ON THE INTEGRAL ANALOGUE OF TWO NEW TYPE HILBERT GENERALIZATION INEQUALITIES

ZHAO Chang-Jian* and Mihaly Bencze†

May 22, 2002

Abstract

It is well-known that the Hilbert double series inequality plays an important role in mathematical analysis and its applications. In 1998, two new inequalities similar to Hilbert double series inequality were given by B.G.Pachpatte. Recently, these two new inequalities were generalized in [4]. The main purpose of the present article is to give the integral analogue of these two generalization inequalities by Holder integral inequality, Tchebychef integral inequality and Jensen inequality.

Key words: Hilbert integral inequality , Holder integral inequality, Tchebychef integral inequality
MR 1991 Subject Classification: 26D15

In 1998, B.G.Pachpatte gave two new inequalities similar to Hilbert double series inequality(see[2] P.226) in [1]. These two new inequalities were generalized in [4]. In this paper we will give the integral analogue of these two generalization inequalities.

Our main results are given in the following theorems.

THEOREM 1 Let $p \geq 1, q \geq 1$ and $f(\sigma) \geq 0, g(\tau) \geq 0$ for $\sigma \in (0,x), \tau \in (0,y)$, where x,y are positive real numbers and define $F(s) = \int_o^s f(\sigma)d\sigma$ and $G(t) = \int_0^t g(\tau)d\tau$ for $s \in (0,x), t \in (0,y)$ and l is the natural number.Then

$$\int_0^x \int_0^y \frac{F^p(s)G^q(t)(st)^{2/l}}{(s\cdot t^{1/l})^2 + (t\cdot s^{1/l})^2} dsdt \leq \frac{1}{2}pq(xy)^{(l-1)/l} \left(\int_0^x \Big(x-s\Big)\Big(F^{p-1}(s)f(s)\Big)^l ds\right)^{1/l}$$

$$\times \left(\int_0^y \Big(y-t\Big)\Big(G^{q-1}(t)g(t)\Big)^l dt\right)^{1/l} \tag{1}$$

Proof From the hypotheses,it is easy to observe that

$$F^p(s) = p\int_0^s F^{p-1}(\sigma)f(\sigma)d\sigma, s \in (0,x)$$

$$G^q(t) = q\int_0^t G^{q-1}(\tau)g(\tau)d\tau, t \in (0,y)$$

*Dept. of Math., Binzhou Teachers College, Shandong Binzhou 256604, China
†Str. Harmanului Nr. 6, 2212 Sacele, Brasov, RUMANIA.

Therefore
$$F^p(s)G^q(t) = pq\Big(\int_0^s F^{p-1}(\sigma)f(\sigma)d\sigma\Big)\Big(\int_0^t G^{q-1}(\tau)g(\tau)d\tau\Big) \tag{2}$$
On the other hand, by using special case of Tchebychef integral inequality(see[3])(when all f_i are equal) we have
$$\int_0^s F^{p-1}(\sigma)f(\sigma)d\sigma \le s^{(l-1)/l}\Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l} \tag{3}$$
$$\int_0^t G^{q-1}(\tau)g(\tau)d\tau \le t^{(l-1)/l}\Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)^{1/l} \tag{4}$$
By (2),(3) and (4) yield that
$$\begin{aligned}
F^p(s)G^q(t) &\le pq(st)^{(l-1)/l}\Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l}\Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)^{1/l} \\
&\le \frac{1}{2}pq(s^{2(l-1)/l} + t^{2(l-1)/l})\Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l} \\
&\quad \times\Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)^{1/l}
\end{aligned}$$
Thus
$$\frac{F^p(s)G^q(t)(st)^{1/l}}{(s\cdot t^{1/l})^2 + (t\cdot s^{1/l})^2} \le \frac{1}{2}pq\Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l}\Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)^{1/l}$$
Integrating over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using again the special case of Tchebychef integral inequality, we observe that
$$\begin{aligned}
\int_0^x\int_0^y \frac{F^p(s)G^q(t)(st)^{2/l}}{(s\cdot t^{1/l})^2 + (t\cdot s^{1/l})^2}dsdt &\le \frac{1}{2}pq\Bigg(\int_0^x \Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l} ds\Bigg) \\
&\quad \times\Bigg(\int_0^y \Big(\int_0^t \Big(G^{q-1}(\tau)f(\tau)\Big)^l d\tau\Big)^{1/l} dt\Bigg) \\
&\le \frac{1}{2}pq\cdot x^{(l-1)/l}\Bigg(\int_0^x \Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)ds\Bigg)^{1/l} \\
&\quad \times y^{(l-1)/l}\Bigg(\int_0^y \Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)dt\Bigg)^{1/l} \\
&= \frac{1}{2}pq(xy)^{(l-1)/l}\Big(\int_0^x \Big(x-s\Big)\Big(F^{p-1}(s)f(s)\Big)^l ds\Big)^{1/l} \\
&\quad \times\Big(\int_0^y \Big(y-t\Big)\Big(G^{q-1}(t)g(t)\Big)^l dt\Big)^{1/l}
\end{aligned}$$
The proof is complete.

The is just a integral analogue of following Theorem A wich was given in [4].

THEOREM A Let $p \geq 1, q \geq 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$ where k, r and l are natural numbers and define $A_m = \sum_{s=1}^{m} a_s$ and $B_n = \sum_{t=1}^{n} b_t$. Then

$$\sum_{m=1}^{k}\sum_{n=1}^{r} \frac{A_m^p B_n^q (mn)^{2/l}}{(m \cdot n^{1/l})^2 + (n \cdot m^{1/l})^2} \leq \frac{1}{2} pq (kr)^{(l-1)/l} \times \left(\sum_{m=1}^{k} \Big(k - m + 1\Big)\Big(a_m A_m^{p-1}\Big)^l \right)^{1/l}$$

$$\times \left(\sum_{n=1}^{r} \Big(r - n + 1\Big)\Big(b_n B_n^{q-1}\Big)^l \right)^{1/l}$$

THEOREM 2 Let f,g,F,G be as in Theorem 1 .Let $p(\sigma)$ and $q(\tau)$ be two positive functions defined for $\sigma \in (0, x), \tau \in (0, y)$ and define $P(s) = \int_0^s p(\sigma) d\sigma$, $Q(t) = \int_0^t q(\tau) d\tau$ for $s \in (0, x), t \in (0, y)$ where x , y are positive real numbers and l is the natural number and p, q are real numbers and $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. Let ϕ and ψ be two real-valued nonnegative, conves, and submultiplicative

functions defined on $R_+ = [0, +\infty)$ Then

$$\int_0^x \int_0^y \frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)(st)^{2/l}}{(s \cdot t^{1/l})^2 + (t \cdot s^{1/l})^2} ds dt \leq L(x, y, p) \left(\int_0^x \left(\left(\int_0^s \Big(p(\sigma)\phi(\frac{f(\sigma)}{p(\sigma)}) \Big)^l d\sigma \right)^{1/l} \right)^q ds \right)^{1/q}$$

$$\times \left(\int_0^y \left(\left(\int_0^t \Big(q(\tau)\psi(\frac{g(\tau)}{q(\tau)}) \Big)^l d\tau \right)^{1/l} \right)^q dt \right)^{1/q} \tag{5}$$

where

$$L(x, y, p) = \frac{1}{2} \left(\int_0^x \left(\frac{\phi\Big(P(s)\Big)}{P(s)} \right)^p ds \right)^{1/p} \left(\int_0^y \left(\frac{\psi\Big(Q(t)\Big)}{Q(t)} \right)^p dt \right)^{1/p}$$

Proof From the hypotheses and by using Jensen inquality and the special case of tchebychef integral inequality , it is easy to observe that

$$\phi\left(F(s)\right) = \phi\left(\frac{P(s) \int_0^s p(\sigma) \frac{f(\sigma)}{p(\sigma)} d\sigma}{\int_0^s p(\sigma) d\sigma} \right) \leq \frac{\phi\left(P(s)\right)}{P(s)} \int_0^s p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right) d\sigma$$

$$\leq \left(\frac{\phi\Big(P(s)\Big)}{P(s)} \right) s^{(l-1)/l} \left(\int_0^s \Big(p(\sigma)\phi(\frac{f(\sigma)}{p(\sigma)}) \Big)^l d\sigma \right)^{1/l} \tag{6}$$

and similarly,

$$\psi\Big(G(t)\Big) \leq \left(\frac{\psi\Big(Q(t)\Big)}{Q(t)} \right) t^{(l-1)/l} \left(\int_0^t \Big(q(\tau)\psi(\frac{g(\tau)}{q(\tau)}) \Big)^l d\tau \right)^{1/l} \tag{7}$$

By (6) and (7),we get that

$$\frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)(st)^{2/l}}{(s \cdot t^{1/l} + t \cdot s^{1/l})^2} \leq \frac{1}{2} \left(\frac{\phi\Big(P(s)\Big)}{P(s)} \right) \left(\frac{\psi\Big(Q(t)\Big)}{Q(t)} \right) \left(\int_0^s \Big(p(\sigma)\phi(\frac{f(\sigma)}{p(\sigma)}) \Big)^l d\sigma \right)^{1/l}$$

$$\times \left(\int_0^t \Big(q(\tau)\psi(\frac{g(\tau)}{q(\tau)}) \Big)^l d\tau \right)^{1/l} \tag{8}$$

Integrating two sides of (8) over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using Holder integral inequality(see[3]) we observe that

$$\int_0^x\int_0^y \frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)(st)^{2/l}}{(s\cdot t^{1/l}+t\cdot s^{1/l})^2}dsdt \le \frac{1}{2}\left(\int_0^x \frac{\phi\Big(P(s)\Big)}{P(s)}\left(\int_0^s \Big(p(\sigma)\phi\Big(\frac{f(\sigma)}{p(\sigma)}\Big)\Big)^l d\sigma\right)^{1/l} ds\right)$$

$$\times\left(\int_0^y \frac{\psi\Big(Q(t)\Big)}{Q(t)}\left(\int_0^t \Big(q(\tau)\psi\Big(\frac{g(\tau)}{q(\tau)}\Big)\Big)^l d\tau\right)^{1/l} dt\right)$$

$$\le \frac{1}{2}\left(\int_0^x \Big(\frac{\phi\Big(P(s)\Big)}{P(s)}\Big)^p ds\right)^{1/p}\left(\int_0^x \left(\left(\int_0^s \Big(p(\sigma)\phi\Big(\frac{f(\sigma)}{p(\sigma)}\Big)\Big)^l d\sigma\right)^{1/l}\right)^q ds\right)^{1/q}$$

$$\times\left(\int_0^y \Big(\frac{\psi\Big(Q(t)\Big)}{Q(t)}\Big)^p dt\right)^{1/p}\left(\int_0^y \left(\left(\int_0^t \Big(q(\tau)\psi\Big(\frac{g(\tau)}{q(\tau)}\Big)\Big)^l d\tau\right)^{1/l}\right)^q dt\right)^{1/q}$$

$$= L(x,y,p)\left(\int_0^x \left(\left(\int_0^s \Big(p(\sigma)\phi\Big(\frac{f(\sigma)}{p(\sigma)}\Big)\Big)^l d\sigma\right)^{1/l}\right)^q ds\right)^{1/q}$$

$$\times\left(\int_0^y \left(\left(\int_0^t \Big(q(\tau)\psi\Big(\frac{g(\tau)}{q(\tau)}\Big)\Big)^l d\tau\right)^{1/l}\right)^q dt\right)^{1/q}$$

Q.E.D.

This is just a integral analogue of following Theorem B wich was given in [4].

THEOREM B Let $\{a_m\}$, b_n, A_m, B_n be defined as Theorem A. let $\{p_m\}$ and $\{q_n\}$ be two positive sequences for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$ and l is a natural number and $P_m = \sum_{s=1}^m p_s, Q_n = \sum_{t=1}^n q_t$. Let ϕ and ψ be two real-valued nonnegative, convex and submultiplicative functions defined on $R_+ = [0, +\infty)$. Let p, q are two real numbers and $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. Then

$$\sum_{m=1}^k\sum_{n=1}^r \frac{\phi(A_m)\psi(B_n)(mn)^{2/l}}{(m\cdot n^{1/l})^2+(n\cdot m^{1/l})^2} \le M(k,r,p)\left(\sum_{m=1}^k\left(\Big(\sum_{s=1}^m (p_s\phi(\frac{a_s}{p_s}))^l\Big)^{1/l}\right)^q\right)^{1/q}$$

$$\times\left(\sum_{n=1}^r\left(\Big(\sum_{t=1}^n (q_t\psi(\frac{b_t}{q_t}))^l\Big)^{1/l}\right)^q\right)^{1/q}$$

where

$$M(k,r,p) = \frac{1}{2}\left(\sum_{m=1}^k\Big(\frac{\phi\Big(P_m\Big)}{P_m}\Big)^p\right)^{1/p}\left(\sum_{n=1}^r\Big(\frac{\psi\Big(Q_n\Big)}{Q_n}\Big)^p\right)^{1/p}$$

Acknowledgment I wish to express my gratitude to professor Li Wenrong for his valuable help in writing this paper.

References

[1] B.G.Pachpatte. On some new inequalities similar to Hilbert's inequality[J]. J.Math.Anal.Appl. 1998,226(1): 166∼179.

[2] G.H.Hardy. J.E.Littlewood. G.Polya. Inequalities[M], Cambridge Univ.Press, Cambridge,1952.

[3] D.S.Mitrinovic. Analytic inequalities[M], Springer-verlag,1970.

[4] Zhao Changjian. Generalization on two new Hilbert type inequalities. J. of Math. (PRC), wait for publication.

[5] Gao Minzhe. On Hilbert inequality and its Applications[J]. J.Math.Anal.Appl. 1997,212: 316∼323.

[6] HuKe. On Hilbert inequality and its application[J], Advances Mathernatics.1993, 22(2): 160∼163.