A KALLMAN-ROTA INEQUALITY FOR EVOLUTION SEMIGROUPS

C. BUŞE AND S.S. DRAGOMIR

Abstract. A Kallman-Rota type inequality for evolution semigroups and applications for real valued functions are given.

1. Introduction

Let X be a real or complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norms in X and in $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$.

Let $\mathbb{R}_+$ the set of all non-negative real numbers and $\mathbf{J} \in \{\mathbb{R}_+, \mathbb{R}\}$. The set $\{(t,s) : t \geq s \in \mathbf{J}\}$ will be denoted by $\Delta_{\mathbf{J}}$. A family

$$\mathcal{U}_{\mathbf{J}} = \{U(t,s) : (t,s) \in \Delta_{\mathbf{J}}\} \subset \mathcal{L}(X)$$

is called an *evolution family* of bounded linear operators on X if $U(t,t) = I$ (the identity operator on X) and $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r \in \mathbf{J}$. Such a family is said to be *strongly continuous* if for each $x \in X$, the maps

$$(t,s) \mapsto U(t,s)x : \Delta_{\mathbf{J}} \to X$$

are continuous. A strongly continuous evolution family is said to be *exponentially bounded* if there exist $\omega \in \mathbb{R}$ and $K_\omega \geq 1$ such that

$$||U(t,s)|| \leq K_\omega e^{\omega(t-s)} \text{ for all } (t,s) \in \Delta_{\mathbf{J}}$$

and *uniformly stable* if there exists $M \in \mathbb{R}_+$ such that

$$\sup_{(t,s)\in\Delta_{\mathbf{J}}} ||U(t,s)|| \leq M < \infty. \tag{1.1}$$

We remind that a family $\mathbf{T} = \{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ is called *one-parameter semigroup* if $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t \geq s \geq 0$. An one-parameter semigroup is called *strongly continuous* or C_0-semigroup if for each $x \in X$ the maps $t \mapsto T(t)x$ are continuous on $\mathbb{R}_+$. For a C_0-semigroup $\mathbf{T}$, its infinitesimal generator A with the domain $D(A)$ is defined by

$$D(A) := \left\{ x \in X : \text{ there exists in } X, \quad \lim_{t\to 0} \frac{T(t)x - x}{t} =: Ax \right\}.$$

It is easy to see that if $\mathbf{T} = \{T(t) : t \geq 0\}$ is a strongly continuous semigroup then the family $\mathcal{U}_{\mathbf{J}} = \{U(t,s) := T(t-s) : (t,s) \in \Delta_{\mathbf{J}}\}$ is a strongly continuous and exponentially bounded evolution family. Conversely, if $\mathcal{U}_{\mathbf{J}}$ is a strongly continuous evolution family and $U(t,s) = U(t-s,0)$ for all $(t,s) \in \Delta_{\mathbf{J}}$ then the family $\mathbf{T} := \{T(t) = U(t,0) : t \geq 0\}$ is a strongly continuous one-parameter semigroup. For

1991 *Mathematics Subject Classification.* 47A30.
Key words and phrases. Kallman-Rota Inequality, Evolution Semigroups.

more details about the strongly continuous semigroups and evolution families we refer to [3].

Lemma 1. *Let* $\mathbf{T}:=\{T(t): t\geq 0\}$ *be a strongly continuous one-parameter semigroup and* $A: D(A)\subset X\to X$ *its infinitesimal generator. If* $\mathbf{T}$ *is uniformly stable, that is, there is a positive constant* M *such that* $\sup_{t\geq 0}||T(t)||\leq M$, *then*

$$\|Ax\|^2\leq 4M^2\left\|A^2x\right\|\|x\|, \quad \text{for all } x\in D\left(A^2\right). \tag{1.2}$$

Proof. See [4]. ∎

We are recalling the notion of evolution semigroup. For more details we refer to [1], [2] and references therein. We will consider the both cases, i.e., the evolution semigroups for evolution families on $\Delta_{\mathbb{R}_+}$ and on $\Delta_{\mathbb{R}}$.

Let $\mathcal{U}_{\mathbb{R}_+}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on X. Let us consider the following spaces:

- $C_{00}(\mathbb{R}_+, X)$ is the space consisting by all X-valued, continuous functions on $\mathbb{R}_+$, such that
$$f(0)=\lim_{t\to\infty} f(t)=0,$$
endowed with the sup-norm.
- $L_p(\mathbb{R}_+, X), 1\leq p<\infty$ is the usual Lebesgue-Bochner space of all measurable functions $f:\mathbb{R}_+\to X$, identifying functions which are equal almost everywhere, such that
$$||f||_p:=\left(\int_0^\infty ||f(s)||^p ds\right)^{\frac{1}{p}}<\infty.$$

Let $\mathcal{X}$ be either $C_{00}(\mathbb{R}_+, X)$ or $L_p(\mathbb{R}_+, X)$ and $f\in\mathcal{X}$.

It is easy to see that for each $t\geq 0$, the function $T(t)f$ given by

$$(T(t)f)(s):=\begin{cases} U(s,s-t)f(s-t), & s\geq t\\ 0, & 0\leq s<t\end{cases} \tag{1.3}$$

belongs to $\mathcal{X}$, and the family $\mathbf{T}=\{T(t): t\geq 0\}$ is an one-parameter semigroup of bounded linear operators acting on $\mathcal{X}$. Moreover, the following result, holds:

Lemma 2. *The semigroup* $\mathbf{T}$ *defined in (1.3) is strongly continuous. If* $(A, D(A))$ *is the generator of* $\mathbf{T}$ *with its domain then for every* u, f *in* $\mathcal{X}$ *the following statements are equivalent:*

(i) $u\in D(A)$ *and* $Au=-f$;
(ii) $u(t)=\int_0^t U(t,s)f(s)ds$;

Proof. See [7]. ∎

The strongly continuous semigroup $\mathbf{T}$ defined in (1.3) is called *evolution semigroup* associated to $\mathcal{U}_{\mathbb{R}_+}$ on the space $\mathcal{X}$.

We will state here our first result.

Theorem 1. *Let* $\mathcal{U}_{\mathbb{R}_+}$ *be a strongly continuous uniformly stable evolution family of bounded linear operators acting on* X, *and let* $g\in\mathcal{X}$. *Suppose that the following conditions are fulfilled:*

(i) $\int_0^{\cdot} U(\cdot,s)g(s)ds$ *belongs to* $\mathcal{X}$;
(ii) $\int_0^{\cdot}(\cdot-s)U(\cdot,s)g(s)ds$ *belongs to* $\mathcal{X}$.

Then the following inequality holds:

$$\left\|\int_0^{\cdot} U(\cdot,s)g(s)ds\right\|_{\mathcal{X}}^2 \le 4M^2 \|g\|_{\mathcal{X}} \times \left\|\int_0^{\cdot} (\cdot - s)U(\cdot,s)g(s)ds\right\|_{\mathcal{X}}, \tag{1.4}$$

where M is the constant from the estimation (1.1).

$BUC(\mathbb{R},X)$ is the space of all X-valued, bounded and uniformly continuous functions on the real line endowed with the sup-norm. The following three spaces are closed subspaces of $BUC(\mathbb{R},X)$:

- $C_0(\mathbb{R},X)$ is the space of all X-valued, continuous functions on $\mathbb{R}$ such that $\lim\limits_{t\to\infty} f(t) = 0$.
- $AP(\mathbb{R},X)$ is the space of all almost periodic functions, that is, the smallest closed subspace of $BUC(\mathbb{R},X)$ containing the functions of the form

  $$t \mapsto e^{i\mu t}x, \quad \mu \in \mathbb{R} \text{ and } x \in X,$$

  see e.g. [6].
- $AAP(\mathbb{R},X)$ is the space of all X-valued asymptotically almost periodic functions on $\mathbb{R}$, i.e., the space consisting in all functions f for which there exist $g \in C_0(\mathbb{R},X)$ and $h \in AP(\mathbb{R},X)$ such that $f = g + h$.

Let $\mathcal{Y}$ one of the spaces described before and $f \in \mathcal{Y}$. If $\mathcal{U}_{\mathbb{R}}$ satisfies certain conditions, which will be outlined in Lemma 3 below, then for each $t \ge 0$ the function given by

$$s \mapsto (T(t)f)(s) := U(s,s-t)f(s-t) : \mathbb{R} \to X \tag{1.5}$$

belongs to $\mathcal{Y}$, and the family $\mathbf{T} := \{T(t) : t \ge 0\}$ is an one-parameter semigroup of bounded linear operators on $\mathcal{Y}$. The semigroup $\mathbf{T}$ can be not strongly continuous. However, in certain cases, this semigroup is strongly continuous, and is called *evolution semigroup* associated to $\mathcal{U}_{\mathbb{R}}$ on the space $\mathcal{Y}$.

Lemma 3. *Let $\mathcal{U}_{\mathbb{R}}$ be a strongly continuous evolution family of bounded linear operators on X, and q be a fixed positive real number.*

(i) *If $\mathcal{Y} = C_0(\mathbb{R},X)$, and $\mathcal{U}_{\mathbb{R}}$ is exponentially bounded, then the semigroup associated to $\mathcal{U}_{\mathbb{R}}$, defined in (1.5), is a strongly continuous one-parameter semigroup of bounded linear operators on $\mathcal{Y}$;*

(ii) *If $\mathcal{Y}$ is either the spaces $AP(\mathbb{R},X)$ or $AAP(\mathbb{R},X)$ and $\mathcal{U}_{\mathbb{R}}$ is q-periodic, that is, $U(t+q,s+q) = U(t,s)$ for all $(t,s) \in \Delta_{\mathbb{R}}$, then the semigroup given in (1.5), is a strongly continuous semigroup on $\mathcal{Y}$.*

Let $(B,D(B))$ the generator of the evolution semigroup given in (1.5). If u and g belongs to $\mathcal{Y}$ then the following statements are equivalent:

(iii) *$u \in D(B)$ and $Bu = -g$;*

(iv)

$$u(t) = U(t,s)u(s) + \int_s^t U(t,s)g(s)ds, \tag{1.6}$$

for all $t \ge s$.

Proof. See [5], [9] for evolution semigroups defined on $C_0(\mathbb{R},X)$ and [8] for evolution semigroups on $AP(\mathbb{R},X)$ or $AAP(\mathbb{R},X)$. ∎

Let $\mathcal{Y}$ be one of the spaces $C_0(\mathbb{R}, X), AP(\mathbb{R}, X), AAP(\mathbb{R}, X)$ and let $\mathcal{Y}_0$ be the set of all functions $f \in \mathcal{Y}$ such that $\lim_{t\to(-\infty)} f(t) = 0$. It is clearly that $\mathcal{Y}_0$ is a closed subspace of $\mathcal{Y}$.

We may now state our second result.

Theorem 2. *Let $\mathcal{U}_{\mathbb{R}}$ be a strongly continuous uniformly stable evolution family of bounded linear operators on X and $q > 0$, fixed. The following statements hold:*

- (j) *If $\mathcal{Y} = C_0(\mathbb{R}, X)$, then the evolution semigroup given in (1.5) is defined on $\mathcal{Y}_0$;*
- (jj) *If $\mathcal{Y}$ is one of the both spaces $AP(\mathbb{R}, X)$ or $AAP(\mathbb{R}, X)$ and $\mathcal{U}_{\mathbb{R}}$ is q-periodic then the evolution semigroup given in (1.4) is defined on $\mathcal{Y}_0$.*

If $(C, D(C))$ is the generator of the evolution semigroup on $\mathcal{Y}_0$, given in (1.5), and v, h belongs to $\mathcal{Y}_0$, then the following statements are equivalent:

- (jjj) *$v \in D(C)$ and $Cv = -h$;*
- (jv)

$$v(t) = \int_{-\infty}^{t} U(t,s)h(s)ds, \tag{1.7}$$

for every real number t. Moreover, the following inequality holds:

$$\left\| \int_{-\infty}^{\cdot} U(\cdot,s)h(s)ds \right\|_{\mathcal{Y}}^{2} \le 4M^2 \|h\|_{\mathcal{Y}} \times \left\| \int_{-\infty}^{\cdot} (\cdot - s)U(\cdot,s)h(s)ds \right\|_{\mathcal{Y}}. \tag{1.8}$$

2. Proofs

Proof of Theorem 1. Let $\mathbf{T}$ be the evolution semigroup associated to $\mathcal{U}_{\mathbb{R}_+}$ on the space $\mathcal{X}$ and $(A, D(A))$ its infinitesimal generator. From Lemma 2 it follows that the function $t \mapsto u(t) := \int_0^t U(t,s)g(s)ds$ belongs to $D(A)$ and $Au = -g$. The function $t \mapsto v(t) := \int_0^t U(t,r)u(r)dr$ belongs to $\mathcal{X}$. Indeed, using the Fubini Theorem, we have:

$$\begin{aligned} v(t) &= \int_0^t \left[U(t,r) \int_0^r U(r,s)g(s)ds \right] dr \\ &= \int_0^t \left[\int_0^r U(t,s)g(s)ds \right] dr \\ &= \int_0^t \left[\int_0^t 1_{[0,r]}(s)U(t,s)g(s)ds \right] dr \\ &= \int_0^t \left[\int_s^t U(t,s)g(s)dr \right] ds \\ &= \int_0^t (t-s)U(t,s)g(s)ds, \end{aligned}$$

where $1_{[0,r]}$ is the characteristic function of the interval $[0, r]$. Using again Lemma 2 follows that $v \in D(A^2)$ and $A^2 v = A(Av) = -Au = g$.

Now the inequality (1.4) follows by Lemma 1, if we replace x with v in (1.2). ∎

Proof of Theorem 2. Firstly we prove that $\mathcal{Y}_0$ is an invariant subspace for each operator $T(t), t \ge 0$, given in (1.5). By Lemma 3 it suffices to prove that $\lim\limits_{s\to(-\infty)} (T(t)f)(s) =$

0 for each $t \geq 0$ and every $f \in \mathcal{Y}_0$, and this fact is an easy consequence of the following estimations:

$$\|(T(t)f)(s)\| \leq \|U(s, s-t)\| \, \|f(s-t)\| \leq M \, \|f(s-t)\| \to 0 \text{ as } s \to (-\infty),$$

where M is the positive constant from (1.1). Now, the implication $(jjj) \Rightarrow (jv)$ follows from Lemma 3, passing to the limit for $s \to (-\infty)$. The converse implication $(jv) \Rightarrow (jjj)$ can be obtained on the following way.

Let v as in (1.7) and $t > 0$. Simple calculus gives

$$\frac{T(t)v - v}{t} = -\frac{\int_0^t T(r)h dr}{t} \to -h \text{ in } \mathcal{X}$$

when $t \to 0$, that is $v \in D(C)$ and $Cv = -h$. Now the inequality (1.8), can be established as in the proof of Theorem 1 and we omit the details. ∎

3. Applications

In this section some scalar inequalities are presented.

Corollary 1. *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that $g(0) = g(\infty) := \lim_{t\to\infty} g(t) = 0$. Suppose that the functions:*

$$t \mapsto h(t) := \int_0^t g(s)ds \text{ and } t \mapsto u(t) := \int_0^t (t-s)g(s)ds$$

verifies the condition $h(\infty) = u(\infty) = 0$.

Then the following inequality holds:

$$\sup_{t\geq 0} \left| \int_0^t g(s)ds \right|^2 \leq 4 \cdot \sup_{t\geq 0} |g(t)| \times \sup_{t\geq 0} \left| \int_0^t (t-s)g(s)ds \right|.$$

Proof. We apply Theorem 1 for $\mathcal{X} = C_{00}(\mathbb{R}_+, \mathbb{R})$ and for $U(t,s)x = x$, where $t \geq s \geq 0$ and $x \in \mathbb{R}$. ∎

Corollary 2. *Let g, h, u as in Corollary 1 and f be a continuous, positive and nondecreasing function on $\mathbb{R}_+$. The following inequality holds:*

$$\sup_{t\geq 0} \left[\frac{\left| \int_0^t f(s)g(s)ds \right|^2}{f(t)^2} \right] \leq 4 \sup_{t\geq 0} |g(t)| \sup_{t\geq 0} \left[\frac{\left| \int_0^t (t-s)f(s)g(s)ds \right|}{f(t)} \right].$$

Proof. Follows by Theorem 1 for $\mathcal{X} = C_{00}(\mathbb{R}_+, \mathbb{R})$ and $U(t,s) = \frac{f(s)}{f(t)}$. ∎

Corollary 3. *Let $1 \leq p < \infty$ and $f \in L_p(\mathbb{R}_+, \mathbb{R})$. If the functions*

$$t \mapsto g(t) := \int_0^t f(s)ds \text{ and } t \mapsto h(t) := \int_0^t (t-s)f(s)ds$$

belongs to $L^p(\mathbb{R}_+, \mathbb{R})$, then the following inequality, holds:

$$\|g\|_p^2 \leq 4 \, \|f\|_p \times \|h\|_p .$$

Proof. Follows by Theorem 1 for $\mathcal{X} = L_p(\mathbb{R}_+, \mathbb{R})$ and for $U(t,s)x = x$ where $t \geq s \geq 0$ and $x \in \mathbb{R}$. ∎

Corollary 4. *Let $g:\mathbb{R}\to\mathbb{R}$ be an almost periodic or asymptotically almost periodic function such that $g(-\infty)=0$. Then*

$$\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t}\frac{1+\sin^2 s}{1+\sin^2 t}g(s)ds\right|^2 \le 16\sup_{t\in\mathbb{R}}|g(t)|\times\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t}(t-s)\frac{1+\sin^2 s}{1+\sin^2 t}g(s)ds\right|.$$

Proof. Follows by Theorem 2 for $\mathcal{Y}=AP(\mathbb{R},\mathbb{R})$ or $\mathcal{Y}=AAP(\mathbb{R},\mathbb{R})$ and $U(t,s)x=\frac{1+\sin^2 s}{1+\sin^2 t}x$ where $t\ge s$ and $x\in\mathbb{R}$. It is clear that $\mathcal{U}=\{U(t,s);t\ge s\}$ is a π-periodic family consisting in operators acting on $\mathbb{R}$, and $\sup_{t\ge s}U(t,s)\le 2$. ∎

References

[1] C. Chicone, Yu. Latushkin, "Evolution Semigroups in Dynamical Systems and Differential Equations", *Amer. Math. Soc., Math. Surv. and Monographs*, **70**, 1999.

[2] S. Clark, Yu Latushkin, S. Montgomery-Smith and T. Randolph, Stability radius and internal versus external stability in Banach spaces: An evolution semigroup approach*, SIAM J. Contr. and Optim.*, **38** (2000), 1757-1793.

[3] K. Engel and R. Nagel, "*One-parameter semigroups for linear evolution equations*", Springer-Verlag, New-York, 2000.

[4] R. R. Kallman and G. C. Rota, On the inequality $\|f'\|\le 4\|f\|\|f''\|$, *Inequalities II*, O. Shisha, Ed., Academic Press, New-York, 1970, pp. 187-192.

[5] Yu. Latushkin and S. Montgomery-Smith, Evolutionary semigroups and Lyapunov theorems in Banach spaces, *J. Func. Anal.*, **127**(1995), 173-197.

[6] B. M. Levitan and V. V. Zhikov, "*Almost Periodic Functions and Differential Equations*", Moscow Univ. Publ. House, 1978. English translation by Cambridge Univ. Press, Cambridge U.K., 1982.

[7] Nguyen Van Minh, Frank Räbiger and Roland Schnaubelt, Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line, *Integral Equations Operator Theory*, **32**,(1998), 332-353.

[8] T. Naito, Nguyen Van Minh, Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations, *J. Differential Equations,* **152**(1999), 338-376.

[9] R. Rau, Hyperbolic evolution semigroups on vector valued-functions, *Semigroup Forum,* **48** (1994), 107-118.

Department of Mathematics, West University of Timisoara, Timisoara, 1900, Bd. V. Parvan. Nr. 4, Romania

E-mail address: buse@tim1.uvt.ro

URL: http://rgmia.vu.edu.au/BuseCVhtml/

School of Communications and Informatics, Victoria University of Technology, P.O. Box 14428, Melbourne City MC, Victoria 8001,, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html