Chapter 1

Introduction

1.1 Shape Matching

The task of *shape matching* is to determine the similarity of geometric shapes and to transform shapes such that they become most similar. The geometric shapes typically are modeled as points, curves, or surfaces, discrete or continuous. The similarity is determined by a *similarity measure*. Often a *distance measure* is used to measure the dissimilarity of the shapes, i.e., shapes with small distance are considered similar. In this thesis, we will measure the similarity of curves and surfaces by the Fréchet distance, the weak Fréchet distance, and the Hausdorff distance.

For solving the task of shape matching, i.e., transforming shapes such that they become most similar, two subproblems need to be solved: the *decision problem* and the *computation problem*. The decision problem asks: Are these shapes similar? and the computation problem: How similar are these shapes? More formally, the decision, computation, and matching problem can be formulated as follows:

Decision problem:

Given two shapes A, B, a distance measure δ, a real value $\varepsilon > 0$

Decide $\delta(A, B) \leq \varepsilon$?

Computation problem:

Given two shapes A, B, a distance measure δ

Compute $\delta(A, B)$

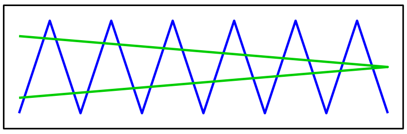
Matching problem:

Given two shapes A, B, a distance measure δ, a set of transformations T

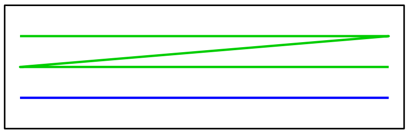
Compute $\min\limits_{t \in T} \delta(t(A), B)$

The solutions of the above problems naturally depend on the distance measure. A commonly used distance measure is the Hausdorff distance [7]. It is defined for point sets but can be generalized to curves and surfaces by interpreting these as point sets. The Hausdorff distance has an intuitive definition: it is the largest nearest neighbor distance between the two point sets. It can be computed for discrete geometric shapes, e.g., simplices, in any dimension in polynomial time [4].

For curves and surfaces, however, the Hausdorff distance is not always a natural or suitable distance measure. Figure 1.1 (a) shows two curves that have small



(a) Small Hausdorff distance but large (weak) Fréchet distance



(b) Small weak Fréchet distance but large Fréchet distance

Figure 1.1: Example of curves

Hausdorff distance but may be considered dissimilar by human perception and in applications. In this case, the Fréchet distance is a more suitable distance measure. Overviews of shape matching algorithms are given by Alt and Guibas [7] and by Veltkamp and Hagedoorn [51].

Applications

Shape matching has many applications, among others in computer vision, pattern recognition, computer aided design, medical imaging, robotics, traffic control, and molecular biology. We briefly describe two examples in which the Fréchet distance has been successfully applied.

The problem of *vehicle tracking* is the following: a moving vehicle is equipped with a geographical positioning system (GPS) and its position is stored at discrete time intervals. Based on this data and a map of the area the vehicle was traveling in, one wants to determine the route that the vehicle took. Due to limited accuracy of GPS the data is noisy, i.e., the stored positions of the vehicle do not precisely lie on roads of the map. The problem of vehicle tracking can be solved as follows: the map is interpreted as a geometric graph G and the sequence of positions of the vehicle as polygonal curve C (the vertices of the curve are the stored positions of the vehicle). Then we determine the route of the vehicle as path P in the graph G with smallest Fréchet distance to the curve C. The Fréchet distance is a well suited distance measure for this because it takes into account the movement of the vehicle. This approach was proposed by [5] and has been succesfully applied to large sets of vehicle tracking data [13].

In molecular biology the Fréchet distance has been successfully applied to *protein structure alignment*. A *protein backbone* consists of amino acids linked by peptide bonds. It can be modeled as a polygonal chain in $\mathbb{R}^3$ with the amino acids modeled as vertices and peptide bonds as edges. A natural distance measure for aligning, i.e., matching protein backbones is the *discrete Fréchet distance*. The discrete Fréchet distance measures the similarity of polygonal curves based on the distances between vertices and taking into account the order of the vertices given by the edges. Protein structure alignment by matching under the discrete Fréchet distance has been successfully applied to protein data [11, 33].

In this thesis, we are interested in the theoretical complexity of shape matching. As distance measure we will consider the Fréchet distance and we will analyze the complexity of the decision and computation problem for the Fréchet distance between surfaces.

1.2 Fréchet Distance

The Fréchet distance is a distance measure for continuous shapes such as curves and surfaces. By taking parameterizations of the shapes into account it is a finer and sometimes more appropriate distance measure than the Hausdorff distance.

The Fréchet distance was introduced for curves by Fréchet [25] and later generalized to surfaces [26]. As a similarity measure for shape matching it was first proposed by Alt and Godau [6].

Definition

The Fréchet distance between two curves or surfaces is defined as the infimum over all reparameterizations and supremum over all pointwise distances along the curves or surfaces, respectively. As reparameterizations usually orientation-preserving homeomorphisms are used. Thus the Fréchet distance between two curves or surfaces $f, g\colon [0,1]^k \to \mathbb{R}^d,\ k = 1,2$ and $k \leq d$, is

$$\delta_F(f,g) := \inf_{\sigma \text{ hom}} \sup_{t \in [0,1]^k} \operatorname{dist}\bigl(f(t), g(\sigma(t))\bigr).$$

where $\sigma\colon [0,1]^k \to [0,1]^k$ ranges over all orientation preserving homeomorphisms and $\operatorname{dist}(\cdot,\cdot)$ denotes an underlying metric in $\mathbb{R}^d$.

The Fréchet distance between curves can be illustrated by a man and a dog walking on the curves: assume the man walks on one curve and the dog on the other and the man holds the dog on a leash. Both may choose their speed and may stop but not walk backwards. Then the Fréchet distance is the shortest leash length that allows them to walk on the two curves from beginning to end. Based on this illustration the Fréchet distance is sometimes also called the *man-dog* or *dogleash distance*.

A variant of the Fréchet distance is the *weak Fréchet distance* where both curves or surfaces are reparameterized by surjective continuous maps. In the man dog illustration this means that the man and dog may also walk backwards. This distance measure is weaker than the Fréchet distance and stronger than the Hausdorff distance in the following sense. It is always greater or equal than the Hausdorff distance and smaller or equal than the Fréchet distance and there are examples of curves where it differs by much from these two distance measures, see Figure 1.1. Another variant of the Fréchet distance is the *discrete Fréchet distance*. Here, vertices are assigned to vertices and distances are measured only between vertices. To this the man-dog illustration no longer applies (one could instead imagine frogs leaping from vertex to vertex).

The man-dog illustration applies only to the Fréchet distance between curves and not between surfaces. For curves, the parameter spaces are intervals of $\mathbb{R}$ which can be interpreted as time axis. For surfaces, the parameter spaces are two-dimensional and do not allow the same interpretation.

Computability

For the Fréchet distance between polygonal curves, polynomial time algorithms for solving the decision, computation and matching problem are known [6, 8, 21]. For surfaces, it is known that the decision problem is NP-hard [29], and it is not known whether it is computable. The NP-hardness proof constructs triangulated surfaces which are mapped non-injectively into $\mathbb{R}^2$. These can be “unfolded” in $\mathbb{R}^4$. Thus, the NP-hardness holds for intersecting triangulated surfaces in $\mathbb{R}^2$ and non-intersecting triangulated surfaces in $\mathbb{R}^4$. It is not known whether the problem is NP-hard for non-intersecting surfaces in $\mathbb{R}^3$.

The reason that the Fréchet distance is more difficult to compute for surfaces than for curves lies in having to take the infimum over all homeomorphisms. For curves, these are the orientation preserving homeomorphisms on the unit interval. These can be characterized as the continuous, surjective and strictly monotone increasing functions on the unit interval. For surfaces we consider the homeomorphisms on the unit square (or the unit k-cube for k-dimensional shapes). These homeomorphisms do not have a similar mathematical characterization and are not as easy to handle algorithmically.

1.3 Overview of the Thesis

In this thesis we investigate the computability of the Fréchet distance between surfaces. For this, we tackle the problem of handling the homeomorphisms in several ways and give three partial answers to the question of the computability of the Fréchet distance between triangulated surfaces.

First, in Chapter 2 we discuss preliminaries that will be used in several of the following chapters. In Chapter 3 we show that the Fréchet distance between triangulated surfaces is semi-computable by approximating the homeomorphisms by discrete maps. Semi-computability is a weaker notion of computability for real-valued numbers and functions. In Chapter 4 we consider the weak Fréchet distance between triangulated surfaces and show that it is polynomial time computable. For this, we give a characterization of the weak Fréchet distance in a geometric data structure called *free space diagram* and show how to compute the characterization. In Chapter 5 we restrict the triangulated surfaces to simple polygons. For these, we show that it suffices to consider a restricted class of realizing maps and give a polynomial time algorithm for computing the Fréchet distance between simple polygons. In Chapter 6 we consider an average or summed Fréchet distance between curves. We show that several intuitive definitions do not satisfy the triangle inequality and are therefore not metrics.