The Distribution of Branches in River Networks

GEOLOGICAL SURVEY PROFESSIONAL PAPER 422-G

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By ENNIO V. GIUSTI *and* WILLIAM J. SCHNEIDER

PHYSIOGRAPHIC AND HYDRAULIC STUDIES OF RIVERS

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THE DISTRIBUTION OF BRANCHES IN RIVER NETWORKS

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ABSTRACT

Bifurcation ratios derived from streams ordered according to the Strahler system are not wholly independent of the stream orders from which they are computed and, within a basin, tend to decrease in a downstream direction. The bifurcation ratios computed from two successive constant orders of streams within equal-order basins increase with the area of the basin but tend to become constant where the basin reaches a certain size.

The distribution of the number of major tributaries (streams of one order-lag) is exponential with a maximum frequency of two and can be expressed as the probability function

$$p=e^{-\frac{1}{2}N_{u-1}}.$$

The distribution of the number of smaller tributaries is both unimodal and skewed and there appears to be an orderly succession of distributions from exponential to normal from the major tributaries to the smallest fingertip branches.

INTRODUCTION

Many geomorphic studies make use of an ordering system of stream branches proposed by Horton (1945). A fundamental parameter of this ordering system is the bifurcation ratio, which is defined as the ratio of the number of stream branches of a given order to the number of stream branches of the next higher order. This ratio can be expressed by

$$R_b=\frac{N_u}{N_{u+1}} \tag{1}$$

where R_b=bifurcation ratio,

N_u=number of streams of given order,

and N_{u+1}=number of streams of next higher order.

According to Horton (1945, p. 290), the bifurcation ratio varies from a minimum of 2 in "flat or rolling drainage basins" to 3 or 4 in "mountainous or highly dissected drainage basins"; it is a parameter used in equations giving the number of streams in a basin. As expressed by Horton, the equation is

$$N_u=R_b^{(s-u)} \tag{2}$$

where s=order of main stream

and u=given order.

Strahler (1957) expresses the equation as

$$\log N_u=a-bu \tag{3}$$

where the antilog of b is the bifurcation ratio. He further states that the bifurcation ratio is "highly stable and shows a small range of variation from region to region." The average (mean) is about 3.5.

This paper discusses the distribution of bifurcation ratios and further analyzes the distribution of the number of any order tributary in a basin.

SOURCES OF DATA

Several sources of data were used for this study. Special photogrammetrically prepared drainage maps of 130 square miles of the Yellow River basin in the Piedmont province of Georgia were analyzed to determine the distribution of streams in the drainage network of the basin. The maps, compiled at the scale of 1:24,000 delineated all drainage courses visible on aerial photographs taken at a flight height of 7,200 feet. Additional data were obtained from 108 standard topographic maps, ranging in scale from 1:24,000 to 1:250,000, of the Piedmont province. Data published by Melton (1957), Coates (1958), and Leopold and Langbein (1962) were used also for analyses. Streams were classified according to a system of ordering proposed by Horton (1945) and modified by Strahler (1957).

A summary of the distribution of number of streams of a given order within the Yellow River drainage system is shown in table 1. The 130-square-mile area of the Yellow River basin consisted of two seventh-order streams. The table lists separately the distribution of streams in each seventh-order basin.

VARIABILITY OF BIFURCATION RATIOS

The number of streams of any order in each of the two seventh-order basins that comprise the part of the Yellow River drainage system analysed here are plotted against their order in figure 1. The figure is typical in that some curvature exists in the range of higher order subbasins. This effect is also shown in

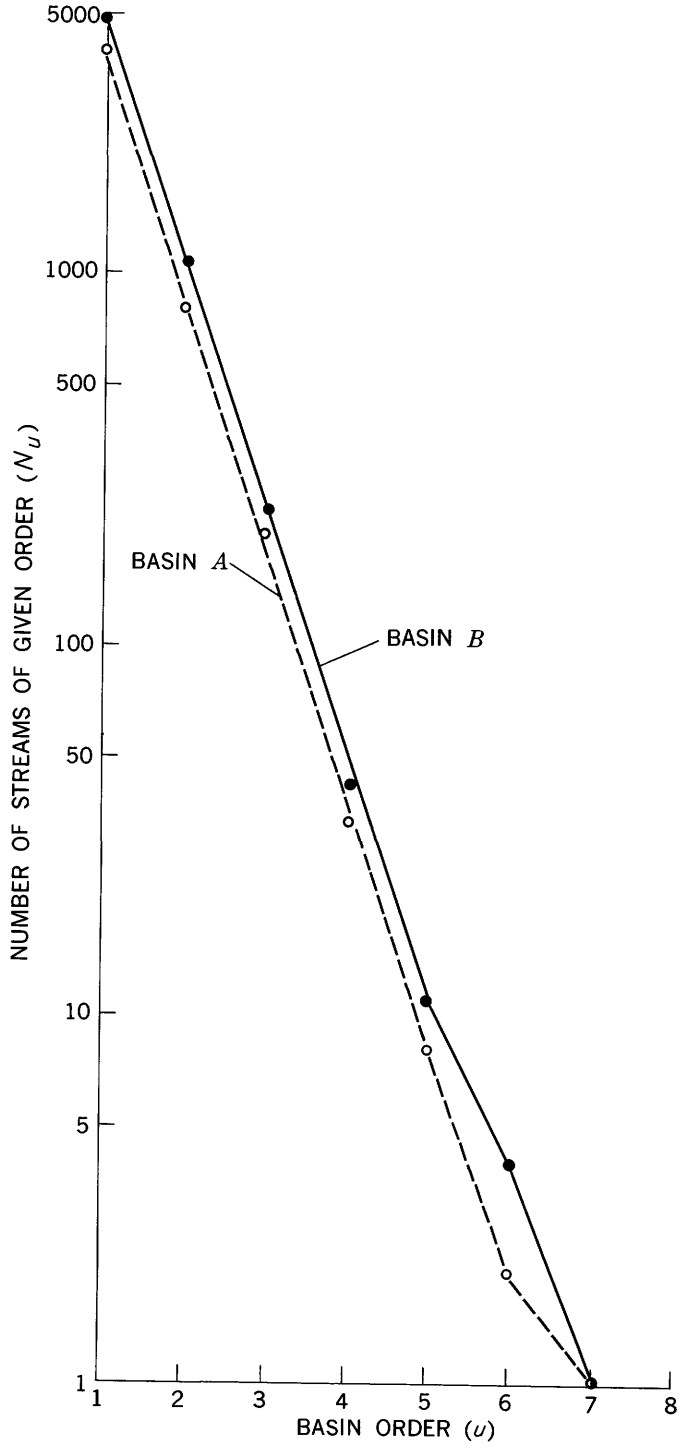

FIGURE 1.—Relation between number of streams and stream order for the Yellow River.

figure 2 where the bifurcation ratios computed as N_1/N_2, N_2/N_3 * * * N_6/N_7 from the values of table 1 are plotted against the basin order. The position of the point suggests that the bifurcation ratio is highly variable. In order to clarify this variability, mean bifurcation ratios and their standard deviation were computed from 26 subbasins of fifth, sixth, and seventh order

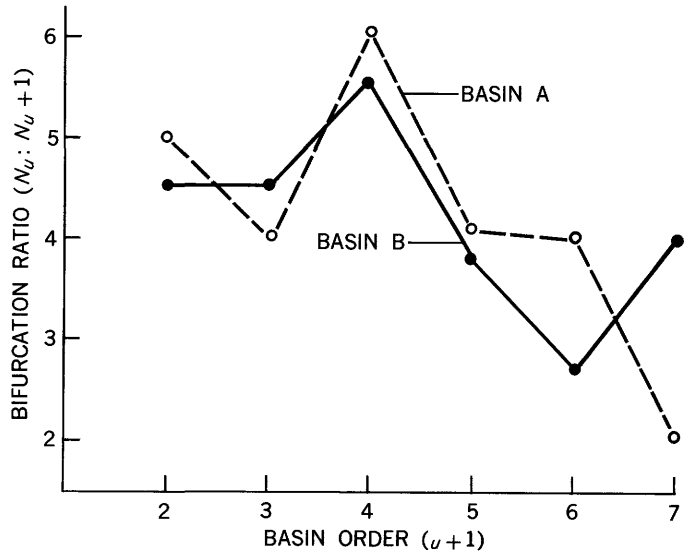

FIGURE 2.—Relation between bifurcation ratios and order for two seventh-order basins.

within the Yellow River drainage system. These data are shown in table 2. Both figure 2 and table 2 suggest (1) that bifurcation ratios expressed as the number of next lower order tributaries or N_{s-1} are smaller than bifurcation ratios computed from lower order tributaries and (2) that bifurcation ratios tend to become constant and less variable (smaller standard deviation) when expressed as ratios between the numbers of lower order branches.

Figure 3 suggests a relation between the bifurcation ratio and the drainage area. This relation, however, depends entirely upon the relation between order and area and can be expressed as follows: Basins of equal order but variable areas tend to have the smallest bifurcation ratios in the smallest areas; the ratios increase with increasing areas up to a certain size, beyond which the bifurcation ratios tend to become constant.

TABLE 1.—*Distribution of streams, by order, within the Yellow River drainage system*

Subbasin	Number of streams (N_u) of order—						
	1	2	3	4	5	6	7
A	4,026	812	200	33	8	2	1
B	4,823	1,060	233	42	11	4	1

TABLE 2.—*Means and standard deviations of bifurcation ratios for Yellow River drainage*

	$\frac{N_{s-1}}{N_s}$	$\frac{N_{s-2}}{N_{s-1}}$	$\frac{N_{s-3}}{N_{s-2}}$	$\frac{N_{s-4}}{N_{s-3}}$	$\frac{N_{s-5}}{N_{s-4}}$	$\frac{N_{s-6}}{N_{s-5}}$
Mean (R_b)	3.2	4.5	4.7	4.8	[1] 4.6	[2] 4.75
Standard deviation (σ)	1.3	1.5	.75	.54		

[1] Computed from 6 basins of 6th order and 2 basins of 7th order.
[2] Computed from 2 basins of 7th order.

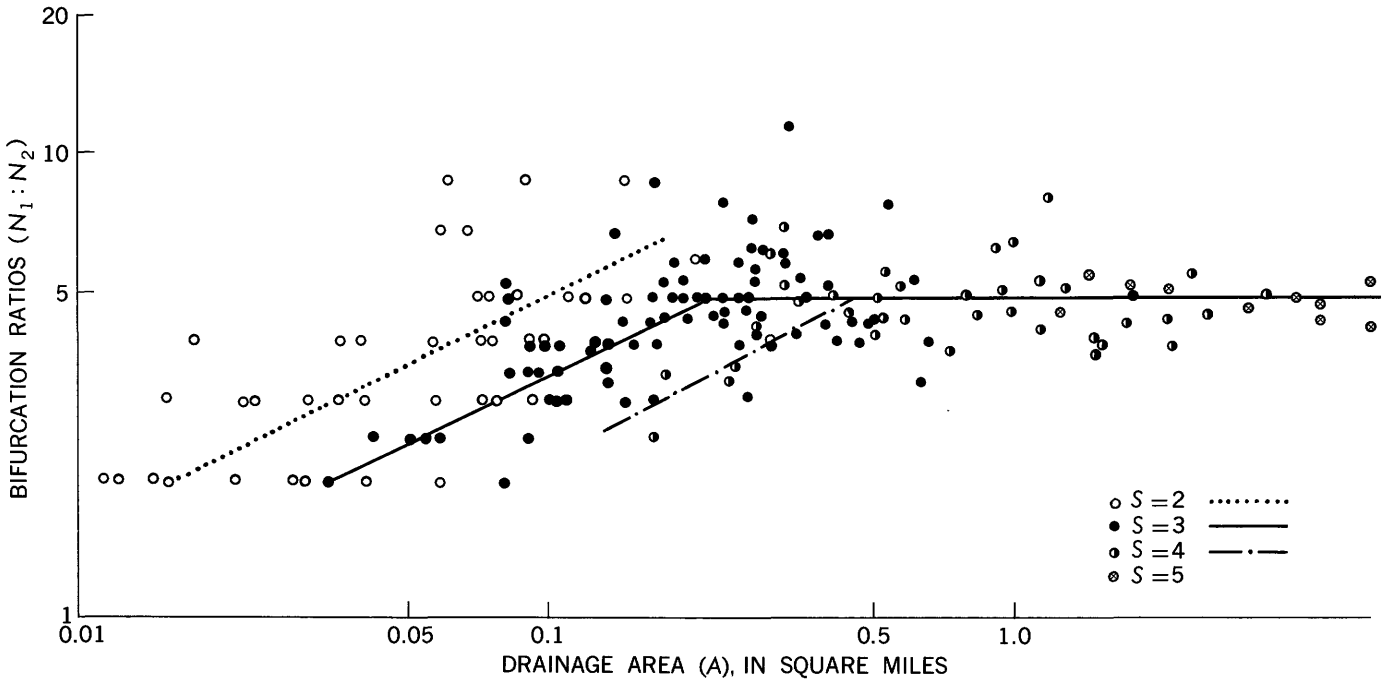

FIGURE 3.—Relation between drainage area and bifurcation ratios N_1:N_2 for subbasins within the Yellow River basin.

The fact that bifurcation ratios become smaller as they are computed from higher order subbasins is basically due to the branching process. Consider figure 4 where a fourth-order basin derived by a random-walk process is shown. For clarity, the same basin is shown successively in figure 4 with the number of streams of increasing order. It is apparent that as the stream order increases the percentage of streams that coalesce into a higher order tributary also increases and that this increase is due to the diminishing amount of area available. In figure 5, all the percentages of coalescing streams for the Yellow River and for the random-walk model developed by Leopold and Langbein (1962, p. A18) have been plotted against their order. The increase in a downstream (with higher order) direction, particularly for the Yellow River, is evident.

Thus, bifurcation ratios $\frac{N_{s-1}}{N_s}$ tend to become constant for values of $u \geq s-2$, where s is the order of the main stem.

DISTRIBUTION OF BIFURCATION RATIOS

Frequency distributions of bifurcation ratios are shown in figures 6 and 7. The distributions are derived from data on all fourth- and higher order subbasins and random sampling of first-, second-, and third-order subbasins within the Yellow River drainage system. Data obtained by Melton (1957) from arid-to-humid mountainous basins in Arizona, New Mexico, Colorado, and Utah, and data obtained by Coates (1958) for small basins in the humid Interior Low Plateaus province are included also.

According to Horton (1945, p. 296), equation 2 becomes

$$R_b = N_{s-1} \tag{4}$$

if $u=s-1$. This relation indicates that the bifurcation ratio, R_b, is equal to the number of streams of the next to the highest order for a given drainage basin. Thus, the bifurcation ratio as defined by equation 4 can be equated to the number of major tributaries to a given stream. The major tributaries can also be defined as streams of one order-lag. Thus, it follows that a drainage system will have streams of one order-lag (major tributaries), two order-lags (smaller tributaries), and so on, and the number of tributaries may be defined as N_{s-1}, N_{s-2}, * * * N_{s-n}, the subscripts indicating the relative position within the drainage system.

Figures 4 and 5 show distributions of bifurcation ratios which are grouped according to their order-lag. Thus ratios $\frac{N_2}{N_3}$ in basins of order 4 can be written as ratios $\frac{N_{s-2}}{N_{s-1}}$ and can be compared to ratios $\frac{N_1}{N_2}$ from third-order basins which are also expressed as $\frac{N_{s-2}}{N_{s-1}}$. The distributions of number of major tributaries (streams of one order-lag) are shown in figure 8. These distributions differ considerably from those of figures 6 and 7. Statistical parameters of the distributions of figures 6, 7, and 8 are tabulated in table 3.

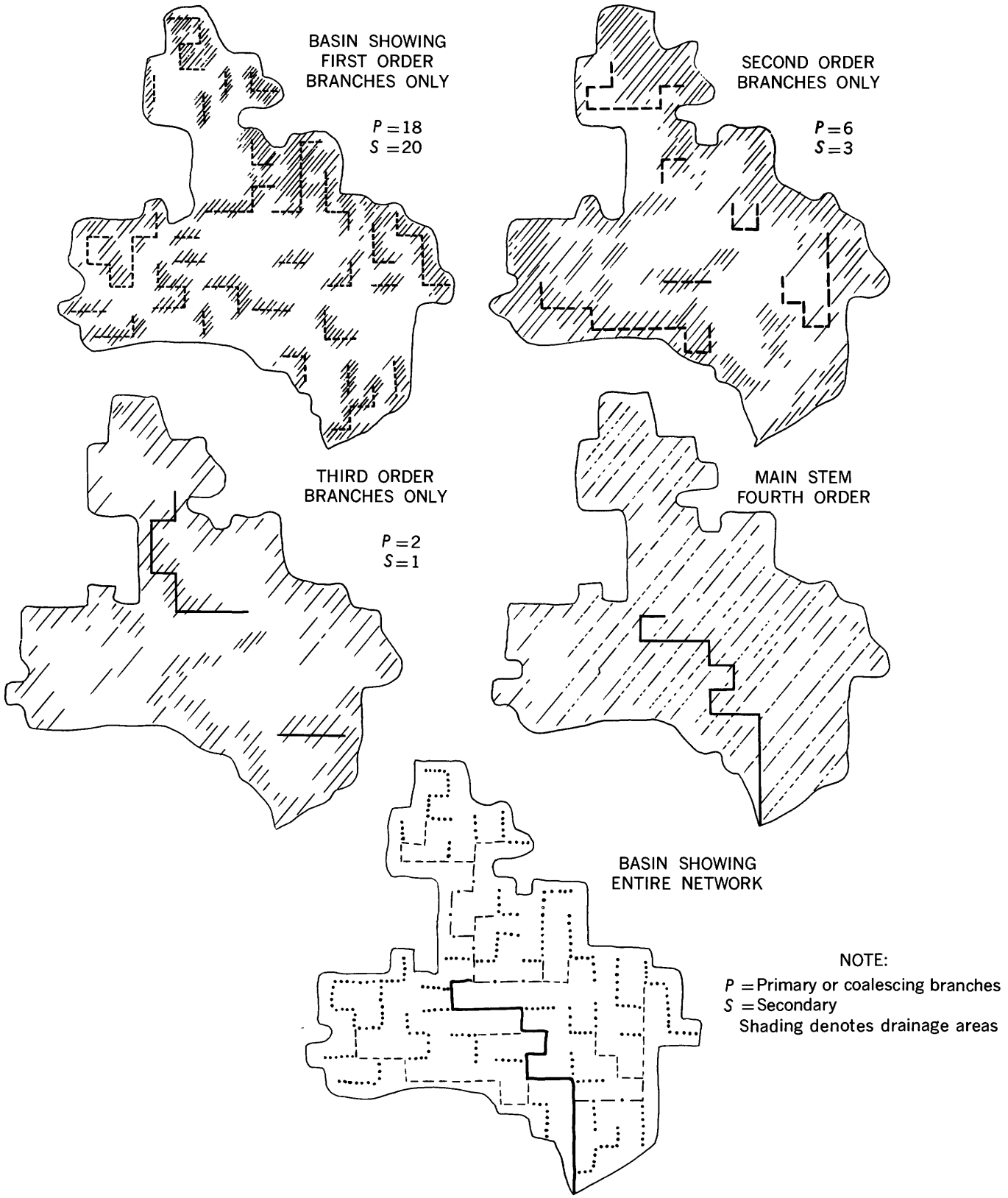

FIGURE 4.—Random-walk model of a fourth order basin (from Leopold and Langbein, 1962, fig. 8, p. A18).

TABLE 3—*Selected statistical data for distributions shown in figures 6, 7, and 8*

Statistic	Source of data										
	Yellow River				Melton (1958)			Coates (1959)		Leopold and Langbein (1962)	Topographic maps (blue lines)
Sample size	270	84	84	84	132	45	45	60	60	92	269
Basin order, s	2-7	3	4	4	3, 4	4	4	3	3	2, 3	2-6
Definition, R_b	N_{s-1}	$\frac{N_{s-1}}{N_s}$	$\frac{N_{s-2}}{N_{s-1}}$	$\frac{N_{s-3}}{N_{s-2}}$	N_{s-1}	$\frac{N_{s-2}}{N_{s-1}}$	$\frac{N_{s-3}}{N_{s-2}}$	N_{s-1}	$\frac{N_{s-2}}{N_{s-1}}$	N_{s-1}	N_{s-1}
Mean bifurcation ratios, $\bar{R}_b$	3.48	4.36	3.88	4.72	3.04	4.04	6.02	3.40	4.38	3.23	3.27
Standard deviation of bifurcation ratios, $S_{(R_b)}$	1.79	1.97	1.09	1.05	1.43	1.17	3.24	1.62	1.17	1.83	1.66
Skewness [1] of bifurcation ratios, $S_{k(R_b)}$	.83	.18	−.11	.21	.73	.034	.31	.86	.32	.67	.77

[1] Skewness calculated by the approximate formula, skewness$=\frac{\text{mean}-\text{mode}}{\text{standard deviation}}$.

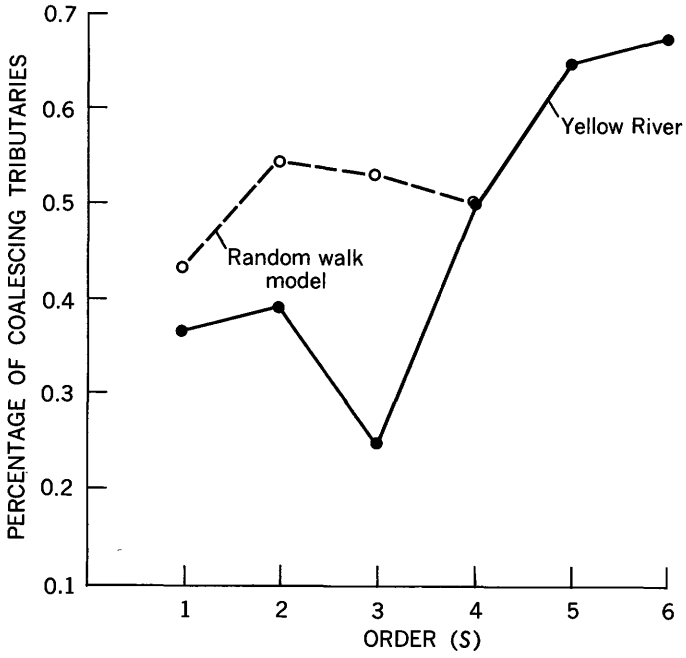

FIGURE 5.—Relation between percentage of coalescing tributaries and order.

DISTRIBUTION OF NUMBER OF MAJOR TRIBUTARIES

Because the distributions for the various data shown in figure 8 did not differ significantly from each other, the variations among the moments of these distributions were assumed to be due to sampling errors, and all data of figure 8 were combined. The resulting distribution is shown in figure 9 as a semilogarithmic function of the form

$$f=ke^{-k'N_{(s-1)}} \tag{5}$$

where f=the frequency of occurrence, in percent,
$N_{(s-1)}$=number of streams of one order-lag,
and e=base of Naperian logarithms.

The computed regression equation is

$$f=99.48e^{-0.517N_{s-1}} \tag{6}$$

which can be simplified to

$$f=100e^{-\frac{1}{2}N_{s-1}}. \tag{6a}$$

Because equation 6a represents a distribution, it may become a probability function as

$$p=e^{-\frac{1}{2}N_{s-1}} \tag{7}$$

from which the probability, p, can be computed for any given number of major tributaries.

Probabilities of occurrences for values of N_{s-1} up to 10 are shown in table 4. This table shows that more than one out of three streams similar to those studied here will have two major tributaries, and about one out of five three major tributaries. A second-order stream will most frequently have two first-order streams as branches; a third-order stream two second-order branches, and so on.

TABLE 4—*Probability of occurrence of number of major tributaries to a stream*

Number of major tributaries (N_{s-1})	*Probability of occurrence* (p)
2	0.368
3	.233
4	.135
5	.082
6	.050
7	.030
8	.018
9	.011
10	.007

Melton (1958) defines a "conservative drainage system" as "one having the minimum number of channel segments necessary for the highest order of the system." His measure of conservancy for any order u is

$$Su=\frac{R_{bu:u+1}}{2}-1 \tag{8}$$

and maximum conservancy corresponds to $Su=0$.

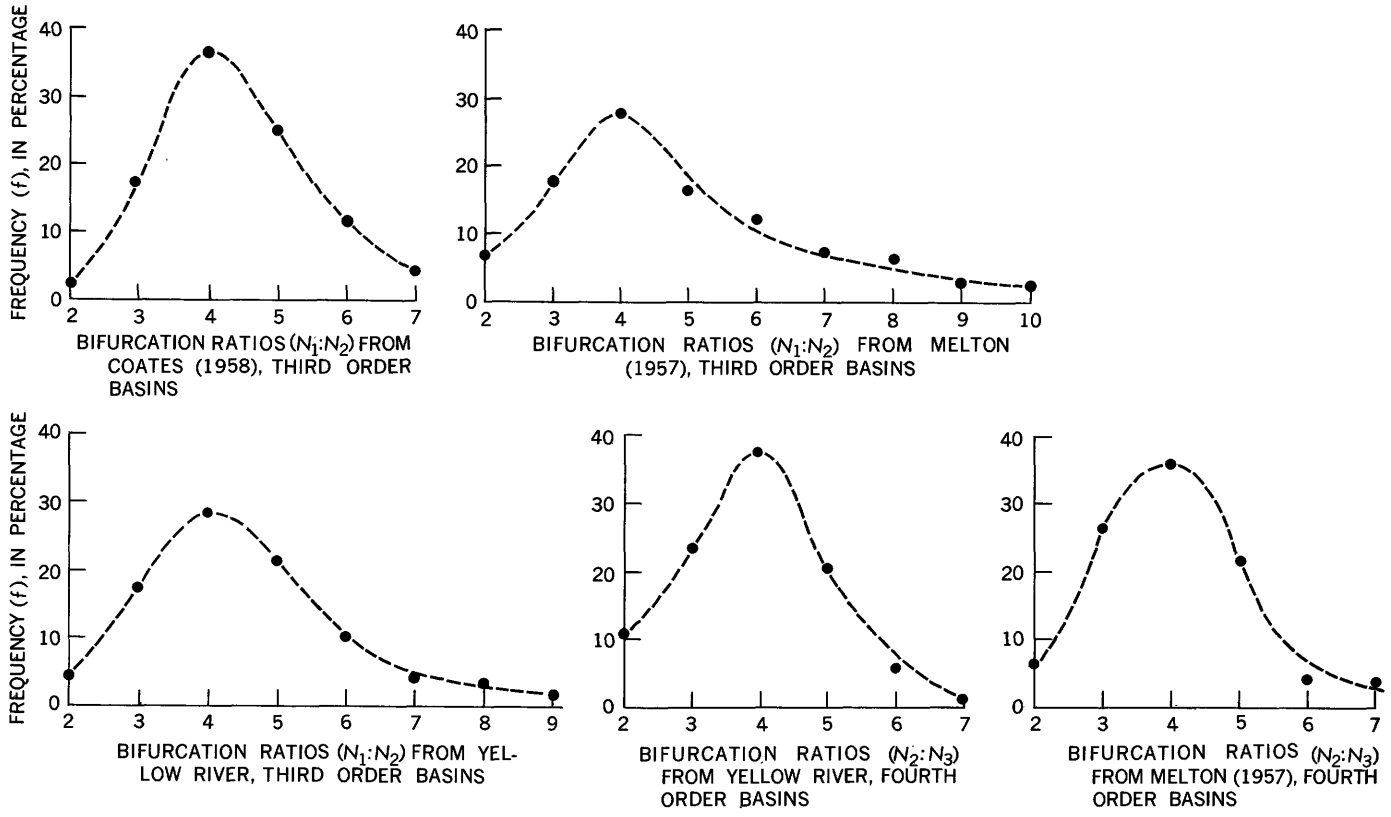

FIGURE 6.—Distributions of bifurcation ratios defined as $N_{s-2} \div N_{s-1}$.

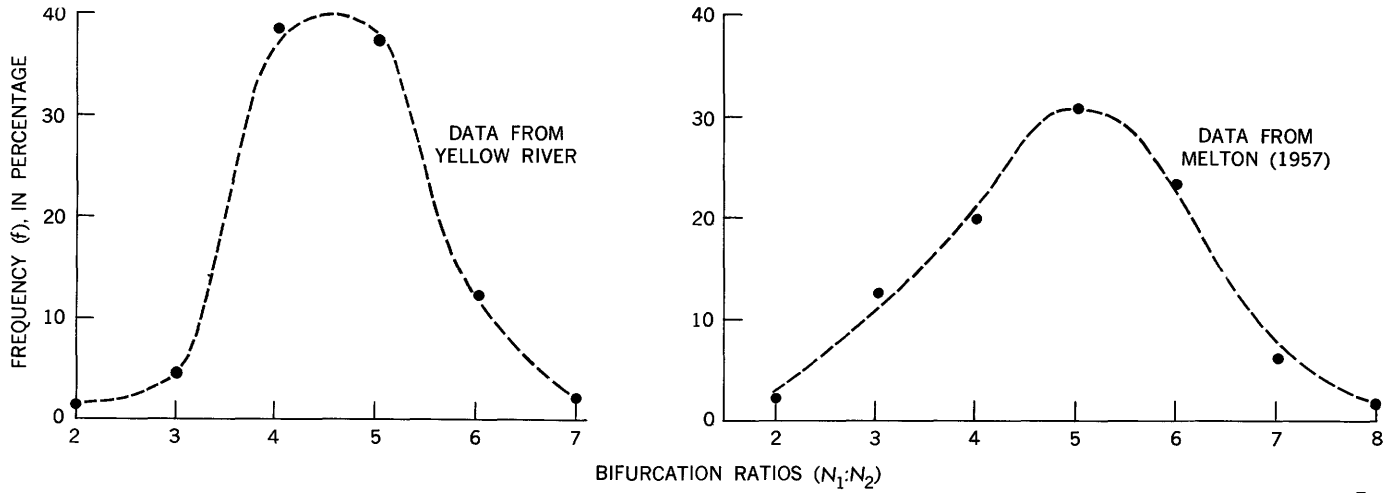

FIGURE 7.—Distribution of bifurcation ratios $N_{s-3} \div N_{s-2}$.

This concept applied to the major tributaries only gives

$$Su = \frac{N_{s-1}}{2} - 1 \qquad (8a)$$

where maximum conservancy refers to streams having only two major tributaries; this relation, according to the probabilities computed, has a one-in-three occurrence. The manner and the number of joining tributaries no doubt greatly affects the hydrology of the drainage system, and the number of major tributaries probably bears some relation to the shape of the basin.

DISTRIBUTION OF NUMBER OF SMALLER TRIBUTARIES

The subbasins of the Yellow River were further investigated in terms of the smaller tributaries present

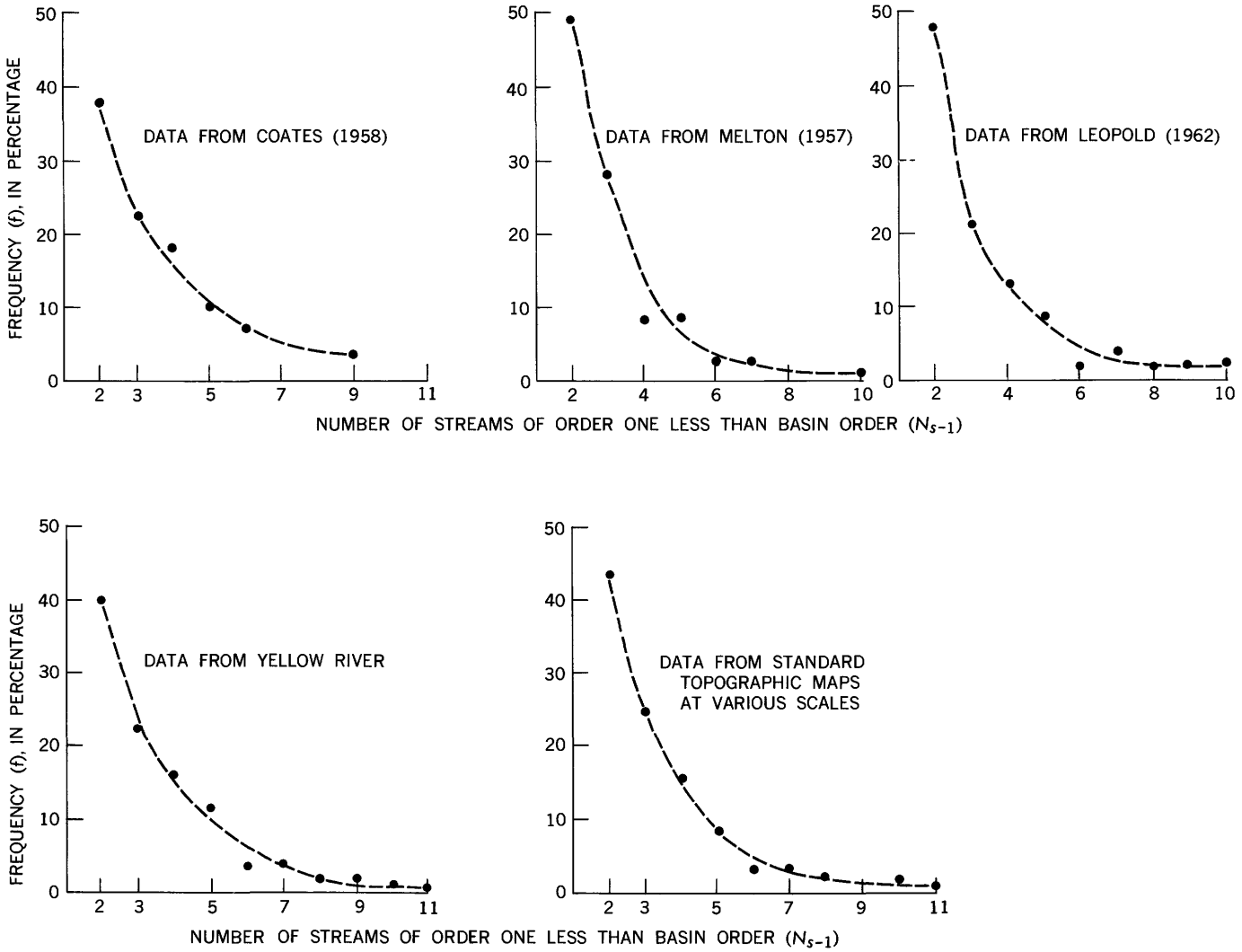

FIGURE 8.—Distributions of number of major tributaries or bifurcation ratios defined as N_{s-1} (data from various sources as shown).

therein by combining the number of branches of two order-lags (such as first-order branches in third-order basins, and second-order branches in fourth-order basins) into one distribution. Similarly, branches of three order-lags (such as first-order branches in fourth-order basin and second in fifth) and four order-lags (such as first in fifth and second in sixth) were combined into one distribution for each. Figure 10 shows the distributions for various order-lags; the progression from exponential to unimodal skewed distribution with increasing order-lag is apparent. As a measure of the symmetry of the distributions, skewness was calculated for each distribution in figure 10. Results are shown in table 5. Skewness appears to decrease with increasing order-lag, and only the number of the smallest fingertip branches in higher order basins may be normally distributed.

TABLE 5.—*Skewness of distributions for given order lags*

Order lag	*Skewness*
$s-1$	0.83
$s-2$	.93
$s-3$	.88
$s-4$	.67

NUMBER OF BRANCHES IN RIVER NETWORK

The minimum number of branchings of a stream is given by Melton (1957) in the equation

$$N_u=2^{(s-u)}. \quad (9)$$

This equation states that a second order stream will have at least two first-order branches, a third-order stream at least two second-order branches and four first-order branches, and so on. A graph of this equation is shown in figure 11 together with a straight line fitted through the modal values of the distributions of figure 10. The stream branches of the Yellow

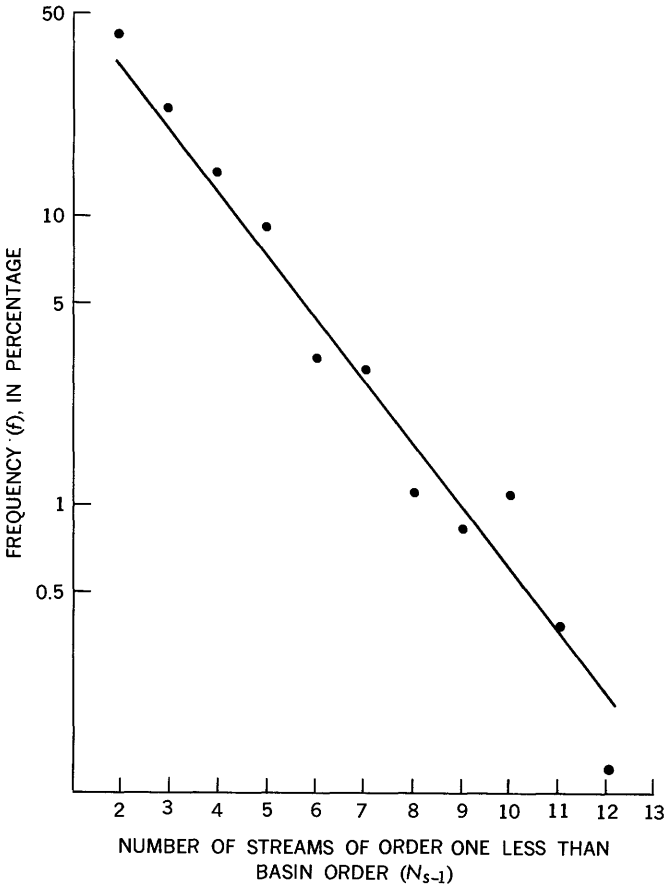

FIGURE 9.—Combined distribution of number of major tributaries, from figure 6.

River obviously multiply much more rapidly than the minimum growth. In a previous study, Giusti (1963) showed that the number of subbasins, N_a, of any given area, a, within a region or basin of a given area, A, could be expressed by the equation

$$N_a=0.3\frac{A}{a}. \tag{10}$$

Similarly, from equation 2 or 3, the number of stream branches can be expressed in terms of the order of the subbasins with respect to the order of the main stem. The fact that the relation based on area is hyperbolic indicates a geometric progression. However, by definition, the number of branches follow a geometric progression and the order (or order-lags) an arithmetic progression. Consequently the relation between number of branches and their order is exponential.

A further equation of Melton (1958) gives the number of channel segments in a basin as

$$N=0.8147\frac{L^{1.75}}{R^{0.25}} \tag{11}$$

where L is the total length of channels and R is the relative relief. There are then several equations from which the number of channel segments in a basin can be computed, and usage of one or another will depend on the preference of the user. However, because of the variability of the bifurcation ratios, the time required for arranging the stream segments of a drainage system into orders, and the lack of true portrayal of the drainage systems on topographic maps, equations 10 and 11 will possibly be more applicable in practice.

SUMMARY AND CONCLUSIONS

River networks have been analyzed in terms of the number of branches ordered according to a system proposed by Horton (1945) and modified by Strahler (1957). All networks were dendritic and in different climatic and geologic environments. The bifurcation ratios computed from two successive constant orders in any stream network vary according to the two stream orders from which they are computed, and decrease in a downstream direction. Bifurcation ratios for equal-order basins increase somewhat with the area of the basins but tend to become constant for basins beyond a certain size. These two factors—the order used for computation and the size of the area—also affect the parameters of the distributions of bifurcation ratios. Bifurcation ratios tend to become constant for ratios made between number of branches which are two or more orders removed from the order of the main stem (two order-lags).

The distribution of the number of one order-lag branches (considered the major tributaries to a given stream) was found to be exponential, with a maximum frequency at two. On an average, one out of three rivers of any given order will have two major tributaries, and one out of five rivers will have three. The distribution of number of higher order-lag branches or number of smaller tributaries was found to be both unimodal and skewed, and only the number of smallest fingertip branches may approach a normal distribution.

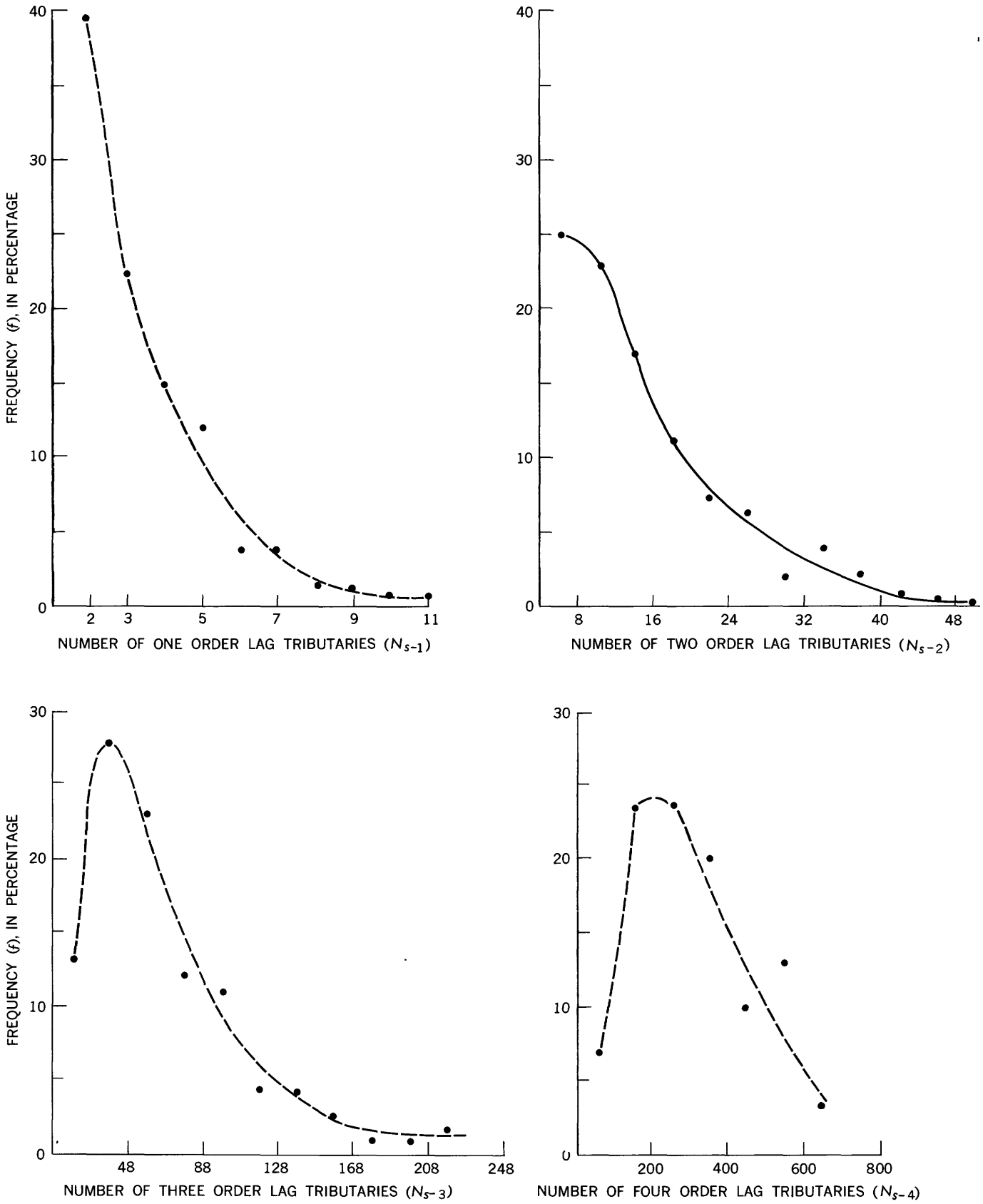

FIGURE 10.—Distributions of number of tributaries of varying order lag in Yellow River basin.

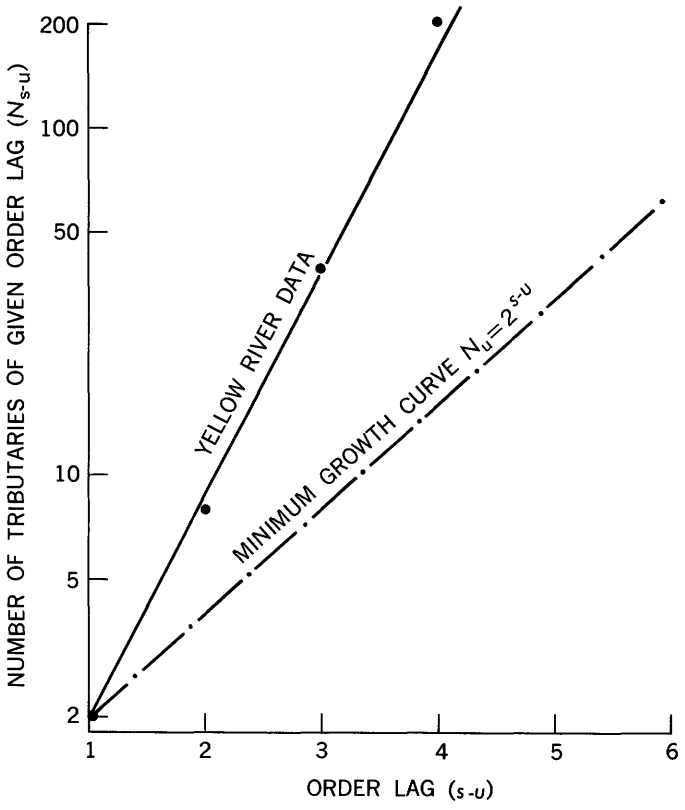

FIGURE 11.—Growth curves of stream branching. Lower curve is minimum growth, upper curve from model values of distributions of figure 10.

The variability of bifurcation ratios found in this study indicates that bifurcation ratios must be carefully defined in terms of the two successive orders from which they are computed and the order of the main stem. Comparisons of undefined bifurcation ratios may lead to erroneous conclusions.

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