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Fault-Tolerant Path-Embedding of Twisted Hypercube-Like Networks ($THLNs$)

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Abstract: It is known widely that an interconnection network can be denoted by a graph $G = (V, E)$, where V denotes the vertex set and E denotes the edge set. Investigating structures of G is necessary to design a suitable topological structure of interconnection network. One of the critical issues in evaluating an interconnection network is graph embedding, which concerns whether a host graph contains a guest graph as its subgraph. Linear arrays (i.e., paths) and rings (i.e., cycles) are two ordinary guest graphs (or basic networks) for parallel and distributed computation. In the process of large-scale interconnection network operation, it is inevitable that various errors may occur at nodes and edges. It is significant to find an embedding of a guest graph into a host graph where all faulty nodes and edges have been removed. This is named as fault-tolerant embedding. The twisted hypercube-like networks ($THLNs$) contain several important hypercube variants. This paper is concerned with the fault-tolerant path-embedding of n-dimensional (n-D) $THLNs$. Let G_n be an n-D $THLN$ and F be a subset of $V(G_n) \cup E(G_n)$ with $|F| \leq n - 2$. We show that for two different arbitrary correct vertices u and v, there is a faultless path P_{uv} of every length l with $2^{n-1} - 1 \leq l \leq 2^n - f_v - 1 - \alpha$, where $\alpha = 0$ if vertices u and v form a normal vertex-pair and $\alpha = 1$ if vertices u and v form a weak vertex-pair in $G_n - F$ ($n \geq 5$).

Keywords: combinatorics; multiprocessor interconnection networks; computer network reliability; network topology; hypercubes; twisted hypercube-like networks $THLNs$; fault tolerance; path-embedding

1. Introduction

As the infrastructure of cloud computing and the innovation platform of next generation network technology, the research of data center networks has become a hot topic in the academic and industrial circles in recent years. The performance of data center networks determine the performance of cloud computing to a large extent. Data center networks require a cluster of large number of switches, servers, and links. So what kind of topological network structure is used to connect these thousands of network components to make the system have the best performance and the lowest cost? What method is used to design such an interconnected network?

It is widely known that an interconnection network can be denoted by a graph $G = (V, E)$, where V denotes the vertex set and E denotes the edge set. Investigating structures of G is necessary to design a suitable topological structure of an interconnection network.

The n-dimensional hypercube [1], which possesses many outstanding properties such as recursive structure, relatively small degree, high symmetry, effective routing, and broadcasting algorithms [2], is one of the most efficient, versatile interconnection networks and, thus, becomes the preferred topological structure of parallel processing and parallel computing systems [3,4]. Thus, a hypercube is also one of the topological structures of data center networks [5]. Although hypercube networks have many excellent properties, it is well known that they also have inherent shortcomings, such as a large diameter. Therefore, many scholars have proposed some hypercube variants, aiming at improving the defects of hypercubes, such as Efe's crossed cubes [6], Cull's and Larson's Möbius cubes [7], Hilbers's twisted cubes [8], and Yang's locally twisted cubes [9]. These hypercube variants retain the good properties of hypercubes, but also have many properties superior to hypercubes, such as the diameter of hypercube variants being almost half of the diameter of hypercubes.

One of the critical issues in evaluating an interconnection network is how well other existing networks can be embedded into this network. This problem can be modeled by the following graph embedding problem: Given a host graph $G_2 = (V_2, E_2)$, which denotes the network into which other networks are to be embedded, and a guest graph $G_1 = (V_1, E_1)$, which denotes the network to be embedded, the problem is to find a mapping from each node of G_1 to a node of G_2, and a mapping from each edge of G_1 to a path in G_2. Graph embedding has good applications in allocating concurrent processes to processors in the network, and transplanting parallel algorithms developed for one network to a different one [10,11]. Linear arrays (i.e., paths) and rings (i.e., cycles) are two ordinary guest graphs (or fundamental networks) for parallel and distributed computing.

The hypercubes and hypercube variants can embed paths [12–14], cycles [15,16], trees [17,18], and meshes [19–21]. A path (respectively, cycle) is a Hamiltonian path (respectively, Hamiltonian cycle) if it passes through every vertex of graph G once and only once. If a graph contains a Hamiltonian cycle, then it is Hamiltonian. A graph G is Hamiltonian connected if for any pair of distinct vertices u and v, there exists a Hamiltonian path P_{uv}.

The data center network stores a large amount of important data information and requires high reliability. Because of the large number of switches, servers, and links in the data center network, it is difficult to avoid failures. Fault tolerance ensures that a variety of tasks that are being performed, such as information processing or algorithms, can run normally when some resources (servers, switches, or links) in a data center network fail. A good network can ensure that the remaining subnets will function properly even if some nodes or edges have errors. Therefore, it is of great practical significance to study the fault-tolerant performance of the network. Fault-tolerant embedding is to find an embedding of a guest graph into a host graph where all faulty nodes and edges have been removed.

Much work has been done on the fault-tolerant embedding [22–42]. In 2007, Fan et al. [13] proved that twisted cubes TQ_n can embed a path of length l between any two different nodes for any faulty set $F \subset V(TQ_n) \cup E(TQ_n)$ with $|F| \leq n-3$ and any integer l with $2^{n-1} - 1 \leq l \leq |V(TQ_n - F)| - 1$ ($n \geq 3$). In 2008, Ma et al. [33] proved the same result of path-embedding in crossed cubes CQ_n. A survey paper of Xu and Ma [23] lists almost all results on this topics until 2009. In 2012, Ye et al. [35] proved that a path of length l can be embedded between any two different nodes in n-dimensional locally twisted cubes LTQ_n for any faulty set $F \subset V(LTQ_n) \cup E(LTQ_n)$ with $|F| \leq n-3$ and any integer l with $2^{n-1} - 1 \leq l \leq |V(LTQ_n - F)| - 1$ ($n \geq 3$). In 2018, we [37] proved the fault-tolerant path-embedding in augmented cubes AQ_n with up to $(2n-4)$-faults. However, if there are $n-2$ faulty elements in TQ_n, CQ_n, LTQ_n respectively which are adjacent to one vertex, then the vertex must be a 2-degree vertex. It is extremely difficult to find the path with some length in this extreme case. Therefore, we [26] put forward the concept of weak 2-degree vertex and weak vertex-pair, where we simultaneously proved that the weak 2-degree vertex and weak vertex-pair are unique if they exist in a graph. Afterward we improved the fault-tolerant Hamiltonian path embedding with $n-2$ fault elements, excluding only the weak vertex-pair in twisted hypercube-like networks ($THLNs$).

The hypercube-like networks ($HLNs$) are a large class of interconnection networks [24,25,43]. Twisted hypercube-like networks ($THLNs$) proposed by Yang [40] in 2011, are a subclass in $HLNs$.

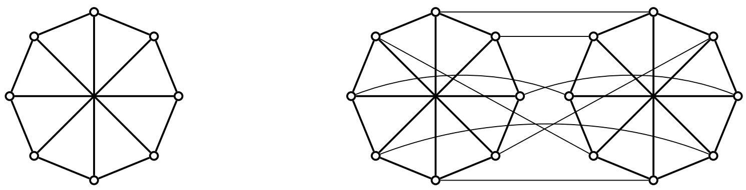
Definition 1 ([40])**.** *An n ($n \geq 3$)-dimensional (n-D) twisted hypercube-like network ($THLN$) is a graph defined recursively as follows.*

(1) A 3-D $THLN$ is isomorphic to the graph depicted in Figure 1a.

(2) For $n \geq 4$, an n-D THLN G_n is obtained from two vertex-disjoint $(n-1)$-D THLNs, denoted by G^0_{n-1} and G^1_{n-1}, in this way:

$$\begin{aligned} V(G_n) &= V(G^0_{n-1}) \cup V(G^1_{n-1}), \\ E(G_n) &= E(G^0_{n-1}) \cup E(G^1_{n-1}) \cup \\ &\quad \{(u, \phi(u)) : u \in V(G^0_{n-1})\}, \end{aligned}$$

where $\phi : V(G^0_{n-1}) \rightarrow V(G^1_{n-1})$ is a bijective mapping. In the following, we will denote this graph G_n as $G_n = \oplus_\phi(G^0_{n-1}, G^1_{n-1})$. Figure 1b plots a 4D THLN.



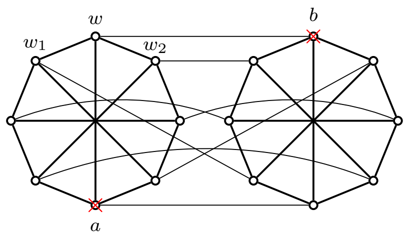
(a) $3D$ THLN (b) $4D$ THLN

Figure 1. Example of weak vertex-pair.

Specifically, the previously-mentioned hypercube variant networks (crossed cubes CQ_n, Möbius cubes MQ_n, twisted cubes TQ_n, and locally twisted cubes LTQ_n) are all $THLNs$. In 2005, Park et al. [24] demonstrated that all n-D $THLNs$ are Hamiltonian with at most $n-2$ faulty elements and Hamiltonian connected with at most $n-3$ faulty elements. Furthermore, using the Hamiltonian connectivity of $THLNs$, some scholars [13,33,35] have improved the lower bound of the path length in each $THLNs$ with $n-3$ faults. In 2018, Zhang et al. [26] improved the upper bound of fault tolerant Hamiltonian connectivity to $n-2$ excepting only a pair of vertices and gave the definitions of weak vertex-pair and normal vertex-pair as follows.

Definition 2 ([26])**.** *Let $F \subset V(G_n) \cup E(G_n)$ with $|F| = n-2$. If $G_n - F$ contains a vertex w such that $N_{G_n-F}(w) = \{w_1, w_2\}$, then w is called as a weak 2-degree vertex and (w_1, w_2) is called as a w-weak vertex pair (short for weak vertex pair).*

If $F = \{a, b\}$, for instance, then w is a weak 2-degree vertex and (w_1, w_2) is a weak vertex-pair in $G_4 - F$ (See Figure 2).



$4D$ $THLN(G_4)$

Figure 2. Example of weak vertex-pair.

Unquestionably, for the weak vertex-pair (w_1, w_2), any correct path $P_{w_1w_2}$ of length $l \geq 3$ cannot include the weak 2-degree vertex w. It follows there is no correct Hamiltonian path joining vertices w_1 and w_2 in $G_n - F$ [26]. However, we proved that $G_n - F$ $(n \geq 5)$ contains at most one weak 2-degree vertex w and one w-weak vertex-pair for any $F \subset V(G_n) \cup E(G_n)$ with $|F| \leq n-2$ in reference [26].

We first give the useful Definition and Theorem in reference [26] as follows.

Definition 3 ([26])**.** *If (w_1, w_2) is not a weak vertex-pair for any vertex $w \in V(G_n - F)$, then (w_1, w_2) is called as a normal vertex pair.*

Theorem 1 ([26]). *Let G_n be an n-dimensional THLN and $F \subset V(G_n) \cup E(G_n)$ with $|F| \leq n-2$. Then for any vertex-pair (u,v) in $G_n - F$, there is a $(n-2)$-fault-tolerant Hamiltonian path P_{uv} connecting vertices u and v except (u,v) being a weak vertex-pair.*

In the paper, we studied the path-embedding in a $THLN$ with $n-2$ faulty elements and showed that if $F \subset V(G_n) \cup E(G_n)$ and $|F| \leq n-2$, then for arbitrary two different correct vertices u and v, there is a fault-free path P_{uv} of every length l with $2^{n-1} - 1 \leq l \leq 2^n - f_v - 1 - \alpha$, where $\alpha = 0$ if vertices u and v form a normal vertex-pair and $\alpha = 1$ if vertices u and v form a weak vertex-pair in $G_n - F$ $(n \geq 5)$.

To do this simply, we can denote $G_n = L \oplus R$, where $L = G^0_{n-1}$ and $R = G^1_{n-1}$. For any vertex $x \in L$ (or R), let x^R (or x^L) be the sole vertex adjacent to vertex x in R (or L), and $N_L(x)$ (or $N_R(x)$) be the set of vertices that are adjacent to vertex x in L (or R). Let E^C be the set of edges that join L to R and $E_L(x)$ (or $E_R(x)$) be the set of edges incident to vertex x in L (or R).

We use P_{uv} to represent the path from vertex u to vertex v. If $P_{uw} = (u, u_1, \ldots, u_s, w)$, $P_{wv} = (w, w_1, \ldots, w_t, v)$ and $V(P_{uw}) \cap V(P_{wv}) = \{w\}$, we use $P_{uw} + P_{wv}$ to denote the path $P_{uv} = (u, u_1, \ldots, u_s, w, w_1, \ldots, w_t, v)$, $P_{uv}(u_1, w_1)$ to represent the subpath of P_{uv} which is from vertex u_1 to vertex w_1, l_{uv} to denote the length of P_{uv}, d_{uv} to denote the distance between vertex u to vertex v. We denote $F^L = F \cap L$, $F^R = F \cap R$, $F^C = F \cap E^C$, $F_v = F \cap V(G_n)$, $F_e = F \cap E(G_n)$, $f_v = |F_v|$, $f^L_v = |F_v \cap V(L)|$, $f^R_v = |F_v \cap V(R)|$. We have $f_v = f^L_v + f^R_v$.

This paper is organized as follows. Section 2 demonstrates the main result. Section 3 concludes the paper.

2. Main Result

In this section, we will establish the main result of the paper. We depict Theorem 2 as follows.

Theorem 2. *If $F \subset V(G_n) \cup E(G_n)$ and $|F| \leq n-2$, then for any two distinct fault-free vertices u and v, there exists a fault-free path P_{uv} of every length l with $2^{n-1} - 1 \leq l \leq 2^n - f_v - 1 - \alpha$, where $\alpha = 0$ if vertices u and v form a normal vertex-pair and $\alpha = 1$ if vertices u and v form a weak vertex-pair in $G_n - F$ $(n \geq 5)$.*

Proof. We prove the theorem by induction on $n \geq 5$. The result holds for $n = 5$ by developing computer program (https://github.com/ZhangHeidi/Hypercubes/blob/master/vcn02.c) using the depth-first searching technique combining with backtracking and a branch and bound algorithm. Assume that the theorem holds for $n-1$ with $n \geq 6$, then we must show the theorem holds for n. In general, we assume $|F^R| \leq |F^L|$. Then $|F^R| \leq \lfloor \frac{n-2}{2} \rfloor \leq n-4$. Since for any vertex $x \in R$, $|N_R(x)| = n-1$. Because of $|F^R| \leq n-4$, we have $|N_{R-F^R}(x)| \geq 3$. Then there is no weak vertex-pair in $R - F^R$.

Let u, v be any two distinct fault-free vertices in $G_n - F$. By Theorem 1, there is a faultless path P_{uv} of length $l = 2^n - f_v - 1$ if vertices u and v form a normal vertex-pair in $G_n - F$. Then we only need to find each length l with $2^{n-1} - 1 \leq l \leq 2^n - f_v - 2$ between arbitrary different vertices u and v in $G_n - F$. We divide the proof to two cases: (1). $|F^L| \leq n-3$; (2). $|F^L| = n-2$.

Case 1. $|F^L| \leq n-3$. We discuss this case by the following two cases: (1). $u, v \in V(L - F^L)$ or $u, v \in V(R - F^R)$; (2). $u \in V(L - F^L)$ and $v \in V(R - F^R)$.

Case 1.1. $u, v \in V(L - F^L)$ or $u, v \in V(R - F^R)$. Firstly, we prove the case of $u, v \in V(L - F^L)$.

Since $|F^L| \leq n-3$, by induction hypothesis, there is a faultless path P_{uv} of each length l with $2^{n-2} - 1 \leq l \leq 2^{n-1} - f^L_v - 2$ in $L - F^L$. Notice that there exist $\lfloor \frac{l+1}{2} \rfloor$ vertex-pairs in P_{uv}. Since $\lfloor \frac{l+1}{2} \rfloor - (n-2) \geq \frac{2^{n-2}}{2} - (n-2) \geq 4$ $(n \geq 6)$, there is a faultless edge $ab \in E(P_{uv})$ with $a^R, b^R, aa^R, bb^R \notin F$. Since $|F^R| \leq n-4$, by induction hypothesis, there is a faultless path $P_{a^Rb^R}$ of each length $l_{a^Rb^R}$ with $2^{n-2} - 1 \leq l_{a^Rb^R} \leq 2^{n-1} - f^R_v - 1$ in $R - F^R$. Let $P^1_{uv} = P_{uv}(u,a) + aa^R + P_{a^Rb^R} + b^Rb + P_{uv}(b,v)$. Then P^1_{uv} is a faultless path of length l^1_{uv} with $2^{n-1} - 1 \leq l^1_{uv} \leq 2^n - f_v - 2$ in $G_n - F$ (see Figure 3a).

For $u, v \in V(R - F^R)$, by a similar discussion, we can get a faultless path P^1_{uv} of each length l^1_{uv} with $2^{n-1} - 1 \leq l^1_{uv} \leq 2^n - f_v - 2$ in $G_n - F$.

Case 1.2. $u \in V(L - F^L)$ and $v \in V(R - F^R)$.

By the definition of G_n, $|E^C| = 2^{n-1}$. Since $2^{n-1} - (n-2) \geq 28$ ($n \geq 6$), there is a faultless edge ab with $ab \in E^C$, $a, b \notin \{u, v\}$ and $a, b \notin F$. By induction hypothesis, there is a faultless path P_{ua} of each length l_{ua} with $2^{n-2} - 1 \leq l_{ua} \leq 2^{n-1} - f^L_v - 2$ in $L - F^L$ and a faultless path P_{bv} of each length l_{bv} with $2^{n-2} - 1 \leq l_{bv} \leq 2^{n-1} - f^R_v - 1$ in $R - F^R$. Let $P_{uv} = P_{ua} + ab + P_{bv}$. Then P_{uv} is a faultless path of each length l_{uv} with $2^{n-1} - 1 \leq l_{uv} \leq 2^n - f_v - 2$ in $G_n - F$ (see Figure 3b).

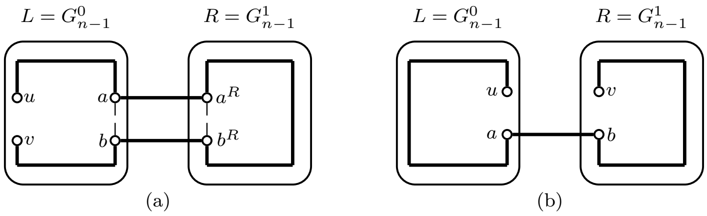


Figure 3. (**a**) Illustration of proof of Case 1.1; (**b**) Illustration of proof of Case 1.2.

Case 2. $|F^L| = n - 2$. Then $|F^R| = |F^C| = 0$. We divide the proof into two cases: (1). $|F^L \cap V(L)| \geq 1$; (2). $|F^L \cap V(L)| = 0$.

Case 2.1. $|F^L \cap V(L)| \geq 1$. Let $x \in F^L \cap V(L)$. In this situation, there exist three cases as follows: (1). $u, v \in V(L - F^L)$; (2). $u \in V(L - F^L)$ and $v \in V(R - F^R)$; (3). $u, v \in V(R)$.

Case 2.1.1. $u, v \in V(L - F^L)$.

We mark the faulty vertex x as faultless temporarily. Let $F^L_1 = F^L - x$, then $|F^L_1| = |F^L| - 1 = n - 3$. By induction hypothesis, there is a faultless path P_{uv} of each length l_{uv} with $2^{n-2} - 1 \leq l_{uv} \leq 2^{n-1} - (f^L_v - 1) - 2 = 2^{n-1} - f^L_v - 1$ in $L - F^L_1$. If the path P_{uv} contains the faulty vertex x, let $a, b \in N_{P_{uv}}(x)$; otherwise, we can arbitrarily select a vertex c from the path P_{uv}. Let $a, b \in N_{P_{uv}}(c)$. Since $|F^R| = 0$, by induction hypothesis, there is a faultless path $P_{a^R b^R}$ of each length $l_{a^R b^R}$ with $2^{n-2} - 1 \leq l_{a^R b^R} \leq 2^{n-1} - 1$ in R. Let $P^1_{uv} = P_{uv}(u, a) + aa^R + P_{a^R b^R} + b^R b + P_{uv}(b, v)$. Then P^1_{uv} is a faultless path of each length l^1_{uv} with $2^{n-1} - 1 \leq l^1_{uv} \leq 2^n - f_v - 2$ in $G_n - F$ (see Figure 4).

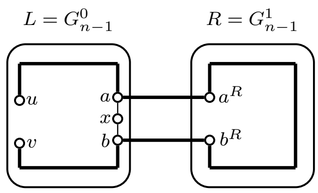


Figure 4. Illustration of proof of Case 2.1.1.

Case 2.1.2. $u \in V(L - F^L)$ and $v \in V(R)$.

We mark the faulty vertex x as faultless temporarily. Let $F^L_1 = F^L - x$, then $|F^L_1| = |F^L| - 1 = n - 3$. By induction hypothesis, there is a faultless path P_{ux} of each length l_{ux} with $2^{n-2} - 1 \leq l_{ux} \leq 2^{n-1} - (f^L_v - 1) - 2 = 2^{n-1} - f^L_v - 1$ in $L - F^L_1$. Let $x_1 \in N_{P_{ux}}(x)$. There are the following two cases: (1). $x^R_1 = v$; (2). $x^R_1 \neq v$.

Case 2.1.2.1. $x^R_1 = v$.

Let $ab \in E(P_{ux})$ with $a, b \notin \{u, x_1, x\}$. We mark the correct vertex v as faulty temporarily. Let $F^R_1 = F^R + v$, then $|F^R_1| = |F^R| + 1 \leq n - 4$ ($n \geq 6$). By induction hypothesis, there is a faultless path $P_{a^R b^R}$ of each length $l_{a^R b^R}$ with $2^{n-2} - 1 \leq l_{a^R b^R} \leq 2^{n-1} - 2$ in $R - F^R_1$. Let $P_{uv} = P_{ux}(u, a) + aa^R + P_{a^R b^R} + b^R b + P_{ux}(b, x_1) + x_1 v$. Then P_{uv} is a faultless path of each length l_{uv} with $2^{n-1} - 1 \leq l_{uv} \leq 2^n - f_v - 2$ in $G_n - F$ (see Figure 5a).

Case 2.1.2.2. $x^R_1 \neq v$.

By induction hypothesis, there is a faultless path $P_{x_1^R v}$ of each length $l_{x_1^R v}$ with $2^{n-2} - 1 \leq l_{x_1^R v} \leq 2^{n-1} - 1$ in R. Let $P_{uv} = P_{ux}(u, x_1) + x_1 x_1^R + P_{x_1^R v}$. Then P_{uv} is a faultless path of each length l_{uv} with $2^{n-1} - 1 \leq l_{uv} \leq 2^n - f_v - 2$ in $G_n - F$ (see Figure 5b).

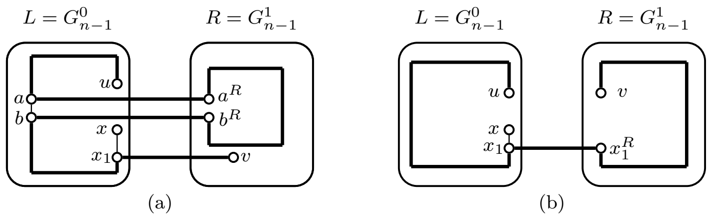


Figure 5. (**a**) Illustration of proof of Case 2.1.2.1; (**b**) Illustration of proof of Case 2.1.2.2.

Case 2.1.3. $u, v \in V(R)$.

Since $|F^R| = 0$, by induction hypothesis, there is a faultless path P_{uv} of length $l = 2^{n-1} - 1$ in R. Thus, we only need to consider each length l with $2^{n-1} \leq l_{uv} \leq 2^n - f_v - 2$. We prove this case by the following two cases: (1). $|\{u^L, v^L\} \cap F_v| \geq 1$; (2). $|\{u^L, v^L\} \cap F_v| = 0$.

Case 2.1.3.1. $|\{u^L, v^L\} \cap F_v| \geq 1$. In general, assume $u^L \in F_v$. We mark the faulty vertex u^L as faultless temporarily. Let $F_1^L = F^L - u^L$, then $|F_1^L| = |F^L| - 1 = n - 3$.

Let $S = N_R(v) - u$. Then $|S| \geq n - 2$. Since $|F_1^L| = n - 3$, there is a vertex $v_1 \in S$ with $v_1^L \notin F$. By induction hypothesis, there is a faultless path $P_{u^L v_1^L}$ of each length $l_{u^L v_1^L}$ with $2^{n-2} - 1 \leq l_{u^L v_1^L} \leq 2^{n-1} - (f_v^L - 1) - 2 = 2^{n-1} - f_v^L - 1$ in $L - F_1^L$. Let $u_1 \in N_{P_{u^L v_1^L}}(u^L)$.

If $u_1^R \neq v$, let $F_1^R = F^R + \{v_1, v\}$, then $|F_1^R| = |F^R| + 2 = 2 \leq n - 4 (n \geq 6)$. By induction hypothesis, there is a faultless path $P_{uu_1^R}$ of each length $l_{uu_1^R}$ with $2^{n-2} - 1 \leq l_{uu_1^R} \leq 2^{n-1} - 3$ in $R - F_1^R$. Let $P_{uv} = P_{uu_1^R} + u_1^R u_1 + P_{u^L v_1^L}(u_1, v_1^L) + v_1^L v_1 + v_1 v$. Then P_{uv} is a faultless path of each length l_{uv} with $2^{n-1} \leq l_{uv} \leq 2^n - f_v - 2$ in $G_n - F$ (See Figure 6a).

If $u_1^R = v$, let $F_1^R = F^R + v$, then $|F_1^R| = 1 \leq n - 4$ $(n \geq 6)$. By induction hypothesis, there is a faultless path P_{uv_1} of each length l_{uv_1} with $2^{n-2} - 1 \leq l_{uv_1} \leq 2^{n-1} - 2$ in $R - F_1^R$. Let $P_{uv} = P_{uv_1} + v_1 v_1^L + P_{u^L v_1^L}(v_1^L, u_1) + u_1 v$. Then P_{uv} is a fault-free path of each length l_{uv} with $2^{n-1} \leq l_{uv} \leq 2^n - f_v - 2$ in $G_n - F$ (See Figure 6b).

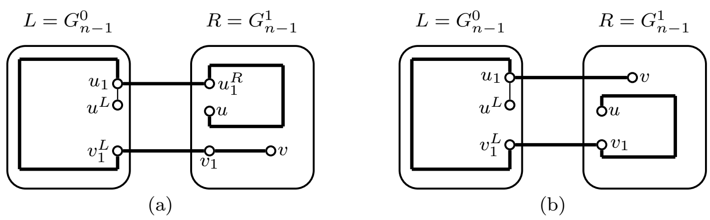


Figure 6. (**a**) Illustration of proof of $u_1^R \neq v$ in Case 2.1.3.1; (**b**) Illustration of proof of $u_1^R = v$ in Case 2.1.3.1.

Case 2.1.3.2. $|\{u^L, v^L\} \cap F_v| = 0$. We mark the faulty vertex x as faultless temporarily. Let $F_1^L = F^L - x$, then $|F_1^L| = |F^L| - 1 = n - 3$.

Since $|F_1^L| = n - 3$, by induction hypothesis, there is a faultless path P_{xv^L} of each length $2^{n-2} - 1 \leq l_{xv^L} \leq 2^{n-1} - (f_v^L - 1) - 2 = 2^{n-1} - f_v^L - 1$ in $L - F_1^L$. Let $x_1 \in N_{P_{xv^L}}(x)$.

If $x_1^R = u$, let $F_1^R = F^R + \{u, v\}$, then $|F_1^R| = |F^R| + 2 = 2 \leq n - 4$ $(n \geq 6)$. Let $ab \in E(P_{xv^L})$ with $a, b \notin \{x, x_1, v^L\}$. By induction hypothesis, there is a faultless path $P_{a^R b^R}$ of each length $l_{a^R b^R}$ with $2^{n-2} - 1 \leq l_{a^R b^R} \leq 2^{n-1} - 3$ in $R - F_1^R$. Let $P_{uv} = ux_1 + P_{xv^L}(x_1, a) + aa^R + P_{a^R b^R} + b^R b + P_{xv^L}(b, v^L) + v^L v$. Then P_{uv} is a faultless path of each length l_{uv} with $2^{n-1} \leq l_{uv} \leq 2^n - f_v - 2$ in $G_n - F$ (See Figure 7a).

If $x_1^R \neq u$, let $F_1^R = F^R + v$, then $|F_1^R| = |F^R| + 1 = 1 \leq n - 4$ ($n \geq 6$). By induction hypothesis, there is a faultless path $P_{ux_1^R}$ of each length $l_{ux_1^R}$ with $2^{n-2} - 1 \leq l_{ux_1^R} \leq 2^{n-1} - 2$ in $R - F_1^R$. Let $P_{uv} = P_{ux_1^R} + x_1^R x_1 + P_{xv^L}(x_1, v^L) + v^L v$. Then P_{uv} is a faultless path of each length l_{uv} with $2^{n-1} \leq l_{uv} \leq 2^n - f_v - 2$ in $G_n - F$ (See Figure 7b).

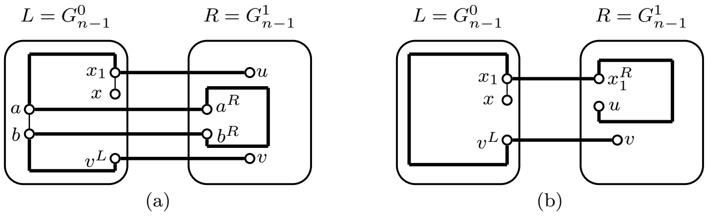


Figure 7. (**a**) Illustration of proof of $x_1^R = u$ in Case 2.1.3.2; (**b**) Illustration of proof of $x_1^R \neq u$ in Case 2.1.3.2.

Case 2.2. $|F^L \cap V(L)| = 0$. Then $F^L = F_e = F$.

Let $x \in V(G_n)$ with $d_{G_n - F}(x) = \delta(G_n - F)$. There are the following two cases: (1). $x \notin \{u, v\}$; (2). $x \in \{u, v\}$.

Case 2.2.1. $x \notin \{u, v\}$.

Let e be an edge with $e \in F_e$, $F^1 = F - e + \{x\}$, then $|F^1| = n - 2$ and $|F_v^1| = 1$. We show that (u, v) is a normal vertex pair in $G_n - F^1$ as follows.

If $\delta(G_n - F) \geq 4$, we discuss $\delta(G_n - F^1)$ in the following four cases.

(1) For any correct vertex $x_1 \in N_{G_n - F}(x)$ with $e \notin E_{G_n}(x_1)$. Notice that $\delta(G_n - F) \geq 4$, then $d_{G_n - F^1}(x_1) = d_{G_n - F}(x_1) - 1 \geq 3$.

(2) For any correct vertex $x_1 \in N_{G_n - F}(x)$ with $e \in E_{G_n}(x_1)$. Since $F^1 = F - e + \{x\}$, we have $d_{G_n - F^1}(x_1) = d_{G_n - F}(x_1) \geq 4$.

(3) For any correct vertex $x_1 \notin N_{G_n - F}(x)$ with $e \in E_{G_n}(x_1)$. Notice that $\delta(G_n - F) \geq 4$, then $d_{G_n - F^1}(x_1) = d_{G_n - F}(x_1) + 1 \geq 5$.

(4) For any correct vertex $x_1 \notin N_{G_n - F}(x)$ with $e \notin E_{G_n}(x_1)$, Since $F^1 = F - e + \{x\}$, we have $d_{G_n - F^1}(x_1) = d_{G_n - F}(x_1) \geq 4$.

Above all, we conclude that $\delta(G_n - F^1) \geq 3$.

If $\delta(G_n - F) \leq 3$, then $|E_{G_n}(x) \cap F| \geq n - 3$. For any $z \in V(G_n - F^1)$, since $|F| = n - 2$ and $|E_{G_n}(x) \cap E_{G_n}(z)| \leq 1$, we have $|(E_{G_n}(z) \cup N_{G_n}(z)) \cap F^1| \leq 3$. It follows that $\delta(G_n - F^1) \geq n - 3 \geq 3$ ($n \geq 6$).

Hence, there is no weak vertex-pair in $G_n - F^1$, *i.e.*, (u, v) is a normal vertex pair in $G_n - F^1$. By the proof of Case 2.1 and Theorem 1, there is a faultless path P_{uv} of every length l with $2^{n-1} - 1 \leq l \leq 2^n - |F_v^1| - 1 = 2^n - 2$ in $G_n - F$ ($n \geq 5$).

Case 2.2.2. $x \in \{u, v\}$. In general, assume that $x = u$.

Let e be an edge with $e = uy \in F_e$ with $y \neq v$ and $F^1 = F - e + \{y\}$, then $|F^1| = n - 2$ and $|F_v^1| = 1$. We show that (u, v) is a normal vertex pair in $G_n - F^1$ as follows.

Let z be an arbitrary vertex of $V(G_n - F^1) - \{u, v\}$.

If $\delta(G_n - F) \geq 4$, similar to the above discussion in Case 2.2.1, we have $\delta(G_n - F^1) \geq 3$. It means that $d_{G_n - F^1}(z) \geq 3$.

If $\delta(G_n - F) \leq 3$, then $|E_{G_n}(u) \cap F| \geq n - 3$. Since $|F| = n - 2$ and $|E_{G_n}(u) \cap E_{G_n}(z)| \leq 1$, we have $|(E_{G_n}(z) \cup N_{G_n}(z)) \cap F^1| \leq 3$. It follows that $d_{G_n - F^1}(z) \geq n - 3 \geq 3$ ($n \geq 6$).

Hence, (u, v) can not be a z-weak vertex pair in $G_n - F^1$, i.e., (u, v) is a normal vertex pair in $G_n - F^1$. By the proof of Case 2.1 and Theorem 1, there is a faultless path P_{uv} of every length l with $2^{n-1} - 1 \leq l \leq 2^n - |F_v^1| - 1 = 2^n - |F_v^1| - 1 = 2^n - 2$ in $G_n - F$ ($n \geq 5$). □

3. Concluding Remarks

This paper improved the previous result of Hamiltonian connectivity in twisted hypercube-like networks ($THLNs$) with $n-2$ fault elements and extended the path-embedding in an n-D $THLN$ ($n \geq 5$) with a set F of up to $n-2$ faulty elements. We have proved that for arbitrary two different correct vertices u and v, there exists a fault-free path P_{uv} of every length l with $2^{n-1} - 1 \leq l \leq 2^n - f_v - 1 - \alpha$, where $\alpha = 0$ if vertices u and v form a normal vertex-pair and $\alpha = 1$ if vertices u and v form a weak vertex-pair in $G_n - F$ ($n \geq 5$).

The lower bound of the path length cannot be uniformly improved in $THLNs$. We designed an efficient algorithm (https://github.com/ZhangHeidi/Hypercubes/blob/master/vcn02.c) to find a lower bound d-rank path in each $THLN$ with up to $n-2$ fault elements (where d represents the shortest distance between any two vertices in the graph) and the lower bound of path is different in each graph of $THLNs$. It is worthwhile to investigate the lower bound of path individually in every $THLN$.

By the discussion in reference [26], $MQ_n, LTQ_n, TQ_n, CQ_n \in THLNs$. The proposed theorem in the paper can be applied to several multiprocessor systems, including n-dimensional Möbius cubes MQ_n [7], n-dimensional locally twisted cubes LTQ_n [9], n-dimensional twisted cubes TQ_n [8] for odd n, and n-dimensional crossed cubes CQ_n [6]. In this paper, we apply our strategy to these four network topologies (MQ_n, LTQ_n, TQ_n, CQ_n). In future work, we will extend our strategy to other graphs of hypercube-like networks.

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