A Note on the Entropy Inequality

J. Rooin, M. Tahghighi, and H. Teimoori *

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Abstract

In this note we generalize the Entropy inequality for the case of $n \geq 2$.

There are two gaps in the previous work concerning the Entropy inequality [1]. First, they didn't generalize that inequality for the case $n > 2$, and it seems that their method doesn't easily work in this case. The second point is that they didn't consider the equality cases. The aim of this article is the generalization of the Entropy inequality for $n \geq 2$, as follows:

$$1 \leq \sum_{i=1}^{n} \frac{p_i - 1}{\ln p_i} \leq \frac{n-1}{\ln n}, \tag{1}$$

in which p_i $(1 \leq i \leq n)$ are nonnegative real numbers with $\sum_{i=1}^{n} p_i = 1$. We also consider the equality cases in the above inequalities (we define $\frac{p_i - 1}{\ln p_i}$ for the special cases $p_i = 0$ and $p_i = 1$ as $\lim_{p_i \to 0} \frac{p_i - 1}{\ln p_i} = 0$ and $\lim_{p_i \to 1} \frac{p_i - 1}{\ln p_i} = 1$, respectively).

Proof of the inequalities in (1). We start by defining the function f as following:

$$f(x) = \begin{cases} \frac{x-1}{\ln x} & 0 < x < 1, \\ 0 & x = 0, \\ 1 & x = 1. \end{cases} \tag{2}$$

Considering [1], we observe that f is an strictly concave function. Now, for proving the left-hand side inequality in (1) we proceed as follows:

$$f(p_i) \quad = \quad f(p_1.0 + \cdots + p_i.1 + \cdots + p_n.0)$$

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\begin{aligned}
&\geq p_1 f(0) + \cdots + p_i f(1) + \cdots + p_n f(0) \\
&= p_i f(1) = p_i \qquad\qquad (1 \leq i \leq n).
\end{aligned}
$$

Thus, we conclude that

$$\sum_{i=1}^{n} f(p_i) \geq \sum_{i=1}^{n} p_i = 1,$$

or equivalently

$$1 \leq \sum_{i=1}^{n} \frac{p_i - 1}{\ln p_i}.$$

The equality holds iff one of $p_i s$ is equal to one (therefore the rest of them are equal to zero). Now, we are at the position to prove the right-hand side inequality as follows:

$$\sum_{i=1}^{n} \frac{1}{n} f(p_i) \leq f\left(\frac{\sum_{i=1}^{n} p_i}{n}\right) = f\left(\frac{1}{n}\right) = \frac{\frac{1}{n} - 1}{\ln \frac{1}{n}} = \frac{n-1}{n \ln n},$$

and therefore $\sum_{i=1}^{n} \frac{p_i - 1}{\ln p_i} \leq \frac{n-1}{\ln n}$. Equality holds iff $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$. This completes our proof.

REFERENCES

1. M. Bahramgiri and O. Naghshineh Arjomand, "A Simple Proof Of The Entropy Inequality, Revisited", RGMIA Research Report Collection, **3**(4), Article 12, 2000.

Institute for Advanced Studies in Basic Sciences,
P.O. Box 45195-159,
Gaveh-Zang, Zanjan,
IRAN.
Email: Rooin@iasbs.ac.ir