ON A PAPER OF MIHALY BENCZE ON THE SEIFFERT MEAN

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Abstract. In this short note, the author shows that among the 65 inequalities presented in [1], some are trivial or known, or follow from each other. The author also proves some inequalities to be incorrect.

1. Introduction

Let x, y be positive real numbers. The logarithmic and identric means of x and y are defined by

$$L = L(x,y) = \frac{x-y}{\log x - \log y}, \quad (x \neq y); \quad L(x,x) = x;$$

and

$$I = I(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}} \qquad (x \neq y); \quad I(x,x) = x$$

respectively.

Let $A = A(x,y) = \frac{x+y}{2}$, $G = G(x,y) = \sqrt{xy}$ respectively denote the arithmetic and geometric means of x and y.

In 1993, H.J. Seiffert [4] introduced the mean $P = P(x,y) = \frac{x-y}{4\arctan\left(\sqrt{\frac{x}{y}}\right)-\pi}$ $(x \neq y)$; $P(x,x,) = x$. The mean P can be written also in the equivalent form $P(x,y) = \frac{x-y}{2\arcsin\frac{x-y}{x+y}}$ $(x \neq y)$. Recently, the author (see [2] and [3]) has discovered that P is the limit of an algorithm introduced by Pfaff, and using this algorithm deduced certain inequalities which improve the earlier results, and in fact the best possible relations are obtainable. Some of them were discovered by Bencze [1], who in a long paper proved 65 inequalities for this mean.

The aim of this note is to show that among these 65 inequalities some are trivial or known (see [5], [2]) or some of them follow each other. We have found also inequalities which are not correct.

2. Findings

2.1. **Identical results.** First note certain known inequalities.

(1) Relation (1.6), i.e., $P^3 > A^2G$ appears in our paper [2] and [3] (as the left side of relation (20) there).
(2) The left side of (1.16) is the left side of relation (17) in [3]: $P > \frac{A+G}{2}$ (see also [2]).
(3) Inequality (1.15) is due to Seiffert [5], for improvements, see [2, 3].

Key words and phrases. Seiffert mean.

(4) The left side of (1.61) written in the form $P > \left[A \cdot \left(\frac{A+G}{2}\right)^2\right]^{\frac{1}{3}}$ is relation (23) of our paper [3] (see also [2]). In fact, much stronger inequalities are obtainable by the sequential method.

2.2. **Trivial results.** Some trivial inequalities are the following:

(1) (1.4) is equivalent to $G < P < A$, as can be easily seen;
(2) (1.7), i.e., $P > \frac{3}{\pi}G$ follows from $P > G$, since $1 > \frac{3}{\pi}$;
(3) The right side of (1.16) can be written as $P < A + G$, which is trivial, by $P < A$;
(4) Since $\frac{G^2}{A} = H$ – harmonic mean, by $P > G > H$, (1.11) is trivial;
(5) (1.13) follows from (1.1), since $L > G$. In fact (1.6) implies (1.13) since $P^3 > A^2G \Longrightarrow P^2 > AG$ by $A > G$;
(6) The left side of (1.3) implies (1.11). Indeed, $\frac{3G}{A+2G} > \frac{G^2}{A^2}$ is equivalent to $3A^2 > 2G^2 + GA$ which is trivial by $A > G$.
(7) The left side of (1.3) implies also (1.5). Indeed, $\frac{3G}{A+2G} > \frac{2G}{2A+G}$ is equivalent to $4A > G$.
(8) Relation (1.14) is implied by $P > \sqrt{AG} > \frac{2}{\left(\frac{1}{A}+\frac{1}{G}\right)}$, i.e. by inequality (1.13) (which in turn follows from (1.6)).
(9) The right side of (1.17), written in the form $P^2 > G\sqrt{A^2 - G^2}$ follows at once from (1.13), since $P^2 > GA > G\sqrt{A^2 - G^2}$.
(10) Relation (1.19) can be written also as $\frac{A-G}{\ln A - \ln G} < P$, or $L(A, G) < P$. This is immediate, since $L(A, G) < \frac{A+G}{2}$ and $\frac{A+G}{2} < P$ by the left side of (1.16).
(11) For relation (1.39) remark that, written in the form $3P^2(P - G) > G \cdot \left(A^2 - G^2\right)$, it follows from (1.13) and the left side of (1.17)
(12) Since $P < A$, by $\frac{A+P}{2} < A$ it is immediate that (1.40) implies (1.38).
(13) Inequality (1.41) written in the form $\frac{3}{P} < \frac{2}{A} + \frac{1}{G}$ is the same as inequality (1.15).
(14) Now, by (1.6) one has $\frac{P^2}{A^2} > \frac{G}{P}$, so $\frac{P^2}{A^2} + \frac{P}{G} > \frac{G}{P} + \frac{P}{G} > 2$, improving relation (1.42).
(15) Inequality (1.50) is the same as the left side of (1.16)
(16) Inequality (1.54) is the same as (1.6).
(17) Inequality (1.58) is trivial, since $\frac{2}{P} > \frac{2}{A} > \frac{1}{A} + \frac{1}{A+G}$.

2.3. **Incorrect Inequalities.** Among the 65 inequalities, the following simple relations do not occur:

(1)

$$P < \frac{G + 2A}{3} \quad \text{(see [2] or [3], right side of relation (20)).} \tag{*}$$

We prove that this implies the right side of (1.3): $\frac{P}{A} < \frac{3A}{4A-G}$. Indeed, $\frac{G+2A}{3} < \frac{3A^2}{4A-G}$ is equivalent to $A^2 + G^2 > 2AG$, which is trivial.

A stronger inequality that the left side of (1.3) is

$$P^3 > \left(\frac{A + G}{2}\right)^2 A \quad \text{(see [2] or [3], relation (23)).} \tag{**}$$

We will prove that (see (**)) $\left(\frac{A+G}{2}\right)^2 A > \frac{27A^3G^3}{(A+2G)^3}$. Let $\frac{A}{G} = t$, so the above inequality is equivalent to $(t + 1)^2 (t + 2)^3 > 108t^2$, where $t > 1$.

By considering $P(t) = (t+1)^2 (t+2)^3 - 108t^2$ for $t > 1$, and using e.g. derivatives, easily follows $P(t) \geq 0$ with equality only for $t = 1$.

In the same manner, (*) improves relation (1.60): $\frac{A^3}{P^3} + \frac{G}{A} > 2$. Written in the form $P^3 < \frac{A^4}{2A-G}$, we have to prove $\left(\frac{G+2A}{3}\right)^3 < \frac{A^4}{2A-G}$. By letting $t = \frac{A}{G}$, this becomes $(1+2t)^3 (2t-1) < 27t^4$, or $P(t) = 11t^4 - 16t^3 + 4t + 1 \geq 0$ for $t \geq 1$. One has $P(1) = 0$, $P'(t) = 44t^3 - 48t^2 + 4$, $P'(1) = 0$, $P''(t) = 132t^2 - 96t > 0$ for $t \geq 1$, so $P(t) \geq 0$ for $t \geq 1$.

(2) We now prove that inequality (1.52) is not correct. This can be written as $P > \frac{2A(A+G)}{2A+G}$. By (*) we would have $\frac{2A\cdot(A+G)}{2A+G} < \frac{G+2A}{3}$ or $6A^2 + 6AG < 4AG + 4A^2 + G^2$ or $2A^2 + 2AG < G^2$, which is impossible.

The left side of (1.17) can be written as $P > G + \frac{\sqrt{A^2-G^2}}{3}$. We will prove that this is not correct. Indeed, by (*) we would have $G + \frac{\sqrt{A^2-G^2}}{3} < \frac{G+2A}{3}$, or $3G + \sqrt{A^2-G^2} < G + 2A \iff \sqrt{A^2-G^2} < 2(A-G)$ or $3A^2 + 5G^2 - 8AG > 0$. Put $t = \frac{A}{G}$. Then $3t^2 - 8t + 5 = (t-1)(3t-5)$ which is not positive for all t, since $t = \frac{A}{G} > \frac{5}{3}$ is valid only for certain particular cases of x and y.

2.4. **Conclusion.** We have not analysed all the inequalities in the paper by Bencze [1], but from the above considerations a conclusion can be stated: an author must study the connections and implications of the obtained results, and he must likewise study the light nature in a given domain, before publication of a paper.

References

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