ldots, c_r)$ are always tuples of the correct length.

For a detailed proof, why this formula satisfies the properties required by Lemma 5.3 under the additional condition that $\approx$ has at least two different equivalence classes, we refer to [Rön18, Proposition 3.7]. In order to get rid of the additional requirement of having at least two equivalence classes, we put:

$$\psi_1 \underset{\bar{w}_1,\ldots,\bar{w}_n}{\vee} \psi_2 := [\forall x \forall y(x \approx y) \wedge (\psi_1 \vee \psi_2)] \vee \big[\exists x \exists y(x \not\approx y) \wedge \big(\psi_1 \underset{\bar{w}_1,\ldots,\bar{w}_n}{\vee'} \psi_2\big)\big]. \qquad \square$$

We can now complete the inductive definition of $\varphi^\star$ by:

$$(\vartheta_1 \vee \vartheta_2)^\star := \vartheta_1^\star \underset{\bar{x},\bar{v}_1,\ldots,\bar{v}_k}{\vee} \vartheta_2^\star$$

We can now establish the following two lemmata.

Lemma 5.4. *For every $\mathfrak{A} \in (\tau, \approx)$ and every non-empty team Y with $\bar{x}, \bar{v}_1, \ldots, \bar{v}_k \in$ $\mathrm{dom}(Y)$,*

$$\mathfrak{A} \models_Y \varphi^\star(\bar{x}, \bar{v}_1, \ldots, \bar{v}_k) \Longrightarrow (\mathfrak{A}, Y(\bar{x}), R_1^Y, \ldots, R_k^Y) \models \varphi(X, R_1, \ldots, R_k)$$

where $R_i^Y := (Y(\bar{v}_i))_\approx$ for $i = 1, \ldots, k$, i.e. R_i^Y is defined as the $\approx$-closure of $Y(\bar{v}_i)$.

Proof. Let λ be a witness for $\mathfrak{A} \models_Y \varphi^\star(\bar{x}, \bar{v}_1, \ldots, \bar{v}_k)$. For every $\gamma \in \mathrm{subf}(\varphi)$ with $\lambda(\gamma^\star) \neq \emptyset$, we have that $(\lambda(\gamma^\star))(\bar{z}) \approx (\lambda(\varphi^\star))(\bar{z}) = Y(\bar{z})$ for all $\bar{z} \in \{(\bar{x}), (\bar{v}_1), \ldots, (\bar{v}_k)\}$, because we have replaced disjunction in φ by disjunctions which preserve values up to equivalence. In particular, we have that the introduced in-/exclusion atoms precisely describe the membership relation of X (which occurs only $\approx$-guarded in φ) and of the $\approx$-closed relations $R_1, \ldots, R_k$ as long as a non-empty team arrives at the atom. This and the fact that Y is non-empty is the reason, why $\lambda(\gamma) := \lambda(\gamma^\star)\restriction_{\mathrm{free}(\gamma)}$ for every $\gamma \in \mathrm{subf}(\varphi)$ defines a witness for $(\mathfrak{A}, Y(\bar{x}), R_1^Y, \ldots, R_k^Y) \models_{\{\emptyset\}} \varphi$. Thus, $(\mathfrak{A}, Y(\bar{x}), R_1^Y, \ldots, R_k^Y) \models \varphi(X, R_1, \ldots, R_k)$ follows as desired. $\square$

Lemma 5.5. *Let X be a team with $\mathrm{dom}(X) = \{\bar{x}\}$ and let $\bar{R} = (R_1, \ldots, R_k)$ be a tuple of non-empty $\approx$-closed relations such that $(\mathfrak{A}, X(\bar{x})_\approx, \bar{R}) \models \varphi(X, R_1, \ldots, R_k)$. Then $\mathfrak{A} \models_Y$ $\varphi^\star(\bar{x}, \bar{v}_1, \ldots, \bar{v}_k)$ where $Y := X[\bar{v}_1 \mapsto R_1, \ldots, \bar{v}_k \mapsto R_k]$.*

Proof. Since $(\mathfrak{A}, X(\bar{x}), \bar{R}) \models \varphi(X, R_1, \ldots, R_k)$, there exists a witness λ for

$$(\mathfrak{A}, X(\bar{x}), \bar{R}) \models_{\{\emptyset\}} \varphi(X, R_1, \ldots, R_k).$$

By defining $\lambda^\star(\gamma^\star) := \lambda(\gamma)[\bar{x} \mapsto X(\bar{x})_\approx, \bar{v}_1 \mapsto R_1, \ldots, \bar{v}_k \mapsto R_k]$ for all $\gamma \in \mathrm{subf}(\varphi)$ we basically obtain a witness for $\mathfrak{A} \models_Y \varphi^\star$.[2] $\square$

The non-emptiness requirement of $R_1, \ldots, R_k$ does not create a serious problem, because by rewriting the formula φ it can be assumed w.l.o.g. that $\exists_\approx R_1 \ldots \exists_\approx R_k \varphi$ is satisfied in a structure $\mathfrak{A}$ if, and only if, there are *non-empty* $\approx$-closed relations $R_1, \ldots, R_k$ such that $(\mathfrak{A}, \bar{R}) \models \varphi$. For example, one could simulate R_i by two different relations R_i^+ and R_i^- with $R_i^+ := R_i \cup \{\bar{a}\}_\approx$ and $R_i^+ := R_i \cup \{\bar{b}\}_\approx$ where $\bar{a}$ and $\bar{b}$ are non-equivalent tuples which leads to $R_i^+ \cap R_i^- = R_i$ and $R_i^- \neq \emptyset \neq R_i^-$.

[2]Notice that we have defined $\lambda^\star$ only for those formulae that have the shape $\gamma^\star$, but, technically, $\lambda^\star$ must be defined also for the subformulae that occur in the construction for the value preserving disjunctions. However, by Lemma 5.3, it is clear that the definition $\lambda^\star$ can be completed in the desired way.

Corollary 5.6. *For every team X with* $\mathrm{dom}(X) = \{\bar{x}\}$*, we have*

$$\mathfrak{A} \models_X \psi := \exists \bar{v}_1 \dots \exists \bar{v}_k \varphi^\star(\bar{x}, \bar{v}_1, \dots, \bar{v}_k) \iff (\mathfrak{A}, X(\bar{x})) \models \exists_\approx R_1 \dots \exists_\approx R_k \varphi(X, R_1, \dots, R_k).$$

Proof. We prove the two directions separately.

"$\Longrightarrow$": If $\mathfrak{A} \models_X \psi$, then $\mathfrak{A} \models_Y \varphi^\star$ where Y is an extension of X by $\bar{v}_1, \dots, \bar{v}_k$, which, by Lemma 5.4, leads to $(\mathfrak{A}, Y(\bar{x}), R_1^Y, \dots, R_k^Y) \models \varphi(X, R_1, \dots, R_k)$. Because of $Y(\bar{x}) = X(\bar{x})$, this implies $(\mathfrak{A}, X(\bar{x})) \models \exists_\approx R_1 \dots \exists_\approx R_k \varphi(X, R_1, \dots, R_k)$.

"$\Longleftarrow$": Now suppose that $(\mathfrak{A}, X(\bar{x})) \models \exists_\approx R_1 \dots \exists_\approx R_k \varphi(X, R_1, \dots, R_k)$. Then there are $\approx$-closed relations $R_1, \dots, R_k$ such that $(\mathfrak{A}, X(\bar{x}), R_1, \dots, R_k) \models \varphi(X, R_1, \dots, R_k)$. W.l.o.g. we assume that these relations are non-empty. Since X occurs only $\approx$-guarded in φ, this implies $(\mathfrak{A}, X(\bar{x})_\approx, R_1, \dots, R_k) \models \varphi(X, R_1, \dots, R_k)$ and, hence, Lemma 5.5 is applicable. So we obtain $\mathfrak{A} \models_Y \varphi^\star(\bar{x}, \bar{v}_1, \dots, \bar{v}_k)$ where $Y := X[\bar{v}_1 \mapsto R_1, \dots, \bar{v}_k \mapsto R_k]$. Because the R_i are non-empty, this proves that $\mathfrak{A} \models_X \psi(\bar{x})$. □

Please observe that the variable tuple $\bar{x}$ occurs in $\varphi^\star$ only in atoms of the form $\bar{w} \subseteq_\approx \bar{x}$ or $\bar{w} \mid_\approx \bar{x}$, which are the surrogates for $X_\approx \bar{w}$ resp. $\neg X_\approx \bar{w}$. As a result, the constructed formula $\psi(\bar{x})$ has the semantical property of being unable to distinguish between teams X, X' with $X(\bar{x}) \approx X'(\bar{x})$. Furthermore, if X does not occur in φ, then $\bar{x}$ are no longer occurring in $\varphi^\star$ and, thus, the sentence $\exists \bar{v}_1 \dots \exists \bar{v}_k \varphi^\star$ that results in this special case is equivalent to the sentence $\exists_\approx R_1 \dots \exists_\approx R_k \varphi$.

5.2.2 From Inclusion-Exclusion Logic to ESO up to Equivalences

Up to this point we only know that $\Sigma_1^1(\approx) \leq \mathrm{FO}(\subseteq_\approx, |_\approx)$. In this section we prove that these two logics have in fact the same expressive power. Towards this end, we demonstrate how a given formula $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq_\approx, |_\approx)$ can be translated into $\Sigma_1^1(\approx)$. There are two obstacles that we need to overcome:

1. When viewed as relations, teams usually are not $\approx$-closed, so we cannot use the quantifier $\exists_\approx$ to fetch the subteams we would need to satisfy the subformulae of e.g. a disjunction.

2. Unlike in Σ_1^1, where a formula of the form $\forall x \exists Y(\dots)$ is equivalent to formula like $\exists Y' \forall x(\dots)$ where $\mathrm{ar}(Y') = \mathrm{ar}(Y) + 1$, there seems to be no obvious way to perform a similar syntactic manipulation in $\Sigma_1^1(\approx)$. Thus we have to be content with the limited quantification that $\Sigma_1^1(\approx)$ allows us.

The main idea of the construction, which is inspired by [Rön18], is to replace every inclusion and exclusion atom ϑ by a separate new relation symbol R_ϑ that contains certain values enabling us to express the semantics of φ in $\Sigma_1^1(\approx)$.

First we describe how this approach deals with exclusion atoms. Let $\vartheta_1, \dots, \vartheta_k$ be an enumeration of all occurrences of exclusion atoms $\vartheta_i = \bar{u}_i \mid_\approx \bar{w}_i$ in $\varphi \in \mathrm{FO}(\subseteq_\approx, |_\approx)$. We assume w.l.o.g. that the tuples $\bar{u}_1, \dots, \bar{u}_k, \bar{w}_1, \dots, \bar{w}_k$ are pairwise different. We use new relation symbols $R_{\vartheta_1}, \dots, R_{\vartheta_k}$ that are intended to separate the sets of possible values for $\bar{v}_i$ and $\bar{w}_i$ (up to equivalence). The desired translation $\varphi^\star$ of φ is now obtained by replacing the exclusion atoms $\vartheta_i = \bar{u}_i \mid_\approx \bar{w}_i$ by $R_{\vartheta_i}\bar{u}_i \wedge \neg R_{\vartheta_i}\bar{w}_i$. This construction leads to the following result.

Theorem 5.7. *For every formula $\varphi(\bar{x}) \in \mathrm{FO}(\mid_\approx, \subseteq_\approx)$ with signature τ there exists a formula $\varphi^\star(\bar{x}) \in \mathrm{FO}(\subseteq_\approx)$ with signature $\tau \cup \{\bar{R}\}$, where $\bar{R}$ is a tuple of new relation symbols, such that for every $\mathfrak{A} \in (\tau, \approx)$ and every team X the following are equivalent:*

1. $\mathfrak{A} \vDash_X \varphi$
2. *There are $\approx$-closed relations $\bar{R}$ over $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{R}) \vDash_X \varphi^\star$.*

Proof. Straightforward adaptation of the proof of [Rön18, Theorem 4.1]. □

After this elimination of the exclusion atoms we still need to cope with $\subseteq_\approx$-atoms. Towards this end, let $\varphi \in \mathrm{FO}(\subseteq_\approx)$ and $\vartheta_1, \dots, \vartheta_k$ be an enumeration of all occurrences of inclusion atoms in φ. Let $\vartheta_i := \bar{x}_i \subseteq_\approx \bar{y}_i$ for every $i \in \{1, \dots, k\}$. Again following the ideas of [Rön18], we use new relation symbols $R_{\vartheta_1}, \dots, R_{\vartheta_k}$ with the intended semantics that $R_{\vartheta_i} \subseteq X(\bar{y}_i)_\approx$ where X is the team that "arrives" at ϑ_i. This will allow us to replace the subformulae ϑ_i by the formula $R_{\vartheta_i}\bar{x}_i$. However, this formula alone does not verify that $R_{\vartheta_i} \subseteq X(\bar{y}_i)_\approx$ really holds. Additional formulae $\varphi^{(1)}(\bar{z}_1), \dots, \varphi^{(k)}(\bar{z}_k)$ are required for the verification that values from R_{ϑ_i} could occur (up to equivalence) as a value for $\bar{y}_i$ in the team X that arrives at the corresponding inclusion atom. More precisely, $\varphi^{(i)}$ is constructed such that

$$(\mathfrak{A}, R_{\vartheta_1}, \dots, R_{\vartheta_k}) \vDash_{s[\bar{z}_i \mapsto \bar{a}]} \varphi^{(i)}(\bar{z}_i)$$

implies that the assignment s also satisfies φ and, more importantly, leads to an assignment s' that satisfies $s'(\bar{z}_i) \approx \bar{a}$ and that could be part of the team that satisfies the inclusion atom. Formally, we will have the property

$$\begin{aligned}\mathfrak{A} \vDash_X \varphi \iff &\text{there are } \approx\text{-closed relations } R_{\vartheta_1}, \dots, R_{\vartheta_k} \text{ such that } (\mathfrak{A}, \bar{R}) \vDash_X \varphi^\star \text{ and}\\ &\text{for every } \bar{a} \in R_{\vartheta_i} \text{ there is an } s \in X \text{ with } (\mathfrak{A}, \bar{R}) \vDash_{s[\bar{z}_i \mapsto \bar{a}]} \varphi^{(i)}(\bar{z}_i).\end{aligned}$$

As already pointed out, $\varphi^\star$ results from φ by replacing every inclusion atom $\vartheta_i = \bar{x}_i \subseteq_\approx \bar{y}_i$ by $R_{\vartheta_i}\bar{x}_i$, while $\varphi^{(i)}$ is defined as in [Rön18] by induction (for every $i \in \{1, \dots, k\}$). Let ϑ be a subformula of φ. First-order literals are unchanged, i.e. $\vartheta^\star := \vartheta =: \vartheta^{(i)}$ if ϑ is such a literal. The inclusion atoms are translated as follows:

$$(\bar{x}_j \subseteq \bar{y}_j)^{(i)} := \begin{cases} R_{\vartheta_i}\bar{x}_i \wedge \bar{y}_i \approx \bar{z}_i, & \text{if } i = j \\ R_{\vartheta_j}\bar{x}_j, & \text{if } i \neq j \end{cases}$$

Conjunctions and existential quantifiers are handled by defining

$$\begin{aligned}(\exists x\, \tilde{\vartheta})^{(i)} &:= \exists x\, \tilde{\vartheta}^{(i)} \text{ and}\\ (\tilde{\vartheta}_1 \wedge \tilde{\vartheta}_2)^{(i)} &:= \tilde{\vartheta}_1^{(i)} \wedge \tilde{\vartheta}_2^{(i)}.\end{aligned}$$

However, the translation of universal quantifiers or disjunctions is more complex:

$$(\tilde{\vartheta}_1 \vee \tilde{\vartheta}_2)^{(i)} := \begin{cases} \tilde{\vartheta}_j^{(i)}, & \text{if } \bar{x}_i \subseteq_\approx \bar{y}_i \text{ occurs in } \tilde{\vartheta}_j \\ (\tilde{\vartheta}_1 \vee \tilde{\vartheta}_2)^\star, & \text{otherwise} \end{cases}$$
$$(\forall x\, \tilde{\vartheta})^{(i)} := \exists x\, \tilde{\vartheta}^{(i)} \wedge (\forall x\, \tilde{\vartheta})^\star.$$

By construction we have that $(\forall x\, \vartheta)^\star$ is implied by $(\forall x\, \vartheta)^{(i)}$, because of the conjunction, while $\exists x\, \vartheta^{(i)}$ fetches the correct extension of the current assignment such that we end up with an assignment satisfying $\bar{y}_i \approx \bar{z}_i$ when arriving at the translation of $\bar{x}_i \subseteq_\approx \bar{y}_i$. The next lemma states that this construction actually captures the intuition that we have described above.

Lemma 5.8. *Let $\varphi \in \mathrm{FO}(\subseteq_\approx)$ and $\varphi^\star, \varphi^{(1)}, \dots, \varphi^{(k)}$ be as above. Let $\mathfrak{A} \in (\tau, \approx)$ and X be a team over $\mathfrak{A}$ with $\mathrm{free}(\varphi) = \mathrm{dom}(X)$. Then the following are equivalent:*

1. $\mathfrak{A} \models_X \varphi$
2. *There are $\approx$-closed relations $\bar{R} = (R_{\vartheta_1}, \dots, R_{\vartheta_k})$ over $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{R}) \models_X \varphi^\star$ and for every $i \in \{1, \dots, k\}$, $\bar{a} \in R_{\vartheta_i}$ there exists some $s \in X$ such that $(\mathfrak{A}, \bar{R}) \models_{s[\bar{z}_i \mapsto \bar{a}]} \varphi^{(i)}$.*

Proof. Straightforward adaptation of the proof of [Rön18, Theorem 4.2]. □

We are now ready to show how inclusion atoms are translated into $\Sigma^1_1(\approx)$.

Theorem 5.9. *For every formula $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq_\approx)$ there exists a sentence $\varphi'(X) \in \Sigma^1_1(\approx)$ such that $\mathfrak{A} \models_X \varphi(\bar{x}) \iff (\mathfrak{A}, X) \models \varphi'(X)$ for every structure $\mathfrak{A}$ and every team X.*

Proof. Let

$$\varphi' := \exists_\approx R_{\vartheta_1} \dots \exists_\approx R_{\vartheta_k} \big(\forall \bar{x}(X\bar{x} \longrightarrow \varphi^\star(\bar{x})) \wedge \bigwedge_{i=1}^{k} \forall \bar{z}_i (R_{\vartheta_i} \bar{z}_i \longrightarrow \exists \bar{x}(X\bar{x} \wedge \varphi^{(i)}(\bar{x}, \bar{z}_i)))\big).$$

By construction, $(\mathfrak{A}, X) \models \varphi'$ if, and only if, there exist $\approx$-closed relations $\bar{R}$ over $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{R}) \models_s \varphi^\star$ for every $s \in X$, and for every $\bar{a} \in R_i$ there exists some $s \in X$ with $(\mathfrak{A}, \bar{R}) \models_{s[\bar{z}_i \mapsto \bar{a}]} \varphi^{(i)}$. Since $\varphi^\star$ is a first-order formula, $\mathfrak{A} \models_s \varphi^\star$ for every $s \in X$ if, and only if, $(\mathfrak{A}, \bar{R}) \models_X \varphi^\star$. Hence, by Lemma 5.8, we can conclude that $(\mathfrak{A}, X) \models \varphi' \iff \mathfrak{A} \models_X \varphi$. □

In particular, every sentence $\varphi \in \mathrm{FO}(\subseteq_\approx)$ can be translated into an equivalent sentence $\varphi' \in \Sigma^1_1(\approx)$. If we additionally demand that $\varphi(\bar{x})$ fulfils the semantical property of being unable to distinguish between teams X, X' with $X(\bar{x}) \approx X'(\bar{x})$, then this semantic property carries over to $\varphi'(X)$ and, consequently, this formula is then equivalent to $\varphi'(X_\approx)$, which results from $\varphi'(X)$ by replacing every occurrence of X by its $\approx$-guarded version. It is worth mentioning that this semantic property can also characterized on a syntactic level: $\varphi(\bar{x})$ cannot distinguish between $\approx$-equivalent teams, if and only if $\varphi(\bar{x})$ is logically equivalent to $\exists \bar{y}(\bar{x} \approx \bar{y} \wedge \varphi(\bar{y}))$ where $\bar{y}$ is new tuple of variables.

5.3 Inclusion Logic up to Equivalences vs. GFP

An important result on logics with team semantics is the tight connection between inclusion logic and GFP^+, established by Galliani and Hella [GH13]. In this section we prove a similar result for $\mathrm{FO}(\subseteq_{\approx})$ by defining a fragment of GFP^+ which has the same expressive power as $\mathrm{FO}(\subseteq_{\approx})$ on the level of sentences.

In Section 4.1, we have already pointed out that the well-known fact that the simultaneous variant of GFP^+ has the same expressive powers as GFP^+ [GKL⁺07]. This is why, we define $\mathrm{GFP}^+_{\approx}$ in the form of its simultaneous variant.

Definition 5.10 ($\mathrm{GFP}^+_{\approx}$)**.** The logic $\mathrm{GFP}^+_{\approx}$ is defined as an extension of FO in negation normal form by the following formula formation rule. Let $k \geq 1$ and $\bar{R} = (R_1, \dots, R_k)$ be a tuple of unused relation symbols of arity $n_1, \dots, n_k$ respectively and let $(\varphi_i(\bar{R}, \bar{x}_i))_{i=1,\dots,k}$ be a tuple of $\mathrm{FO}(\tau \cup \{R_1, \dots, R_k\})$-formulae in negation normal form where $|\bar{x}_i| = n_i$ and every R_i occurs only positively in $\varphi_1, \dots, \varphi_k$. Furthermore, let $j \in \{1, \dots, k\}$ and $\bar{v}$ be a n_j-tuple of variables. Then

$$\varphi(\bar{v}) := [\mathrm{sGFP}_{\approx}\, S]_j\, (\bar{v})$$

is a $\mathrm{GFP}^+_{\approx}$-formula where S is of the form

$$S := \begin{cases} R_1\bar{x}_1 & : \varphi_1(R_1, \dots, R_n, \bar{x}_1) \\ R_2\bar{x}_2 & : \varphi_2(R_1, \dots, R_n, \bar{x}_2) \\ & \vdots \\ R_n\bar{x}_n & : \varphi_n(R_1, \dots, R_n, \bar{x}_n). \end{cases}$$

On every structure $\mathfrak{A} \in (\tau, \approx)$, the system $(\varphi_i(\bar{R}, \bar{x}_i))_{i=1,\dots,k}$ defines a simultaneous update operator $\Gamma^{\mathfrak{A}} : \mathcal{P}(A^{n_1}) \times \cdots \times \mathcal{P}(A^{n_k}) \longrightarrow \mathcal{P}(A^{n_1}) \times \cdots \times \mathcal{P}(A^{n_k})$, by

$$\begin{aligned} \Gamma(\bar{R}) &:= \big(\Gamma_1(\bar{R}), \dots, \Gamma_k(\bar{R})\big) \text{ where} \\ \Gamma_i(\bar{R}) &:= [\![\varphi_i(\bar{R})]\!]^{\mathfrak{A}}_{\approx} = \{\bar{a} \in A^{n_i} : (\mathfrak{A}, \bar{R}) \models \varphi_i(\bar{R}, \bar{a})\}_{\approx} \end{aligned}$$

A tuple $(\mathfrak{A}, s)$ where $\mathfrak{A} \in (\tau, \approx)$ and $s : \{\bar{v}\} \longrightarrow A$ is called a model of φ (and we write $\mathfrak{A} \models_s \varphi$ in this case) if, and only if, for the greatest fixed-point $\bar{S} = (S_1, \dots, S_k)$ of $\Gamma^{\mathfrak{A}}$ we have that $s(\bar{v}) \in S_j$.

The non-simultaneous variant $\mathrm{GFP}^+_{\approx}$, where it is only allowed to use the operator $\mathrm{sGFP}_{\approx}$ in a non-simultaneous way, i.e. only in the following shape

$$[\mathrm{GFP}_{\approx}\, R\bar{x} \ : \ \varphi(R, \bar{x})]\, (\bar{y}) := [\mathrm{sGFP}\, R\bar{x} \ : \ \varphi(R, \bar{x})]_1\, (\bar{y}),$$

has exactly the same expressive power as $\mathrm{GFP}^+_{\approx}$. The following lemma is a variant of the well-known Knaster-Tarski Theorem and gives a characterization of the fixed-points of Γ:

Lemma 5.11 (Knaster-Tarski-Theorem for $\mathrm{GFP}^+_\approx$)**.** *Let $\varphi(\bar{v}) = [\mathrm{sGFP}\ S]_j\,(\bar{v})$ be a $\mathrm{GFP}^+_\approx$-formula where S consists of $\varphi_1, \ldots, \varphi_k$, $\mathfrak{A} \in (\tau, \approx)$ and $\Gamma (= \Gamma^{\mathfrak{A}})$ be the corresponding simultaneous update operator w.r.t. $\varphi_1, \ldots, \varphi_k$. For two given k-tuples $\bar{R}, \bar{S}$ of relations, we write $\bar{R} \subseteq \bar{S}$ if, and only if $R_i \subseteq S_i$ for every $i \in \{1, \ldots, k\}$.*

Let $X := \{\bar{S} : \bar{S} \subseteq \Gamma(\bar{S})\}$. Then $\bigcup X := (Y_1, \ldots, Y_k)$ where for every $j \in \{1, \ldots, k\}$, $Y_j := \bigcup_{\bar{S} \in X} S_j$ is the greatest fixed-point of Γ. Furthermore, these Y_j are $\approx$-closed.

5.3.1 From Inclusion Logic up to Equivalences to GFP

Theorem 5.12. *For every formula $\varphi(\bar{x}) \in \mathrm{FO}(\subseteq_\approx)$ there exists a sentence $\varphi^+(X) \in \mathrm{GFP}^+_\approx$ such that $\mathfrak{A} \models_X \varphi \iff (\mathfrak{A}, X) \models \varphi^+$ for every structure $\mathfrak{A} \in (\tau, \approx)$ and every team X over $\mathfrak{A}$.*

Proof. In Section 5.2.2 we have presented the FO-formulae $\varphi^\star(\bar{R}, \bar{x})$ and $\varphi^{(i)}(\bar{R}, \bar{z}_i)$ (for $i \in \{1, \ldots, k\}$) using new relation symbols $\bar{R} = (R_1, \ldots, R_k)$ such that for every $\mathfrak{A} \in (\tau, \approx)$ and every team X over $\mathfrak{A}$ with $\mathrm{dom}(X) \supseteq \mathrm{free}(\varphi)$ the following are equivalent:

(1) $\mathfrak{A} \models_X \varphi$

(2) There are $\approx$-closed relations $\bar{R}$ over $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{R}) \models_X \varphi^\star$ and for every $i \in \{1, \ldots, k\}$, $\bar{a} \in R_i$ there exists some $s_{i,\bar{a}} \in X$ such that $(\mathfrak{A}, \bar{R}) \models_{s_{i,\bar{a}}[\bar{z}_i \mapsto \bar{a}]} \varphi^{(i)}$.

Furthermore, the relation symbols $R_1, \ldots, R_k$ occur only positively in $\varphi^\star$ and $\varphi^{(i)}$ and the tuple $\bar{z}_i$ occurs exactly once in a subformula of the form $\bar{x}_i \approx \bar{z}_i$ in $\varphi^{(i)}$. Let $\tilde{\varphi}^\star$ and the $\tilde{\varphi}^{(i)}$ be the formulae that result from $\varphi^\star, \varphi^{(i)}$ by replacing every occurrence of the form $R_i\bar{v}$ by its guarded version $(R_i)_\approx \bar{v} := \exists \bar{w}(\bar{v} \approx \bar{w} \wedge R_i\bar{w})$. This allows us to drop the requirement that the relations $\bar{R}$ are $\approx$-closed.

Claim 5.13. For every $\mathfrak{A}$ and every team X over $\mathfrak{A}$, (1) and (2) are equivalent to:

(3) There are relations $\bar{R}$ over $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{R}) \models_X \tilde{\varphi}^\star$ and for every $i \in \{1, \ldots, k\}$ and $\bar{a} \in R_i$ there exists some $s_{i,\bar{a}} \in X$ such that $(\mathfrak{A}, \bar{R}) \models_{s_{i,\bar{a}}[\bar{z}_i \mapsto \bar{a}]} \tilde{\varphi}^{(i)}$.

To prove this, one has to exploit the fact that every R_j ($j \in \{1, \ldots, k\}$) occurs only $\approx$-guarded in $\tilde{\varphi}^\star, \tilde{\varphi}^{(1)}, \ldots, \tilde{\varphi}^{(k)}$ and the variables $\bar{z}_i$ occur (exactly once) in a subformula of the form $\bar{w} \approx \bar{z}_i$ in $\varphi^{(i)}$. By expressing (3) in existential second-order logic, we obtain the following equivalent statement:

(4) $(\mathfrak{A}, X) \models \exists \bar{R}\left(\forall \bar{x}(X\bar{x} \longrightarrow \tilde{\varphi}^\star(\bar{R}, \bar{x})) \wedge \psi\right)$ where $\psi := \bigwedge_{i=1}^{k} \forall \bar{z}_i(R_i\bar{z}_i \longrightarrow \eta_i(\bar{R}, \bar{z}_i))$ and $\eta_i(\bar{R}, \bar{z}_i) := \exists \bar{x}(X\bar{x} \wedge \tilde{\varphi}^{(i)}(\bar{R}, \bar{z}_i, \bar{x}))$.

Let $\Gamma(\bar{R}) := \left(\Gamma_1(\bar{R}), \ldots, \Gamma_k(\bar{R})\right)$ where

$$\Gamma_i(\bar{R}) := [\![\eta_i(\bar{R}, \bar{z}_i)]\!]^{(\mathfrak{A},X)} = \{\bar{a} \in A^{\mathrm{ar}(R_i)} : (\mathfrak{A}, X(\bar{x}), \bar{R}) \models \eta_i(\bar{a})\}.$$

Note that $[\![\eta_i(\bar{R}, \bar{z}_i)]\!]^{(\mathfrak{A},X(\bar{x}))} = [\![\eta_i(\bar{R}, \bar{z}_i)]\!]_\approx^{(\mathfrak{A},X(\bar{x}))}$, because the free variables $\bar{z}_i$ occur exactly once in a subformula of the form $\bar{w} \approx \bar{z}_i$. This is the reason why Γ is the $\mathrm{GFP}^+_\approx$-update operator w.r.t. $\eta_1, \ldots, \eta_k$.

Furthermore, $(\mathfrak{A}, X, \bar{R}) \models \forall \bar{z}_i(R_i\bar{z}_i \longrightarrow \eta_i(\bar{R}, \bar{z}_i))$ if, and only if, $R_i \subseteq \Gamma_i(\bar{R})$. Consequently, we have $(\mathfrak{A}, X, \bar{R}) \models \psi$ if, and only if, $\bar{R} \subseteq \Gamma(\bar{R})$.

Claim 5.14. For $j \leq k$, let $\vartheta_j(\bar{z}_j) := [\text{sGFP}_\approx\, S]_j\,(\bar{z}_j)$ where

$$S := \begin{cases} R_1\bar{z}_1 & : \eta_1(\bar{R}, \bar{z}_1) \\ R_2\bar{z}_2 & : \eta_2(\bar{R}, \bar{z}_2) \\ & \vdots \\ R_k\bar{z}_k & : \eta_k(\bar{R}, \bar{z}_k). \end{cases}$$

and let γ result from $\tilde{\varphi}^\star$ by replacing every occurrence of $R_j(\bar{w})$ by $\vartheta_j(\bar{w})$. Then, for every $\mathfrak{A} \in (\tau, \approx)$ and every team X, (4) is equivalent to

(5) $(\mathfrak{A}, X) \models \forall \bar{x}(X\bar{x} \longrightarrow \gamma)$.

Proof. (4) $\Longrightarrow$ (5): Let $(\mathfrak{A}, X) \models \exists \bar{R}\left(\forall \bar{x}(X\bar{x} \longrightarrow \tilde{\varphi}^\star(\bar{R}, \bar{x})) \wedge \psi\right)$. Then there are relations $\bar{R}$ such that $(\mathfrak{A}, X, \bar{R}) \models \forall \bar{x}(X\bar{x} \longrightarrow \tilde{\varphi}^\star(\bar{R}, \bar{x}))$ and $(\mathfrak{A}, X, \bar{R}) \models \psi$. As observed above, it follows that $\bar{R} \subseteq \Gamma(\bar{R})$. So, by Lemma 5.11, $\bar{R} \subseteq \bar{S}$ where $\bar{S}$ is the greatest fixed-point of Γ. Since we have $(\mathfrak{A}, X, \bar{R}) \models \forall \bar{x}(X\bar{x} \longrightarrow \tilde{\varphi}^\star(\bar{R}, \bar{x}))$ and the relations symbols $R_1, \dots, R_k$ occur only positively in $\tilde{\varphi}^\star$, we can conclude that $(\mathfrak{A}, X, \bar{S}) \models \forall \bar{x}(X\bar{x} \longrightarrow \tilde{\varphi}^\star(\bar{S}, \bar{x}))$. Because $\bar{S}$ is the greatest fixed-point of Γ, it follows that $S_i = [\![\vartheta_i(\bar{z}_i)]\!]^{(\mathfrak{A},X)}$ and, by construction of γ, we obtain that $(\mathfrak{A}, X) \models \forall \bar{x}(X\bar{x} \longrightarrow \gamma)$.

(5) $\Longrightarrow$ (4): Let $(\mathfrak{A}, X) \models \forall \bar{x}(X\bar{x} \longrightarrow \gamma)$ and let $\bar{S}$ be the greatest fixed-point of Γ. Then $(\mathfrak{A}, X, \bar{S}) \models \forall \bar{x}(X\bar{x} \longrightarrow \tilde{\varphi}^\star)$ and $\bar{S} = \Gamma(\bar{S})$. Therefore, we have $(\mathfrak{A}, X, \bar{S}) \models \forall \bar{z}_i(S_i\bar{z}_i \longrightarrow \eta_i(\bar{z}_i))$ for every $i \in \{1, \dots, k\}$ and, hence, $(\mathfrak{A}, X, \bar{S}) \models \psi(\bar{S})$. ☐

So we have $\mathfrak{A} \models_X \varphi \Longleftrightarrow \mathfrak{A} \models_X \varphi^+ := \forall \bar{x}(X\bar{x} \longrightarrow \gamma) \in \text{GFP}^+_\approx$. ☐

5.3.2 The Translation in the Other Direction

In order to translate a given sentence $\varphi \in \text{GFP}^+_\approx$ into a $\text{FO}(\subseteq_\approx)$-formula, we assume that φ is in a normal form which is given by the following lemma. By using adaptations of ideas from [GH13] we then show that such a sentence can be expressed in $\text{FO}(\subseteq_\approx)$.

Lemma 5.15. *For every sentence $\varphi \in \text{GFP}^+_\approx$ there exists a formula $\psi(R, \bar{x}) \in \text{FO}$, in which R occurs only positively and only $\approx$-guarded, such that φ is equivalent to*

$$\exists \bar{v}[GFP_\approx R\bar{x}\,.\,\psi(R, \bar{x})](\bar{v}).$$

Our next lemma shows that we can eliminate the relation symbol R in ψ by introducing $\subseteq_\approx$-atoms and encoding R in a tuple $\bar{x}$ of variables. The next lemma is a straightforward adaptation of [GH13, Theorem 16].

Lemma 5.16. *Let R be a relation symbol of arity n, let $\bar{x}, \bar{y}$ be tuples of variables where $|\bar{x}| = n$ (whereas $\bar{y}$ is of arbitrary length and can also be empty). Furthermore, let*

$\psi(R, \bar{x}, \bar{y}) \in \mathrm{FO}(\tau \cup \{R\})$ be a first-order formula in which R occurs only positively and $\approx$-guarded, and with $\mathrm{free}(\psi) \subseteq \{\bar{x}, \bar{y}\}$ such that the variables in $\bar{x}$ are never quantified in ψ. Then there exists a formula $\psi^\star(\bar{x}, \bar{y}) \in \mathrm{FO}(\subseteq_\approx)$ of signature τ such that for every $\mathfrak{A} \in (\tau, \approx)$ and every team X we have that

$$\mathfrak{A} \models_X \psi^\star(\bar{x}, \bar{y}) \iff (\mathfrak{A}, X(\bar{x})) \models_s \psi(R, \bar{x}, \bar{y}) \text{ for every } s \in X.$$

As done in proof of [GH13, Theorem 16], this lemma can be shown by induction over the structure of ψ. Now we are able to express $[\mathrm{GFP}_\approx R\bar{x} \,.\, \varphi(R, \bar{x})]$ in $\mathrm{FO}(\subseteq_\approx)$.

Theorem 5.17. *Let $\psi(R, \bar{x}) \in \mathrm{FO}$ where $\mathrm{ar}(R) = |\bar{x}|$, R occurs only positively and $\approx$-guarded in ψ, and the variables in $\bar{x}$ are never quantified in ψ. Then there exists a formula $\psi^+(\bar{x}) \in \mathrm{FO}(\subseteq_\approx)$ such that for every $\mathfrak{A} \in (\tau, \approx)$ and every team X we have that*

$$\mathfrak{A} \models_X \psi^+(\bar{x}) \iff \mathfrak{A} \models_s [\mathit{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{x}) \text{ for every } s \in X.$$

Proof. Let $\psi^+(\bar{x}) := \exists\bar{y}(\bar{x} \subseteq_\approx \bar{y} \wedge \exists\bar{z}(\bar{y} \approx \bar{z} \wedge \psi^\star(\bar{z})))$ where $\psi^\star$ stems from Lemma 5.16.

"$\Longrightarrow$": First we assume that $\mathfrak{A} \models_X \psi^+(\bar{x})$. Then there exists a function $F : X \longrightarrow \mathcal{P}^+(A^n)$ such that $\mathfrak{A} \models_Y \bar{x} \subseteq_\approx \bar{y} \wedge \exists\bar{z}(\bar{y} \approx \bar{z} \wedge \psi^\star(\bar{z}))$ where $Y := X[\bar{y} \mapsto F]$. So there exists a function $G : Y \longrightarrow \mathcal{P}^+(A^n)$ satisfying $\mathfrak{A} \models_Z \bar{y} \approx \bar{z} \wedge \psi^\star(\bar{z})$ where $Z := Y[\bar{z} \mapsto G]$. By Lemma 5.16, it follows that

$$(\mathfrak{A}, Z(\bar{z})) \models_s \psi(R, \bar{z}) \text{ for every } s \in Z.$$

So we have $Z(\bar{z}) \subseteq [\![\psi(R, \bar{z})]\!]^{(\mathfrak{A}, Z(\bar{z}))} \subseteq [\![\psi(R, \bar{z})]\!]_\approx^{(\mathfrak{A}, Z(\bar{z}))} = \Gamma_\psi(Z(\bar{z}))$ where Γ_ψ is the $\mathrm{GFP}^+_\approx$-update operator w.r.t. ψ and $\mathfrak{A}$. It follows that $Z(\bar{z}) \subseteq \mathrm{gfp}(\Gamma_\psi)$ (by Lemma 5.11). Since $\mathrm{gfp}(\Gamma_\psi)$ is $\approx$-closed and $X(\bar{x}) \subseteq_\approx Y(\bar{y}) \approx Z(\bar{z})$, we have that $X(\bar{x}) \subseteq \mathrm{gfp}(\Gamma_\psi)$. Hence, we obtain that $\mathfrak{A} \models_s [\mathrm{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{x})$ for every $s \in X$.

"$\Longleftarrow$": Now we assume that $\mathfrak{A} \models_s [\mathrm{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{x})$ for every $s \in X$. If $X = \emptyset$, then $\mathfrak{A} \models_X \psi^+(\bar{x})$ follows from the empty team property. Henceforth, let $X \neq \emptyset$. Let Γ_ψ be the $\mathrm{GFP}^+_\approx$-update operator defined w.r.t. $\psi(R)$. From our assumption follows that $X(\bar{x}) \subseteq \mathrm{gfp}(\Gamma_\psi)$. Since $X \neq \emptyset$, it follows that $\mathrm{gfp}(\Gamma_\psi) \neq \emptyset$. Our goal is to prove that $\mathfrak{A} \models_X \psi^+(\bar{x})$. Towards this end, we define $F : X \longrightarrow \mathcal{P}^+(A^n)$, $F(s) := \mathrm{gfp}(\Gamma_\psi)$ and $Y := X[\bar{y} \mapsto F]$ and claim that $\mathfrak{A} \models_Y \bar{x} \subseteq_\approx \bar{y} \wedge \exists\bar{z}(\bar{y} \approx \bar{z} \wedge \psi^\star(\bar{z}))$. Since $Y(\bar{x}) = X(\bar{x}) \subseteq \mathrm{gfp}(\Gamma_\psi) = Y(\bar{y})$ it is clear that $\mathfrak{A} \models_Y \bar{x} \subseteq_\approx \bar{y}$.

We still need to prove that $\mathfrak{A} \models_Y \exists\bar{z}(\bar{y} \approx \bar{z} \wedge \psi^\star(\bar{z}))$. By definition of Y, we know that $Y(\bar{y}) = \mathrm{gfp}(\Gamma_\psi) = \Gamma_\psi(\mathrm{gfp}(\Gamma_\psi)) = [\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]_\approx^{\mathfrak{A}}$. This implies that for every $s \in Y$ there exists some $\bar{a} \in [\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}}$ such that $\bar{a} \approx s(\bar{y})$.

Let $G : Y \longrightarrow \mathcal{P}^+(A^n)$ be given by

$$G(s) := \{\bar{a} \in [\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}} : s(\bar{y}) \approx \bar{a}\}$$

and $Z := Y[\bar{z} \mapsto G]$. Clearly it holds that $Z(\bar{z}) \subseteq [\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}}$. We claim that even $Z(\bar{z}) = [\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}}$ is true. To see this, let $\bar{a} \in [\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}}$. Since

$[\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}} \subseteq [\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}}_{\approx} = Y(\bar{y})$, there exists an $s \in Y$ with $s(\bar{y}) \approx \bar{a}$. Hence, we have that $\bar{a} \in G(s)$ and, consequently, $\bar{a} \in Z(\bar{z})$.

It is the case that $\mathfrak{A} \models_Z \bar{y} \approx \bar{z}$, because this follows from the definition of G. Now we prove that $\mathfrak{A} \models_Z \psi^\star(\bar{z})$. By Lemma 5.16, we need to verify that $(\mathfrak{A}, Z(\bar{z})) \models_s \psi(R, \bar{z})$ for every $s \in Z$. In other words, we need to verify that $Z(\bar{z}) \subseteq [\![\psi(Z(\bar{z}), \bar{z})]\!]^{\mathfrak{A}}$. Since $Z(\bar{z}) = [\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}}$, we can conclude that

$$[\![\psi(Z(\bar{z}), \bar{z})]\!]^{\mathfrak{A}} = [\![\psi([\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}}, \bar{z})]\!]^{\mathfrak{A}}$$

Due to the fact that R occurs only $\approx$-guarded in ψ, we can observe that

$$\begin{aligned}[\![\psi([\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}}, \bar{z})]\!]^{\mathfrak{A}} &= [\![\psi([\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{x})]\!]^{\mathfrak{A}}_{\approx}, \bar{z})]\!]^{\mathfrak{A}} \\ &= [\![\psi(\Gamma_\psi(\mathrm{gfp}(\Gamma_\psi)), \bar{z})]\!]^{\mathfrak{A}} \\ &= [\![\psi(\mathrm{gfp}(\Gamma_\psi), \bar{z})]\!]^{\mathfrak{A}} = Z(\bar{z})\end{aligned}$$

Therefore, we have $Z(\bar{z}) = [\![\psi(Z(\bar{z}), \bar{z})]\!]^{\mathfrak{A}}$ which implies that $Z(\bar{z}) \subseteq [\![\psi(Z(\bar{z}), \bar{z})]\!]^{\mathfrak{A}}$. So we have $(\mathfrak{A}, Z(\bar{z})) \models_s \psi(R, \bar{z})$ for every $s \in Z$, which concludes the proof of $\mathfrak{A} \models_Z \psi^\star$ and of $\mathfrak{A} \models_X \psi^+$. □

Corollary 5.18. *For every* $\mathrm{GFP}^+_{\approx}$*-sentence φ there is an equivalent sentence $\vartheta \in \mathrm{FO}(\subseteq_\approx)$.*

Proof. Let $\varphi \in \mathrm{GFP}^+_{\approx}$ be a sentence. By Lemma 5.15, there exists a first-order formula $\psi(R, \bar{x})$ where the n-ary relation symbol R occurs only positively and only $\approx$-guarded in ψ such that

$$\varphi \equiv \exists \bar{v}[\mathrm{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{v}).$$

W.l.o.g. we can assume that the variables in $\bar{x}$ are never quantified in ψ. So, by Theorem 5.17, it follows that there exists some $\psi^+(\bar{x}) \in \mathrm{FO}(\subseteq_\approx)$ such that for every $\mathfrak{A} \in (\tau, \approx)$ and every team X over $\mathfrak{A}$ with $\mathrm{dom}(X) \supseteq \{\bar{x}\}$ holds

$$\mathfrak{A} \models_X \psi^+(\bar{x}) \iff \mathfrak{A} \models_s [\mathrm{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{x}) \text{ for every } s \in X$$

Let $\vartheta := \exists \bar{v} \psi^+(\bar{v})$ and $\mathfrak{A} \in (\tau, \approx)$. Our goal is to prove that $\mathfrak{A} \models \varphi \iff \mathfrak{A} \models \vartheta$.

"$\Longleftarrow$": Let $\mathfrak{A} \models \vartheta$. Then there exists a function $F : \{\emptyset\} \to \mathcal{P}^+(A^{|\bar{v}|})$ such that $\mathfrak{A} \models_Y \psi^+(\bar{v})$ where $Y = \{\emptyset\}[\bar{v} \mapsto F]$. Then we have $\mathfrak{A} \models_s [\mathrm{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{v})$ for every $s \in Y$ and, since Y is non-empty, it follows that $\mathfrak{A} \models \exists \bar{v}[\mathrm{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{v})$.

"$\Longrightarrow$": Now let $\mathfrak{A} \models \varphi \equiv \exists \bar{v}[\mathrm{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{v})$. Then there exists some $\bar{a} \in A$ such that $\mathfrak{A} \models [\mathrm{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{a})$. Let $Y = \{s\}$ be the singleton team consisting only of s with $s(\bar{v}) = \bar{a}$. Then it follows that $\mathfrak{A} \models_s [\mathrm{GFP}_\approx R\bar{x} \,.\, \psi(R, \bar{x})](\bar{v})$ for every $s \in Y$ and, consequently, $\mathfrak{A} \models_Y \psi^+(\bar{v})$, proving that $\mathfrak{A} \models_{\{\emptyset\}} \exists \bar{v} \psi^+(\bar{v}) = \vartheta$. □

5.4 ESO up to Equivalences on Restricted Classes of Structures

In this section we compare $\Sigma^1_1(\approx)$ with FO and Σ^1_1 and study how restrictions imposed on the given equivalence influence the expressive power of $\Sigma^1_1(\approx)$. Our first result is that the expressive power of $\Sigma_1^1(\approx) \equiv \mathrm{FO}(\subseteq_\approx, |_\approx)$ lies strictly between FO and Σ_1^1. Furthermore, we also have $\mathrm{FO} < \mathrm{FO}(\subseteq_\approx, |_\approx) < \Sigma_1^1$ on the class of structures with only a bounded number of non-trivial equivalence classes and on the class of structures where each equivalence class is of size $\leq k$ (for some fixed $k > 1$). However, when restricting both the size of the equivalence classes and the number of non-trivial equivalence classes, then $\mathrm{FO}(\subseteq_\approx, |_\approx)$ has the same expressive power as Σ_1^1. To prove these results, we use an adaptation of the Ehrenfeucht-Fraïssé method for $\mathrm{FO}(\subseteq_\approx, |_\approx)$, which relies on the games presented in [Vää07].

Definition 5.19. Let $\mathfrak{A}, \mathfrak{B} \in (\tau, \approx)$, $n \in \mathbb{N}$ and $\Omega_\approx \subseteq \{\mathrm{dep}_\approx, \perp^\approx, \subseteq_\approx, |_\approx\}$. The game $\mathcal{G}_{\Omega_\approx,n}(\mathfrak{A}, \mathfrak{B})$ is played by two players which are called Duplicator and Spoiler. The positions of the game are tuples (X, Y) of teams over $\mathfrak{A}, \mathfrak{B}$ with $\mathrm{dom}(X) = \mathrm{dom}(Y)$. Unless stated otherwise the game starts at position $(\{\emptyset\}, \{\emptyset\})$ and then n moves are played. In each move Spoiler always chooses between one of the following 3 moves to continue the game:

1. Move $\vee$:
   - Spoiler represents X as a union $X = X_0 \cup X_1$.
   - Duplicator replies with a representation of Y as $Y = Y_0 \cup Y_1$.
   - Spoiler chooses $i \in \{0, 1\}$ and the game continues at position (X_i, Y_i).
2. Move $\exists$:
   - Spoiler chooses a function $F : X \longrightarrow \mathcal{P}^+(A)$.
   - Duplicator replies with a function $G : Y \longrightarrow \mathcal{P}^+(B)$.
   - The game continues at position $(X[v \mapsto F], Y[v \mapsto G])$ where v is a new variable.
3. Move $\forall$:
   - The game continues at position $(X[v \mapsto A], Y[v \mapsto B])$ where v is a new variable.

Positions (X, Y) with $\mathfrak{A} \models_X \vartheta$ but $\mathfrak{B} \not\models_Y \vartheta$ for some literal $\vartheta \in \mathrm{FO}(\Omega_\approx)$ are Spoiler's winning position. Duplicator wins, if such positions are avoided for n moves.

The game $\mathcal{G}_{\Omega_\approx}(\mathfrak{A}, \mathfrak{B})$ is played similarly: first Spoiler chooses a number $n \in \mathbb{N}$ and then $\mathcal{G}_{\Omega_\approx,n}(\mathfrak{A}, \mathfrak{B})$ is played.

These games characterize semi-equivalences of $\mathfrak{A}$ and $\mathfrak{B}$ (up to a certain depth). The depth of $\varphi \in \mathrm{FO}(\Omega_\approx)$, denoted as $\mathrm{depth}(\varphi)$, is defined inductively:

$$\begin{aligned}
\mathrm{depth}(\vartheta) &:= 0 \text{ for every literal } \vartheta \in \mathrm{FO}(\Omega_\approx)\\
\mathrm{depth}(\exists v \varphi') &:= \mathrm{depth}(\varphi') + 1 =: \mathrm{depth}(\forall v \varphi')\\
\mathrm{depth}(\varphi_1 \vee \varphi_2) &:= \max(\mathrm{depth}(\varphi_1), \mathrm{depth}(\varphi_2)) + 1\\
\mathrm{depth}(\varphi_1 \wedge \varphi_2) &:= \max(\mathrm{depth}(\varphi_1), \mathrm{depth}(\varphi_2))
\end{aligned}$$

Definition 5.20 (Semi-equivalence, [Vää07]). Let $\mathfrak{A}, \mathfrak{B} \in (\tau, \approx)$ and X, Y be teams over $\mathfrak{A}, \mathfrak{B}$ with $\mathrm{dom}(X) = \mathrm{dom}(Y)$. We write $\mathfrak{A}, X \Rrightarrow_{\Omega_\approx, n} \mathfrak{B}, Y$ (and say that $\mathfrak{A}, X$ is semi-equivalent to $\mathfrak{B}, Y$ up to depth n), if $\mathfrak{A} \models_X \varphi$ implies $\mathfrak{B} \models_Y \varphi$ for every $\varphi \in \mathrm{FO}(\Omega_\approx)$ with $\mathrm{depth}(\varphi) \leq n$. Furthermore, we write $\mathfrak{A}, X \Rrightarrow_{\Omega_\approx} \mathfrak{B}, Y$, if $\mathfrak{A}, X \Rrightarrow_{\Omega_\approx, n} \mathfrak{B}, Y$ for every $n \in \mathbb{N}$. When $\Omega_\approx$ is clear from the context, we sometimes omit it as a subscript.

In first-order logic, the concept of semi-equivalence coincides with the usual equivalence concept between structures, but this is not the case in logics with team semantics. For example $\mathfrak{A}, X \Rrightarrow \mathfrak{B}, \emptyset$ follows from the empty team property, but $\mathfrak{B}, \emptyset \Rrightarrow \mathfrak{A}, X$ is not true in general. We write $\mathfrak{A}, X \equiv_n \mathfrak{B}, Y$, if $\mathfrak{A}, X \Rrightarrow_n \mathfrak{B}, Y$ and $\mathfrak{B}, Y \Rrightarrow_n \mathfrak{A}, X$. $\mathfrak{A}, X \equiv \mathfrak{B}, Y$ is defined analogously.

Theorem 5.21. *Let τ be a finite signature and $\mathfrak{A}, \mathfrak{B} \in (\tau, \approx)$. Duplicator has a winning strategy for $\mathcal{G}_{\Omega_\approx, n}(\mathfrak{A}, \mathfrak{B})$ from position (X, Y) if, and only if $\mathfrak{A}, X \Rrightarrow_{\Omega_\approx, n} \mathfrak{B}, Y$.*

Having these games at our disposal, we can prove that $\mathrm{FO}(\subseteq_\approx, |_\approx)$ is strictly less powerful than Σ^1_1. Consider the following problem:

$$\mathcal{C}_{\mathrm{even}} := \{\mathfrak{A} \in (\tau, \approx) : \text{there is some } a \in A \text{ such that } |[a]_\approx| \text{ is even}\}.$$

Theorem 5.22. *$\mathcal{C}_{even}$ is not expressible in* $\mathrm{FO}(\subseteq_\approx, |_\approx)$.

We just give a short sketch of the proof: Consider $\mathfrak{A}_m := (A_m, \approx^{\mathfrak{A}_m})$ and $\mathfrak{B}_m := (B_m, \approx^{\mathfrak{B}_m})$ where $|A_m| = 2m$, $|B_m| = 2m + 1$, $\approx^{\mathfrak{A}_m} := A_m \times A_m$ and $\approx^{\mathfrak{B}_m} := B_m \times B_m$. Then $\mathfrak{A}_m \in \mathcal{C}_{\mathrm{even}}$ while $\mathfrak{B}_m \notin \mathcal{C}_{\mathrm{even}}$. It is not difficult to prove that Duplicator wins the games $\mathcal{G}_m(\mathfrak{A}_m, \mathfrak{B}_m)$ and $\mathcal{G}_m(\mathfrak{B}_m, \mathfrak{A}_m)$ by maintaining as an invariant that the equality types induced by the assignments in the two teams are always equal.

On the other hand, it is easy to see that $\mathrm{FO}(\mathrm{dep}_\approx)(\leq \mathrm{FO}(\subseteq_\approx, |_\approx))$ can express that the number of equivalence classes is even, but this is not definable in first-order logic.

Corollary 5.23. $\mathrm{FO} < \mathrm{FO}(\subseteq_\approx, |_\approx) < \Sigma^1_1$.

Next we study whether restrictions imposed on the given equivalence influence the expressive power of Σ^1_1. Consider the class $\mathcal{K}_{\leq p}$ of structures $\mathfrak{A} \in (\tau, \approx)$ where every equivalence class of $\mathfrak{A}$ is of size $\leq p$. On $\mathcal{K}_{\leq 1}$, $\Sigma^1_1(\approx)$ has the same expressive power as Σ^1_1, because every relation over $\mathfrak{A} \in \mathcal{K}_{\leq 1}$ is $\approx^{\mathfrak{A}}$-closed. However, this is not the case for $p \geq 2$ as the next theorem shows.

Theorem 5.24. *Let $p \geq 2$. $\mathrm{FO} < \mathrm{FO}(\subseteq_\approx, |_\approx) < \Sigma^1_1$ holds on the class $\mathcal{K}_{\leq p}$ of structures $\mathfrak{A} \in (\tau, \approx)$ with $|[a]_\approx| \leq p$ for every $a \in A$.*

Proof. It suffices to prove this for $p = 2$. Let $\tau = \{E, \approx\}$. Consider the following problem: $C := \{\mathfrak{A} \in \mathcal{K}_{\leq 2} : (A, E^{\mathfrak{A}}) \text{ is not connected}\}$. By using the method of Ehrenfeucht-Fraïssé we will show that C is not definable in $\mathrm{FO}(\subseteq_\approx, |_\approx)$.

For every $m > 3$ let $\mathfrak{A}_m := (A_m, E^{\mathfrak{A}_m}, \approx)$ and $\mathfrak{B}_m := (B_m, E^{\mathfrak{B}_m}, \approx)$ where $A_m := \{0, \ldots, m-1\} \cup \{0', \ldots, (m-1)'\} =: B_m$ and $E^{\mathfrak{A}_m} := E^{\mathfrak{A}_m}_+ \cup E^{\mathfrak{A}_m}_-$ with

$$E^{\mathfrak{A}_m}_+ := \{(i, j), (i', j') : j = i + 1 \ (\mathrm{mod}\ m)\}$$

and $E_-^{\mathfrak{A}_m} := \{(w, v) : (v, w) \in E_+^{\mathfrak{A}_m}\}$. Similary, $E^{\mathfrak{B}_m} := E_+^{\mathfrak{B}_m} \cup E_-^{\mathfrak{B}_m}$ where $E_+^{\mathfrak{B}_m} := \{(0,1), (1,2), \dots, (m-2, m-1), (m-1, 0'), (0', 1'), \dots, ((m-2)', (m-1)'), ((m-1)', 0)\}$ and $E_-^{\mathfrak{B}_m} := \{(w, v) : (v, w) \in E_+^{\mathfrak{B}_m}\}$. $\approx$ is in both structures defined such that $[i]_\approx = \{i, i'\}$ for every $i \in \{0, \dots, m-1\}$. In other words, $\mathfrak{A}_m$ consists of two cycles $(0, 1, \dots, m-1, 0)$ and $(0', 1', \dots, (m-1)', 0')$ of length m, while $\mathfrak{B}_m$ is a single cycle $(0, 1, \dots, m-1, 0', 1', \dots, (m-1)', 0)$ of length $2m$.

For every $v \in \{0, 1, \dots, m-1, 0', 1', \dots, (m-1)'\}$ there are uniquely determined $s^{\mathfrak{A}_m}(v)$ and $s^{\mathfrak{B}_m}(v)$ such that $(v, s^{\mathfrak{A}_m}(v)) \in E_+^{\mathfrak{A}_m}$ and $(v, s^{\mathfrak{B}_m}(v)) \in E_+^{\mathfrak{B}_m}$. Similarly, there are exists uniquely determined predecessors $(s^{\mathfrak{A}_m})^{-1}(v)$ and $(s^{\mathfrak{B}_m})^{-1}(v)$ with $(v, (s^{\mathfrak{A}_m})^{-1}(v)) \in E_-^{\mathfrak{A}_m}$ and $(v, (s^{\mathfrak{A}_m})^{-1}(v)) \in E_-^{\mathfrak{B}_m}$. We define for every $v \in A_m$, every $w \in B_m$ and every $k \in \mathbb{Z}$

$$v +_{\mathfrak{A}_m} k := (s^{\mathfrak{A}_m})^k(v) \text{ and } w +_{\mathfrak{B}_m} k := (s^{\mathfrak{B}_m})^k(w).$$

We are going omit $\mathfrak{A}_m$ and $\mathfrak{B}_m$ as a subscript, when it is clear from the context that v belongs to $\mathfrak{A}_m$ resp. $\mathfrak{B}_m$.

For $v, w \in A_m$ we define $\mathrm{dist}_{\mathfrak{A}_m}(v, w)$ to be the minimal number $n \in \mathbb{N}$ such that $v + n = w$ or $v - n = w$, or ∞, if no such number $n \in \mathbb{N}$ exists. $\mathrm{dist}_{\mathfrak{B}_m}(v, w)$ is defined analogously. Please note, that $\mathrm{dist}_{\mathfrak{A}_m}(v, w) = \mathrm{dist}_{\mathfrak{A}_m}(w, v)$ and $\mathrm{dist}_{\mathfrak{B}_m}(v, w) = \mathrm{dist}_{\mathfrak{B}_m}(w, v)$. Furthermore, for every $a \in \{0, 1, \dots, m-1, 0', 1', \dots, (m-1)'\}$ and every $b, c \in \mathbb{Z}$ holds,

$$(a +_{\mathfrak{A}_m} b) +_{\mathfrak{A}_m} c = a +_{\mathfrak{A}_m} (b + c) \text{ and } (a +_{\mathfrak{B}_m} b) +_{\mathfrak{B}_m} c = a +_{\mathfrak{B}_m} (b + c).$$

It is easy to see that $\mathrm{dist}(v_1, v_3) \leq \mathrm{dist}(v_1, v_2) + \mathrm{dist}(v_2, v_3)$ for every v_1, v_2, v_3 from A_m or B_m. Furthermore, $v \approx w$ implies that $s^{\mathfrak{A}_m}(v) \approx s^{\mathfrak{B}_m}(v)$ and $(s^{\mathfrak{A}_m})^{-1}(v) \approx (s^{\mathfrak{B}_m})^{-1}(v)$. This observation leads to the following claim.

Claim 5.25. Let $v \in A_m$, $w \in B_m$ with $v \approx w$. Then $v + k \approx w + k$ for every $k \in \mathbb{Z}$.

For every $i, j, q \in \mathbb{N}$ we write $i \approx_q j$ if, and only if $i = j$ or $i \geq q \leq j$. Given two assignments $s : \{x_1, \dots, x_\ell\} \longrightarrow A_m$ and $t : \{x_1, \dots, x_\ell\} \longrightarrow B_m$, we write $s \approx_q t$ if, and only if $s(x_i) \approx t(x_i)$ (which is equivalent to: $s(x_i), t(x_i) \in \{n, n'\}$ for some $n \in \{0, \dots, m-1\}$) and $\mathrm{dist}_{\mathfrak{A}_m}(s(x_i), s(x_j)) \approx_q \mathrm{dist}_{\mathfrak{B}_m}(t(x_i), t(x_j))$ holds for every $i, j \in \{1, \dots, \ell\}$.

Lemma 5.26. *Let $m > 2^{n+2}$ and $0 \leq \ell \leq k < n$. Furthermore, let $s : \{x_1, \dots, x_\ell\} \longrightarrow A_m$ and $t : \{x_1, \dots, x_\ell\} \longrightarrow B_m$ be two assignments with $s \approx_{2^{n+1-k}} t$. Then:*

(1) For every $a \in A_m$ there exists some $b = b(s, t, a) \in B_m$ such that

$$s' := s[x_{\ell+1} \mapsto a] \approx_{2^{n-k}} t[x_{\ell+1} \mapsto b] =: t'.$$

(2) For every $b \in B_m$ there exists some $a = a(s, t, b) \in A_m$ such that

$$s' := s[x_{\ell+1} \mapsto a] \approx_{2^{n-k}} t[x_{\ell+1} \mapsto b] =: t'.$$

Furthermore, for two teams X, Y over $\mathfrak{A}_m, \mathfrak{B}_m$ with $\mathrm{dom}(X) = \{x_1, \dots, x_\ell\} = \mathrm{dom}(Y)$ we write $X \approx_q Y$ if, and only if for every $s \in X$ there exists some $t \in Y$ and, conversely, for every $t \in Y$ there exists some $s \in X$ such that $s \approx_q t$.

Claim 5.27. Let $n, m \in \mathbb{N}$ with $m > 2^{n+2}$. Duplicator has a winning strategy in $\mathcal{G}_n(\mathfrak{A}_m, \mathfrak{B}_m)$.

Thus we have $\mathfrak{A}_m \Rrightarrow_n \mathfrak{B}_m$ for every $m > 2^{n+2}$. Using very similar arguments, it is possible to prove that $\mathfrak{B}_m \Rrightarrow_n \mathfrak{A}_m$. Furthermore, we have $\mathfrak{A}_m \in C$ and $\mathfrak{B}_m \notin C$. This proves that C is not definable in $\mathrm{FO}(\subseteq_\approx, |_\approx)$ (because φ is unable to distinguish between $\mathfrak{A}_m$ and $\mathfrak{B}_m$ for every $m > 2^{\mathrm{depth}(\varphi)+2}$). On the other hand, C is definable in Σ^1_1 by the sentence

$$\exists X \exists x \exists y (Xx \wedge \neg Xy \wedge \forall u \forall v (Xu \wedge Euv \longrightarrow Xv)).$$

This concludes the proof of $\mathrm{FO}(\subseteq_\approx, |_\approx) < \Sigma^1_1$. $\mathrm{FO} < \mathrm{FO}(\subseteq_\approx, |_\approx)$ follows from the fact that $\mathrm{FO}(|_\approx) \equiv \mathrm{FO}(\mathrm{dep}_\approx)$ and that the sentence

$$\forall x \exists y \forall x' \exists y' \big(\mathrm{dep}_\approx(x, y) \wedge \mathrm{dep}_\approx(x', y') \wedge x \not\approx y \wedge (x \not\approx x' \vee y \approx y') \wedge (x \not\approx y' \vee y \approx x') \big)$$

expresses that the number of equivalence classes is even, but this is not definable in first-order logic (this can be proven by using the method of Ehrenfeucht-Fraïssé for FO). □

Restricting the number of equivalence classes is not really interesting, because this leads to a situation where $\Sigma^1_1(\approx)$ has the same expressive power as FO, because there are only $2^{(k^r)}$ many $\approx$-closed relations of arity r when k is the number of $\approx$-classes, which can be simulated in first-order logic.

Another possible restriction is to admit only a bounded number of non-trivial equivalence classes (which consist of more than one element). Let $\mathcal{K}_{\mathrm{NT} \le p}$ be the class of all $\mathfrak{A} \in (\tau, \approx)$ with at most p many non-trivial equivalence classes (for some $p \ge 1$).

But then again, $\mathcal{C}_{\mathrm{even}} \cap \mathcal{K}_{\mathrm{NT} \le p}$ is not definable in $\mathrm{FO}(\subseteq_\approx, |_\approx)$ on $\mathcal{K}_{\mathrm{NT} \le p}$. Hence, we also have $\mathrm{FO} < \Sigma^1_1(\approx) < \Sigma^1_1$ on $\mathcal{K}_{\mathrm{NT} \le p}$.

However, combining the conditions imposed on the number of non-trivial equivalence and their size, leads to an interesting situation: $\Sigma^1_1(\approx) \equiv \Sigma^1_1$ on the class $\mathcal{K}_{\mathrm{NT} \le p_1, \le p_2} := \mathcal{K}_{\mathrm{NT} \le p_1} \cap \mathcal{K}_{\le p_2}$. The reason for this is that at most $p_1 \cdot p_2$ many elements are located inside non-trivial equivalence classes, while all the other elements are only equivalent to themselves. Since $\Sigma^1_1(\approx)$ allows us to obtain a linear order on the equivalence classes, it is possible to encode arbitrary relations and, hence, to simulate Σ^1_1.

In this chapter we have defined logics with dependency concepts up to given equivalences and analysed their expressive power by comparing these logics with certain fragments of Σ^1_1 and GFP. Furthermore, we have studied the behaviour of $\mathrm{FO}(\subseteq_\approx, |_\approx)$ on certain restricted classes of structures.

6 Concluding Remarks

We have addressed two open problems that have been presented at the Dagstuhl seminar 2019 [GKKV19] and we have recapitulated dependency concepts up to equivalence from [GH18].

For the union closed fragment, different characterisations have been presented. On the level of Σ^1_1, this fragment has been characterised as the myopic fragment of Σ^1_1, while on the side of team semantics union closure is captured by the $\bar{x}$-myopic fragment of FO($\subseteq$, $|$) and the logic FO($\cup$−game). Furthermore, a novel class of games, the inclusion-exclusion game, have been introduced and we have seen that these games are indeed the model-checking games for sentences $\varphi(X) \in \Sigma^1_1$. The union games, which have been defined as a restricted variant of these games, are not only the model-checking games for myopic Σ^1_1-sentences, but have also led to insights regarding the essence of the union closed fragment. Given that Σ^1_1 is a rather strong logic, it is conceivable that other variants of these games are useful in other scenarios.

Regarding the question of Galliani and Hella [GH13], the atom $\cup$−game that captures the union closed fragment and syntactic restrictions of inclusion-exclusion formulae have been introduced in Chapter 3 and in [HW19, HW20]. In terms of “naturalness” the atom $\cup$−game might be considered be cumbersome to use, because the logic FO($\cup$−game) is not really well-suited for writing down concrete formulae. In contrast to that, it is a much better idea to use either myopic Σ^1_1-sentence or, equivalently, the $\bar{x}$-myopic fragment of inclusion-exclusion logic for writing concrete formulae. A possible future question could be whether or not there is a “more natural” (whatever this means) atom instead of $\cup$−game. However, whatever other atom one proposes in place of $\cup$−game, it must solve a problem not expressible in first-order logic, because Galliani and Hella showed that inclusion logic is able to express all first-order definable union closed properties.

Regarding Rönnholm’s question [GKKV19] of whether FO($\subseteq$)$[k]$ corresponds to some fragment of GFP$^+$, we have defined GFP$^+[k]$ and showed how formulae between these different fragments can be translated into the corresponding other one. An interesting open question for future research is whether or not an analogue of Theorem 3.7 holds for GFP$^+$-sentences: is the union closed fragment of GFP$^+$ captured by myopic GFP$^+$-sentences?

Finally, in Chapter 5 we have explored dependency concepts up to equivalence. We have compared logics using these concepts with certain fragments of Σ^1_1 and GFP$^+$. Our results are mostly about the expressive power of these logics, while future work in this area might center around finding concrete applications for these or similar dependency concepts.

All in all, we have investigated different fragments of Σ_1^1 and of logics with team semantics, syntactical characterisations of semantical properties were presented and new concepts like the inclusion-exclusion games were introduced, that might aid future research.

Bibliography

[Ajt83] M. Ajtai. Σ^1_1-Formulae on finite structures. *Annals of Pure and Applied Logic*, 24(1):1 – 48, 1983.

[BG86] A. Blass and Y. Gurevich. Henkin Quantifiers and Complete Problems. *Annals of Pure and Applied Logic 32 (1986), 1-16*, 32:1–16, April 1986.

[Blu18] A. Blumensath. Set theory. In *Logic, Algebra and Geometry*. Digital version, https://www.fi.muni.cz/ blumens/st.pdf, 2018. Book in preparation.

[End70] H. Enderton. Finite Partially-Ordered Quantifiers. *Z. Math. Logik*, 16:393–397, 1970.

[Eng12] F. Engström. Generalized Quantifiers in Dependence Logic. *Journal of Logic, Language, and Information*, 2012.

[Gal12] P. Galliani. Inclusion and Exclusion Dependencies in Team Semantics: On Some Logics of Imperfect Information. *Annals of Pure and Applied Logic*, 163(1):68–84, 2012.

[GH13] P. Galliani and L. Hella. Inclusion Logic and Fixed Point Logic. In Simona Ronchi Della Rocca, editor, *Computer Science Logic 2013 (CSL 2013)*, volume 23 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 281–295, Dagstuhl, Germany, 2013. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[GH18] E. Grädel and M. Hoelzel. Dependency Concepts up to Equivalence. In Dan Ghica and Achim Jung, editors, *27th EACSL Annual Conference on Computer Science Logic (CSL 2018)*, volume 119 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 25:1–25:21, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[GKKV19] E. Grädel, P. G. Kolaitis, J. Kontinen, and H. Vollmer. Logics for Dependence and Independence (Dagstuhl Seminar 19031). *Dagstuhl Reports*, 9(1):28–46, 2019.

[GKL+07] E. Grädel, P. G. Kolaitis, L. Libkin, M. Marx, J. Spencer, M. Y. Vardi, Y. Venema, and S. Weinstein. *Finite Model Theory and Its Applications*. Springer-Verlag, Berlin Heidelberg, 2007.

[Grä13] E. Grädel. Model-Checking Games for Logics of Incomplete Information. *Theoretical Computer Science, Special Issue dedicated to GandALF 2011*, pages 2–14, 2013.

[Grä16] E. Grädel. Games for Inclusion Logic and Fixed-Point Logic. In Samson Abramsky and Others, editors, *Dependence Logic: Theory and Applications*, pages 73–98. Birkhäuser, 2016.

[GV13] E. Grädel and J. Väänänen. Dependence and Independence. *Studia Logica*, 101(2):399–410, 2013.

[Han15] M. Hannula. Hierarchies in Inclusion Logic with Lax Semantics. In M. Banerjee and S. N. Krishna, editors, *Logic and Its Applications*, pages 100–118, Berlin, Heidelberg, 2015. Springer Berlin Heidelberg.

[Hen61] L. Henkin. Some Remarks on Infinitely Long Formulas. In *Journal of Symbolic Logic*, pages 167–183. Pergamon Press, 1961.

[Hod97a] W. Hodges. *A Shorter Model Theory*. Cambridge University Press, New York, NY, USA, 1997.

[Hod97b] W. Hodges. Compositional Semantics for a Language of Imperfect Information. *Logic Journal of IGPL*, 5:539–563, 1997.

[HS89] J. Hintikka and G. Sandu. Informational Independence as a Semantical Phenomenon. In *Studies in Logic and Foundations of Mathematics*, volume 126, pages 571–589. North-Holland, 1989.

[HW19] M. Hoelzel and R. Wilke. On the Union Closed Fragment of Existential Second-Order Logic and Logics with Team Semantics. *arXiv e-prints*, page arXiv:1910.06057, Oct 2019.

[HW20] M. Hoelzel and R. Wilke. On the Union Closed Fragment of Existential Second-Order Logic and Logics with Team Semantics. In M. Fernández and A. Muscholl, editors, *28th EACSL Annual Conference on Computer Science Logic (CSL 2020)*, volume 152 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 25:1–25:16, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[KV09] J. Kontinen and J. Väänänen. On Definability in Dependence Logic. *Journal of Logic, Language and Information*, 18(3):317–332, 2009.

[MSS12] A. Mann, G. Sandu, and M. Sevenster. *Independence-Friendly Logic. A Game-Theoretic Approach*, volume 386 of *London Mathematical Society Lecture Notes Series*. Cambridge University Press, 2012.

[Rön18] R. Rönnholm. Capturing k-ary existential second-order logic with k-ary inclusion-exclusion logic. *Ann. Pure Appl. Logic*, 169(3):177–215, 2018.

[Vää07] J. Väänänen. *Dependence Logic: A New Approach to Independence Friendly Logic*, volume 70. Cambridge University Press, 2007.

[vB76] Johan van Benthem. *Modal Correspondence Theory*. PhD dissertation, University of Amsterdam, 1976.

[Wal70] W. Walkoe. Finite Partially-Ordered Quantification. *Journal of Symbolic Logic*, 35:535–555, 1970.