COMBINATORIAL INTERPRETATIONS OF CONGRUENCES FOR THE SPT-FUNCTION

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ABSTRACT. Let $\text{spt}(n)$ denote the total number of appearances of the smallest parts in all the partitions of n. In 1988, the second author gave new combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11 in terms of a crank for weighted vector partitions. In 2008, the first author found Ramanujan-type congruences for the spt-function mod 5, 7 and 13. We give new combinatorial interpretations of the spt-congruences mod 5 and 7. These are in terms of the same crank but for a restricted set of vector partitions. The proof depends on relating the spt-crank with the crank of vector partitions and the Dyson rank of ordinary partitions. We derive a number of identities for spt-crank modulo 5 and 7. We prove the surprising result that all the spt-crank coefficients are nonnegative.

1. INTRODUCTION

In [4], the function $\text{spt}(n)$ was defined as the total number of appearances of the smallest parts in the partitions of n, and the following congruences were proved

$$\text{spt}(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$\text{spt}(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$\text{spt}(13n+6) \equiv 0 \pmod{13}. \tag{1.3}$$

For example, the partitions of 4 are

$$\dot{4}, \quad 3+\dot{1}, \quad \dot{2}+\dot{2}, \quad 2+\dot{1}+\dot{1}, \quad \dot{1}+\dot{1}+\dot{1}+\dot{1},$$

so that $\text{spt}(4) = 10 \equiv 0 \pmod{5}$. In this paper, we prove new combinatorial interpretations of (1.1) and (1.2). The congruences (1.1)–(1.3) are reminiscent of Ramanujan's

Date: January 11, 2012 (Minor corrections made February 23, 2012).

2010 *Mathematics Subject Classification.* 05A17, 05A19, 11F33, 11P81, 11P82, 11P83, 11P84, 33D15.

Key words and phrases. spt-function, partitions, rank, crank, vector partitions, Ramanujan's Lost Notebook, congruences, basic hypergeometric series.

The first author was supported in part by NSA Grant H98230-12-1-0205.

The second author was supported in part by NSA Grant H98230-09-1-0051.

The third author was supported by the Summer Research Experience for Rising Seniors (SRRS) program of the University of Florida with funding from the Howard Hughes Medical Institute the Science for Life Program.

partition congruences

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.4}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.5}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.6}$$

Dyson [10] defined the rank of a partition as the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m. Let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. Atkin and Swinnerton-Dyer [8] proved Dyson's conjectures that

$$N(k,5,5n+4) = \frac{p(5n+4)}{5} \qquad \text{for } 0 \le k \le 4, \tag{1.7}$$

$$N(k,7,7n+5) = \frac{p(7n+5)}{7} \qquad \text{for } 0 \le k \le 6. \tag{1.8}$$

Let $\mathscr{P}$ denote the set of partitions and $\mathscr{D}$ denote the set of partitions into distinct parts. Following [12], the set of vector partitions V is defined by the cartesian product

$$V = \mathscr{D} \times \mathscr{P} \times \mathscr{P}.$$

For $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V$, we define the weight $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}$, the crank$(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$, and $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$, where $|\pi_j|$ is the sum of the parts of π_j. The number of vector partitions of n with crank m counted according to the weight ω is denoted by $N_V(m,n)$, so that

$$N_V(m,n) = \sum_{\substack{\vec{\pi} \in V, |\vec{\pi}|=n \\ \text{crank}(\vec{\pi})=m}} \omega(\vec{\pi}).$$

Then

$$\sum_{\vec{\pi} \in V, |\vec{\pi}|=n} \omega(\vec{\pi}) = \sum_m N_V(m,n) = p(n),$$

the number of partitions of n. Let $N_V(m,t,n)$ denote the number of vector partitions of n with crank congruent to m modulo t counted according to the weight ω. In [12], it was proved that

$$N_V(k,5,5n+4) = \frac{p(5n+4)}{5} \qquad \text{for } 0 \le k \le 4, \tag{1.9}$$

$$N_V(k,7,7n+5) = \frac{p(7n+5)}{7} \qquad \text{for } 0 \le k \le 6, \tag{1.10}$$

$$N_V(k,11,11n+6) = \frac{p(11n+6)}{11} \qquad \text{for } 0 \le k \le 10. \tag{1.11}$$

For a partition π, define $s(\pi)$ as the smallest part in the partition with $s(-) = \infty$ for the empty partition. We define the following subset of vector partitions,

$$S := \{\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V \,:\, 1 \le s(\pi_1) < \infty \text{ and } s(\pi_1) \le \min(s(\pi_2), s(\pi_3))\}.$$

For $\vec{\pi} \in S$ we define the weight ω_1 by $\omega_1(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$. The number of vector partitions of n in S with crank m counted according to the weight ω_1 is denoted by $N_S(m,n)$, so that

$$N_S(m,n) = \sum_{\substack{\vec{\pi}\in S,\,|\vec{\pi}|=n \\ \mathrm{crank}(\vec{\pi})=m}} \omega_1(\vec{\pi}). \tag{1.12}$$

It turns out that

$$\sum_{\vec{\pi}\in S, |\vec{\pi}|=n} \omega_1(\vec{\pi}) = \sum_m N_S(m,n) = \mathrm{spt}(n). \tag{1.13}$$

See Corollary 2.2. The number of vector partitions of n in S with crank congruent to m modulo t counted according to the weight ω_1 is denoted by $N_S(m,t,n)$, so that

$$N_S(m,t,n) = \sum_{k=-\infty}^{\infty} N_S(kt+m,n) = \sum_{\substack{\vec{\pi}\in S, |\vec{\pi}|=n \\ \mathrm{crank}(\vec{\pi})\equiv m \pmod{t}}} \omega_1(\vec{\pi}). \tag{1.14}$$

There is an involution $\iota : S \longrightarrow S$ given by

$$\iota(\pi_1,\pi_2,\pi_3) = (\pi_1,\pi_3,\pi_2), \tag{1.15}$$

that preserves the weight ω_1,

$$\omega_1(\iota(\vec{\pi})) = \omega_1(\vec{\pi}).$$

Thus we have

$$N_S(m,n) = N_S(-m,n), \tag{1.16}$$

so that

$$N_S(m,t,n) = N_S(t-m,t,n). \tag{1.17}$$

One of our main results is the following theorem.

Theorem 1.1.

$$N_S(k,5,5n+4) = \frac{\mathrm{spt}(5n+4)}{5} \qquad \textit{for } 0 \le k \le 4, \tag{1.18}$$

$$N_S(k,7,7n+5) = \frac{\mathrm{spt}(7n+5)}{7} \qquad \textit{for } 0 \le k \le 6. \tag{1.19}$$

We illustrate Theorem 1.1 with an example. Below is a table of the 16 vector partitions $\vec{\pi} \in S$ with $|\vec{\pi}| = 4$.

	weight	crank
$\vec{\pi}_1 = (1, 1+1+1, -)$	$+1$	3
$\vec{\pi}_2 = (1, -, 1+1+1)$	$+1$	-3
$\vec{\pi}_3 = (1, 1+1, 1)$	$+1$	1
$\vec{\pi}_4 = (1, 1, 1+1)$	$+1$	-1
$\vec{\pi}_5 = (1, 1+2, -)$	$+1$	2
$\vec{\pi}_6 = (1, -, 1+2)$	$+1$	-2
$\vec{\pi}_7 = (1, 2, 1)$	$+1$	0
$\vec{\pi}_8 = (1, 1, 2)$	$+1$	0
$\vec{\pi}_9 = (1, 3, -)$	$+1$	1
$\vec{\pi}_{10} = (1, -, 3)$	$+1$	-1
$\vec{\pi}_{11} = (1+2, 1, -)$	-1	1
$\vec{\pi}_{12} = (1+2, -, 1)$	-1	-1
$\vec{\pi}_{13} = (1+3, -, -)$	-1	0
$\vec{\pi}_{14} = (2, 2, -)$	$+1$	1
$\vec{\pi}_{15} = (2, -, 2)$	$+1$	-1
$\vec{\pi}_{16} = (4, -, -)$	$+1$	0

From the table, we have

$$\begin{aligned} N_S(0,5,4) &= \omega_1(\vec{\pi}_7) + \omega_1(\vec{\pi}_8) + \omega_1(\vec{\pi}_{13}) + \omega_1(\vec{\pi}_{16}) \\ &= 1 + 1 - 1 + 1 = 2. \end{aligned}$$

Similarly,

$$N_S(0,5,4) = N_S(1,5,4) = N_S(2,5,4) = N_S(3,5,4) = N_S(4,5,4) = 2 = \frac{\operatorname{spt}(4)}{5}.$$

In Section 2, we obtain generating function identities for the spt-crank. We express the generating function in terms of the crank of vector partitions and the Dyson rank of partitions using Bailey's Lemma. In Section 3, we sketch the proof of Theorem 1.1. In Section 4, we obtain identities for the spt-crank modulo 5 and 7 using known identities for the rank and crank mod 5 and 7. In Section 5, we prove the amazing inequality

$$N_S(m,n) \geq 0, \tag{1.20}$$

for all m and n. In Section 6, we close the paper with a few problems.

Notation. We will use the standard q-notation.

$$(z;q)_n = (z)_n = \begin{cases} \displaystyle\prod_{j=0}^{n-1}(1 - zq^j), & n > 0, \\ 1, & n = 0, \end{cases}$$

and

$$(z;q)_\infty = (z)_\infty = \lim_{n\to\infty}(z;q)_n = \prod_{n=1}^{\infty}(1 - zq^{n-1}),$$

where $|q| < 1$.

2. Generating Function for the spt-crank

Define

$$S(z,q) := \sum_{n=1}^{\infty}\sum_{m} N_S(m,n) z^m q^n. \tag{2.1}$$

Theorem 2.1.

$$S(z,q) = \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_\infty}{(zq^n;q)_\infty (z^{-1}q^n;q)_\infty}. \tag{2.2}$$

Proof. Let $k \geq 1$, then

$$\begin{aligned}
&\frac{q^k (q^{k+1};q)_\infty}{(zq^k;q)_\infty (z^{-1}q^k;q)_\infty} \\
&= \left(\sum_{\substack{\pi_1\in\mathscr{D}\\ s(\pi_1)=k}} (-1)^{\#(\pi_1)-1} q^{|\pi_1|}\right)\left(\sum_{\substack{\pi_2\in\mathscr{P}\\ k\leq s(\pi_2)}} z^{\#(\pi_2)} q^{|\pi_2|}\right)\left(\sum_{\substack{\pi_3\in\mathscr{P}\\ k\leq s(\pi_3)}} z^{-\#(\pi_3)} q^{|\pi_3|}\right) \\
&= \sum_{\substack{\vec{\pi}=(\pi_1,\pi_2,\pi_3)\in S\\ s(\pi_1)=k}} \omega_1(\vec{\pi}) z^{\mathrm{crank}(\vec{\pi})} q^{|\vec{\pi}|}.
\end{aligned}$$

Hence,

$$\begin{aligned}
S(z,q) &= \sum_{n=1}^{\infty}\sum_{m} N_S(m,n) z^m q^n \\
&= \sum_{\vec{\pi}=(\pi_1,\pi_2,\pi_3)\in S} \omega_1(\vec{\pi}) z^{\mathrm{crank}(\vec{\pi})} q^{|\vec{\pi}|} \\
&= \sum_{k=1}^{\infty} \sum_{\substack{\vec{\pi}\in S\\ s(\pi_1)=k}} \omega_1(\vec{\pi}) z^{\mathrm{crank}(\vec{\pi})} q^{|\vec{\pi}|} \\
&= \sum_{k=1}^{\infty} \frac{q^k (q^{k+1};q)_\infty}{(zq^k;q)_\infty (z^{-1}q^k;q)_\infty}.
\end{aligned}$$

□

Corollary 2.2. *For $n \geq 1$,*

$$\sum_{\vec{\pi} \in S, |\vec{\pi}|=n} \omega_1(\vec{\pi}) = \sum_m N_S(m,n) = \mathrm{spt}(n).$$

Proof. In (2.1) we let $z = 1$,

$$\begin{aligned} S(1,q) &= \sum_{n=1}^{\infty} \left(\sum_m N_S(m,n) \right) q^n \\ &= \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_\infty}{(q^n;q)_\infty^2} \\ &= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_\infty} \\ &= \sum_{n=1}^{\infty} \mathrm{spt}(n) q^n, \end{aligned}$$

by [4]. The result follows. □

A pair of sequences $(\alpha_n(a,q), \beta_n(a,q))$ is called a Bailey pair with parameters (a,q) if

$$\beta_n(a,q) = \sum_{r=0}^{n} \frac{\alpha_r(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}}$$

for all $n \geq 0$. We will need

Lemma 2.3 (Bailey's Lemma)**.** *Suppose $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameters (a,q). Then $(\alpha'_n(a,q), \beta'_n(a,q))$ is another Bailey pair with parameters (a,q), where*

$$\alpha'_n(a,q) = \frac{(\rho_1;q)_n (\rho_2;q)_n (\frac{aq}{\rho_1 \rho_2})^n}{(\frac{aq}{\rho_1};q)_n (\frac{aq}{\rho_2};q)_n} \alpha_n(a,q),$$

$$\beta'_n(a,q) = \sum_{j=0}^{n} \frac{(\rho_1;q)_j (\rho_2;q)_j (\frac{aq}{\rho_1\rho_2};q)_{n-j} (\frac{aq}{\rho_1\rho_2})^j}{(q;q)_{n-j} (\frac{aq}{\rho_1};q)_n (\frac{aq}{\rho_2};q)_n} \beta_j(a,q).$$

We will apply Bailey's Lemma using the following Bailey pairs

$$\alpha_n = \begin{cases} 1, & n = 0, \\ (-1)^n q^{n(n-1)/2}(1+q^n), & n \geq 1, \end{cases} \qquad \beta_n = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases} \tag{2.3}$$

and

$$\alpha_n = \begin{cases} 1, & n = 0, \\ (-1)^n q^{n(3n-1)/2}(1+q^n), & n \geq 1, \end{cases} \qquad \beta_n = \frac{1}{(q)_n}. \tag{2.4}$$

For more information on these Bailey pairs, Bailey's Lemma, and its applications see [2, Ch.3].

Theorem 2.4.

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-zq^n)(1-z^{-1}q^n)}\cdot\frac{(q^{n+1};q)_\infty}{(zq^{n+1};q)_\infty(z^{-1}q^{n+1};q)_\infty}$$

$$=\frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}-\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-1}q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}\right). \tag{2.5}$$

Proof. From Bailey's Lemma with $a=1$, $\rho_1=z$, and $\rho_2=z^{-1}$; we have

$$\beta'_n(1,q)=\sum_{r=0}^{n}\frac{\alpha'_r(1,q)}{(q;q)_{n-r}(q;q)_{n+r}},$$

$$\sum_{j=0}^{n}\frac{(z)_j(z^{-1})_jq^j\beta_j}{(zq)_n(z^{-1}q)_n}=\sum_{r=0}^{n}\frac{1}{(q)_{n-r}(q)_{n+r}}\frac{(z)_r(z^{-1})_rq^r\alpha_r}{(zq)_r(z^{-1}q)_r}.$$

We divide both sides by $(1-z)(1-z^{-1})$ and let $n\to\infty$ to obtain

$$\frac{1}{(zq)_\infty(z^{-1}q)_\infty}\sum_{n=0}^{\infty}\frac{(qz)_n(qz^{-1})_nq^n\beta_n}{(1-zq^n)(1-z^{-1}q^n)}=\frac{1}{(q)_\infty^2}\sum_{n=0}^{\infty}\frac{q^n\alpha_n}{(1-zq^n)(1-z^{-1}q^n)}, \tag{2.6}$$

assuming certain convergence conditions.

Now we substitute Bailey pair (2.3) into (2.6),

$$\frac{1}{(zq)_\infty(z^{-1}q)_\infty(1-z)(1-z^{-1})}=\frac{1}{(q)_\infty^2(1-z)(1-z^{-1})}+\frac{1}{(q)_\infty^2}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}. \tag{2.7}$$

Next we substitute Bailey pair (2.4) into (2.6),

$$\frac{1}{(zq)_\infty(z^{-1}q)_\infty(1-z)(1-z^{-1})}+\frac{1}{(zq)_\infty(z^{-1}q)_\infty}\sum_{n=1}^{\infty}\frac{(qz)_n(qz^{-1})_nq^n}{(1-zq^n)(1-z^{-1}q^n)(q)_n} \tag{2.8}$$

$$=\frac{1}{(q)_\infty^2(1-z)(1-z^{-1})}+\frac{1}{(q)_\infty^2}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}.$$

By (2.7) and (2.8), we have

$$\frac{1}{(zq)_\infty(z^{-1}q)_\infty}\sum_{n=1}^{\infty}\frac{(qz)_n(qz^{-1})_nq^n}{(1-zq^n)(1-z^{-1}q^n)(q)_n}$$

$$=\frac{1}{(q)_\infty^2}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(3n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right).$$

We multiply both sides by $(q)_\infty$, simplify, and obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{q^n}{(1-zq^n)(1-z^{-1}q^n)} \cdot \frac{(q^{n+1};q)_\infty}{(zq^{n+1};q)_\infty (z^{-1}q^{n+1};q)_\infty} \\
&= \frac{1}{(q)_\infty} \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)} - \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)} \right).
\end{aligned}$$

□

Let

$$C_1(z,q) := \sum_{n=0}^{\infty} \sum_{m} N_V(m,n) z^m q^n. \tag{2.9}$$

Then

$$\begin{aligned}
C_1(z,q) &= \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty} \\
&= \frac{1}{(q)_\infty} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (1+q^n)(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \right] \\
&= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)},
\end{aligned} \tag{2.10}$$

by [12, eq. (7.15), p.70].

Let

$$R_1(z,q) := \sum_{n=0}^{\infty} \sum_{m} N(m,n) z^m q^n. \tag{2.11}$$

Then

$$\begin{aligned}
R_1(z;q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n} \\
&= \frac{1}{(q)_\infty} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2} (1+q^n)(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \right] \\
&= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2} (1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)},
\end{aligned} \tag{2.12}$$

by [3, eq. (1.8)].

The following Corollary of Theorem 2.5 now follows using (2.9)–(2.12).

Corollary 2.5.

$$S(z,q)=\frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}-\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-1}q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}\right)$$
$$=\frac{-1}{(1-z)(1-z^{-1})}\left[\sum_{n=0}^{\infty}\sum_m N_V(m,n)z^mq^n-\sum_{n=0}^{\infty}\sum_m N(m,n)z^mq^n\right].$$

3. Proof of Theorem 1.1

For $t\geq 5$ prime, let δ_t be the reciprocal of 24 modulo t. By using an argument analogous to Lemma (2.2) in [12], we find that Theorem 1.1 is equivalent to showing that the coefficient of $q^{tn+\delta_t}$ in

$$S(\zeta_t,q)=\sum_{n=1}^{\infty}\sum_m N_S(m,n)\zeta_t^mq^n=\sum_{n=1}^{\infty}\left(\sum_{r=0}^{t-1}N_S(r,t,n)\zeta_t^m\right)q^n \tag{3.1}$$

is zero, where $t=5,\ 7$ and $\zeta_t=\exp(2\pi i/t)$.

By Corollary 2.5,

$$S(\zeta_t,q)=\frac{-1}{(1-\zeta_t)(1-\zeta_t^{-1})}\left[\sum_{n=1}^{\infty}\sum_m N_V(m,n)\zeta_t^mq^n-\sum_{n=1}^{\infty}\sum_m N(m,n)\zeta_t^mq^n\right]$$
$$=\frac{-1}{(1-\zeta_t)(1-\zeta_t^{-1})}\left[\sum_{n=1}^{\infty}\left(\sum_{r=0}^{t-1}N_V(r,t,n)\zeta_t^r\right)q^n-\sum_{n=1}^{\infty}\left(\sum_{r=0}^{t-1}N(r,t,n)\zeta_t^r\right)q^n\right],$$

and the result follows from (1.7)–(1.10).

4. Identities for The *spt*-crank Modulo 5 and 7

For $0\leq b,c,d\leq t-1$, we define

$$S_b(d,t,q):=\sum_{n=1}^{\infty}N_S(b,t,tn+d)q^n, \tag{4.1}$$

$$S_{b,c}(d,t,q):=S_b(d,t,q)-S_c(d,t,q). \tag{4.2}$$

Using known results for the rank and crank mod 5 and 7, we derive identities for the spt-crank differences $S_{b,c}(d,t,q)$ for $t=5,\ 7$. These identities are in terms of the following functions:

$$\phi_{a,t}(q):=\frac{q^a}{(q^t;q^t)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{3}{2}tn(n+1)}}{1-q^{tn+a}}=-1+\sum_{n=0}^{\infty}\frac{q^{tn^2}}{(q^a;q^t)_{n+1}(q^{t-a};q^t)_n}, \tag{4.3}$$

$$P_{a,t}(q)=(q^a;q^t)_\infty(q^{t-a};q^t)_\infty \qquad \text{(for } 1\leq a\leq t-1\text{)}, \tag{4.4}$$

$$P_{0,t}(q)=(q^t;q^t)_\infty. \tag{4.5}$$

We note that the second part of (4.3) is a special case of the following identity

$$-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(z)_{n+1}(z^{-1}q)_n}=\frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{3}{2}n(n+1)}}{1-zq^n}.$$

See [13, eq. (4.25)].

4.1. **The** ***spt*****-crank Modulo 5.** In this subsection, $\zeta=\zeta_5=\exp(2\pi i/5)$. We need two identities from Ramanujan's Lost Notebook

$$C_1(\zeta,q)=A(q^5)-q(\zeta+\zeta^{-1})^2B(q^5)+q^2(\zeta^2+\zeta^{-2})C(q^5)-q^3(\zeta+\zeta^{-1})D(q^5), \tag{4.6}$$

$$\begin{aligned}R_1(\zeta,q)=&\left\{A(q^5)+(\zeta+\zeta^{-1}-2)\phi_{1,5}(q^5)\right\}+qB(q^5)+q^2(\zeta+\zeta^{-1})C(q^5)\\&-q^3(\zeta+\zeta^{-1})\left\{D(q^5)-(\zeta^2+\zeta^{-2}-2)\frac{\phi_{2,5}(q^5)}{q^5}\right\},\end{aligned} \tag{4.7}$$

where

$$A(q)=\frac{P_{0,5}(q)P_{2,5}(q)}{P_{1,5}(q)^2}, \tag{4.8}$$

$$B(q)=\frac{P_{0,5}(q)}{P_{1,5}(q)}, \tag{4.9}$$

$$C(q)=\frac{P_{0,5}(q)}{P_{2,5}(q)}, \tag{4.10}$$

$$D(q)=\frac{P_{0,5}(q)P_{1,5}(q)}{P_{2,5}(q)^2}. \tag{4.11}$$

For a proof of (4.6) see [12, §3]. Also in [12, §8] it was shown how (4.7) is equivalent to a result of Atkin and Swinnerton-Dyer [8].

From (4.6)–(4.7) and some calculation, we have the following

Theorem 4.1.

$$\begin{aligned}S(\zeta,q)&=\frac{-1}{(1-\zeta)(1-\zeta^{-1})}\Big(C_1(\zeta,q)-R_1(\zeta,q)\Big)\\&=-\phi_{1,5}(q^5)+qB(q^5)-(\zeta^2+\zeta^{-2})q^2C(q^5)+(\zeta^2+\zeta^{-2})\frac{\phi_{2,5}(q^5)}{q^2}.\end{aligned} \tag{4.12}$$

We may easily recast this theorem in terms of spt-crank differences.

Corollary 4.2.

$$S_{0,1}(0,5,q)=-\phi_{1,5}(q), \tag{4.13}$$

$$S_{0,1}(1,5,q)=B(q), \tag{4.14}$$

$$S_{1,2}(2,5,q)=C(q), \tag{4.15}$$

$$S_{1,2}(3,5,q) = -\frac{\phi_{2,5}(q)}{q}; \tag{4.16}$$

otherwise, $S_{b,b+1}(d,5,q) = 0$ *for* $0 \le b \le 1$ *and* $0 \le d \le 4$.

Proof.

$$\begin{aligned} S(\zeta,q) &= \sum_{n=1}^{\infty}\sum_{m} N_S(m,n)\zeta^m q^n \\ &= \sum_{d=0}^{4}\left(\sum_{r=0}^{4}\left(\sum_{n=0}^{\infty} N_S(r,5,5n+d)q^{5n+d}\right)\zeta^r\right) \\ &= \sum_{d=0}^{4}\left(\sum_{r=0}^{4} S_r(d,5,q^5)\zeta^r\right)q^d \\ &= \sum_{d=0}^{4}\left[S_{0,1}(d,5,q^5) + (\zeta^2+\zeta^{-2})S_{2,1}(d,5,q^5)\right]q^d, \end{aligned}$$

using (1.17) and the fact that $1+\zeta+\zeta^2+\zeta^3+\zeta^4=0$. The results follow from the theorem. ☐

4.2. **The** ***spt*****-crank Modulo 7.** In this subsection, $\zeta = \zeta_7 = \exp(2\pi i/7)$. We need the analogs of (4.6) and (4.7)

(4.17)

$$\begin{aligned} C_1(\zeta,q) &= A_7(q^7) + q(\zeta+\zeta^{-1}-1)B_7(q^7) + q^2(\zeta^2+\zeta^{-2})C_7(q^7) + q^3(\zeta^3+\zeta^{-3}+1)D_7(q^7) \\ &\quad - q^4(\zeta+\zeta^{-1})E_7(q^7) - q^6(\zeta^2+\zeta^{-2}+1)F_7(q^7); \end{aligned}$$

(4.18)

$$\begin{aligned} &R_1(\zeta,q) \\ &= \left\{(\zeta+\zeta^{-1}-1)A_7(q^7) + (2-\zeta-\zeta^{-1})(1+\phi_{1,7}(q^7))\right\} + qB_7(q^7) \\ &+ q^2\left\{(\zeta+\zeta^{-1})C_7(q^7) + (\zeta+\zeta^{-1}-\zeta^2-\zeta^{-2})\frac{\phi_{1,7}(q^7)}{q^7}\right\} + q^3(1+\zeta^2+\zeta^{-2})D_7(q^7) \\ &- q^4(\zeta^2+\zeta^{-2})E_7(q^7) - q^6\left\{(1+\zeta^3+\zeta^{-3})F_7(q^7) + (\zeta^3+\zeta^{-3}-\zeta^2-\zeta^{-2})\frac{\phi_{2,7}(q^7)}{q^7}\right\}, \end{aligned}$$

where

$$A_7(q) = \frac{P_{0,7}(q)P_{3,7}(q)}{P_{1,7}(q)P_{2,7}(q)}, \tag{4.19}$$

$$B_7(q) = \frac{P_{0,7}(q)}{P_{1,7}(q)}, \tag{4.20}$$

$$C_7(q) = \frac{P_{0,7}(q)P_{2,7}(q)}{P_{1,7}(q)P_{3,7}(q)}, \tag{4.21}$$

$$D_7(q) = \frac{P_{0,7}(q)}{P_{2,7}(q)}, \tag{4.22}$$

$$E_7(q) = \frac{P_{0,7}(q)}{P_{3,7}(q)}, \tag{4.23}$$

$$F_7(q) = \frac{P_{0,7}(q)P_{1,7}(q)}{P_{2,7}(q)P_{3,7}(q)}. \tag{4.24}$$

Neither (4.17) nor (4.18) appear in the Lost Notebook. However, according to Berndt, Chan, Chan and Liaw [9], there are clues that Ramanujan had been working on these identities. On page 19 of the Lost Notebook, one sees the definition of $C_1(\zeta_7, q)$ and $R_1(\zeta_7, q)$. On page 71 we find the infinite products $A_7(q)$, $B_7(q)$, …, $F_7(q)$ which appear in both (4.17) and (4.18). Equation (4.17) is a restatement of equation (5.2) in [12, p.62]. Equation (4.18) follows from Theorem 5 in [8, p.103].

From (4.17)–(4.18) and some calculation, we have the following

Theorem 4.3.

$$\begin{aligned} S(\zeta, q) &= \frac{-1}{(1-\zeta)(1-\zeta^{-1})}\Big(C_1(\zeta, q) - R_1(\zeta, q)\Big) \\ &= \Big\{1 - A_7(q^7) + \phi_{1,7}(q^7)\Big\} + qB_7(q^7) \\ &\quad + q^2(1+\zeta+\zeta^{-1})\Big\{C_7(q^7) + \frac{\phi_{3,7}(q^7)}{q^7}\Big\} - q^3(\zeta^3+\zeta^{-3})D_7(q^7) \\ &\quad + q^4(1+\zeta+\zeta^{-1})E_7(q^7) - q^6(\zeta^3+\zeta^{-3})\Big\{F_7(q^7) + \frac{\phi_{2,7}(q^7)}{q^7}\Big\}. \end{aligned} \tag{4.25}$$

Again we can easily recast this theorem in terms of spt-crank differences.

Corollary 4.4.

$$S_{0,1}(0, 7, q) = 1 - A_7(q) + \phi_{1,7}(q), \tag{4.26}$$

$$S_{0,1}(1, 7, q) = B_7(q), \tag{4.27}$$

$$S_{1,2}(2, 7, q) = C_7(q) + \frac{\phi_{3,7}(q)}{q}, \tag{4.28}$$

$$S_{2,3}(3, 7, q) = D_7(q), \tag{4.29}$$

$$S_{1,2}(4, 7, q) = E_7(q), \tag{4.30}$$

$$S_{2,3}(6, 7, q) = F_7(q) + \frac{\phi_{2,7}(q)}{q}; \tag{4.31}$$

otherwise, $S_{b,b+1}(d, 7, q) = 0$ *for* $0 \le b \le 2$ *and* $0 \le d \le 6$.

5. The Nonnegativity Theorem

In [13], it was proved that

$$N_V(m,n) \geq 0, \tag{5.1}$$

for all $(m,n) \neq (0,1)$. This was the clue to completing the solution of Dyson's so-called Crank Conjecture [6]. Here we have a similar situation. In this section, we prove the surprising result that all spt-crank coefficients are nonnegative.

Recall from (2.1) and (2.2) that

$$\begin{aligned} S(z,q) &= \sum_{n=1}^{\infty}\sum_{m} N_S(m,n) z^m q^n \\ &= \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_\infty}{(zq^n;q)_\infty (z^{-1}q^n;q)_\infty}. \end{aligned}$$

By (1.16), we can write

$$S(z,q) = A_0 + \sum_{i=1}^{\infty} A_i (z^i + z^{-i}),$$

where the A_i are power series in q with integer coefficients.

Theorem 5.1.

$$N_S(m,n) \geq 0, \tag{5.2}$$

for all (m,n).

Proof. It suffices to prove that for $i \geq 0$, the coefficients in the A_i are all nonnegative.

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_\infty}{(zq^n;q)_\infty (z^{-1}q^n;q)_\infty} \\ &= \sum_{n=1}^{\infty} \frac{q^n}{(zq^n;q)_\infty} \sum_{j=0}^{\infty} \frac{(zq;q)_j \, (q^n/z)^j}{(q;q)_j} \qquad \text{(by [1, p.17, eq.(2.2.1)])} \\ &= \sum_{n=1}^{\infty} q^n (zq;q)_{n-1} \sum_{j=0}^{\infty} \frac{(q^n/z)^j}{(q;q)_j (zq^{j+1};q)_\infty} \\ &= \sum_{n=1}^{\infty} q^n \sum_{h=0}^{n-1} \begin{bmatrix} n-1 \\ h \end{bmatrix} (-1)^h q^{\binom{h+1}{2}} z^h \sum_{i,j\geq 0} \frac{(q^n/z)^j \, (zq^{j+1})^i}{(q;q)_j (q;q)_i} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by [1, p.36, eq.(3.3.6)])} \\ &= \sum_{i=-\infty}^{\infty} z^i \sum_{n=1}^{\infty} q^n \sum_{h=0}^{n-1} \begin{bmatrix} n-1 \\ h \end{bmatrix} (-1)^h q^{\binom{h+1}{2}} \sum_{j=0}^{\infty} \frac{q^{nj+(i+j-h)(j+1)}}{(q;q)_j (q;q)_{i+j-h}} \end{aligned}$$

(after replacing i by $i+j-h$)

As noted earlier, we need only deal with $i \geq 0$. So we have

$$\begin{aligned} A_i &= \sum_{n=1}^{\infty} q^n \sum_{h=0}^{n-1} \begin{bmatrix} n-1 \\ h \end{bmatrix} (-1)^h q^{\binom{h+1}{2}} \sum_{j=0}^{\infty} \frac{q^{nj+(i+j-h)(j+1)}}{(q;q)_j (q;q)_{i+j-h}} \\ &= \sum_{j=0}^{\infty} \frac{q^{j^2+ij+2j+i+1}}{(q;q)_j (q;q)_{i+j}} \sum_{n=0}^{\infty} q^n p(i,j,n), \end{aligned}$$

where

$$\begin{aligned} p(i,j,n) &:= \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix} (-1)^h q^{\binom{h}{2}+j(n-h)} \frac{(q;q)_{i+j}}{(q;q)_{i+j-h}} \\ &= q^{jn} \lim_{\tau \to 0} {}_2\phi_1 \left(\begin{matrix} q^{-n}, & q^{-i-j}; & q, & \frac{q^{n+i}}{\tau} \\ & \tau^{-1} & & \end{matrix} \right) \\ &= q^{jn} \lim_{\tau \to 0} \frac{(q^i/\tau)_n}{(1/\tau)_n} {}_2\phi_1 \left(\begin{matrix} q^{-j}, & q^{-n}; & q, & \frac{q^n}{\tau} \\ & q^i \tau^{-1} & & \end{matrix} \right) \qquad \text{(by [14, p.241, eq.(III.2)])} \\ &= q^{(i+j)n} \sum_{h=0}^{j} \frac{(q;q)_j}{(q;q)_{j-h}} \begin{bmatrix} n \\ h \end{bmatrix} (-1)^h q^{\binom{h}{2}-ih-jh}. \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} q^n p(i,j,n) = \sum_{h=0}^{j} \frac{(q;q)_j (-1)^h q^{\binom{h+1}{2}}}{(q;q)_{j-h} (q^{i+j+1};q)_{h+1}} \qquad \text{(by [1, p.36, eq.(3.3.7)])}.$$

To conclude our proof we need the following identity

$$\sum_{h=0}^{j} \frac{(-1)^h q^{\binom{h+1}{2}}}{(q;q)_{j-h} (zq^{j+1};q)_{h+1}} = \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h (1-zq^{h+j+1})}. \tag{5.3}$$

The left-hand side of (5.3) has simple poles at $z = q^{-h-j-1}$ for $h = 0, 1, \ldots j$. Hence the left-hand side has the following partial fraction decomposition

$$\sum_{h=0}^{j} \frac{C_h}{1-zq^{h+j+1}},$$

and for $0 \leq s \leq j$,

$$\begin{aligned} C_s &= \lim_{z \to q^{-s-j-1}} (1-zq^{s+j+1}) \sum_{h=0}^{j} \frac{(-1)^h q^{\binom{h+1}{2}}}{(q;q)_{j-h} (zq^{j+1};q)_{h+1}} \\ &= \lim_{z \to q^{-s-j-1}} (1-zq^{s+j+1}) \sum_{h=s}^{j} \frac{(-1)^h q^{\binom{h+1}{2}}}{(q;q)_{j-h} (zq^{j+1};q)_{h+1}} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{z\to q^{-s-j-1}} \sum_{h=0}^{j-s} \frac{(-1)^{h+s} q^{\binom{h+s+1}{2}}}{(q;q)_{j-h-s}(zq^{j+1};q)_s(zq^{s+j+2};q)_h} \\
&= \sum_{h=0}^{j-s} \frac{(-1)^{h+s} q^{\binom{h+s+1}{2}}}{(q;q)_{j-h-s}(q^{-s};q)_s(q;q)_h} \\
&= \frac{q^{s^2+s}}{(q;q)_s(q;q)_{j-s}} \sum_{h=0}^{j-s} \begin{bmatrix} j-s \\ h \end{bmatrix} (-1)^h q^{\binom{h+1}{2}+hs} \\
&= \frac{q^{s^2+s}}{(q;q)_s(q;q)_{j-s}} (q^{s+1};q)_{j-s} \qquad \text{(by [1, p.36, eq.(3.3.6)])} \\
&= q^{s^2+s} \begin{bmatrix} j \\ s \end{bmatrix} \frac{1}{(q;q)_s},
\end{aligned}
$$

and thus (5.3) is proved.

If we now put $z = q^i$ in (5.3), we see that

$$
\sum_{n=0}^{\infty} q^n p(i,j,n) = (q;q)_j \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{i+j+h+1})}.
$$

Consequently for $i \geq 0$

$$
\begin{aligned}
A_i &= \sum_{j=0}^{\infty} \frac{q^{j^2+ij+2j+i+1}}{(q;q)_j(q;q)_{i+j}} (q;q)_j \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{i+j+h+1})} \\
&= \sum_{j=0}^{\infty} \sum_{h=0}^{\infty} \frac{q^{j^2+ij+2hj+2j+i+hi+2h^2+3h+1}}{(q^{j+h+1};q)_i(q;q)_h^2(q;q)_j(1-q^{i+j+2h+1})}.
\end{aligned}
\tag{5.4}
$$

Thus A_i clearly has nonnegative coefficients and our theorem is proved. □

6. Conclusion

We pose the following problems

(1) Find a statistic on partitions that explains (5.2) combinatorially. More precisely, find a statistic s-rank : $\mathscr{P} \longrightarrow \mathbb{Z}$ and a weight function $\varphi : \mathscr{P} \longrightarrow \mathbb{N}$ such that

$$
\sum_{\pi\in\mathscr{P},\, |\pi|=n} \varphi(\pi) = \mathrm{spt}(n), \quad \text{and} \tag{6.1}
$$

$$
\sum_{\substack{\pi\in\mathscr{P},\, |\pi|=n \\ \text{s-rank}(\pi)=m}} \varphi(\pi) = N_S(m,n), \tag{6.2}
$$

for $m \in \mathbb{Z}$ and $n \geq 1$.

(2) Find a crank-type result that explains the congruence $\mathrm{spt}(13n+6) \equiv 0 \pmod{13}$.

It is straightforward to interpret the generating function in (5.4) in terms of Durfee squares and rectangles for fixed i. The problem is to interpret the result so that something like (6.1) and (6.2) hold. Unfortunately the spt-crank does not work for $\mathrm{spt}(13n+6)$. The 13-analog of (1.18), (1.19) does not even hold for the first case $n=0$. At present the mod 13 congruence (1.3) remains mysterious.

We say a vector partition is self-conjugate if it is a fixed point of the involution ι (1.15). In our next paper [7], we prove that the number of self-conjugate vector partitions in S is related to the coefficients of a certain mock theta function studied by the first author, Dyson and Hickerson [5]. One byproduct is an elementary q-series proof of Folsom and Ono's results [11] for the parity of $\mathrm{spt}(n)$.

7. Table

For reference we include values of the spt-crank coefficients $N_S(m,n)$ for small m and n.

n \ m	0	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0	0
4	2	2	1	1	0	0	0	0	0	0	0
5	2	2	2	1	1	0	0	0	0	0	0
6	4	4	3	2	1	1	0	0	0	0	0
7	5	4	4	3	2	1	1	0	0	0	0
8	7	7	6	5	3	2	1	1	0	0	0
9	10	9	8	6	5	3	2	1	1	0	0
10	13	13	11	10	7	5	3	2	1	1	0
11	17	16	15	12	10	7	5	3	2	1	1
12	24	24	21	18	14	11	7	5	3	2	1
13	31	29	27	23	19	14	11	7	5	3	2
14	40	40	36	32	26	21	15	11	7	5	3
15	53	51	48	41	35	27	21	15	11	7	5
16	69	68	62	56	46	38	29	22	15	11	7
17	88	85	80	70	61	49	39	29	22	15	11
18	113	112	104	94	80	67	52	41	30	22	15
19	144	139	132	118	103	85	70	53	41	30	22
20	183	181	169	154	133	113	91	73	55	42	30

Note added: Since this paper was submitted, Freeman Dyson ("Partitions and the Grand Canonical Esemble," this volume) has found a simpler and more elementary proof of Theorem 5.1. The proof depends on a new expression for the generating function of $N_S(m,n)$ which follows easily from Corollary 2.5.

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