Chapter 6

Summed and Average Fréchet Distance between Curves

6.1 Introduction

In this chapter we consider variations of the Fréchet distance between curves, which we call *average* and *summed Fréchet distance*. The Fréchet distance is defined as the maximum pointwise distance minimized over all reparameterizations. All other pointwise distances on the curves have no influence on the Fréchet distance. Consider for example the curves shown in Figure 6.1. Assume one wants to match the curve f either to the curve g or h. Intuitively, it seems that g is the better match. This is however not reflected by the Fréchet distance which is equal for both pairs of curves (f, g) and (f, h). By slight perturbation, the Fréchet distance can be made larger for either pair of curves. Note that both pairs of curves are already matched optimally under rigid motions.

The aim of an average or summed Fréchet distance is to distinguish these kinds of curves by taking into account all pointwise distances. Two natural questions arise:

1. How to define an average or summed Fréchet distance?

2. How to compute the average or summed Fréchet distance?

In this chapter we consider the first question, i.e., we consider the question of defining an average or summed Fréchet distance. A related notion is *Dynamic Time Warping* [48]. A continuous variant of dynamic time warping has been considered in [20] and an average Fréchet distance in [13].

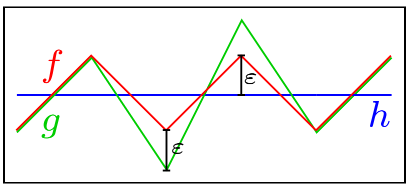

Figure 6.1: The pairs of curves f, g and f, h have the same Fréchet distance

6.1.1 Triangle Inequality

As discussed in the introduction of this thesis, Chapter 1, in shape matching often a distance measure is used for measuring the dissimilarity of shapes. An important property of a distance measure usually is that it is a *metric*. In mathematics, distance measure and metric are often used synonymously.

A metric on a set X is a function $\delta\colon X \times X \to \mathbb{R}_+$ fulfilling

1. $\delta(x, y) = 0 \;\Leftrightarrow\; x = y$ (identity of indiscernibles)
2. $\delta(x, y) = \delta(x, y)$ (symmetry)
3. $\forall x, y, z \in X :\; \delta(x, z) \leq \delta(x, y) + \delta(y, z)$ (triangle inequality).

The triangle inequality is useful for matching shapes stored in data bases. Assume a database of geometric objects given. A typical query is to find the k best matches in the database to a given query shape. For a large data base, the time complexity can be significantly reduced by presorting the objects into clusters of similar shapes.

For this purpose, also a *relaxed triangle inequality* would suffice, namely

$$\exists c\; \forall x, y, z : \quad \delta(x, z) < c(\delta(x, y) + \delta(y, z)).$$

Fagin and Stockmeyer [24] have shown that the relaxed triangle inequality holds for a variant of the distance measure *nonlinear elastic matching (NEM)*. NEM is a similar distance measure to the discrete Fréchet distance. Both are defined based on couplings of the vertices, cf. Section 2.3.6. Instead of the distance between points, NEM is defined based on difference in angles.

In cognitive psychology, similarity measured by human perception is studied. Two important theories are the feature-based approach of Tversky [50] and the transformation-based approach of Hahn et al. [31]. The feature-based approach suggests that the similarity of two objects is a function of their common and distinctive features. The transformation-based approach suggests that similarity depends on the number of operations required to transform one object into the other. Mumford [43] argues, based on Tverskys work, that human perception does not fulfill the triangle inequality.

6.1.2 Further Properties

Some further properties that seem reasonable for an average or summed Fréchet distance, denoted by δ_{avgF} and δ_{sumF}, respectively, are:

1. An average Fréchet distance should not be larger than the Fréchet distance: $\delta_{avgF}(f, g) \;\leq\; \delta_F(f, g)$
2. If an average or summed Fréchet distance is zero, then so should the Fréchet distance: $\delta_{avgF}(f, g) = 0 \;\vee\; \delta_{sumF}(f, g) = 0 \;\Rightarrow\; \delta_F(f, g) = 0$
3. An average or summed Fréchet distance should not depend on the given parameterizations of the curves, but should be invariant under reparameterization: $\forall\, \hom \sigma :\; \delta_{avgF}(f, g) = \delta_{avgF}(f, g \circ \sigma) \;\wedge \delta_{sumF}(f, g) = \delta_{sumF}(f, g \circ \sigma)$.

In the next section, Section 6.2, we develop several definitions for an average or summed Fréchet distance. We show, in Section 6.3, that none of these definitions fulfill the triangle inequality and are in particular not metrics. More specifically, we show that one definition is not symmetric and does not fulfill the triangle inequality. All other definitions are symmetric but do not even fulfill the relaxed triangle inequality for any constant $C > 0$.

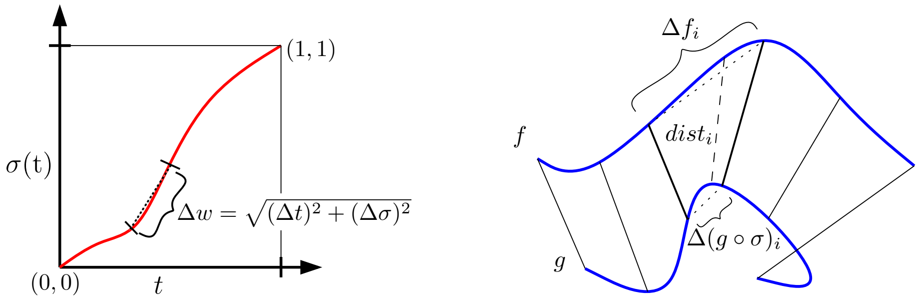

Figure 6.2: Definitions for an average Fréchet distance using the curve integral in parameter space (right) and in image space (left).

6.2 Definitions for an Average or Summed Fréchet Distance

In the following, we develop possible definitions for an average or summed Fréchet distance. For both we accumulate the pointwise distances of all points on the two curves. This gives a summed Fréchet distance. The value of a summed Fréchet distance depends on the measure of the points over which the pointwise distances have been accumulated. For an average Fréchet distance we normalize the summed Fréchet distance by dividing it by the measure of these points.

We will give the definitions both in continuous and discrete form, i.e., we consider discrete sums and their continuous limits. The discrete versions may be called *discrete summed or average Fréchet distance.*

We will develop most definitions using the *path integral.* The path integral of a function $F\colon \mathbb{R}^d \to \mathbb{R}$ over a path $\gamma\colon [0,1] \to \mathbb{R}^d$, which is piecewise continuously differentiable, is

$$\int_\gamma F(z)\; dz = \int_0^1 F(\gamma(t))\; |\dot\gamma(t)|\; dt,$$

where $\dot\gamma$ denotes the derivative of γ. We will use $L(\gamma)$ to denote the length of the path γ, i.e., $L(\gamma) = \int_0^1 |\dot\gamma(t)|\; dt$.

6.2.1 Integrating over the Path in Free Space

One intuitive definition for a summed Fréchet distance is to take the path integral over the monotone path in the free space diagram. That is, we compute the integral $\int_\gamma F(z)\; dz$ over the function $F(s,t) = \|f(s) - g(t)\|$ and the curve $\gamma(t) = (t, \sigma(t))$. As $|\dot\gamma(t)| = \sqrt{1 + \dot\sigma^2(t)}$ this gives the definition

$$\delta_a(f,g) := \inf_{hom\, \sigma} \int_0^1 \; \|f(t) - g(\sigma(t))\|\; \sqrt{1 + \dot\sigma^2(t)}\; dt,$$

which is illustrated in Figure 6.2 right.

A discrete version of this definition is achieved by summing up the discrete Fréchet distance:

$$\min_{C \text{ coupling}} \sum_{(p_i, q_j) \in C} \|p_i - q_j\|.$$

Definition δ_a is a summed Fréchet distance and yields values depending also on the length of the path in free space. For example, for two parallel segments of distance 1, as in Figure 6.3 (a), definition δ_a gives the value $\sqrt{2}$. For an average

Fréchet distance we can divide definition δ_a by the length of the path in free space. This can be done in several ways. We consider the following three:

1. Infimum over the path integral divided by the shortest path length which is $\sqrt{2}$
$$\delta_{a1}(f,g) := \frac{1}{\sqrt{2}} \inf_{hom\,\sigma} \int_0^1 \|f(t) - g(\sigma(t))\| \sqrt{1 + (\dot{\sigma}(t))^2}\, dt$$
2. Infimum over the path integral divided by the path length
$$\delta_{a2}(f,g) := \inf_{hom\,\sigma} \frac{\int_0^1 \|f(t) - g(\sigma(t))\| \sqrt{1 + (\dot{\sigma}(t))^2}\, dt}{\int_0^1 \sqrt{1 + (\dot{\sigma}(t))^2}\, dt}$$
3. Infimum over the path integral divided by the minimum length of a path achieving the infimum over the path integral
$$\begin{array}{lrcl} \text{Let} & \delta(f,g,\sigma) & = & \int_0^1 \|f(t) - g(\sigma(t))\| \sqrt{1 + (\dot{\sigma}(t))^2}\, dt, \\ & l(\sigma) & = & \int_0^1 \sqrt{1 + (\dot{\sigma}(t))^2}\, dt, \qquad \text{and} \\ & M & = & \{\, \sigma \mid \forall\, \text{hom}\, \rho : \ \delta(f,g,\sigma) \le \delta(f,g,\rho)\}. \\ \text{Then} & \delta_{a3}(f,g) & = & \dfrac{\delta(f,g,\sigma)}{l(\sigma)} \quad \text{where } \sigma \in M \text{ fulfills } \forall \rho \in M : l(\sigma) \le l(\rho). \end{array}$$

The first two definitions still yield values depending on the length of the path in free space. Consider the examples in Figure 6.3 (b) and (c). For the curves in Figure 6.3 (b), walking first on f and then on g minimizes the pointwise distance for all points. This walk corresponds to the path in free space along the lower and left boundary, which has length 2 under the Euclidean norm. The average distance along the walk, i.e., the path integral over this path divided by the length of the path, is $\frac{1}{2}$. Definition δ_{a1}, however, divides by the minimum path length $\sqrt{2}$, thus $\delta_{a1} = \frac{1}{\sqrt{2}}$. Thus, definition δ_{a1} "punishes" paths that deviate from the diagonal path in free space.

For the curves in Figure 6.3 (b), an intuitively good walk is walking on the two curves at equal, constant speed. This gives $\delta_a = \frac{1}{4}\sqrt{2}$ for the summed distance. This walk corresponds to the diagonal path in free space, which has length $\sqrt{2}$. Dividing by this length gives $\frac{1}{4}$ as (potential) average distance. For definition δ_{a2}, however, we can achieve a smaller value using the path in free space which walks in infinitely small zig-zags along the first half of the diagonal. Thus, it achieves the same summed Fréchet distance distance, but the path length is larger, $\frac{1}{2}(2 + \sqrt{2})$, and thus the quotient smaller: $\frac{1}{2(1+\sqrt{2})} < \frac{1}{4}$.

Observation 6.1. *Definitions $\delta_a, \delta_{a1}, \delta_{a2}$, and δ_{a3} are symmetric, i.e., for all curves f, g, and $\delta \in \{\delta_a, \delta_{a1}, \delta_{a2}, \delta_{a3}\}$ it holds $\delta(f,g) = \delta(g,f)$.*

Proof. We substitute in the integral t for $\sigma^{-1}(t)$, the inverse of σ of t. Then we use that $\dot{\sigma}(\sigma^{-1}(t)) = \frac{1}{\dot{\sigma}^{-1}(t)}$ (*derivative of the inverse function*).

$$\begin{aligned} \delta_a(f,g) &= \inf_{hom\,\sigma} \int_0^1 \|f(t) - g(\sigma(t))\| \sqrt{1 + (\dot{\sigma}(t))^2}\, dt \\ &= \inf_{hom\,\sigma} \int_0^1 \|f(\sigma^{-1}(t)) - g(t)\|\, \dot{\sigma}^{-1}(t) \sqrt{1 + (\dot{\sigma}(\sigma^{-1}(t)))^2}\, dt \\ &= \inf_{hom\,\sigma} \int_0^1 \|f(\sigma^{-1}(t)) - g(t)\| \sqrt{(\dot{\sigma}^{-1}(t))^2 + 1}\, dt \qquad = \quad \delta_a(g,f) \end{aligned}$$

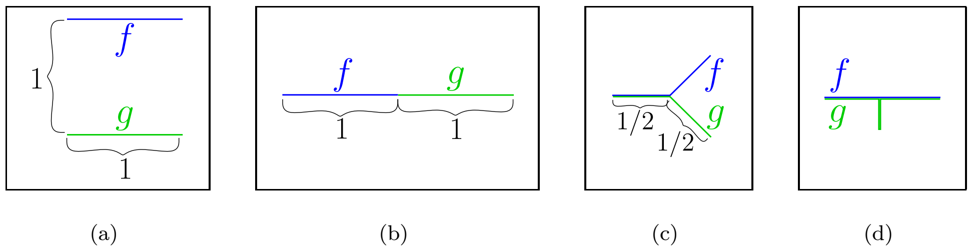

Figure 6.3: Examples of curves

In the free space we can formulate this as: integrating over a monotone path p in the free space of f, g is equivalent to integrating over the path p', which is p reflected at the diagonal from $(0,0)$ to $(1,1)$, in the free space of g, f.

The proof for the average definitions δ_{a1}, δ_{a2}, and δ_{a3} is analogue. □

6.2.2 Integrating over the Curves in Image Space

Instead of working in the free space diagram, i.e., in parameter space, we can also define a summed and average Fréchet distance based on the path integral over the curves in image space. For this, we will assume that the curves f, g are both piecewise continuously differentiable. The curves we have in mind are polygonal or smooth curves, which fulfill this condition.

We develop a summed Fréchet distance where the path integral is taken simultaneously over both curves. That is, we consider the discrete sums

$$\sum \mathrm{dist}_i \cdot \big(\Delta f_i + \Delta(g \circ \sigma)_i\big),$$

which are illustrated in Figure 6.2 (left). The limit of this sum is

$$\delta_b(f,g) \;:=\; \inf_{hom\,\sigma} \int_0^1 \|f(t) - g(\sigma(t))\| \;\big(\|\dot{f}(t)\| + \|\dot{g}(\sigma(t))\,\dot{\sigma}(t)\|\big)\,dt.$$

This gives a definition for a summed Fréchet distance. For an average Fréchet distance we can normalize it in two ways:

1. divide by the sum of the lengths of the two curves

$$\delta_{b1}(f,g) \;:=\; \inf_{hom\,\sigma} \frac{1}{L(f)+L(g)} \int_0^1 \|f(t) - g(\sigma(t))\| \big(\|\dot{f}(t)\| + \|\dot{g}(\sigma(t))\dot{\sigma}(t)\|\big)\,dt$$

2. normalize the weight of both curves to one and divide by the sum of the weights, i.e., divide by two

$$\delta_{b2}(f,g) \;:=\; \inf_{hom\,\sigma} \frac{1}{2} \int_0^1 \|f(t) - g(\sigma(t))\| \left(\frac{\|\dot{f}(t)\|}{L(f)} + \frac{\|\dot{g}(\sigma(t))\,\dot{\sigma}(t)\|}{L(g)}\right)dt.$$

In the first case a longer curve receives more weight than a shorter curve.

If f, g are of length one and parameterized with constant speed one, as will be the case for the counterexamples in the next section, both definitions simplify and coincide to

$$\inf_{hom\,\sigma} \frac{1}{2} \int_0^1 \|f(t) - g(\sigma(t))\| \;(1 + \dot{\sigma}(t))\;dt.$$

In particular, these definitions for summed and average Fréchet distance differ only by the scalar factor $\frac{1}{2}$.

Another way of writing the simplified definition is

$$\inf_{hom\,\sigma} \frac{1}{2}\Big(\int_0^1 \|f(t)-g(\sigma(t))\|\ dt\ +\ \int_0^1 \|f(\sigma^{-1}(t))-g(t)\|\ dt\Big).$$

That is, we take the equally weighted medium of the one-sided definitions using for both sides the same reparameterization.

Alternatively to taking the sum of the two one-sided definitions, the maximum could be taken, similar to the Hausdorff distance. And instead of using the same reparameterization for both sides, one could use different parameterizations for the two directions, i.e., exchange the sum or maximum with the infimum.

Observation 6.2. *Definitions δ_b, δ_{b1}, and δ_{b2} are symmetric, i.e., for all curves f, g, and $\delta \in \{\delta_b, \delta_{b1}, \delta_{b2}\}$ holds $\delta(f,g) = \delta(g,f)$.*

The proof Observation 6.2 is analogue to the proof of Observation 6.1.

6.2.3 General One-sided Definition

Perhaps the simplest way to define a summed or average Fréchet distance based on the path integral, is to take the path integral along the path $\gamma(t) = t$ over the function $F(t) = \|f(t) - g(\sigma(t))\|$. This gives the integral

$$\inf_{hom\,\sigma} \int_0^1 \ \|f(t)-g(\sigma(t))\|\ dt.$$

This expression is one-sided, as it allows to "jump" over parts of the curve g. That is, in the infimum over the homeomorphism parts of g can be traversed with infinite speed. Therefore, for example for the curves f, g in Figure 6.3 (d) the above expression is 0. In other words, it measures the distance of f to parts of g and is not symmetric.

We can rewrite the expression as $\inf_{hom\,\sigma} \|f - g\circ\sigma\|_1$, i.e., the 1-norm of $(f - g\circ\sigma)$. Similarly the Fréchet distance can be expressed as the ∞-norm of $(f - g\circ\sigma)$:

$$\delta_F(f,g) = \inf_{hom\,\sigma} \ \|f - g\circ\sigma\|_\infty.$$

We define a general one-sided definition by replacing the ∞-norm of $(f - g\circ\sigma)$ by the p-norm for $0 < p < \infty$. That is,

$$\delta_p(f,g)\ =\ \inf_{hom\,\sigma}\ \|f-g\circ\sigma\|_p = \inf_{hom\,\sigma} \sqrt[p]{\int_0^1 \ \|f(t)-g(\sigma(t))\|^p\ dt},$$

for $0 < p < \infty$.

As for $p = 1$, this general one-sided definition allows to "jump" on the curve g and is not symmetric. In fact, $\delta_p(f,g)$ can be arbitrarily much larger than $\delta_p(g,f)$.

Observation 6.3. *For all $p > 0$, for all $C > 0$, exist curves f, g s.t. $\delta_p(f,g) > C\cdot\delta_p(g,f)$. In particular, definition δ_p is not symmetric for any $0 < p < \infty$.*

Proof. Consider the curves f and $g_{\delta\eta}$ in Figure 6.4. Both lie on the x-axis. f runs from 0 to 1. g runs from $0 < \delta < \frac{1}{2}$ to 1 and then zig-zags for a length of δ between 1 and $1-\eta$ for $0 < \eta < \frac{1}{2}$. It is $\delta_b(f, g_{\delta\eta}) = \frac{\delta^2}{2}$ and $\delta_b(g_{\delta\eta}, f) = \frac{\delta\,\eta}{2}$. Thus, for $\delta < \frac{\eta}{C}$ holds $\delta_b(f, g_{\delta\eta}) > C\cdot\delta_b(g_{\delta\eta}, f)$. □

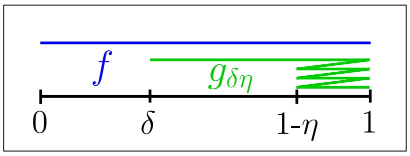

Figure 6.4: Curves for which the one-sided definition is not symmetric.

6.2.4 Comparison of the Definitions

We essentially considered the following three definitions.

1. Integral over the path in free space:

$$\delta_a(f,g) \;=\; \inf_{hom\,\sigma} \int_0^1 \|f(t)-g(\sigma(t))\| \,\sqrt{1+(\dot\sigma(t))^2}\; dt$$

2. Integral over the curves in image space, for curves of length one and parameterized with constant speed one:

$$\delta_b(f,g) \;=\; \inf_{hom\,\sigma} \int_0^1 \|f(t)-g(\sigma(t))\| \,(1+\dot\sigma(t))\; dt$$

3. General one-sided definition:

$$\delta_p(f,g) \;=\; \inf_{hom\,\sigma} \sqrt[p]{\int_0^1 \|f(t)-g(\sigma(t))\|^p\; dt} \qquad \text{for } 0<p<\infty.$$

The first two definitions differ only in that the first uses the 2-norm of the path length and the second the 1-norm. Under the 1-norm all paths have length two in the free space. Therefore normalization by the path length is canonical: division by two. Under the 2-norm the path lengths vary from $\sqrt{2}$ to 2. Therefore normalization is not canonical and the definitions for an average Fréchet distance give values depending on the path length. The third definition is one-sided and not symmetric.

6.3 Relaxed Triangle-Inequality is not fulfilled

In this section, we show that the general one-sided definition does not fulfill the triangle-inequality and all other definitions do not fulfill the relaxed triangle inequality for any constant C. For this, we give counter-examples of curves where the (relaxed) triangle inequality is violated. In particular, this implies that none of the definitions are metrics.

The curves we consider are shown in Figure 6.5. For illustration purposes they have been slightly perturbed. Unperturbed they would all lie on the x-axis. As previously assumed, all curves have length one and are parameterized with constant speed one.

All curves $f_\theta, g_\theta, h_\theta$ are parameterized from left to right and depend on a value θ which we will specify later. For now, let $0<\theta\leq\frac{1}{2}$. The curve f_θ first zig-zags for a length of $1-\theta$ in the the point $-(\theta+\theta^2)$ and $-\theta$ on the real axis. Then it runs straight from $-\theta$ to 0. The curve h_θ is f_θ mirrored at 0, but traversed also from left to right. The curve g_θ runs from $-\theta$ to 0, then zig-zags for a length of $1-2\theta$ in the θ^2-neighborhood of 0 and then runs from 0 to θ. We choose $f_\theta, g_\theta, h_\theta$ to be parameterized with constant speed one.

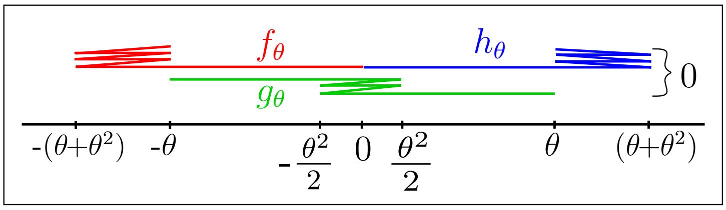

Figure 6.5: Curves not fulfilling the relaxed triangle inequality

Because f_θ and h_θ and g_θ itself are symmetric, the distances between f_θ and g_θ and between g_θ and h_θ are equal.

We will now show, that for all definitions developed in the previous section, we can choose $0 < \theta \leq \frac{1}{2}$ s.t. the curves $f_\theta, g_\theta, h_\theta$ violate the relaxed triangle inequality.

Lemma 6.1. *For $\delta \in \{\delta_a, \delta_{a1}, \delta_{a2}, \delta_{a3}\}$, for all constants $C > 0$, there exist curves f, g, h s.t. $\delta(f, h) > C\big(\delta(f, g) + \delta(g, h)\big)$ holds.*

Proof. We first show that the claim holds for δ_a and then that it holds also for δ_{a1}, δ_{a2}, and δ_{a3}. For this we consider the curves $f_\theta, g_\theta, h_\theta$ as above. We bound the distance between f_θ and h_θ from below, and the distance between f_θ and g_θ, which equals the distance between g_θ and h_θ, from above. Then we consider the quotient of these bounds which is larger then $\frac{a}{\theta}$ for some constant a, i.e.,

$$\frac{\delta(f_\theta, h_\theta)}{\delta(f_\theta, g_\theta)} \geq \frac{a}{\theta}.$$

By choosing $\theta < \frac{a}{2C}$ the claim follows for all constants $C > 0$.

Bounding $\delta_a(f_\theta, h_\theta)$ from below:

$$\begin{aligned}
\delta_a(f_\theta, h_\theta) &= \inf_{hom\,\sigma} \int_0^1 \|f_\theta(t) - h_\theta(\sigma(t))\| \sqrt{1 + (\dot{\sigma}(t))^2}\, dt \\
&\geq \inf_{hom\,\sigma} \int_0^1 \|f_\theta(t) - h_\theta(\sigma(t))\|\, dt \\
&\geq \int_0^1 \|f_\theta(t)\|\, dt \\
&= \int_0^{1-\theta} \|f_\theta(t)\|\, dt \;+\; \int_{1-\theta}^1 \|f_\theta(t)\|\, dt \\
&\geq (1-\theta)\theta \;+\; \theta\frac{\theta}{2} \;=\; \theta(1 - \frac{\theta}{2}).
\end{aligned}$$

In the first step we use that all homeomorphisms are monotone increasing and therefore $\dot{\sigma}$ is positive and $\sqrt{1 + (\dot{\sigma}(t))^2} \geq 1$. In the next step we bound for all points $t \in [0, 1]$ the distance $\|f_\theta(t) - h(\sigma(t))\|$ from below by the distance of $f_\theta(t)$ to its closest neighbor on the curve h_θ, which is the point 0 for all points t. Then we split the integral and evaluate the two integrals from 0 to $1 - \theta$ and from $1 - \theta$ to 1. For all points from 0 to $1 - \theta$ the distance of $f_\theta(t)$ to 0 is at least θ. Thus the integral is at least $(1 - \theta)\theta$. In the interval from $1 - \theta$ to 1 it is $f_\theta(t) = 1 - t$ and thus the integral evaluates to $\theta\frac{\theta}{2}$.

Bounding $\delta_a(f_\theta, g_\theta)$ and $\delta_a(g, h_\theta)$ from above Since $\delta_a(f_\theta, g_\theta) = \delta_a(g, h_\theta)$ it suffices to consider $\delta_a(f_\theta, g_\theta)$. We bound $\delta_a(f_\theta, g_\theta)$ from below by evaluating the

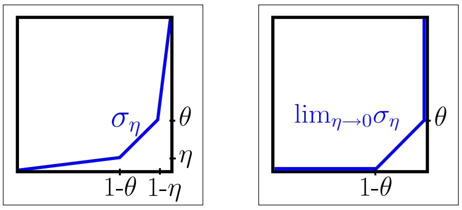

Figure 6.6: Paths in free space bounding the distance between the curves f_θ and g_θ

path integral over the path shown in Figure 6.6 (left). This path is not monotone but it is the limit of monotone paths, e.g., it is the limit of the graphs of homeomorphisms $\lim_{\eta\to 0}\sigma_\eta$ which are shown in Figure 6.6 (right).

Informally, the path integral over the limit path is bounded as follows: On the first segment from $(0,0)$ to $(1-\theta,0)$ we are traversing the zig-zag of f and standing in the starting point of g. The distance is at most θ^2 and the length of the segment is $1-\theta$ giving a value of at most $(1-\theta)\theta^2$. Then we run on both f and g simultaneously from $-\theta$ to 0. The distance on this segment is 0 for a length of $\sqrt{2}\theta$ giving a value of $0\cdot\sqrt{2}\theta = 0$. Then we traverse the rest of g while standing in the endpoint of f. First we traverse the zig-zag of g, which has a length of $1-2\theta$ and where the distance to the endpoint of f is at most θ^2. This gives the value $(1-2\theta)\theta^2$. Traversing the final segment of g gives an average distance of $\frac{\theta}{2}$ for a length of θ. Thus, in total we get a bound of $(1-\theta)\theta^2 + 0 + (1-2\theta)\theta^2/2 + \theta^2/2 \le \theta^2(2-\theta)$.

More formally, we evaluate the integral for the homeomorphisms σ_η and consider the limit $\eta \to 0$ of the integral. The homeomorphisms σ_η, for $\eta < \theta$, are given as

$$\sigma_\eta(t) = \begin{cases} \frac{\eta}{1-\theta}t & \text{for } 0 \le t \le 1-\theta \\ \eta + (t-(1-\theta)) & \text{for } 1-\theta \le t \le 1-\eta \\ \theta + \frac{1-\theta}{\eta}(t-(1-\eta)) & \text{for } 1-\eta \le t \le 1 \end{cases} .$$

Their derivative is $\frac{\eta}{1-\theta}$ for $t < 1-\theta$, 1 for $1-\theta < t < 1-\eta$, and $\frac{1-\theta}{\eta}$ for $t > 1-\eta$.

We evaluate the integral $\int_0^1 \|f_\theta(t) - g(\sigma_\eta(t))\| \sqrt{1+(\dot{\sigma_\eta}(t))^2}\, dt$ by splitting it into the three integrals from 0 to $1-\theta$, from $1-\theta$ to $1-\eta$, and from $1-\eta$ to 1 and evaluating these separately. We assume that $\eta \le \theta^2$.

The first integral we evaluate using $\|f_\theta(t) - g(\sigma_\eta(t))\| \le \theta^2 + \eta$ for $t \le 1-\theta$:

$$\begin{aligned} & \int_0^{1-\theta} \|f_\theta(t) - g(\sigma_\eta(t))\| \sqrt{1+(\dot{\sigma_\eta}(t))^2}\; dt \\ \le\ & \sqrt{1+\left(\frac{\eta}{1-\theta}\right)^2} \int_0^{1-\theta} (\theta^2+\eta)\; dt \\ =\ & \sqrt{1+\left(\frac{\eta}{1-\theta}\right)^2}\, (1-\theta)\, (\theta^2+\eta) \quad \underset{\eta\to 0}{\to} \quad (1-\theta)\, \theta^2 \end{aligned}$$

The second integral we evaluate using $\|f_\theta(t) - g(\sigma_\eta(t))\| = \eta$ and $\sigma_\eta^{'}(t) = 1$ for $1-\theta < t < 1-\eta$:

$$\begin{aligned} & \int_{1-\theta}^{1-\eta} \|f_\theta(t) - g(\sigma_\eta(t))\| \sqrt{1+(\dot{\sigma_\eta}(t))^2}\; dt \\ =\ & \sqrt{2}\int_{1-\theta}^{1-\eta} \eta\; dt \quad = \quad \sqrt{2}\,\eta\,(\theta-\eta) \quad \underset{\eta\to 0}{\to} \quad 0. \end{aligned}$$

The third integral we evaluate by substituting s for $\frac{t-(1-\eta)}{\eta}$. Then we split the integral into two at $s = \frac{1-2\theta}{1-\theta}$. The first of these integrals we evaluate using that the distance is bounded by $\eta + \frac{\theta^2}{2}$. In the second we substitute r for $s - (\frac{1-2\theta}{1-\theta})$. Further, we use that $\eta^2 + (1-\theta)^2 \leq 1 - \theta(2 - \theta - \theta^3) \leq 1$ for $\eta \leq \theta^2$ and $0 < \theta \leq \frac{1}{2}$ and therefore $\sqrt{\eta^2 + (1-\theta)^2} \leq 1$.

$$\begin{aligned}
&\int_{1-\eta}^{1} \|f_\theta(t) - g(\sigma_\eta(t))\| \sqrt{1 + (\dot{\sigma_\eta}(t))^2}\, dt \\
= &\int_0^1 \|f((1-\eta) + s\eta) - g(\theta + (1-\theta)s)\| \, \eta \sqrt{1 + (\frac{1-\theta}{\eta})^2}\, ds \\
= &\sqrt{\eta^2 + (1-\theta)^2} \Big(\int_0^{\frac{1-2\theta}{1-\theta}} \|f((1-\eta) + s\eta) - g(\theta + (1-\theta)s)\|\, ds \\
&\qquad + \int_{\frac{1-2\theta}{1-\theta}}^{1} \|f((1-\eta) + s\eta) - g(\theta + (1-\theta)s)\|\, ds \Big) \\
\leq &\int_0^{\frac{1-2\theta}{1-\theta}} (\eta + \frac{\theta^2}{2})\, ds \; + \; \int_0^{\frac{\theta}{1-\theta}} (\eta + r)\, dr \\
= &\Big(\frac{1-2\theta}{1-\theta}\Big)(\eta + \frac{\theta^2}{2}) \; + \; \frac{\theta}{1-\theta}\eta \; + \; \frac{1}{2}(\frac{\theta}{1-\theta})^2 \; \leq \; 2\eta + \theta^2 \quad \underset{\eta\to 0}{\to} \quad \theta^2
\end{aligned}$$

Putting the results for the three integrals together, we get $\delta_a(f_\theta, g) \;\leq\; \theta^2(2-\theta)$.

Bounding the Quotient: Now we consider the quotient

$$\frac{\delta_a(f_\theta, h_\theta)}{\delta_a(f_\theta, g)} \quad \geq \quad \frac{\theta\,\big(1 - \frac{\theta}{2}\big)}{\theta^2\,(2-\theta)} \quad \geq \quad \frac{1}{\theta}\frac{3}{8}.$$

Thus, by choosing $\theta < \frac{3}{16C}$ the claim follows for all constants $C > 0$.

Definitions $\delta_{a1}, \delta_{a2}, \delta_{a3}$: The definitions δ_{a1}, δ_{a2} and δ_{a3} differ from δ_a in that they divide by a path length. But since the path lengths vary between $\sqrt{2}$ and 2, definitions δ_{a1}, δ_{a2} and δ_{a3} do not vary from δ_a by more than a constant scalar factor. That is, for $i = 1, 2, 3$ and curves f, g, holds

$$\frac{\delta_a(f,g)}{2} \;\leq\; \delta_{ai}(f,g) \;\leq\; \frac{\delta_a(f,g)}{\sqrt{2}}.$$

Thus the same bounds as before hold but for a scalar factor of $\sqrt{2}$ or 2, respectively. In particular, the following bound on the quotient holds:

$$\frac{\delta_{ai}(f_\theta, h_\theta)}{\delta_{ai}(f_\theta, g)} \;\geq\; \frac{\sqrt{2}\;\delta_a(f_\theta, h_\theta)}{2\;\;\delta_a(f_\theta, g)} \quad \geq \quad \frac{1}{\theta}\frac{3}{\sqrt{2}\,8}$$

Now, by choosing $\theta < \frac{3}{\sqrt{2}\,16\,C}$ the claim follows as before for all constants $C > 0$. □

Lemma 6.2. *For $\delta \in \{\delta_b, \delta_{b1}, \delta_{b2}\}$, for all constants $C > 0$, there exist curves f, g, h s.t. $\delta_b(f,h) > C\big(\delta_b(f,g) + \delta_b(g,h)\big)$ holds.*

Proof. Since for curves of length one and parametrized with constant speed one the definitions δ_{b1} and δ_{b2} differ from δ_b only by a constant scalar factor, it suffices to show the claim for δ_b with such curves.

The proof for δ_b is analog to the proof for δ_a in the previous lemma. The same lower and upper bounds on $\delta_b(f_\theta, h_\theta)$ and $\delta_b(f_\theta, g_\theta)$, respectively, hold. The lower bound for $\delta_b(f_\theta, h_\theta)$ carries over directly. In the upper bound for $\delta_b(f_\theta, g_\theta)$ the metric in which the path length is measured changes from the Euclidean metric d_2

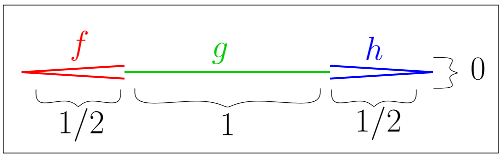

Figure 6.7: Curves for which the one-sided definition violates the triangle-inequality

to the Manhattan metric d_1. But for the path we consider, this makes a difference only for the diagonal segment of the path where the distance is (nearly) zero and which therefore does not contribute to the bound.

Thus, we get the same lower bound on the quotient $\delta_b(f_\theta, h_\theta)/\delta_b(f_\theta, g_\theta)$ and the claim follows as before. $\square$

Lemma 6.3. *For all $0 < p < \infty$, there exist curves f, g, h s.t. $\delta_p(f,h) > \delta_p(f,g) + \delta_p(g,h)$ holds.*

Proof. We show that the curves in Figure 6.7 violate the triangle inequality for the one-sided definition. The curves lie on the x-axis, but are slightly perturbed in the figure for illustration purpose. For all three pairs of curves, the limit of homeomorphisms that traverses the curves sequentially, are optimal, because they minimize the term $\|f(t) - g(\sigma(t))\|$ for all points t. Thus, we can compute the values $\delta_p(f,g), \delta_p(g,h), \delta_p(f,h)$ by computing the integrals along this path. This can be done analogously as in the proof of Lemma 6.1, i.e., by computing the path integral for a homeomorphism σ_θ in a sequence of homeomorphisms whose graphs in free space converge to this path. The result of this is

$$\begin{aligned}
\delta_p(f,g) &= \left(2\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}} &&= \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
\delta_p(g,h) &= \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} &&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
\delta_p(f,h) &= \left(2\int_0^{\frac{1}{2}} (1+t)^p dt\right)^{\frac{1}{p}} &&= \frac{3}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} 3^{\frac{1}{p}}.
\end{aligned}$$

It holds $\delta_p(f,h) > \delta_p(f,g) + \delta_p(g,h)$ because $3^{\frac{1}{p}} > 1$ for $p > 0$ and therefore $\delta_p(f,h) = \frac{3}{2}(p+1)^{-\frac{1}{p}} 3^{\frac{1}{p}} > \frac{3}{2}(p+1)^{-\frac{1}{p}} = \delta_p(f,g) + \delta_p(g,h)$. $\square$

6.4 Discussion

In this chapter we derived and analyzed several definitions for an average or summed Fréchet distance. These definitions have intuitive meanings in parameter or image space. We showed, however, that none of the definitions fulfill the triangle inequality.

This is only a first step in analyzing this problem. An immediate open problem is the computation of any of these definitions. Naturally, all discrete versions for fixed discretization are computable. In particular, the discrete versions based on the free space diagram can be computed with the same techniques as the discrete Fréchet distance. We do not, however, know how to compute the continuous versions. For this, techniques of variational calculus seem promising.

It also remains open, if other definitions than the ones considered in this chapter for an average Fréchet distance exist, which have the desired properties, in particular which fulfill the triangle inequality and are polynomial time computable.