Slope space integrals for specular next event estimation

Supplemental material

GUILLAUME LOUBET, École Polytechnique Fédérale de Lausanne (EPFL)

TIZIAN ZELTNER, École Polytechnique Fédérale de Lausanne (EPFL)

NICOLAS HOLZSCHUCH, Inria, Univ. Grenoble-Alpes, CNRS, LJK

WENZEL JAKOB, École Polytechnique Fédérale de Lausanne (EPFL)

Abstract: This document provides additional derivations concerning integration and sampling of GGX distribution on polygonal domains, building up to C++ fragments. We also include a brief background discussion of off-center microfacet models.

1 MASKING-SHADOWING TERMS FOR OFF-CENTER MICROFACET MODELS

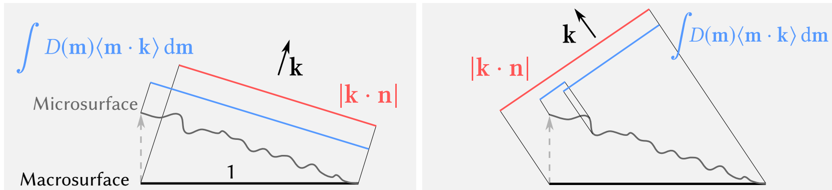
Masking-shadowing terms in microfacet-based BSDFs ensure that these model are physically plausible and do not create energy. They are generally computed from Smith's monostatic terms $G_1(\mathbf{i})$ and $G_1(\mathbf{o})$ [Heitz 2014]. In the case of off-center distributions of microfacet normals, that is, distributions whose average normal is different from the geometric normal of the surface, Walter et al. [2015] and Dupuy [2015] define the masking-shadowing term $G_1(\mathbf{k})$ for a direction $\mathbf{k}$ as the ratio between the projected macrosurface in the direction $\mathbf{k}$ and the total projected area of front-facing microfacets:

$$G_1(\mathbf{k}) = \min\left(1, \frac{|\mathbf{k}\cdot\mathbf{n}|}{\int D(\mathbf{m})\,\langle\mathbf{m}\cdot\mathbf{k}\rangle\,\mathrm{d}\mathbf{m}}\right). \tag{1}$$

This definition does not require the definition of a *mesosurface*, which could be ill-defined for arbitrary distributions of slopes. In the case of a translated distribution of microfacet normals in the space of slopes, Dupuy [2015] showed that G_1 can be written as a function of the masking-shadowing term of the centered distribution:

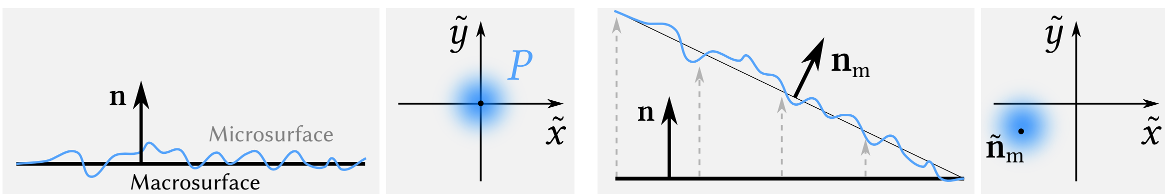
$$G_1(\mathbf{k}) = \min\left(1, \frac{|\mathbf{k}\cdot\mathbf{n}|\,|\mathbf{n}_m\cdot\mathbf{n}|}{|\mathbf{n}_m\cdot\mathbf{k}|} G_{1\,\mathrm{std}}(\mathbf{k})\right) \tag{2}$$

where $\mathbf{n}_m$ is the mean normal of the sheared microsurface and $G_{1\mathrm{std}}$ is the standard masking-shadowing term for the centered distribution [Walter et al. 2007]. The clamping of shadowing terms to $[0, 1]$ is missing in the



(a) (b)

Fig. 1. The projected area in a direction **k** of a microsurface with an off-center distribution of slopes can be either larger (a) or smaller (b) than the projected area of the macrosurface. The ratio of these projected areas defines a shadowing probability. In the case (b), the shadowing term must be clamped to the interval $[0, 1]$ to ensure energy conservation.



(a) Centered distribution of slopes (b) Off-center distribution of slopes

Fig. 2. (a): A microsurface with a centered distribution of microfacet slopes. (b): The same microsurface deformed by a vertical shear transformation resulting in a mean normal $\mathbf{n}_m$ that differs from the normal of the macrosurface, and a translated distribution of microfacet slopes.

work of Dupuy [2015] and it is important since the projected macrosurface can exceed the total projected area of microfacets in the direction **k** in some configurations, as illustrated in Fig. 1.

2 EQUIVALENCE OF THE ELLIPSOID NDF AND OFF-CENTER GGX DISTRIBUTIONS

Dupuy, Heitz et al. [2013] noted that applying a vertical shear deformation to a surface corresponds to a translation of microfacet slopes, as shown in Fig. 2. The center of translated distributions corresponds to the slopes of a *mean normal* also referred to as the *normal of the mesosurface*.

Walter et al. [2015] define their off-center normal distribution as the distribution of normals of an arbitrary ellipsoid, which reduces to a centered GGX distribution when the ellipsoid is aligned with **n** as mentioned in their article. In fact, the ellipsoid NDF is always equivalent to a translated GGX distribution in the space of slopes: sheared ellipsoids are also ellipsoids, and arbitrary ellipsoids can be written as sheared axis-aligned ellipsoids, which means that their distribution of slopes is exactly an off-center (potentially anisotropic) GGX distribution. To our knowledge, the equivalence of the ellipsoid NDF and the model used by Dupuy, Heitz et al. has not been mentioned before.

3 INTEGRATING GGX DISTRIBUTIONS IN ARBITRARY POLYGONAL DOMAINS

Importance sampling triangles for Specular Next Event Estimation (SNEE) requires integrating GGX distributions inside arbitrary triangular domains in slope space. We show that such integrals have a very simple closed-form expression.

3.1 The GGX distribution of slopes

In slope space, the axis-aligned GGX distribution [Walter et al. 2007] with roughness parameters α_x and α_y is given by

$$P(x,y) = \frac{1}{\pi\alpha_x\alpha_y\left(1+\frac{x^2}{\alpha_x^2}+\frac{y^2}{\alpha_y^2}\right)^2}. \tag{3}$$

Integrals of this distribution over finite 2D domains can be reduced to integrals of the isotropic distribution with roughness $\alpha_x = \alpha_y = 1$ using the linear change of variables

$$\begin{cases} X = \dfrac{x}{\alpha_x} \\ Y = \dfrac{y}{\alpha_y} \end{cases}. \tag{4}$$

Indeed:

$$\int P(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int P(X\alpha_x, Y\alpha_y)\,\alpha_x\alpha_y\,\mathrm{d}X\,\mathrm{d}Y \tag{5}$$

$$= \int \frac{1}{\pi\,(1+X^2+Y^2)^2}\,\mathrm{d}X\,\mathrm{d}Y \tag{6}$$

$$= \int P_{\mathrm{std}}(X,Y)\,\mathrm{d}X\,\mathrm{d}Y, \tag{7}$$

where $\alpha_x\alpha_y$ is the Jacobian determinant of the change of variables. We can easily verify that P_{std} is a normalized distribution by integrating in polar coordinates:

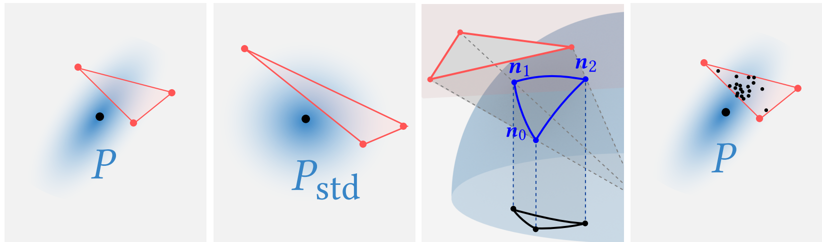
$$P_{\mathrm{std}}(r) = \frac{1}{\pi(1+r^2)^2}, \tag{8}$$

$$\iint P_{\mathrm{std}}(r)\,r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\int \frac{r}{\pi(1+r^2)^2}\,\mathrm{d}r = 2\pi\left[\frac{-1}{2\pi(1+r^2)}\right]_0^\infty = 1. \tag{9}$$

3.2 Integration in arbitrary polygonal domains

Let $\tilde{\mathbf{n}}_0$, $\tilde{\mathbf{n}}_1$ and $\tilde{\mathbf{n}}_2$ be the vertices of a 2D triangle $\tilde{\Delta}$ in the space of slopes, and let $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$ denote their corresponding normals. We can obtain a geometric interpretation of the integral of P_{std} over the triangular domain $\tilde{\Delta}$ by writing the integral in the spherical domain. We use a relationship between slopes and polar angles

$$|\mathbf{m}\cdot\mathbf{n}| = \frac{1}{\sqrt{1+\tilde{\mathbf{m}}_x^2+\tilde{\mathbf{m}}_y^2}} \tag{10}$$



(a) (b) (c) (d)

Fig. 3. We show that the integral of GGX distributions of slopes P in triangular domains (a) have simple closed-form expressions. Arbitrary cases reduce to the case of an isotropic distribution with unit roughness (b) through a simple linear transformation. Then, the integral corresponds to the vertically-projected area of a spherical triangle (c), which has a simple analytic expression. We also importance sample the distribution restricted to triangles (d). These mathematical tools are at the core of our algorithms for rendering caustics and high-resolution normal maps.

in order to obtain

$$\int_{\tilde{\Delta}} P_{\text{std}}(\tilde{\mathbf{m}})\, d\tilde{\mathbf{m}} = \int_{\Delta} \frac{1}{\pi(1+\tilde{\mathbf{m}}_x^2+\tilde{\mathbf{m}}_y^2)^2} \frac{1}{|\mathbf{m}\cdot\mathbf{n}|^3}\, d\mathbf{m} \tag{11}$$

$$= \int_{\Delta} \frac{1}{\pi} |\mathbf{m}\cdot\mathbf{n}|\, d\mathbf{m}, \tag{12}$$

where $\frac{1}{|\mathbf{m}\cdot\mathbf{n}|^3} = \left\|\frac{\partial\tilde{\mathbf{m}}}{\partial\mathbf{m}}\right\|$ is the Jacobian of the transformation between normals and slopes [Walter et al. 2007]. Eq. 12 shows that P_{std} corresponds to a constant distribution of normals $D(\mathbf{m}) = 1/\pi$. The new integration domain Δ is the spherical triangle (Fig. 3c), whose vertices are $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$.

We have converted an integral over a triangular domain $\tilde{\Delta}$ into an integral of a cosine lobe restricted to a spherical triangle. This integral has a simple analytic expression:

$$\int_{\Delta} \frac{1}{\pi} |\mathbf{m}\cdot\mathbf{n}|\, d\mathbf{m} = \frac{1}{2\pi} \sum_{i=0}^{2} \operatorname{acos}(\mathbf{n}_i \cdot \mathbf{n}_j) \left(\frac{\mathbf{n}_i \times \mathbf{n}_j}{\|\mathbf{n}_i \times \mathbf{n}_j\|} \cdot \mathbf{n} \right).$$

This is a classical result from the work of Lambert [1760] and a complete derivation can be found in the work of Heitz [2017].

3.3 Code

Here is an example of implementation in C++. This code computes the integral of an isotropic GGX distribution inside an arbitrary triangle in slope space.

```
/* Conversion from slopes to normalized directions */
Vector3f slopes_to_n(const Vector2f &v) {
    return normalize(Vector3f(-v.x(), -v.y(), 1));
}

/* Helper function for the integration of projected spherical triangles */
float edge_integral(const Vector3f &v1, const Vector3f &v2) {
    float cosTheta = dot(v1, v2);
    float theta = acos(cosTheta);
    return cross(v1, v2).z() * ((theta > 0.0001f) ? theta/sin(theta) : 1);
}

/* Integral of a GGX distribution districted to a triangle in slope space */
float triangle_integral_ggx(const Vector2f &p0, const Vector2f &p1, const Vector2f &p2,
                            float alpha_x, float alpha_y) {

    /* Linear transformation of slopes which reduces the problem to the case of
       an isotropic GGX distribution with unit roughness. */
    Vector2f scale(1/alpha_x, 1/alpha_y);
    Vector3f n0 = slopes_to_n(p0 * scale);
    Vector3f n1 = slopes_to_n(p1 * scale);
    Vector3f n2 = slopes_to_n(p2 * scale);

    /* Integral of a cosine-weighted spherical triangle */
    float integral = edge_integral(n0, n1)
                   + edge_integral(n1, n2)
                   + edge_integral(n2, n0);
    return abs(integral) / (2 * M_PI);
}
```

Integral of anisotropic GGX distributions inside arbitrary triangles

4 SAMPLING GGX DISTRIBUTIONS RESTRICTED TO TRIANGLES

4.1 Sampling in polar coordinates

Importance sampling a specular light path with our framework requires sampling a point inside a slope-space triangle $\tilde{\Delta}$ proportionally to the distribution of slopes P as shown in Fig. 3d. A sample (s_r, s_θ) can be obtained in polar coordinates from two random numbers U and V sampled uniformly in the interval $[0, 1]$. The first step consists in sampling an azimuth s_θ from the marginal distribution

$$f_\Theta(\theta) = \frac{\int \mathbb{1}_{(r,\theta)\in\tilde{\Delta}} P(r,\theta)\, r\, \mathrm{d}r}{N_\Theta}, \tag{13}$$

where $\mathbb{1}_{(r,\theta)\in\tilde{\Delta}}$ is an indicator function whose value is 1 if the point (r, θ) belongs to the triangle $\tilde{\Delta}$, and the normalization term N_Θ corresponds to the integral of P in the triangle $\tilde{\Delta}$ (Sec. 3):

$$N_\Theta = \iint \mathbb{1}_{(r,\theta)\in\tilde{\Delta}} P(r,\theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta. \tag{14}$$

The corresponding cumulative distribution function for θ is an integral of f_Θ:

$$F_\Theta(t) = \int_0^t f_\Theta(\theta)\, \mathrm{d}\theta. \tag{15}$$

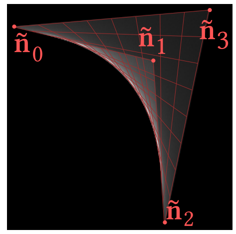


Fig. 4. Bilinear interpolation sometimes leads to singular densities of normals. In this example, bilinear interpolation in the space of slopes results in infinite densities of slopes on a parabola. Slope-space integrals pose additional challenges in such cases.

Since f_Θ is a normalized probability density function, F_Θ is monotonically increasing with $F_\Theta(0) = 0$ and $F_\Theta(2\pi) = 1$. This function is proportional to the integral of P in the triangle $\tilde{\Delta}$ restricted to angles θ belonging to the interval $[0, t]$. Therefore, it has an closed-form expression can be evaluated efficiently as discussed in Sec. 3. Obtaining a sample from the distribution f_Θ requires solving the equation $F_\Theta(s_\theta) = U$. Unfortunately, this equation does not have an analytic solution and must be inverted numerically, *e.g.*, using a Newton-Raphson solver.

Given s_θ, the next step entails sampling a distance s_r on the line $\theta = s_\theta$. Let r_{min} and r_{max} denote the intersection distances of this line with the triangle $\tilde{\Delta}$. For an isotropic GGX distribution, the probability density function for s_r is given by

$$f_R(r) = \frac{\mathbb{1}_{(r,s_\theta)\in\tilde{\Delta}}}{N_r} \frac{ra^2}{\pi(a^2 + r^2)^2} \tag{16}$$

where the normalization term N_R equals

$$\int \mathbb{1}_{(r,s_\theta)\in\tilde{\Delta}} \frac{r\alpha^2}{\pi(\alpha^2 + r^2)^2} dr = \frac{\alpha^2}{2\pi} \left[\frac{1}{\alpha^2 + r_{\text{min}}^2} - \frac{1}{\alpha^2 + r_{\text{max}}^2} \right]. \tag{17}$$

The corresponding cumulative distribution function is given by

$$F_R(t) = \frac{1}{N_R} \frac{\alpha^2}{2\pi} \left[\frac{1}{\alpha^2 + r_{\text{min}}^2} - \frac{1}{\alpha^2 + t^2} \right]. \tag{18}$$

This function can be inverted analytically:

$$F_R(s_r) = V \;\Rightarrow\; s_r = \sqrt{\frac{\alpha^2}{\frac{\alpha^2}{r_0^2 + \alpha^2} - 2\pi N_R Y} - \alpha^2}. \tag{19}$$

5 CHALLENGES INVOLVING BILINEAR INTERPOLATION

Bilinear interpolation is a perfectly valid interpolation method for normals, but the resulting distribution of normals can be singular, as noted in previous work [Yan et al. 2014] and illustrated in Fig. 4. In such situations, integrals in slope space are numerically problematic since the integrand can take on arbitrarily high values. Even when the corresponding distributions of normals are non-singular, they lead to integrals that do not have closed-form expressions in our framework. Similar to previous work [Yan et al. 2014], we use triangle meshes and replace bilinear interpolation in normal maps with barycentric interpolation in normal map triangles.

REFERENCES

Jonathan Dupuy. 2015. *Photorealistic Surface Rendering with Microfacet Theory*. Theses. Université Claude Bernard - Lyon I ; Université de Montréal. https://hal.archives-ouvertes.fr/tel-01291974

Jonathan Dupuy, Eric Heitz, Jean-Claude Iehl, Pierre Poulin, Fabrice Neyret, and Victor Ostromoukhov. 2013. Linear Efficient Antialiased Displacement and Reflectance Mapping. *ACM Transactions on Graphics* 32, 6 (Nov. 2013), Article No. 211. https://doi.org/10.1145/2508363.2508422

Eric Heitz. 2014. Understanding the Masking-Shadowing Function in Microfacet-Based BRDFs. *Journal of Computer Graphics Techniques (JCGT)* 3, 2 (30 June 2014), 48–107. http://jcgt.org/published/0003/02/03/

Eric Heitz. 2017. *Geometric Derivation of the Irradiance of Polygonal Lights*. Research Report. Unity Technologies. https://hal.archives-ouvertes.fr/hal-01458129

Johann Heinrich Lambert. 1760. *Photometria sive de mensura et gradibus luminis, colorum et umbrae.*

Bruce Walter, Zhao Dong, Steve Marschner, and Donald P. Greenberg. 2015. The Ellipsoid Normal Distribution Function, supplemental material of Predicting Appearance from Measured Microgeometry of Metal Surfaces. *ACM Trans. Graph.* 35, 1 (2015), 9:1–9:13.

Bruce Walter, Stephen R. Marschner, Hongsong Li, and Kenneth E. Torrance. 2007. Microfacet Models for Refraction Through Rough Surfaces. In *Proceedings of the 18th Eurographics Conference on Rendering Techniques* (Grenoble, France) *(EGSR'07)*. Eurographics Association, Aire-la-Ville, Switzerland, Switzerland, 195–206. https://doi.org/10.2312/EGWR/EGSR07/195-206

Ling-Qi Yan, Miloš Hašan, Wenzel Jakob, Jason Lawrence, Steve Marschner, and Ravi Ramamoorthi. 2014. Rendering Glints on High-resolution Normal-mapped Specular Surfaces. *ACM Trans. Graph.* 33, 4, Article 116 (July 2014), 9 pages. https://doi.org/10.1145/2601097.2601155