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The Fuzzy Relations between Intervals in a Convolution-based Depiction

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Abstract. This paper is aimed at the proposing a new convolution-based approach to the representation of fuzzy Allen's relations between fuzzy intervals. It refers to the earlier attempt of H-J. Ohlbach to represent these relations is terms of integrals. The next a framework of a theory for the convolution fuzzy Allen's relations is put forward.

Keywords. Depiction, convolution, fuzzy relations.

1 Introduction

In [1], J. Allen introduced the 13 possible relations between intervals – described later in [2] in modal terms. The intervals from the original Allen's work – as compact subsets of a real line $\mathbb{R}$ – form operationally convenient objects and do not fuzzify any Allen's relations between them. The situation may change radically, when these intervals are exchanged for fuzzy intervals. They form two-dimensional objects in $\mathbb{R}^2$ of a (usually) trapezoidal form. These fuzzy intervals sometimes 'fuzzify' Allen's relations between them. It holds, when some points of an initial fuzzy interval remain in a given Allen's relation ('before', 'later', etc.) with the points of a second fuzzy interval, but some points do not. Anyhow, all the situations ellucidate only a qualitative side of both Allen's and fuzzy Allen's relations.

An interesting attempt to grasp quantitative (computational) aspects of Allen's and fuzzy Allen's relations was put forward in such works of De Cock-Schocker's school as: [15, 14, 13][1]. This proposal forms a kind of a sophisticated calculus, in which fuzzy Allen's relations are expressed in terms of minima, maxima, suprema and infima. An alternative approach to the representation of fuzzy Allen's relations was proposed by H-J. Ohlbach in [12, 11, ?]. In the conceptual framework of his approach, fuzzy Allen's relations are represented by normalized integrals. Some ideas of the paper steem from the earlier integral-based approaches to fuzziness from [5, 9, 4]. The Ohlbach's ideas were adopted and referred to the so-called *Simple Temporal Problem under Uncertainty with Preferences* (STPU) in [7] and developed in [3, 8].

[1] Fuzzy-temporal aspects of Allen's relations was also discussed from a more engineering perspective in [6] and implicitly mentioned in [10].

Motivation and Objectives of the Paper. Meanwhile, one has an impression that Ohlbach's integral-based interpretation is not sufficient. It follows from the following reasons: A) fuzzy Allen's relations are viewed here as fuzzy values of integrals, what seems to be an excessive simplification, B) Ohlbach's approach 'escapes' towards reasonings based on probability theory and statistics instead of real-analysis and algebra-based reasonings and C) it seems that this approach does not (completely properly) emphasize a sense of some definitions of fuzzy Allen's relations. It motivates us to propose an alternative approach to the representation of fuzzy Allen's relations in terms of (normalizable) convolutions as a more adequate solution. According to it – the main objective of the paper is to propose an outline of a convolution-based approach to the representation of fuzzy Allen's relations.

The rest of the paper is organized as follows. In Section 2 a terminological background of the paper analysis is put forward – the Ohlbach's approach in particular. In Section 3, the convolution-based approach to the representation of fuzzy Allen's relations is proposed. Section 4 contains concluding remarks and a brief description of further perspectives.

2 Terminological Background

2.1 Ohlbach's Integral-based Approach to Fuzzy Allen's Relations in a Nutshell

. Ohlbach's approach to the representation of fuzzy Allen's interval relations is two-stages. In the first stage, the so-called *fuzzy Allen's relations of the point-interval type* are considered. In the second one – fuzzy Allen's relations are extended to the so-called *fuzzy Allen's relations of the fuzzy interval- fuzzy interval type*[2]. Both types of relations may be briefly specified as follows.

- *Fuzzy Allen's relations of the point- interval* type. They assert that a point, say p, remains in R-relation to a fuzzy interval j. Symbolically: $R_p(j)$, where R is a chosen Allen's relation and may be represented as a distribution function.
- *Fuzzy Allen's relations of the fuzzy interval- fuzzy interval* type arise, if we blow points p's to a new fuzzy interval, say i. Since each $R_p(j)$ may be interpreted as a distribution function, it also has a density-based representation. Finally, such a newly created fuzzy interval-interval relation $R^{Fuzz}(i, j)$ (if i, j are fuzzy intervals) may be interpreted as expected values of the general form:

  $$R(i,j)^{Fuzzy} = \int_{-\infty}^{\infty} i(x)\widehat{R_p}(j)(x)\mathrm{d}x, \qquad (1)$$

  where $\widehat{R_p}(j)(x)$ forms a density-based representation of $R_p(j)(x)$ (see:[12, 11, **?**]).

To cut the long story short, Ohlbach proposes to see fuzzy Allen's relations as *normalized integrals of a single variable.*

[2] Fuzzy intervals are representable here as two-dimentional trapezoidal objects.

2.2 Ohlbach's Integral-based Approach to Fuzzy Allen's Relations in Detail

Fuzzy Allen's relations of the points-fuzzy interval type. Assume that some point p, a fuzzy interval j and one of 13 Allen's relations, say R, are given. Observe that one can put the point-interval relation, $R_p j(x)$, which asserts that p is located in a position defined by R with respect to the interval j, etc.

Example 1 *Taking a point p and an interval j, the relation $B_p(j)$ will asserts that p lies 'before' the interval j and $D_p(j)$ asserts that p is 'during' j.*

Let us preface further considerations by some useful observation that each point-interval relation (of Allen's sort) determines its corresponding function.

Collolary 1 *([12, 11])*[3] *A fuzzy point-interval relation R(t, i) is a function that maps a time point t and the interval i to a fuzzy value. Conversely, if i is a fuzzy interval and R' is a function, then R defined as follows:*

$$R(t,i) =^{def} R'(i)(t), \tag{2}$$

is the corresponding fuzzy point-interval relation.

Fuzzy Allen's relations for two fuzzy intervals. Observe now that each such a point-interval Allen's relation may be extended to its corresponding interval-interval relation over a new fuzzy interval – as depicted in Fig. 22. For example, taking a time point t, an interval (not necessary a fuzzy one) j and a point-interval *R(t, j)* we can put:

$$R(i,j) = i(t)R(t,\ j). \tag{3}$$

It allows us to write: $R(i,j)(t) = f^{i(x)}(t)R(j)(t)$, where $f^{i(x)}(t)$ is a function characterizing an interval i.

Example 2 *In this way one can specify after(i,j) = i(t)after(t, j), before(i,j) = i(t)before(t, j) and all interval-interval Allen's relations.*

Fuzzy Allen's relations for fuzzy intervals. In order to define fuzzy Allen's relations – due to Ohlbach's ideas – let us return to Allen's point-interval relations and consider, say 'before'-relation $B_p(j)(x)$, for a fuzzy interval j and i such that $p \in i$. Due to – [11], this relation may be rendered in terms of the so-called *extend* function E^+ and the complementation operator $N(E^+)$. This function 'behaves' as the functions depicted in Fig. 20 for $L^{\leq}$-relation. Namely, $N(E^+)$ decreases in the right neighborhood of a given point (b-α in Fig.20a) and it increases for arguments being far from it. Let us try to think about $B_p(j)(x)$ determined by $N(E^+)$ in terms of probability theory now.

[3] This corollary was introduced as a definition by Ohlbach in [12].

Allen's point-interval relations in terms of probability theory. Therefore, assume that a probability space Ω of elementary events with a probability measure $P : \Omega \to [0, 1]$ are given. For a given fuzzy interval i we define points of its $\mathbb{R}$-support (See: Figure 18) as elements of Ω. Define also a *random variable* $X : \Omega \to \mathbb{R}$ such that $X(\omega) = X(p) = x \in \mathbb{R}$. In other words, we associate each point p of i-support to a single variable x of a real line. It enables to view $N(E^+)$ as a *distribution* function for i. In fact, $N(E^+)$ in Fig. 20 a) 'represents' a probability the event: $-\infty \leq X = x < b - \alpha$. Formally, $N(E^+)(x) = P(-\infty \leq X = x < b - \alpha))$.

Observe also that such a $N(E^+)$ is a continuous function and $N(E^+) < \infty$. Thus, there exists a function f_X to be called *probability density* – co-definable with $N(E^+)$ as follows:

$$N(E^+)(x) = \int_{-\infty}^{x} f_X(x)\mathrm{d}x. \tag{4}$$

Summing up, the (fuzzy) point-interval relation $B_p(j)$ in terms of $N(E^+)$ may be interpreted as a distribution for the second fuzzy interval i, that contains p's points. It remains to decide, what might represent the fuzzy interva-interval relation $B(i, j)$.

H-J. Ohlbach postulates to consider a unique expected value for X for in this role, although he did not render this postulate explicitely. In a general case, having a function $\phi : \mathbb{R} \to \mathbb{R}$, the expected value $E(\phi(X))$ is defined as:

$$E(\phi(X)) = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}(F(x)), \tag{5}$$

where $F(x)$ is a distribution.

Fuzzy Allen's relations of the interval-interval type. It remains to specify this expected value of (38) in our case, or taking $N(E^+)(x) = F(x)$ (as a distribution) in (38). Therefore:

$$E(\phi(X)) = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}(N(E^+)(x)). \tag{6}$$

Because of (5), we have:

$$\int_{-\infty}^{\infty} \phi(x)\mathrm{d}N(E^+)(x) = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}(\int_{-\infty}^{x} f_X(x)\mathrm{d}x) = \int_{-\infty}^{\infty} \phi(x) f_X(x)\mathrm{d}x,$$

Thus:

$$E(\phi(X)) = \int_{-\infty}^{\infty} \phi(x) f_X(x)\mathrm{d}x. \tag{7}$$

If put $\phi(x) = i(x)$ as a function characterizing the interval i, we can obtain the required form in our case:

$$E(i(x)) = \int_{-\infty}^{\infty} i(x)\widehat{B_p}(x)\mathrm{d}x, \tag{8}$$

where $\widehat{B_p(x)}$ denotes the fuzzy point-interval 'before' as a density function.

Example 3 *If assume that:*

$$F(x) = N(E^+) = \begin{cases} 0, & \text{for } x \leq a, \\ \frac{x-a}{b-a}, & \text{for } a < x \leq b, \\ 1, & \text{for} x > b, \end{cases} \quad \textit{then } f(x) = \begin{cases} 0, & \text{for } x < a, \\ \frac{1}{b-a}, & \text{for } a \leq x \leq b, \\ 1, & \text{for } x > b, \end{cases}$$

$$\textit{and } B(i,j) = E^j_{before}(i(x)) = \frac{1}{b-a} \int_{-\infty}^{\infty} i(x)\mathrm{d}x.$$

Taxonomy of fuzzy Allen's relations in Ohlbach's depiction. We have just emphasized how fuzzy Allen's relations may be rendered in terms of integrals. In addition, a general form of them was elaborated for 'before'-relation. In this moment, a complete taxonomy of fuzzy Allen's relations in Ohlbach's depiction will be introduced – due to [12, 11].

Normalization. Obviously, all the integrals above, in particular (37) and (42), are finite. In particular, (37) holds, if and only if the integral on the right side of (37) is finite. It is warranted by the fact that $N(E^+)$ is assumed to be finite. Whereas finiteness of the integrals constitutes a sufficient condition from a purely mathematical point of view, it is unsatisfactory to consider (42) and similar conditions as the adequate representations for fuzzy Allen's relations. In fact, we expect that these integrals will take fuzzy values from $[0, 1]$, so they should be normalized.

Different methods of normalization is known and used. For 'before'-relation, the factors $|i|$ and $|i|_a^b$ defined as follows:

$$|i| = \int_{-\infty}^{\infty} i(x)\mathrm{d}x, \quad |i|_a^b = \int_a^b i(x)\mathrm{d}x. \tag{9}$$

By contrast, $meet(i,j), start, finishes$ require the normalization factors of the type $N(i,j)$, so as dependent on both i and j. Ohlbach argues in [12], p. 26) for a choice of the following two factors of this type, as they admit also 1 as a possible value:

$$N(i,j) = \max_a \int_{-\infty}^{\infty} i(x-a)j(x)\mathrm{d}x, \quad N(i,j) = \min(|i|, |j|). \tag{10}$$

An outline of Ohlbach's taxonomy. These general assumptions and observations allows us to introduce the whole taxonomy of the integral-based representation of Allen's relations. Putting aside its detailed presentation, we illustrate how fuzzy Allen's relations are defined in this approach in two cases.

1. Meet^{Fuzzy} (see [12, 11]): This definition is based on the observation that 'meet'-relation holds between two fuzzy intervals if and only if there are some functions, say *Fin(i)* and *St(j)*, that 'cut' the initial points from the first interval i and the final ones from the second j interval. It allows us to note the following.

– If i or j are empty, they cannot meet, so the relation yields 0.

– Similarly, if i is $[a,\infty)$-type, j is $(-\infty,a)$ and conversely, for a given fixed a,
– Otherwise, one can define this relation as a statement that the 'end' part *Fin(i)* of the interval i touches the initial part *St(i)* of the interval j. The factor *N(Fin(i), S(j))* normalizes this integral to be smaller than 1.

It leads to the following depiction:

$$meet(i,j)^{Fuzzy} = \begin{cases} 0 & \text{if } i=\emptyset \text{ or } j=\emptyset \\ & \text{or } i=[a,\infty) \ \wedge j=(-\infty,a), \\ & \text{or } j=[a,\infty) \ \wedge i=(-\infty,a), \\ \displaystyle\int_{-\infty}^{\infty} \frac{Fin(i)St(j)dx}{N(Fin(i),St(j))} & \text{otherwise.} \end{cases}$$

2. Before. A similar way of reasoning enables of defining other Allen's relations. For example, 'before' $B(i,j)^{Fuzzy}$ is defined as follows (a detailed justification may be found in [12, 11, **?**]. As earlier, $(i\cap j)\widehat{B(j)}$ forms a density-based representation of $B_p j(x)$ blown up over the intersection of fuzzy intervals i and j.):

$$B(i,j)^{Fuzzy} = \begin{cases} 0 & i=\emptyset \text{ or } i=[a,\infty) \text{ or } j=\emptyset, \\ 1 & i=(-\infty,a] \text{ or } i\cap_{\min} j=\emptyset, \\ \displaystyle\int_{-\infty}^{\infty}(i\cap j)\widehat{B(j)}dx/|i\cap_{\min} j| & i=(-\infty,a], j \text{ is bounded} \\ & \text{or } j=[a,\infty), \\ \displaystyle\int_{-\infty}^{\infty} i\widehat{B(j)}dx/|i| & \text{otherwise.} \end{cases}$$

Further examples of fuzzy integral Allen's relations may be found in [12, 11].

2.3 Conceptual Framework of Further Analysis

The notion of convolution, that we need, requires a new conceptual framework. Its determined by a class of Lebesgue integrable functions on $\mathbb{R}$, denoted by $L(\mathbb{R})$. This class forms a unique example of the so-called *Banach spaces*.

In order to describe both types of spaces, assume that X is a given vector (linear) space. Each vector space is defined over a scalar field, say K. This fact is denoted by $X(K)$. The usual scalar fields are: the field R of real numbers $\mathbb{C}$, or over the field $\mathbb{C}$ of complex numbers. We write then: $X(\mathbb{R})$ and $X(\mathbb{C})$ (*resp.*) to render the fact that X is defined over $\mathbb{R}$ or $\mathbb{C}$.

Assume also that $X(\mathbb{R})$ is given. Let us introduce now a new function $\| \bullet \| : X(\mathbb{R}) \mapsto [0,\infty)$ that respects the following conditions:

$$\|x\| = 0 \iff x = 0, \ \ \|\alpha x\| = |\alpha|\|x\|, \text{ for } \alpha \in \mathbb{R}, \tag{11}$$

$$\|x+y\| \leq \|x\| + \|y\|. \tag{12}$$

This function is to be called *a norm* and the whole space $(X(\mathbb{R}), |\bullet|)$ forms a *normed space*. A *Banach space* is such a *normed vector space* X, which is complete with respect to that norm, that is to say, each Cauchy sequence $\{x_n\}$ in X converges to an element x in X, i.e. $\lim_{n\to\infty} x_n = x$.

Example 1. $(\mathbb{R}, |\bullet|)$ with the norm $\|x\| = |x|$, for each $x \in \mathbb{R}$ is a Banach space.

Some special examples of Banach spaces – that are especially interesting for us – are presented in the table below.

Type of spaces	**Abbr.**	**Elements of the space**	**Norms**
The space of Lebesgue integrable functions on $\mathbb{R}$ 'in square'	$L^2(\mathbb{R})$	Functions $f, g, h, \ldots$ - Lebesgue integrable on $\mathbb{R}$	$\|f\| = \left(\int \|f\|^2 \mathrm{d}x \right)^{\frac{1}{2}}$
The space of Lebesgue integrable functions on $\mathbb{R}$ 'in p'	$L^p(\mathbb{R})$	Functions $f, g, h, \ldots$ - Lebesgue integrable on $\mathbb{R}$	$\|f\| = \left(\int \|f\|^p \mathrm{d}x \right)^{\frac{1}{p}}$, $1 < p < \infty$

Definition 1 ***(Convolution.)*** *Let us assume that functions f and g are Lebesgue integrable in $\mathbb{R}$, i.e they belong to $(L^1(\mathbb{R}, \|\bullet\|)$. Then the convolution of f and g – denoted by $f * g$ – is usually defined as follows:*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt\,, \tag{13}$$

where the right side is an improper (Riemann) integral.

The proofs will also exploit the concept of (Lebesque) measurable functions and the Beppo-Levi's monotone convergence theorem.

Definition 2 ***Beppo-Levi's theorem*** *Assume that a non-decreasing sequence $\{f_k\}$ of measurable non-negative functions $f_k : X \to [0, +\infty]$, for a given measure space (Ω, σ, μ), $X \in \sigma$, is given. If $f(x) = \lim_{k\to\infty} f_k(x)$, for each $x \in X$, then:*

$$\int_X f(x)\mathrm{d}\mu = \lim_{k\to\infty} \int_X f_k(x)\mathrm{d}\mu.$$

3 Fuzzy Allen's Relations in a Computational Depiction

Before we introduce the convolution-based representation of fuzzy Allen's relations in order to grasp the computational aspects of these relations, let us return to the initial case of 'meet'-relation in the Ohlbach's integral depiction. It was already said that this definition is based on two functions: $Fin(i)$ and $St(j)$ that 'cut' the initial part of i and the final part of j (resp.).

In fact, both intervals should be integrated together, but both functions should be considered as 'running' in different directions: $Fin(i)$ running towards the second j interval and $St(j)$ as running in the inverse direction – towards the interval i. It means that $St(j)$ should be rather consider as a function of a new argument, say t, and $Fin(i)$ as $Fin(i)(x-t)$[4]. Meanwhile, such a combination of $Fin(i)$ and $St(j)$ of mutually independent functions is given by a *convolution* of them[5].

Finally, there exists another argument for the convolution-based representation of fuzzy Allen's relations. Namely, it is a known fact of real analysis that convolutions take finite values. It means that their use would make a mathematical discussion on fuzzy Allen's relations less conditional than the Ohlbach's integral approach. In fact, a success of Ohlbach's approach is only possible provided that the appropriate integrals are finite. In the convolution-based approach this problem disappears thanks to this elementary property of convolutions.

3.1 Fuzzy Allen's Relations in the Convolution-based Depiction

In this convention, fuzzy Allen's relations from [12, 11, **?**] should be rather rendered as follows:

$$meet(i,j)^{Fuzzy}(x) = \int_{-\infty}^{\infty} \frac{Fin(i)(x-t)St(j)(t)dt}{N(Fin(i)(x-t), St(j)(t))}. \tag{14}$$

(Note that $\mathrm{meet(i,j)}^{Fuzzy}$ is a function of x-argument as we integrate with respect to the second argument t.) Similarly, one could modify, for example, 'before' -relation:

$$before(i,j)^{Fuzzy} = \int_{-\infty}^{\infty} i(x-t)\widehat{B}(j)(t)\mathrm{d}t/|i|. \tag{15}$$

Fuzzy Allen's relations as norms of convolutions. It seems that an idea to represent fuzzy Allen's relations in terms of convolution is already the appropriate one. As illustrated, convolutions better 'encode' an idea to combine two functions and they are computationally convenient – as they are finite. Finally, they might be also normalized.

Nevertheless, one can argue that convolutions still are not ideal in this role. In order to illustrate this fact, let us consider the following paradoxical dichotomy.

A On one hand, we can have *two different* fuzzy Allen's relations, say $R_1^{Fuzzy}(i,j)$ – R_2^{Fuzzy} given by two normalized convolutions C_1 and C_2 (resp.)– that take the same values $\alpha \in [0,1]$ in some area. (They diagrams are identical in this area).

[4] The argument $x-t$ ensures that i and j meet together. Thus, $Fin(i)$ should be seen as a function of an argument x in t-translation. $|i|$ is a normalization factor

[5] Note that this situation may be seen as a 'combination' of two independent signals running in inverse directions that are represented in physics by convolutions.

B On the other hand, we can have *a single* fuzzy Allen's relation, say $R^{Fuzzy}(i,j)$ – depicted by a normalized convolution C – that may be multiplied by a scalar $\alpha \in \mathbb{R}$. This new convolution αC would take another values than C alone, although it represents the same relation $R^{Fuzzy}(i,j)$.

Obviously, we are willing to consider C and αC as mutually linked (as they represent $R^{Fuzzy}(i,j)$ and $\alpha R^{Fuzz}(i,j)$). Simultaneously, we want to see C_1 and C_2 as mutually independent – as they represent different relations $R_1^{Fuzzy}(i,j)$ and $R_2^{Fuzzy}(i,j)$ – even though their diagrams are partially identical.

It seems that considering *norms from convolutions* as the alternative representation of fuzzy Allen's relations (instead of convolutions themselves) allows us to avoid these difficulties. For a confirmation of this hypothesis let us assume that $R^{Fuzzy}(i,j)$ is represented now by two norms, say $\|\|_1$ and $\|\|_2$. Formally, we postulate:

$$R^{Fuzzy}(i,j) = \|C\|_1, \quad R^{Fuzzy}(i,j) = \|C\|_2. \tag{16}$$

Let us state that $\|C\|_1$ and $\|C\|_2$ may be viewed as *mutually equal* provided that there are such real constants $a, b < \infty$ that:

$$a\|C\|_1 \leq \|C\|_2 \leq b\|C\|_1{}^6. \tag{17}$$

This condition allows us to identify two norm-values associated to a given fuzzy Allen's relation. Since such a mutual equivalence of norms is sufficient in our approach, it delivers an argument to represent fuzzy Allen's relations by *norms of convolutions*. Formally, if $R(i,j)$ denotes a convolution 'basis' of $R^{Fuzzy}(i,j)$, then $R^{Fuzzy}(i,j)$ can be written:

$$R^{fuzzy}(i,j) = \|R(i,j)\|_{L(\mathbb{R}^1)}. \tag{18}$$

Example 2. Assume that Allen's relation 'before' in a convolutive representation $B(i,j) = f^i \star B_p(j)$ is given, for some functions f^i and $B_p(j)$. Then fuzzy 'before' B^{fuzzy}:

$$B^{fuzzy}(i,j) = \|B(i,j)\|_{L(\mathbb{R}^1)}. \tag{19}$$

But $B(i,j)$ is a convolution, so $B(i,j)(x) = \displaystyle\int_{-\infty}^{\infty} f^i(x-t)B_p(j)(t)\mathrm{d}t$. Assuming that $B(i,j)(x) \in L(\mathbb{R})$ with the norm $\|f\| = \displaystyle\int |f(x)|\mathrm{d}u$, for each $f \in \Big(L(\mathbb{R}^1), \|\ \|\Big)$, we have:

$$B^{fuzzy}(i,j) = \| \int_{-\infty}^{\infty} f^i(x-t)B_p(j)(t)\mathrm{d}t\|_{L(\mathbb{R}^1)} = \int | \int_{-\infty}^{\infty} f^i(x-t)B_p(j)(t)\mathrm{d}t|\mathrm{d}\mu. \tag{20}$$

[6] This property is a known property of norms.Note that this condition is satisfied in our case. In fact, it is enough to put $a = b = \alpha$.

The key idea of our proposal is briefly expressed as follows.

Type of relations:	Given by:	Examples:
Fuzzy Allen's relations of the 'interval-interval' type	L($\mathbb{R}$)-norms of convolutions	$R^{fuzzy}(i,j) = \|R(i,j)(x)\|_{L(\mathbb{R})} = \int(\int_{-\infty}^{\infty} f^i(x-t)R_p(j)(t)\mathrm{d}t)\mathrm{d}\mu$

In last part of this section, we intend to prove two computational features of fuzzy Allen's relations in a convolution depiction. The first theorem shows that these relations are normalizable, the second one – that their diagrams are, somehow, predictable as they are uniformly continuous.

Theorem 1. *Convolution-basis of fuzzy Allen's relations are normalizable.*

Proof: It follows from the above fact that:
$\int_{-\infty}^{\infty} f^{i(x)}R(t,j)dt < \infty$ and the fact that – due to Fubini's theorem – this integral belongs to $L^1(\mathbb{R})$. It means that there is an upper bound for it, say M. It is enough now to put a normalization factor N to ensure that $0 \leq \frac{M}{N} \leq 1$. □

Theorem 2. *Let* $1 \leq p < \infty$, $f_i, R(j) \in L_p(\mathbb{R})$, $\|R(j)\| \leq M$, *for some* M *where* i_f *is a characteristic function for a fuzzy interval* i *and* $R(j)$ *is a functional representation of a point-interval relation (of Allen type) with respect to a fuzzy interval* j. *Assume also that* f_i *is uniformly continuous on* $\mathbb{R}$. *Then* $f_i \star R(j)$ *is* ***uniformly continuous*** *on* $\mathbb{R}$, *too:*

Proof: Since f_i depends on $x - t$ and $R(j)$ dependent on t, let us establish $x - t = z$. It easy to see now that there is a $\rho > 0$ such that $z_1, z_2 \in \mathbb{R}$ and $\|z_1 - z_2\| < \rho$ implies $\|f_i(z_1) - f_i(z_2)\| < \epsilon$. However, f_i is assumed to be uniformly continuous in $\mathbb{R}$, i.e. there is a $\rho > 0$ such that for all $z_1, z_2 \in \mathbb{R}$, $|z_1 - z_2| < \rho$ implies $\|f_i(z_1 - f(z_2)\| < \epsilon$. Then $\|z_1 - z_2\| < \rho$ also implies:

$$\|i_f \star R(j)(z_1) - f_i \star R(j)(z_2)\| = \Big(\int(|i_f(z_1) - f_i(z_2)| \bullet |R(j)(t)|)^p dt\Big)^{\frac{1}{p}}$$

$$= \|(f_i(z_1) - f_i(z_2)) \bullet R(j)\|_p \leq \|f_i(z_1) - f_i(z_2\| \bullet \|R(j)\|_p < \epsilon_1,$$

where $\epsilon_1 = M\epsilon$. Obviously, $\epsilon_1 \to 0$. The last inequality follows from the fact that $\|R(j)\|_p \leq M$ and from Schwartz's inequality $\|xy\| \leq \|x\|\|y\|$, for each $x, y \in L^p(\mathbb{R})$, $1 < p < \infty$.

The next theorem illustrates a computational power of the convolution-based approach. In fact, it forms a unique version of *Borel's Convolution Theorem* for convolutions used for defining fuzzy Allen's relations. It will be briefly called: 'Convolution Theorem for fuzzy Allen's relations'.

Theorem 3. *(**Convolution Theorem for fuzzy Allen's relations**). Let* i, j *be fuzzy intervals. Let also* $f^i, R(t,j) \in L^1[-\infty, \infty]$ *be a function characterizing a* *fuzzy interval i and a point-interval Allen relation (resp.). Then their convolution $h = f^i * R$ has the following property:*

$$h(x) = \int_{-\infty}^{\infty} |f^{i(x)}(x-t) * R(j)(t)| \mathrm{d}t = \int_{-\infty}^{\infty} |f^{i(x)}(x-t)| \mathrm{d}x \int_{-\infty}^{\infty} |R(j)(t)| \mathrm{d}t.$$

Simply:

$$\mathcal{F}(f^i * R(j)) = \mathcal{F}(f^i)\mathcal{F}(R(j)). \tag{21}$$

Proof: Assume that $f^i, R \in L^1(\mathbb{R}^1)$ – as in the formulation of this theorem – are given. In order to compute their convolution:

$$h(x) = \int_{-\infty}^{\infty} |f^{i(x)}(x-t) * R(j)(t)| \mathrm{d}t, \tag{22}$$

we exploit Fourier transforms of f^i and R, i.e. $\mathcal{F}(f)$ and $\mathcal{F}(g)$ (respectively) – defined as follows:

$$\mathcal{F}(f^i) = \int_{\mathbb{R}} f^{i(x)}(x-t) e^{-2\pi i x v} \mathrm{d}x \quad \text{and } \mathcal{F}(R) = \int_{\mathbb{R}} R(j)(t) e^{-2\pi i x v} \mathrm{d}t.$$

Therefore, for all $y \in \mathbb{R}$, Fourier transform $F(f * g)$ is as follows.

$$\begin{aligned} \mathcal{F}(f^i * R(j)) = \mathcal{F}(h)(x) = \mathcal{F}(\int_{-\infty}^{\infty} |f^{i(x)}(x-t) R(j)(t)| dt) \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f^{i(x)}(x-t) R(j)(t)| \mathrm{d}t e^{-2\pi i x y} \mathrm{d}x \\ = \int_{-\infty}^{\infty} R(j)(t) \int_{-\infty}^{\infty} f^{i(x)}(x-t) e^{-2\pi i x y} \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{23}$$

By substitution $x - t = u$, or $x = t + u$ (and $\mathrm{d}x = \mathrm{d}u$) we obtain:

$$\begin{aligned} \int_{-\infty}^{\infty} R(j)(t) \int_{-\infty}^{\infty} f^{i(x)}(x-t) e^{-2\pi i x y} \mathrm{d}x \mathrm{d}t \\ = \int_{-\infty}^{\infty} R(j)(t) \Big(\int_{-\infty}^{\infty} f^{i(x)}(x-t) e^{-2\pi i (t+u) y} \mathrm{d}u \Big) \mathrm{d}t. \end{aligned}$$

Applying Fubini theorem in order to interchange the order of limitation we can write:

$$\begin{aligned} \int_{-\infty}^{\infty} R(j)(t) \Big(\int_{-\infty}^{\infty} f^{i(x)}(x-t) e^{-2\pi i (t+u) y} \mathrm{d}u \Big) \mathrm{d}t \\ = \int_{-\infty}^{\infty} R(j)(t) e^{-2\pi i t y} \mathrm{d}t \int_{-\infty}^{\infty} f^{i(x)}(x-t) e^{-2\pi i u y} \mathrm{d}x = \mathcal{F}(f^i)\mathcal{F}(R(j)). \end{aligned}$$

Therefore,

$$\mathcal{F}(f^i * R(j)) = \mathcal{F}(f^i)\mathcal{F}(R(j)), \tag{24}$$

what finishes the proof.

4 Conclusions

It has already been shown how fuzzy Allen's relations may be depicted in a computational, convolution-based approach. It seems that this approach serves a kind of an improvement and a further development of the Ohlbach's integral approach towards a theoretic well-founded calculus. This solution allows us to omit some theoretic difficulties of Ohlbach's approach, such as an unexpected infiniteness of his integrals. Research on fuzzy Allen's relations may be extended in (at least) two different directions. At first, some approximation methods (such as Hardy-Littlewood Theorem) for fuzzy Allen's relations in the convolution-based depiction may be directly adopted. Secondly, the convolution-based approach may be developed towards a new algebraic reinterpretation of Allen's algebra in terms of incidence algebra. Nevertheless, it requires a deeper analysis.

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