A REFINEMENT OF THE GRÜSS INEQUALITY AND APPLICATIONS

P. CERONE AND S.S. DRAGOMIR

Abstract. A sharp refinement of the Grüss inequality in the general setting of measurable spaces and abstract Lebesgue integrals is proven. Some consequential particular inequalities are mentioned.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for $\mu-$ a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is $\mu-$measurable and $\int_\Omega w(x) |f(x)| d\mu(x) < \infty\}$. Assume $\int_\Omega w(x) d\mu(x) > 0$.

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the Čebyšev functional

$$T_w(f,g) := \frac{1}{\int_\Omega w(x) d\mu(x)} \int_\Omega w(x) f(x) g(x) d\mu(x) - \frac{1}{\int_\Omega w(x) d\mu(x)} \int_\Omega w(x) f(x) d\mu(x) \times \frac{1}{\int_\Omega w(x) d\mu(x)} \int_\Omega w(x) g(x) d\mu(x). \tag{1.1}$$

The following result is known in the literature as the Grüss inequality

$$|T_w(f,g)| \leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta), \tag{1.2}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.3}$$

for $\mu-$ a.e. $x \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Note that if $\Omega = \{1, \ldots, n\}$ and μ is the discrete measure on Ω, then we obtain the discrete Grüss inequality

$$\left| \frac{1}{W_n} \sum_{i=1}^n w_i x_i y_i - \frac{1}{W_n} \sum_{i=1}^n w_i x_i \cdot \frac{1}{W_n} \sum_{i=1}^n w_i y_i \right| \leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta), \tag{1.4}$$

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provided $\gamma \leq x_i \leq \Gamma$, $\delta \leq y_i \leq \Delta$ for each $i \in \{1, \ldots, n\}$ and $w_i \geq 0$ with $W_n := \sum_{i=1}^{n} w_i > 0$.

The following result was proved in Cheng and Sun [4].

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\delta \leq g(x) \leq \Delta$ for some constants δ, Δ for all $x \in [a, b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx\right| \leq \frac{\Delta - \delta}{2}\int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy\right| dx. \tag{1.5}$$

They used the result (1.5) to obtain perturbed trapezoidal rules.

In the current paper we obtain bounds for $|T_w(f, g)|$ under the general setting expressed in (1.1). A bound which is shown to be *sharp* is obtained in Section 2. The sharpness of (1.5) was not demonstrated in [4]. Sharp results were obtained for a perturbed interior point rule (Ostrowski-Grüss) inequalities in Cheng [3]. Some particular instances of the results in Section 2 are investigated in Sections 4 and 5, recapturing earlier work. Results are presented in Section 3, for Lebesgue measurable functions and for a discrete weighted Čebyšev functional involving $n-$tuples.

2. An Integral Inequality

With the assumptions as presented in the Introduction and if $f \in L_w(\Omega, \mu)$ then we may define

$$\begin{aligned} D_w(f) &:= D_{w,1}(f) \\ &:= \frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x) \\ &\quad \times \left|f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega w(y)\,f(y)\,d\mu(y)\right| d\mu(x). \end{aligned} \tag{2.1}$$

The following fundamental result holds.

Theorem 2. *Let $w, f, g : \Omega \to \mathbb{R}$ be $\mu-$measurable functions with $w \geq 0$ $\mu-$ a.e. on Ω and $\int_\Omega w(y)\,d\mu(y) > 0$. If $f, g, fg \in L_w(\Omega, \mu)$ and there exists the constants δ, Δ such that*

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \quad \text{for} \quad \mu - a.e.\ x \in \Omega, \tag{2.2}$$

then we have the inequality

$$|T_w(f, g)| \leq \frac{1}{2}(\Delta - \delta)\,D_w(f). \tag{2.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Proof. Obviously, we have

$$\begin{aligned} T_w(f, g) &= \frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x) \\ &\quad \times \left(f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right) g(x)\,d\mu(x). \end{aligned} \tag{2.4}$$

Consider the measurable subsets Ω_+ and Ω_-, of Ω, defined by

$$\Omega_+ := \left\{ x \in \Omega \,\middle|\, f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega w(y) f(y)\,d\mu(y) \geq 0 \right\}$$

and

$$\Omega_- := \left\{ x \in \Omega \,\middle|\, f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega w(y) f(y)\,d\mu(y) < 0 \right\}.$$

Obviously, $\Omega = \Omega_+ \cup \Omega_-$, $\Omega_+ \cap \Omega_- = \emptyset$ and if we define

$$\begin{aligned} I_+(f,g,w) &:= \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_{\Omega_+} w(x) \\ &\quad \times \left(f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega w(y) f(y)\,d\mu(y) \right) g(x)\,d\mu(x) \end{aligned}$$

and

$$\begin{aligned} I_-(f,g,w) &:= \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_{\Omega_-} w(x) \\ &\quad \times \left(f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega w(y) f(y)\,d\mu(y) \right) g(x)\,d\mu(x) \end{aligned}$$

then we have

$$T_w(f,g) = I_+(f,g,w) + I_-(f,g,w). \tag{2.5}$$

Since $-\infty < \delta \leq g(x) \leq \Delta < \infty$ for $\mu-$ a.e. $x \in \Omega$ and $w(x) \geq 0$ for $\mu-$ a.e. $x \in \Omega$, we may write:

$$\begin{aligned} I_+(f,g,w) &\leq \frac{\Delta}{\int_\Omega w(y)\,d\mu(y)} \int_{\Omega_+} w(x) \\ &\quad \times \left(f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega f(y) w(y)\,d\mu(y) \right) d\mu(x) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} I_-(f,g,w) &\leq \frac{\delta}{\int_\Omega w(y)\,d\mu(y)} \int_{\Omega_-} w(x) \\ &\quad \times \left(f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega f(y) w(y)\,d\mu(y) \right) d\mu(x). \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} 0 &= \int_\Omega w(x) \left(f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega f(y) w(y)\,d\mu(y) \right) d\mu(x) \\ &= \int_{\Omega_+} w(x) \left(f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega f(y) w(y)\,d\mu(y) \right) d\mu(x) \\ &\quad + \int_{\Omega_-} w(x) \left(f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega f(y) w(y)\,d\mu(y) \right) d\mu(x) \end{aligned}$$

we get

$$\int_{\Omega_-} w(x)\left(f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right)d\mu(x)$$
$$=-\int_{\Omega_+} w(x)\left(f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right)d\mu(x)$$

and thus, from (2.7), we deduce

$$I_-(f,g,w)\le\frac{-\delta}{\int_\Omega w(y)\,d\mu(y)}\int_{\Omega_+} w(x)\times\left(f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega w(y)\,f(y)\,d\mu(y)\right)d\mu(x). \tag{2.8}$$

Consequently, by adding (2.6) with (2.8), we deduce

$$T_w(f,g)\le\frac{\Delta-\delta}{\int_\Omega w(y)\,d\mu(y)}\int_{\Omega_+} w(x)\times\left(f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega w(y)\,f(y)\,d\mu(y)\right)d\mu(x). \tag{2.9}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega w(x)\left|f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right|d\mu(x)\\
=&\int_{\Omega_+} w(x)\left|f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right|d\mu(x)\\
&+\int_{\Omega_-} w(x)\left|f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right|d\mu(x)\\
=&\int_{\Omega_+} w(x)\left(f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right)d\mu(x)\\
&-\int_{\Omega_-} w(x)\left(f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right)d\mu(x)\\
=&\;2\int_{\Omega_+} w(x)\left(f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right)d\mu(x),
\end{aligned}$$

and thus, by (2.9) we deduce

$$T_w(f,g)\le\frac{1}{2}(\Delta-\delta)\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega w(x)\times\left|f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right|d\mu(x). \tag{2.10}$$

Now, if we write the inequality (2.10) for $-f$ instead of f and taking into account that $T_w(-f,g)=-T_w(f,g)$, we deduce

$$-T(f,g)\le\frac{1}{2}(\Delta-\delta)\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega w(x)\times\left|f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega f(y)\,w(y)\,d\mu(y)\right|d\mu(x), \tag{2.11}$$

giving the desired inequality (2.3).

To prove the sharpness of the constant $\frac{1}{2}$, assume that (2.3) holds for $\Omega=[a,b]$ and $w\equiv 1$, with a constant $C>0$. That is,

$$|T(f,g)|\le C(\Delta-\delta)\frac{1}{b-a}\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(y)\,dy\right|dx, \tag{2.12}$$

where

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx$$

and the integral $\int_a^b$ is the usual Lebesgue integral on $[a,b]$.

Choose in (2.12) $g=f$ and $f:[a,b]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}-1 & \text{if } x\in\left[a,\frac{a+b}{2}\right],\\ 1 & \text{if } x\in\left(\frac{a+b}{2},b\right],\end{cases}$$

then, obviously,

$$T(f,f)=\frac{1}{b-a}\int_a^b f^2(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2=1,$$
$$D(f)=\frac{1}{b-a}\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(y)\,dy\right|dx=1,$$
$$\delta=-1,\quad \Delta=1,$$

and by (2.12) we get $2C\ge 1$ giving $C\ge\frac{1}{2}$. ∎

For $f\in L_{p,w}(\Omega,\mathcal{A},\mu):=\left\{f:\Omega\to\mathbb{R},\int_\Omega w(x)|f(x)|^p\,d\mu(x)<\infty\right\}$, $p\ge 1$ we may also define

$$\begin{aligned}D_{w,p}(f)&:=\left[\frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x)\right.\\ &\qquad\left.\times\left|f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega w(y)\,f(y)\,d\mu(y)\right|^p d\mu(x)\right]^{\frac{1}{p}}\\ &=\frac{\left\|f-\frac{1}{\int_\Omega w\,d\mu}\int_\Omega wf\,d\mu\right\|_{\Omega,p}}{\left[\int_\Omega w(x)\,d\mu(x)\right]^{\frac{1}{p}}}\end{aligned} \tag{2.13}$$

where $\|\cdot\|_{\Omega,p}$ is the usual $p-$norm on $L_{p,w}(\Omega,\mathcal{A},\mu)$, namely,

$$\|h\|_{\Omega,p}:=\left(\int_\Omega w|h|^p\,d\mu\right)^{\frac{1}{p}},\quad p\ge 1.$$

Using Hölder's inequality we get

$$D_{w,1}(f)\le D_{w,p}(f)\quad\text{for } p\ge 1,\ f\in L_{p,w}(\Omega,\mathcal{A},\mu); \tag{2.14}$$

and, in particular for $p=2$

$$D_{w,1}(f)\le D_{w,2}(f)=\left[\frac{\int_\Omega wf^2\,d\mu}{\int_\Omega w\,d\mu}-\left(\frac{\int_\Omega wf\,d\mu}{\int_\Omega w\,d\mu}\right)^2\right]^{\frac{1}{2}}, \tag{2.15}$$

if $f \in L_{2,w}(\Omega, \mathcal{A}, \mu)$.

For $f \in L_\infty(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{R},\ \|f\|_{\Omega,\infty} := ess \sup_{\alpha\in\Omega} |f(x)| < \infty \right\}$ we also have

$$D_{w,p}(f) \leq D_{w,\infty}(f) := \left\| f - \frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right\|_{\Omega,\infty}. \tag{2.16}$$

The following corollary may be useful in practice.

Corollary 1. *With the assumptions of Theorem 2, we have*

$$\begin{aligned} &|T_w(f,g)| \\ &\leq \frac{1}{2}(\Delta - \delta) D_w(f) \\ &\leq \frac{1}{2}(\Delta - \delta) D_{w,p}(f) \quad \textit{if } f \in L_p(\Omega, \mathcal{A}, \mu),\ 1 < p < \infty; \\ &\leq \frac{1}{2}(\Delta - \delta) \left\| f - \frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right\|_{\Omega,\infty} \quad \textit{if } f \in L_\infty(\Omega, \mathcal{A}, \mu). \end{aligned} \tag{2.17}$$

Remark 1. *The inequalities in (2.17) are in order of increasing coarseness. If we assume that $-\infty < \gamma \leq f(x) \leq \Gamma < \infty$ for μ – a.e. $x \in \Omega$, then by the Grüss inequality for $g = f$ we have for $p = 2$*

$$\left[\frac{\int_\Omega w f^2 d\mu}{\int_\Omega w d\mu} - \left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu} \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2}(\Gamma - \gamma). \tag{2.18}$$

By (2.17), we deduce the following sequence of inequalities

$$\begin{aligned} |T_w(f,g)| &\leq \frac{1}{2}(\Delta - \delta) \frac{1}{\int_\Omega w d\mu} \int_\Omega w \left| f - \frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right| d\mu \\ &\leq \frac{1}{2}(\Delta - \delta) \left[\frac{\int_\Omega w f^2 d\mu}{\int_\Omega w d\mu} - \left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu} \right)^2 \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Delta - \delta)(\Gamma - \gamma) \end{aligned} \tag{2.19}$$

for $f, g : \Omega \to \mathbb{R}$, μ – measurable functions and so that $-\infty < \gamma \leq f(x) < \Gamma < \infty$, $-\infty < \delta \leq g(x) \leq \Delta < \infty$ for μ – a.e. $x \in \Omega$. Thus, the inequality (2.19) is a refinement of Grüss' inequality (1.2).

It is well known that if $f \in L_{2,w}(\Omega, \mathcal{A}, \mu)$, then the following Schwartz's type inequality holds:

$$\frac{1}{\int_\Omega w d\mu} \int_\Omega w f^2 d\mu \geq \left(\frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right)^2. \tag{2.20}$$

Using the above results, we may point out the following counterpart result.

Proposition 1. *Assume that the $\mu-$measurable function $f : \Omega \to \mathbb{R}$ satisfies the assumption:*

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty \quad \textit{for} \quad \textit{a.e. } x \in \Omega. \tag{2.21}$$

Then one has the inequality

$$0 \leq \frac{1}{\int_\Omega w d\mu} \int_\Omega w f^2 d\mu - \left(\frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right)^2 \tag{2.22}$$
$$\leq \frac{1}{2} (\Gamma - \gamma) \frac{1}{\int_\Omega w d\mu} \int_\Omega w \left| f - \frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right| d\mu$$
$$\left(\leq \frac{1}{4} (\Gamma - \gamma)^2 \right).$$

The constant $\frac{1}{2}$ *is sharp.*

The proof follows by the inequality (2.3) for $g = f$.

The following proposition also holds.

Proposition 2. *Assume that the measurable functions* $f, g : \Omega \to \mathbb{R}$ *satisfy (1.3) (the condition in Grüss' inequality). Then*

$$|T_w (f, g)| \leq \frac{1}{2} [(\Gamma - \gamma) (\Delta - \delta)]^{\frac{1}{2}} [D_w (f) D_w (g)]^{\frac{1}{2}} \tag{2.23}$$
$$\leq \frac{1}{4} (\Delta - \delta) (\Gamma - \gamma).$$

The constant $\frac{1}{2}$ *in the first inequality and* $\frac{1}{4}$ *in the second inequality are sharp.*

Proof. By (2.19) we have

$$|T_w (f, g)| \leq \frac{1}{2} (\Delta - \delta) D_w (f)$$

and

$$|T_w (f, g)| \leq \frac{1}{2} (\Gamma - \gamma) D_w (g)$$

from which, by multiplication, gives the first part of (2.23).

The second part and the sharpness of the constants are obvious. ∎

3. Some Particular Inequalities

The following particular inequalities are of interest.

1. Let $w, f, g : [a, b] \to \mathbb{R}$ be Lebesgue measurable functions with $w \geq 0$ a.e. on $[a, b]$ and $\int_a^b w(y)\, dy > 0$. If $f, g, fg \in L_w [a, b]$, where

$$L_w [a, b] := \left\{ f : [a, b] \to \mathbb{R} \,\middle|\, \int_a^b w(x) |f(x)| dx < \infty \right\}$$

and

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \text{ for a.e. } x \in [a, b], \tag{3.1}$$

then we have the inequalities

$$\left| \frac{1}{\int_a^b w(x) dx} \int_a^b w(x) f(x) g(x) dx \right. \tag{3.2}$$
$$\left. - \frac{1}{\int_a^b w(x) dx} \int_a^b w(x) f(x) dx \cdot \frac{1}{\int_a^b w(x) dx} \int_a^b w(x) g(x) dx \right|$$

$$\leq \frac{1}{2}(\Delta-\delta)\frac{1}{\int_a^b w(x)\,dx}\int_a^b w(x)\left|f(x)-\frac{1}{\int_a^b w(y)\,dy}\int_a^b w(y)f(y)\,dy\right|dx$$

$$\leq \frac{1}{2}(\Delta-\delta)\left[\frac{\int_a^b w(x)\left|f(x)-\frac{1}{\int_a^b w(y)dy}\int_a^b w(y)f(y)\,dy\right|^p dx}{\int_a^b w(x)\,dx}\right]^{\frac{1}{p}}$$

if $f\in L_{p,w}[a,b]$, $1<p<\infty$,

$$\leq \frac{1}{2}(\Delta-\delta)\,ess\sup_{x\in[a,b]}\left|f(x)-\frac{1}{\int_a^b w(y)\,dy}\int_a^b w(y)f(y)\,dy\right| \quad \text{if } f\in L_\infty[a,b].$$

The constant $\frac{1}{2}$ is sharp in the first inequality in (3.2).

The following counterpart of Schwartz's inequality holds

$$0\leq \frac{1}{\int_a^b w(y)\,dy}\int_a^b w(x)f^2(x)\,dx-\left(\frac{1}{\int_a^b w(y)\,dy}\int_a^b w(x)f(x)\,dx\right)^2 \tag{3.3}$$

$$\leq \frac{1}{2}(\Delta-\gamma)\frac{1}{\int_a^b w(y)\,dy}\int_a^b w(x)\left|f(x)-\frac{1}{\int_a^b w(y)\,dy}\int_a^b w(y)f(y)\,dy\right|dx$$

$$\left(\leq \frac{1}{4}(\Gamma-\gamma)^2\right),$$

provided $-\infty<\gamma\leq f(x)\leq\Gamma<\infty$ for a.e. $x\in[a,b]$. The constant $\frac{1}{2}$ is sharp.

If $w(x)=1$, $x\in[a,b]$, then we recapture the result in [4] as depicted here by (1.5).

2. Let $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$, $\bar{\mathbf{p}}=(p_1,\dots,p_n)$ be $n-$tuples of real numbers with $p_i\geq 0$ $(i\in\{1,\dots,n\})$ and $\sum_{i=1}^n p_i=1$. If

$$b\leq b_i\leq B,\qquad i\in\{1,\dots,n\}, \tag{3.4}$$

then one has the inequality

$$\left|\sum_{i=1}^n p_i a_i b_i-\sum_{i=1}^n p_i a_i\cdot\sum_{i=1}^n p_i b_i\right| \tag{3.5}$$

$$\leq \frac{1}{2}(B-b)\sum_{i=1}^n p_i\left|a_i-\sum_{j=1}^n p_j a_j\right|$$

$$\leq \frac{1}{2}(B-b)\left[\sum_{i=1}^n p_i\left|a_i-\sum_{j=1}^n p_j a_j\right|^p\right]^{\frac{1}{p}} \quad \text{if } 1<p<\infty$$

$$\leq \frac{1}{2}(B-b)\max_{i=\overline{1,n}}\left|a_i-\sum_{j=1}^n p_j a_j\right|.$$

The constant $\frac{1}{2}$ is sharp in the first inequality.

If $p_i = 1,\ i \in \{1, \dots n\}$, the following unweighted inequality may be stated

$$0 \le \frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} b_i \tag{3.6}$$
$$\le \frac{1}{2}(B-b)\frac{1}{n}\sum_{i=1}^{n}\left|a_i - \frac{1}{n}\sum_{j=1}^{n} a_j\right|$$
$$\le \frac{1}{2}(B-b)\left(\frac{1}{n}\sum_{i=1}^{n}\left|a_i - \frac{1}{n}\sum_{j=1}^{n} a_j\right|^p\right)^{\frac{1}{p}}$$
$$\le \frac{1}{2}(B-b)\max_{i=\overline{1,n}}\left|a_i - \frac{1}{n}\sum_{j=1}^{n} a_j\right|.$$

The following counterpart of Schwartz's inequality also holds

$$0 \le \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i\right)^2 \le \frac{1}{2}(A-a)\sum_{i=1}^{n} p_i\left|a_i - \sum_{j=1}^{n} p_j a_j\right| \tag{3.7}$$
$$\left(\le \frac{1}{4}(A-a)^2\right),$$

provided $a \le a_i \le A$ for each $i \in \{1, \dots, n\}$ and $\sum_{i=1}^{n} p_i = 1.$. The constant $\frac{1}{2}$ is sharp.

4. Applications for Ostrowski's Inequality

If $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ such that $\varphi' \in L_\infty [a,b]$, then the following inequality is known in the literature as Ostrowski's inequality

$$\left|\varphi(x) - \frac{1}{b-a}\int_a^b \varphi(t)\,dt\right| \tag{4.1}$$
$$\le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]\|\varphi'\|_\infty (b-a), \quad x \in [a,b],$$

where $\|\varphi'\|_\infty := ess \sup_{\alpha \in [a,b]} |\varphi'(x)|$. The constant $\frac{1}{4}$ is best possible.

A simple proof of this fact, as mentioned in [1], may be accomplished by the use of the Montgomery identity

$$\varphi(x) = \frac{1}{b-a}\int_a^b \varphi(t)\,dt + \frac{1}{b-a}\int_a^b K(x,t)\,\varphi'(t)\,dt, \tag{4.2}$$

where the kernel $K : [a,b]^2 \to \mathbb{R}$ is defined by

$$K(x,t) := \begin{cases} t-a & \text{if } a \le t \le x \\ \\ t-b & \text{if } a \le x < t \le b. \end{cases} \tag{4.3}$$

We will now use the unweighted version of the inequality (3.2), namely, (1.5) (obtained by Cheng and Sun [4]) to procure the next result concerning a perturbed version of Ostrowski's inequality (4.1).

The following result also obtained by Cheng [3] is recaptured in a simpler manner. A weighted version of this result was obtained by Roumeliotis [5].

Theorem 3. *Assume that $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ such that $\varphi' : [a,b] \to \mathbb{R}$ satisfies the condition*

$$-\infty < \gamma \leq \varphi'(x) \leq \Gamma < \infty \quad \text{for a.e. } x \in [a,b]. \tag{4.4}$$

Then we have the inequality

$$\left| \varphi(x) - \frac{1}{b-a} \int_a^b \varphi(t)\, dt - \left(x - \frac{a+b}{2} \right) [\varphi; a, b] \right| \leq \frac{1}{8} (b-a)(\Gamma - \gamma) \tag{4.5}$$

for any $x \in [a,b]$, where $[\varphi; a, b] = \frac{\varphi(b)-\varphi(a)}{b-a}$ is the divided difference. The constant $\frac{1}{8}$ is best possible.

Proof. We apply inequality (3.1) for the choices $w(t) = 1$, $f(t) = K(x,t)$ defined by (4.3), $g(t) = \varphi'(t)$, $t \in [a,b]$ to get

$$\left| \frac{1}{b-a} \int_a^b K(x,t) \varphi'(t)\, dt - \frac{1}{b-a} \int_a^b K(x,t)\, dt \cdot \frac{1}{b-a} \int_a^b \varphi'(t)\, dt \right| \tag{4.6}$$
$$\leq \frac{1}{2} (\Gamma - \gamma) \cdot \frac{1}{b-a} \int_a^b \left| K(x,t) - \frac{1}{b-a} \int_a^b K(x,s)\, ds \right| dt.$$

We obviously have,

$$\frac{1}{b-a} \int_a^b K(x,t)\, dt = x - \frac{a+b}{2}$$

and

$$\frac{1}{b-a} \int_a^b \varphi'(t)\, dt = \frac{\varphi(b) - \varphi(a)}{b-a}.$$

Also

$$\begin{aligned} I(x) &:= \frac{1}{b-a} \int_a^b \left| K(x,t) - \left(x - \frac{a+b}{2} \right) \right| dt \\ &= \frac{1}{b-a} \left[\int_a^x \left| t - a - x + \frac{a+b}{2} \right| dt + \int_x^b \left| t - b - x + \frac{a+b}{2} \right| dt \right] \\ &= \frac{1}{b-a} \left[\int_a^x \left| t - x + \frac{b-a}{2} \right| dt + \int_x^b \left| t - x - \frac{b-a}{2} \right| dt \right]. \end{aligned}$$

Straight forward substitution of $u = t - x + \frac{b-a}{2}$ and $v = t - x - \frac{b-a}{2}$ gives

$$\begin{aligned} I(x) &= \frac{1}{b-a} \left[\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}} |u|\, du + \int_{-\frac{b-a}{2}}^{\frac{a+b}{2}-x} |v|\, dv \right] \\ &= \frac{1}{b-a} \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} |u|\, du = \frac{2}{b-a} \int_0^{\frac{b-a}{2}} u du = \frac{b-a}{4}. \end{aligned}$$

Substitution of the above into (4.6) produces (4.5). The sharpness of the constant was proved in [3]. ∎

5. Application for the Generalised Trapezoid Inequality

If $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ so that $\varphi' \in L_\infty [a,b]$, then the following inequality is known as the generalised trapezoid inequality

$$\left|(x-a)\varphi(a) + (b-x)\varphi(b) - \int_a^b \varphi(t)\,dt\right| \leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|\varphi'\|_\infty \tag{5.1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible.

A simple proof of this fact is accomplished by using the identity [2]

$$\int_a^b \varphi(t)\,dt = (x-a)\varphi(a) + (b-x)\varphi(b) + \int_a^b (x-t)\varphi'(t)\,dt. \tag{5.2}$$

Utilising the inequality (3.1) we may point out the following perturbed version of (5.1).

Theorem 4. *Assume that $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ so that $\varphi' : [a,b] \to \mathbb{R}$ satisfies the condition (4.4). Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b \varphi(t)\,dt - \left[\left(\frac{x-a}{b-a}\right)\varphi(a) + \left(\frac{b-x}{b-a}\right)\varphi(b)\right] - \left(x - \frac{a+b}{2}\right)[\varphi; a, b]\right| \leq \frac{1}{8}(b-a)(\Gamma - \gamma) \tag{5.3}$$

for any $x \in [a,b]$, where $[\varphi; a, b]$ is the divided difference. The constant $\frac{1}{8}$ is sharp.

Proof. We apply inequality (3.2) for the choices $f(t) = (x-t)$, $g(t) = \varphi'(t)$, $w(t) = 1$, $t \in [a,b]$, to get

$$\left|\frac{1}{b-a}\int_a^b (x-t)\varphi'(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)\,dt \cdot \frac{1}{b-a}\int_a^b \varphi'(t)\,dt\right| \leq \frac{1}{2}(\Gamma - \gamma)\frac{1}{b-a}\int_a^b \left|(x-t) - \frac{1}{b-a}\int_a^b (x-s)\,ds\right| dt. \tag{5.4}$$

Since

$$\frac{1}{b-a}\int_a^b (x-t)\,dt = \left(x - \frac{a+b}{2}\right),$$
$$\frac{1}{b-a}\int_a^b \varphi'(t)\,dt = \frac{\varphi(b) - \varphi(a)}{b-a} = [\varphi; a, b]$$

and

$$\frac{1}{b-a}\int_a^b \left|(x-t)-\frac{1}{b-a}\int_a^b (x-s)\,ds\right| dt = \frac{1}{b-a}\int_a^b \left|x-t-x+\frac{a+b}{2}\right| dt$$
$$= \frac{1}{b-a}\int_a^b \left|t-\frac{a+b}{2}\right| dt$$
$$= \frac{b-a}{4},$$

from (5.4) we deduce the desired inequality (5.3).

The sharpness of the constant may be shown on choosing $t = \frac{a+b}{2}$ and $\varphi(t) = \left|t-\frac{a+b}{2}\right|$, $t \in [a,b]$. We omit the details. ∎

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, MCMC 8001, Victoria, Australia

E-mail address: pc@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/cerone

E-mail address: sever@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html