PERTURBED GENERALISED TAYLOR'S FORMULAE WITH SHARP BOUNDS

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Abstract. Sharp bounds are obtained for perturbed generalised Taylor series. The perturbation involves the arithmetic sum of the upper and lower bounds of the $(n+1)^{th}$ derivative. The sharpest bound is in terms of the one norm of the Appell polynomial which constitutes the coefficients of the derivative of the function to be approximated. The results are demonstrated for an application to the logarithm.

1. Introduction

A number of authors have recently obtained generalisations of the traditional Taylor series expansion of a function $f(x)$ about a point a assuming sufficient differentiability. A Taylor series representation is a fundamental mechanism for estimation in problems arising in many applications. Estimates of bounds on the remainder have also been procured.

Before proceeding further with more generalisations, let us introduce some notation. We shall term polynomials of degree k, W_k as Appell type and say $W_k \in \mathcal{A}$ if they satisfy the condition

$$W_k' = \xi_k W_{k-1}(t), \ W_0(t) = 1, \quad t \in \mathbb{R}. \tag{1.1}$$

These are so named since Appell studied (1.1) with $\xi_k = k$ in 1880 (see [2]). Polynomials satisfying (1.1) with $\xi_k = 1$ have been termed harmonic polynomials in Matić et al. [10], however a simple scaling will demonstrate that these are Appell.

The following results were obtained by Matić et al. [10] where $P_n(t)$ satisfy (1.1) with $\xi_k = 1$.

Theorem 1. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials, that satisfy*

$$P_n'(t) = P_{n-1}(t), \ P_0(t) = 1, \ t \in \mathbb{R}, \ n \in \mathbb{N}, \ n \geq 1. \tag{1.2}$$

Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is any function such that for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then for any $x \in I$

$$f(x) = T_n(f; a, x) + R_n(f; a, x) \tag{1.3}$$

where

$$T_n(f; a, x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)\right], \tag{1.4}$$

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$$R_n(f;a,x) = (-1)^n \int_a^x P_n(t)\, f^{(n+1)}(t)\, dt. \tag{1.5}$$

They also pointed out the following bounds for the remainder $R_n(f;a,x)$.

Corollary 1. *With the above assumptions of Theorem 1, the following estimations hold. Namely for $x \geq a$,*

$$|R_n(f;a,x)| \leq \begin{cases} \|P_n\|_\infty \left\|f^{(n+1)}\right\|_1 & \textit{provided} \quad f^{(n+1)} \in L_1[a,x], \\ \|P_n\|_q \left\|f^{(n+1)}\right\|_p & \textit{provided} \quad f^{(n+1)} \in L_p[a,x],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|P_n\|_1 \left\|f^{(n+1)}\right\|_\infty & \textit{provided} \quad f^{(n+1)} \in L_\infty[a,x], \end{cases} \tag{1.6}$$

where $x \geq a$ and $\|\cdot\|_s$ $(1 \leq s \leq \infty)$ are the usual $s-$Lebesgue norms. That is,

$$\|g\|_s := \left(\int_a^x |g(t)|^s\, dt\right)^{\frac{1}{s}}, \quad s \in [1,\infty)$$

and

$$\|g\|_\infty := ess \sup_{t\in[a,x]} |g(t)|.$$

We introduce superscripts for $T_n(f;a,x)$ and $R_n(f;a,x)$ as given in (1.4) and (1.5) respectively to reflect the particular polynomial $P_n(t)$ involved. Let

$$P_n^{c_\lambda}(t) = \frac{(t-\theta(\lambda))^n}{n!},\ \theta(\lambda) = \lambda a + (1-\lambda)x,\quad \lambda \in [0,1], \tag{1.7}$$

$$P_n^B(t) = \frac{(x-a)^n}{n!} B_n\left(\frac{t-a}{x-a}\right), \tag{1.8}$$

and

$$P_n^E(t) = \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right) \tag{1.9}$$

represent polynomials involving; a convex combination of the end points, Bernoulli polynomials and Euler polynomials respectively. We should note that the dependence of the polynomials in (1.7) – (1.9), on x is not shown explicitly.

With the polynomials (1.7) – (1.9) then from (1.4) and (1.5)

$$\begin{aligned} T_n^{c_\lambda}&(f;a,x) \\ &= f(a) + \sum_{k=1}^n (-1)^{k+1} \frac{(x-a)^k}{k!}\left[\lambda^k f^{(k)}(x) + (-1)^{k+1}(1-\lambda)^k f^{(k)}(a)\right], \end{aligned} \tag{1.10}$$

$$\begin{aligned} T_n^B(f;a,x) = f(a) + \frac{x-a}{2}\left[f'(x)+f'(a)\right] \\ - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(x-a)^{2k}}{(2k)!} B_{2k}\left[f^{(2k)}(x) - f^{(2k)}(a)\right], \end{aligned} \tag{1.11}$$

$$T_n^E(f;a,x) = f(a) + 2\sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(x-a)^{2k-1}\left(4^k-1\right)}{(2k)!} B_{2k}\left[f^{(2k-1)}(x) + f^{(2k-1)}(a)\right] \tag{1.12}$$

and

$$R_n^{c_\lambda}(f;a,x) = \frac{(-1)^{n+1}}{n!}\int_a^x (t-\theta(\lambda))^n f^{(n+1)}(t)\,dt, \tag{1.13}$$
$$\theta(\lambda) = \lambda a + (1-\lambda)x, \quad \lambda \in [0,1],$$

$$R_n^B(f;a,x) = (-1)^{n+1}\frac{(x-a)^n}{n!}\int_a^x B_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)\,dt, \tag{1.14}$$

$$R_n^E(f;a,x) = (-1)^{n+1}\frac{(x-a)^n}{n!}\int_a^x E_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)\,dt, \tag{1.15}$$

where $B_n(\cdot)$ are the Bernoulli polynomials, $B_n = B_n(0)$ the Bernoulli numbers and $E_n(\cdot)$ the Euler polynomials.

The above expressions (1.10) – (1.15) were obtained by Matić et al. [10] for the Bernoulli and Euler polynomials but only for the equivalent of $\lambda = 0$ and $\frac{1}{2}$ in (1.7).

Cerone and Dragomir [4] obtained the following theorem which follows directly from Corollary 1 with $P_n^{c_\lambda}(t)$ as given by (1.7) and using (1.10) and (1.13).

Theorem 2. *Assume that f is as in Theorem 1 with $x \geq a$, then we have*

$$\begin{aligned} & \left|f(x) - T_n^{c_\lambda}(f;a,x)\right| \\ = & \left|R_n^{c_\lambda}(f;a,x)\right| \\ \leq & \begin{cases} \frac{1}{n!}(x-a)^n\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n \left\|f^{(n+1)}\right\|_1 & \text{if } f^{(n+1)} \in L_1[a,x]; \\ \frac{1}{n!(nq+1)^{\frac{1}{q}}}(x-a)^{n+\frac{1}{q}}\left[(1-\lambda)^{nq+1}+\lambda^{nq+1}\right]^{\frac{1}{q}}\left\|f^{(n+1)}\right\|_p \\ \qquad \text{if } f^{(n+1)} \in L_p[a,x], \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(n+1)!}(x-a)^{n+1}\left[(1-\lambda)^{n+1}+\lambda^{n+1}\right]\left\|f^{(n+1)}\right\|_\infty, \\ \qquad \text{if } f^{(n+1)} \in L_\infty[a,x]. \end{cases} \end{aligned} \tag{1.16}$$

It was also noted that since $h_1(\lambda) = \left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n$, $h_2(\lambda) = \left[(1-\lambda)^{nq+1}+\lambda^{nq+1}\right]^{\frac{1}{q}}$ and $h_3(\lambda) = (1-\lambda)^{n+1}+\lambda^{n+1}$ are convex and symmetric about $\frac{1}{2}$, then

$$\inf_{\lambda\in[0,1]} h_i(\lambda) = h_i\left(\frac{1}{2}\right), \quad i = 1,2,3.$$

Hence the best inequality possible in the class, in the sense of providing the tightest bound, is

$$\left|f(x)-T_n^{c_{\frac{1}{2}}}(f;a,x)\right| \tag{1.17}$$

$$\leq\begin{cases}\dfrac{1}{2^n n!}(x-a)^n\left\|f^{(n+1)}\right\|_1; \\[2ex] \dfrac{1}{n!\,(nq+1)^{\frac{1}{q}}\,2^n}(x-a)^{n+\frac{1}{q}}\left\|f^{(n+1)}\right\|_p, \\ \qquad\qquad \text{if } \ f^{(n+1)}\in L_p[a,x],\ \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{1}{(n+1)!2^n}(x-a)^{n+1}\left\|f^{(n+1)}\right\|_\infty, \ \text{if } \ f^{(n+1)}\in L_\infty[a,x].\end{cases}$$

Taking $\lambda=0$ in (1.16) produces the classical Taylor series expansion in terms of the $L_p[a,x]$, $p\geq 1$, Lebesgue norms for the bounds (see for example, Dragomir [8]). That is,

$$\left|f(x)-\sum_{k=0}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a)\right| \tag{1.18}$$

$$\leq\begin{cases}\dfrac{(x-a)^n}{n!}\left\|f^{(n+1)}\right\|_1, & f^{(n+1)}\in L_1[a,x]; \\[2ex] \dfrac{(x-a)^{n+\frac{1}{q}}}{n!\,(nq+1)^{\frac{1}{q}}}\left\|f^{(n+1)}\right\|_p, & f^{(n+1)}\in L_p[a,x], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(x-a)^{n+1}}{(n+1)!}\left\|f^{(n+1)}\right\|_\infty, & f^{(n+1)}\in L_\infty[a,x].\end{cases}$$

Recently in [7] Dragomir introduced a perturbed Taylor's formula using the Grüss inequality for the Čebyšev functional. Matić et al. [10] obtained generalised Taylor's formulae involving expansions in terms of general polynomials satisfying (1.2) producing in particular Theorem 1 and Corollary 1 above. They also examined perturbed versions of (1.3), namely

$$\begin{aligned}f(x)&=T_n(f;a,x) \\ &\quad+(-1)^n\left[P_{n+1}(x)-P_{n+1}(a)\right]\left[f^{(n)};a,x\right]+\rho_n(f;a,x),\end{aligned} \tag{1.19}$$

where

$$\left[f^{(n)};a,x\right]:=\frac{f^{(n)}(x)-f^{(n)}(a)}{x-a},\quad\text{the divided difference,} \tag{1.20}$$

$$\rho_n(f;a,x)\quad\text{is the remainder.} \tag{1.21}$$

Dragomir [7] developed an estimate of the remainder using the Grüss inequality for $P_n(t)=\frac{(t-x)^n}{n!}$, Matić et al. [10] used a premature or pre-Grüss argument to procure bounds on $\rho_n^{c_{\frac{1}{2}}}(f;a,x)$, $\rho_n^{c_0}(f;a,x)$, $\rho_n^B(f;a,x)$ and $\rho_n^E(f;a,x)$. Dragomir

[8] obtained tighter bounds for the same polynomial generators of the perturbed Taylor series for $f^{(n+1)} \in L_2(I)$ with $x, a \in I \subseteq \mathbb{R}$. In the paper [4], Cerone and Dragomir procured bounds on $\rho_n(f; a, x)$ in terms of $\Delta-$seminorms resulting from the Čebyšev functional and Korkine's identity which are used to produce (1.18).

In [7], S.S. Dragomir seems to be the first author to introduce the perturbed Taylor formula

$$f(x) = T_n(f; a, x) + \frac{(x-a)^{n+1}}{(n+1)!}\left[f^{(n)}; a, x\right] + \rho_n(f; a, x), \tag{1.22}$$

where

$$T_n(f; a, x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{1.23}$$

and $\left[f^{(n)}; a, x\right]$ is as given in (1.20). Dragomir [7] estimated the remainder $\rho_n(f; a, x)$ by using Grüss and Čebyšev type inequalities.

In [10], the authors generalised and improved the results from [7] via a pre-Grüss inequality (see [10, Theorem 3]).

Theorem 3. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials satisfying (1.2). Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Suppose $f : I \to \mathbb{R}$ is as in Theorem 1. Then for all $x \in I$ we have the perturbed generalised Taylor formula, (1.19), where $x \geq a$, the remainder $\rho(f; a, x)$ satisfies the estimate*

$$|\rho_n(f; a, x)| \leq \frac{x-a}{2}\sqrt{\mathcal{T}(P_n, P_n)}\left[\Gamma(x) - \gamma(x)\right], \tag{1.24}$$

provided that $f^{(n+1)}$ is bounded and

$$\gamma(x) := \inf_{t\in[a,x]} f^{(n+1)}(t) > -\infty, \quad \Gamma(x) := \sup_{t\in[a,x]} f^{(n+1)}(t) < \infty. \tag{1.25}$$

In (1.24), $\mathcal{T}(\cdot,\cdot)$ is the Čebyšev functional on the interval $[a, x]$. That is,

$$\mathcal{T}(g, h) := \frac{1}{x-a}\int_a^x g(t) h(t) dt - \frac{1}{x-a}\int_a^x g(t) dt \cdot \frac{1}{x-a}\int_a^x h(t) dt. \tag{1.26}$$

It is the intention in the current article to produce perturbed generalised Taylor series like (1.19), however the perturbation involves the arithmetic average of the upper and lower bounds of the $f^{(n+1)}(t)$, $t \in I$. The bounds for the expansion involve the norms of the Appell polynomials with the one norm, which is shown to provide the tightest bound.

A novel Čebyšev functional and its bounds are presented in Section 2, the results of which, are applied in Section 3 to perturbed generalised Taylor series, for a selection of Appell polynomials. The approximation of the logarithmic function using the results of Sections 2 and 3 is presented in Section 4.

2. A Novel Čebyšev-like Functional and its Bound

Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions and define the functional

$$T(f, g; a, b) := \mathcal{M}(fg; a, b) - \mathcal{M}(f; a, b)\mathcal{M}(g; a, b), \tag{2.1}$$

where the integral mean is given by

$$\mathcal{M}(f; a, b) := \frac{1}{b-a}\int_a^b f(x) dx. \tag{2.2}$$

The functional $T(f,g;a,b)$ as defined in (2.1) – (2.2) is widely known in the literature as Čebyšev's functional. The reader is referred to [11], Chapters IX and X and, to the work by Dragomir [6] and Fink [9] for extensive treatments of the functional.

We now define a Čebyšev-like functional

$$C(f,g;a,b) := \mathcal{M}(fg;a,b) - \frac{M+m}{2}\mathcal{M}(f;a,b), \tag{2.3}$$

where $-\infty < m \le g(t) \le M < \infty$, for $t \in [a,b]$ and $\mathcal{M}(f;a,b)$ is as given by (2.2).

The following theorem holds providing bounds for the functional $C(f,g;a,b)$.

Theorem 4. *Let $f,g : [a,b] \to \mathbb{R}$ be integrable functions and $-\infty < m \le g(t) \le M < \infty$, then*

$$\begin{aligned} |C(f,g;a,b)| &= \left|\mathcal{M}(fg;a,b) - \frac{M+m}{2}\mathcal{M}(f;a,b)\right| \\ &\le \frac{M-m}{2}\cdot\frac{1}{b-a}\|f\|_1, \quad f \in L_1[a,b] \\ &\le \frac{M-m}{2}\cdot\frac{1}{(b-a)^{\frac{1}{p}}}\|f\|_p, \quad f \in L_p[a,b],\ 1<p<\infty \\ &\le \frac{M-m}{2}\|f\|_\infty = \frac{M-m}{2}\max\{|N|,|n|\}, \quad f \in L_\infty[a,b], \\ &-\infty < n \le f(t) \le N < \infty,\ t \in [a,b], \end{aligned} \tag{2.4}$$

where $\|f\|_p$ are the usual Lebesgue norms for $f \in L_p[a,b]$ defined by

$$\|f\|_p := \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

and

$$\|f\|_\infty := ess \sup_{t\in[a,b]} |f(t)|.$$

The $\frac{1}{2}$ in the three inequalities in (2.4) are sharp.

Proof. From (2.3) and using (2.2) we have the identity

$$\begin{aligned} C(f,g;a,b) &= \mathcal{M}(fg;a,b) - \frac{M+m}{2}\mathcal{M}(f;a,b) \\ &= \mathcal{M}\left(f\left(g - \frac{M+m}{2}\right)\right) \end{aligned}$$

and so

$$C(f,g;a,b) = \frac{1}{b-a}\int_a^b f(t)\left(g(t) - \frac{M+m}{2}\right)dt. \tag{2.5}$$

Taking the modulus from identity (2.5) gives

$$|C(f,g;a,b)| \le \frac{1}{b-a}\int_a^b |f(t)|\left|g(t) - \frac{M+m}{2}\right|dt. \tag{2.6}$$

Now, since $-\infty < m \le g(t) \le M < \infty$, $t \in [a,b]$ then

$$-\frac{M-m}{2} \le g(t) - \frac{m+M}{2} \le \frac{M-m}{2}$$

and so from (2.6)

$$|C(f,g;a,b)| \le \frac{M-m}{2} \cdot \frac{1}{b-a} \int_a^b |f(t)|\, dt \tag{2.7}$$

giving the first inequality in (2.4).

We further have, using Hölder's inequality

$$\begin{aligned} \frac{1}{b-a} \int_a^b |f(t)|\, dt &\le \frac{1}{(b-a)^{1-\frac{1}{q}}} \left(\int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}}, \; f \in L_p[a,b], \; \frac{1}{p}+\frac{1}{q}=1, \; 1<p<\infty \\ &\le ess \sup_{t\in[a,b]} |f(t)|, \; f \in L_\infty[a,b] \end{aligned}$$

producing, from (2.7), the second and third inequalities in (2.4).

For the sharpness of the constant $\frac{1}{2}$, consider

$$|C(f,g;a,b)| \le k(M-m) \cdot \frac{1}{b-a} \int_a^b |f(t)|\, dt. \tag{2.8}$$

If we choose $g = f = f_0$ where $f_0 : [a,b] \to \mathbb{R}$ is given by

$$f_0(t) = \begin{cases} -1, & t \in \left[a, \frac{a+b}{2}\right] \\ 1, & t \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

then

$$|C(f_0, f_0; a, b)| = \frac{1}{b-a} \int_a^b f_0^2(t)\, dt - \left(\frac{1}{b-a} \int_a^b f_0(t)\, dt \right) \left(\frac{M+m}{2} \right) = 1$$

and

$$\frac{1}{b-a} \int_a^b |f_0(t)|\, dt = 1$$

giving from (2.8) since $m=-1$, $M=1$, that $1 \le 2k$ and so $\frac{1}{2} \le k$.

The same function $f_0(t)$ will prove the sharpness of the second and third inequalities or, more directly from the properties of the Hölder inequality. The theorem is now completely proved. ∎

Remark 1. *The inequalities in (2.4) are in the order of increasing coarseness although each of them are sharp for $f \in L_p[a,b]$, $p \ge 1$.*

3. The Čebyšev-like Functional and Perturbed Taylor Results with Bounds

In this section we will now apply the results of Section 2 to obtain sharp bounds for perturbed Taylor-like formulae.

Theorem 5. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials that satisfy (1.2). Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is a function which for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous and $-\infty < \gamma_{n+1}(x) \le f^{(n+1)}(t) \le \Gamma_{n+1}(x) < \infty$, then for any $a \le x \in I$*

$$\begin{aligned} f(x) = T_n(f;a,x) + \left(\frac{\gamma_{n+1}(x) + \Gamma_{n+1}(x)}{2} \right) (-1)^n \frac{[P_{n+1}(x) - P_{n+1}(a)]}{n+1} \\ + G_n(f;a,x) \end{aligned} \tag{3.1}$$

and

$$(3.2)\quad \begin{aligned} &|G_n(f;a,x)| \\ &\leq \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\|P_n\|_{1,[a,x]}, \qquad P_n \in L_1[I] \\ &\leq \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\cdot\frac{1}{(x-a)^{\frac{1}{p}-1}}\|P_n\|_{p,[a,x]},\ P_n\in L_p[I],\ 1<p<\infty \\ &\leq \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\|P_n\|_{\infty,[a,x]}, \qquad P_n \in L_\infty[I], \end{aligned}$$

where

$$(3.3)\qquad G_n(f;a,x) = (-1)^n \int_a^x P_n(t)\left[f^{(n+1)}(t) - \frac{\gamma_{n+1}(x)+\Gamma_{n+1}(x)}{2}\right]dt$$

and $T_n(f;a,x)$ is as given by (1.4).

Proof. If we identify $(-1)^n P_n(t)$ with $f(t)$ and $f^{(n+1)}(t)$ with $g(t)$ in Theorem 4, then from identity (2.5) we have

$$(3.4)\qquad \begin{aligned} G_n(f;a,x) &= (-1)^n \int_a^x P_n(t)\left[f^{(n+1)}(t) - \frac{\gamma_{n+1}(x)+\Gamma_{n+1}(x)}{2}\right]dt \\ &= (x-a)\,C\left((-1)^n P_n(t), f^{(n+1)}(t); a, x\right). \end{aligned}$$

That is,

$$(3.5)\quad \begin{aligned} &G_n(f;a,x) \\ &= (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\,dt \\ &\qquad - (-1)^n \frac{\gamma_{n+1}(x)+\Gamma_{n+1}(x)}{2}\int_a^x P_n(t)\,dt \\ &= R_n(f;a,x) - (-1)^n\left(\frac{\gamma_{n+1}(x)+\Gamma_{n+1}(x)}{2}\right)\frac{[P_{n+1}(x)-P_{n+1}(a)]}{n+1}, \end{aligned}$$

where $R_n(f;a,x)$ satisfies (1.3) – (1.5). Using identity (1.3) in (3.5) produces the stated result (3.1).

For the bound on the remainder $|G_n(f;a,x)|$ we have from the first inequality in (2.4) and (3.4)

$$|G_n(f;a,x)| \leq \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\int_a^x |P_n(t)|\,dt,$$

and hence (3.2). Utilising the second and third inequality in (2.4) produces the respective bounds in (3.2). ∎

Corollary 2. *Let the conditions of Theorem 5 persist, and assume $a \leq x$, then we have from (3.1) and (3.2), for $P_n^{c_\lambda}(t)$ given by (1.7)*

$$\begin{aligned}
&\left| f(x) - T_n^{c_\lambda}(f;a,x) - \frac{1}{(n+1)!}\left[(\theta(\lambda)-a)^{n+1}\right.\right. \\
&\qquad \left.\left. + (-1)^n (x-\theta(\lambda))^{n+1}\right]\left(\frac{\gamma_{n+1}(x)+\Gamma_{n+1}(x)}{2}\right)\right| \\
&\leq \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\cdot\frac{1}{n!}\Psi_1(\lambda;a,x) \\
&\leq \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\cdot\frac{1}{(x-a)^{\frac{1}{p}-1}}\cdot\frac{1}{n!}\left[\Psi_p(\lambda;a,x)\right]^{\frac{1}{p}} \\
&\leq \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\cdot\frac{(x-a)}{n!}\cdot\left[\frac{x-a}{2}+\left|\theta(\lambda)-\frac{a+x}{2}\right|\right]^n,
\end{aligned} \tag{3.6}$$

where

$$\begin{cases} \Psi_p(\lambda;a,x) = \dfrac{(x-\theta(\lambda))^{np+1}+(\theta(\lambda)-a)^{np+1}}{np+1} \\ \\ \theta(\lambda) = \lambda a + (1-\lambda)x, \quad \lambda\in[0,1]. \end{cases} \tag{3.7}$$

Proof. We need to evaluate for $P_n^{c_\lambda}(t)$ given by (1.7)

$$\int_a^x P_n^{c_\lambda}(t)\,dt \quad \text{and} \quad \int_a^x \left|P_n^{c_\lambda}(t)\right|^p dt, \quad p\geq 1.$$

Thus, from (1.7), we have,

$$\int_a^x P_n^{c_\lambda}(t)\,dt = \frac{1}{(n+1)!}\left[(x-\theta(\lambda))^{n+1}-(\theta(\lambda)-a)^{n+1}\right]$$

and

$$\begin{aligned}
\int_a^x \left|P_n^{c_\lambda}(t)\right|^p dt &= \frac{1}{n!}\int_a^x |t-\theta(\lambda)|^n\,dt \\
&= \frac{1}{n!}\left[\int_a^{\theta(\lambda)}(\theta(\lambda)-t)^n\,dt + \int_{\theta(\lambda)}^b (t-\theta(\lambda))^n\,dt\right] \\
&= \frac{1}{(n+1)!}\left[(\theta(\lambda)-a)^{n+1}+(x-\theta(\lambda))^{n+1}\right],
\end{aligned}$$

producing (3.6).

Further,

$$\begin{aligned}
\left\|P_n^{c_\lambda}\right\|_{p,[a,x]}^p = \int_a^x \left|P_n^{c_\lambda}(t)\right|^p dt &= \frac{1}{n!}\int_a^x |t-\theta(\lambda)|^{np}\,dt \\
&= \frac{1}{n!}\left[\int_a^{\theta(\lambda)}(\theta(\lambda)-t)^{np}\,dt + \int_{\theta(\lambda)}^x (t-\theta(\lambda))^{np}\,dt\right] \\
&= \frac{1}{n!}\left[\frac{(\theta(\lambda)-a)^{np+1}+(x-\theta(\lambda))^{np+1}}{np+1}\right]
\end{aligned}$$

and so from (3.2) the second inequality in (3.6) is procured.

The final inequality is obtained from (1.7) and (3.2) giving

$$\begin{aligned} \left\|P_n^{c_\lambda}\right\|_{\infty,[a,x]} &= ess \sup_{t\in[a,x]} \left|P_n^{c_\lambda}(t)\right| \\ &= ess \sup_{t\in[a,x]} \frac{|t-\theta(\lambda)|^n}{n!} \\ &= \frac{1}{n!}\left[\max\left\{x-\theta(\lambda), \theta(\lambda)-a\right\}\right]^n . \end{aligned}$$

∎

Remark 2. *The bounds in (3.6) are in order of increasing coarseness. This was commented upon in Remark 1 referring to the results (2.4), of which (3.6) is a specialisation. For* $\lambda = \frac{1}{2}$, $\theta\left(\frac{1}{2}\right) = \frac{a+x}{2}$ *then from (3.6) and (3.7)*

$$\begin{aligned} &\left| f(x) - T_n^{c_{\frac{1}{2}}}(f;a,x) - \frac{1+(-1)^n}{(n+1)!}\cdot\left(\frac{x-a}{2}\right)^{n+1}\left(\frac{\gamma_{n+1}(x)+\Gamma_{n+1}(x)}{2}\right)\right| \\ &\le \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\cdot\frac{(x-a)^{n+1}}{2^n(n+1)!} \\ &\le \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\cdot\frac{(x-a)^{n+1}}{2^n(np+1)^{\frac{1}{p}}\,n!} \\ &\le \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\cdot\frac{(x-a)^{n+1}}{2^n n!} \end{aligned} \tag{3.8}$$

where, from (1.10),

$$T_n^{c_{\frac{1}{2}}}(f;a,x) = f(a) + \sum_{k=1}^{n}\frac{(x-a)^k}{2^k k!}\left[f^{(k)}(a) + (-1)^{k+1} f^{(k)}(x)\right] \tag{3.9}$$

from which we may confirm, for this case at least, the fact that the bounds are in order of increasing coarseness since $\frac{1}{n+1} < \frac{1}{(np+1)^{\frac{1}{p}}} < 1$, $1 < p < \infty$.

We further note that for n odd, the perturbation in (3.8) vanishes, giving the tightest bound

$$\left| f(x) - T_n^{c_{\frac{1}{2}}}(f;a,x)\right| \le \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\cdot\frac{(x-a)^{n+1}}{2^n(n+1)!}, \quad n \text{ odd.} \tag{3.10}$$

For n odd, the result (3.10) may be compared with the last result in (1.17) demonstrating that it is tighter since

$$\left\|f^{(n+1)}\right\|_\infty \ge \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}$$

where $\gamma_{n+1}(x) \le f(t) \le \Gamma_{n+1}(x)$, $t \in [a,x]$.

For n even, the bound is still tighter in (3.8), however, the perturbation is now present.

If $\lambda = 0$ in (3.6) then $\theta(0) = x$ and we obtain a perturbed version of the traditional Taylor series expansion about a point a. That is, from (1.10),

$$\begin{aligned}(3.11)\qquad &\left|f(x) - T_n^{c_0}(f;a,x) - \frac{(x-a)^{n+1}}{(n+1)!}\left(\frac{\gamma_{n+1}(x)+\Gamma_{n+1}(x)}{2}\right)\right| \\ &= \left|f(x) - \sum_{k=0}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a) - \frac{(x-a)^{n+1}}{(n+1)!}\left(\frac{\gamma_{n+1}(x)+\Gamma_{n+1}(x)}{2}\right)\right| \\ &\le \frac{\Gamma_{n+1}(x)-\gamma_{n+1}(x)}{2}\cdot\frac{(x-a)^{n+1}}{(n+1)!},\end{aligned}$$

where

$$T_n^{c_0}(f;a,x) = f(x) + \sum_{k=0}^{n}(-1)^{k+1}\frac{(x-a)^k}{k!}f^{(k)}(a).$$

We notice that the bound in (3.11) is inferior to that in the first inequality in (3.8) where $T_n^{c_{\frac{1}{2}}}(f;a,x)$ is as given by (3.9). However, (3.9) requires information involving $f^{(k)}(x)$ being available in order to approximate $f(x)$.

The following corollary examines perturbed Taylor series expansions when the polynomials are either Bernoulli or Euler. A similar approach would also be fruitful for other polynomials satisfying the Appell condition (1.2).

Before proceeding any further we require, for the sake of lucidity, to present some properties of the Bernoulli and Euler polynomials. Let $B_n(t)$ and $E_n(t)$ represent the Bernoulli and Euler polynomials respectively. The $B_n(t)$ may be defined by the expansion

$$(3.12)\qquad \frac{xe^{tx}}{e^x-1} = \sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!},\quad |x|<2\pi,\ t\in\mathbb{R}$$

or, alternatively, they are uniquely determined by the properties

$$(3.13)\qquad B_n'(t) = nB_{n-1}(t),\ \ n\in\mathbb{N};\ B_0(t)=1$$

and

$$(3.14)\qquad B_n(t+1) - B_n(t) = nt^{n-1},\ n\in\mathbb{N}.$$

From (3.13) and (3.14) it may be shown that

$$(3.15)\qquad \int_0^1 B_n(t)\,dt = 0\ \text{ and }\ B_n(1) = B_n(0) := B_n,$$

the Bernoulli numbers.

The $B_n(t)$ also satisfy the property (see [1], 23)

$$\int_0^1 B_n(t)B_m(t)\,dt = (-1)^{n-1}\frac{n!m!}{(m+n)!}B_{n+m},\quad n,m\in\mathbb{N}$$

and in particular

$$(3.16)\qquad \int_0^1 B_n^2(t)\,dt = (-1)^{n-1}\frac{(n!)^2}{(2n)!}B_{2n} = \frac{(n!)^2}{(2n)!}|B_{2n}|.$$

The $E_n(t)$ satisfy (see [1], 23)

$$(3.17)\qquad \frac{2e^{tx}}{e^x-1} = \sum_{n=0}^{\infty}E_n(t)\frac{t^n}{n!},\quad |x|<\pi,\ t\in\mathbb{R}$$

and may be uniquely defined by

$$E_n'(t) = nE_{n-1}(t), \quad n \in \mathbb{N}; \; E_0(t) = 1 \tag{3.18}$$

and

$$E_n(t+1) - E_n(t) = 2t^n, \; n \in \mathbb{N}. \tag{3.19}$$

Further, since ([2, 23.1.20])

$$E_n(0) = -E_n(1) = -\frac{2}{n+1}\left(2^{n+1}-1\right)B_{n+1}, \quad n \in \mathbb{N} \tag{3.20}$$

then from (3.18)

$$\int_0^1 E_n(t)\,dt = \frac{E_{n+1}(1) - E_{n+1}(0)}{n+1} = \frac{2E_{n+1}(1)}{n+1} = \frac{4\left(2^{n+2}-1\right)B_{n+2}}{(n+1)(n+2)} \tag{3.21}$$

and

$$\int_0^1 E_n(t)\,E_m(t)\,dt = (-1)^n\,4\left(2^{n+m+2}-1\right)\frac{n!m!}{(n+m+2)!}B_{n+m+2}, \quad n, m \in \mathbb{N},$$

giving

$$\int_0^1 E_n^2(t)\,dt = 4\left(4^{n+1}-1\right)\frac{(n!)^2}{(2n+2)!}\left|B_{2n+2}\right|. \tag{3.22}$$

Corollary 3. *Let the conditions of Theorem 5 be valid, then we have from (3.1) and (3.2), for $P_n^B(t)$, $P_n^E(t)$ as given by (1.8), (1.9),*

$$\begin{aligned} &\left|f(x) - T_n^B(x)\right| \\ &\leq \frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\frac{(x-a)^{n+1}}{n!}\int_0^1 |B_n(u)|\,du \\ &\leq \frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\frac{(x-a)^{n+1}}{n!}\left(\int_0^1 |B_n(u)|^p\,du\right)^{\frac{1}{p}}, \; 1 < p < \infty \\ &\leq \frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\frac{(x-a)^{n}}{n!}\left\|B_n\right\|_{\infty,[0,1]} \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &\Bigg|f(x) - T_n^E(f;a,x) - \left(\frac{\gamma_{n+1}(x) + \Gamma_{n+1}(x)}{2}\right) \\ &\qquad \times (-1)^n\frac{(x-a)^{n+1}}{(n+1)!}\cdot\frac{4\left(4^n-1\right)B_{n+2}}{(n+2)}\Bigg| \\ &\leq \frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\frac{(x-a)^{n+1}}{n!}\int_0^1 |E_n(u)|\,du \\ &\leq \frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\frac{(x-a)^{n+1}}{n!}\left(\int_0^1 |E_n(u)|^p\,du\right)^{\frac{1}{p}}, \; 1 < p < \infty \\ &\leq \frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\frac{(x-a)^{n}}{n!}\left\|E_n\right\|_{\infty,[0,1]}. \end{aligned} \tag{3.24}$$

Proof. From (3.1) it may be seen that with $P_n(t) = P_n^B(t)$, as given by (1.8) and using (3.15), produces the left hand side of (3.23) without a perturbation.

Now, for the bounds. From (1.8), we have

$$\left\|P_n^B\right\|_{1,[a,x]} = \frac{(x-a)^n}{n!}\int_a^x \left|B_n\left(\frac{t-a}{x-a}\right)\right| dt = \frac{(x-a)^{n+1}}{n!}\int_0^1 \left|B_n(u)\right| du$$

producing the first bound in (3.23) on using the first result in (3.2).

For the second, we require

$$\begin{aligned}\left\|P_n^B\right\|_{p,[a,x]} &= \frac{(x-a)^n}{n!}\left(\int_a^x \left|B_n\left(\frac{t-a}{x-a}\right)\right|^p dt\right)^{\frac{1}{p}} \\ &= \frac{(x-a)^{n-\frac{1}{p}}}{n!}\left(\int_0^1 \left|B_n(u)\right|^p du\right)^{\frac{1}{p}},\end{aligned}$$

producing the stated result from (3.2).

Finally, from (3.2),

$$\left\|P_n^B\right\|_{\infty,[a,x]} = \frac{(x-a)^n}{n!} ess \sup_{u\in[0,1]} \left|B_n(u)\right|$$

as required.

The expressions (3.24) are obtained in a similar manner for $P_n(t) = P_n^E(t)$ as defined by (1.9).

For the perturbation we require

$$\begin{aligned}\frac{P_{n+1}^E(x) - P_{n+1}^E(a)}{n+1} &= \frac{(x-a)^{n+1}}{(n+1)!}\left[E_{n+1}(1) - E_{n+1}(0)\right] \\ &= \frac{(x-a)^{n+1}}{(n+1)!}\frac{4}{n+2}\left(4^n-1\right)B_{n+2},\end{aligned}$$

where we have used (3.20) for the final step. ∎

Remark 3. *The bounds in (3.23) and (3.24) involve the norms* $\left\|P_n^B\right\|_{p,[0,1]}$ *and* $\left\|P_n^E\right\|_{p,[0,1]}$. *These are difficult to obtain explicitly although they may be evaluated numerically if sufficient care is taken.*

To obtain explicit bounds then we require knowledge about the zeros of $B_n(x)$ and $E_n(x)$, $u \in [0,1]$. This is in general not known explicitly. The first inequalities in (3.23) and (3.24) are the sharpest, however, we may obtain bounds in terms of $\|B_n\|_{2,[0,1]}$ and $\|E_n\|_{2,[0,1]}$ explicitly (see also [10]).

Thus,

$$\left|f(x) - T_n^B(x)\right| \le \frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\cdot (x-a)^{n+1}\sqrt{\frac{|B_{2n}|}{(2n)!}}$$

and

$$\begin{aligned}&\left| f(x) - T_n^E(f;a,x) - \frac{\gamma_{n+1}(x) + \Gamma_{n+1}(x)}{2} \right.\\ &\qquad\qquad \left. \times (-1)^n\frac{(x-a)^{n+1}}{(n+1)!}\cdot 4\left(4^n-1\right)\frac{B_{n+2}}{(n+2)}\right| \\ &\qquad\qquad \le \frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\cdot (x-a)^{n+1}\cdot 2\sqrt{\frac{\left(4^{n+1}-1\right)|B_{2n+2}|}{(2n+2)!}}.\end{aligned}$$

4. Applications to the Logarithm

We shall apply the results of the previous sections to the logarithm function to illustrate the results.

Let $f:(0,\infty)\to\mathbb{R},\ f(t)=\ln t$ then

$$f^{(k)}(t)=(-1)^{k-1}\frac{(k-1)!}{t^k},\ t>0,\ k\in\mathbb{N}. \tag{4.1}$$

We note that $f^{(n+1)}$ is strictly monotonic on $(0,\infty)$ so that

$$\begin{aligned}&\frac{\gamma_{n+1}(x)\mp\Gamma_{n+1}(x)}{2}\\ &=\frac{1}{2}\left[\max\left\{f^{(n+1)}(a),f^{(n+1)}(x)\right\}\mp\min\left\{f^{(n+1)}(a),f^{(n+1)}(x)\right\}\right]\\ &=\frac{1}{2}\left|f^{(n+1)}(a)\mp f^{(n+1)}(x)\right|\\ &=\frac{n!}{2}\left(\frac{1}{a^{n+1}}\mp\frac{1}{x^{n+1}}\right),\quad\text{for}\quad x\geq a.\end{aligned} \tag{4.2}$$

We shall utilise the first inequality in (3.6), which is the least coarse of the three, to illustrate the results to give

$$\begin{aligned}&\Big|f(x)-T_n^{c_\lambda}(\ln;a,x)\\ &\qquad-\frac{1}{2(n+1)}\left[(\theta(\lambda)-a)^{n+1}+(-1)^n(x-\theta(\lambda))^{n+1}\right]\Big|\\ &\qquad\leq\frac{1}{2(n+1)}\left(\frac{1}{a^{n+1}}-\frac{1}{x^{n+1}}\right)\left[(\theta(\lambda)-a)^{n+1}+(x-\theta(\lambda))^{n+1}\right],\end{aligned} \tag{4.3}$$

where $\theta(\lambda)=\lambda a+(1-\lambda)x,\ \lambda\in[0,1]$, and from (1.10),

$$T_n^{c_\lambda}(\ln;a,x)=\ln a+\sum_{k=1}^{n}\frac{(x-a)^k}{k}\left[\left(\frac{\lambda}{x}\right)^k+(-1)^{k+1}\left(\frac{1-\lambda}{a}\right)^k\right].$$

Simplification of (4.1) gives

$$\begin{aligned}&\Bigg|\ln x-\ln a-\sum_{k=1}^{n}\frac{(x-a)^k}{k}\left[\left(\frac{\lambda}{x}\right)^k+(-1)^{k+1}\left(\frac{1-\lambda}{a}\right)^k\right]\\ &\qquad-\frac{(x-a)^{n+1}}{2(n+1)}\left[(1-\lambda)^{n+1}+\lambda^{n+1}\right]\left(\frac{1}{a^{n+1}}+\frac{1}{x^{n+1}}\right)\Bigg|\\ &\qquad\qquad\leq\frac{(x-a)^{n+1}}{2(n+1)}\left[(1-\lambda)^{n+1}+\lambda^{n+1}\right]\left(\frac{1}{a^{n+1}}-\frac{1}{x^{n+1}}\right).\end{aligned} \tag{4.4}$$

where the sharpest bound results from taking $\lambda=\frac{1}{2}$.

In particular, taking $\lambda=0$ gives

$$\begin{aligned}&\left|\ln x-\ln a-\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\left(\frac{x-a}{a}\right)^k-\frac{(x-a)^{n+1}}{2(n+1)}\left(\frac{1}{a^{n+1}}+\frac{1}{x^{n+1}}\right)\right|\\ &\qquad\qquad\leq\frac{(x-a)^{n+1}}{2(n+1)}\left(\frac{1}{a^{n+1}}-\frac{1}{x^{n+1}}\right):=R,\end{aligned} \tag{4.5}$$

which may be compared with the result (see case in Section 4 of Matić et al. [10])

$$\ln x = \ln a + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \left(\frac{x-a}{a}\right)^{k} + (-1)^{n} \frac{(x-a)^{n}}{n(n+1)} \left(\frac{1}{a^{n}} - \frac{1}{x^{n}}\right) + B, \tag{4.6}$$

where $|B| \le \frac{n}{\sqrt{2n+1}} R$.

It may be seen that the bound, R, obtained here, is much tighter, especially for large n since $\frac{n}{\sqrt{2n+1}} > 1$. It must be remembered, however, that a different perturbation is present in (4.5) than in (4.6). There is very little difference in complexity between the two perturbations. The perturbation used in [10] from the traditional Čebyšev functional (2.1) giving rise to (1.19) rather than the novel Čebyšev functional (2.3) producing (3.1). That is, the perturbation in (1.19) involves $\left[f^{(n+1)}; a, x\right] = \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a}$ whereas that in (3.1) contains $\frac{\gamma_{n+1}(x) + \Gamma_{n+1}(x)}{2}$, where $\gamma_{n+1}(x) \le f^{(n+1)}(t) \le \Gamma_{n+1}(x)$.

5. Concluding Remarks

A new Čebyšev-type functional has been introduced giving sharp bounds involving the upper and lower bounds for one of the functions. The results have been applied to obtain perturbed generalised Taylor series together with sharp bounds of the approximations. The approximation of the logarithmic function is given as an example.

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