Reductio ad Absurdum

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Proof. $\alpha \models \beta$ if and only if $(\alpha \wedge \neg\beta)$ is unsatisfiable.

First, we will show that $\alpha \models \beta$ implies that $(\alpha \wedge \neg\beta)$ is unsatisfiable. Let M be the set of models satisfying α. Since $\alpha \models \beta$, we know that any $m \in M$ satisfies β. Assume for a contradiction that there exists some $\mu \in M$ that satisfies $(\alpha \wedge \neg\beta)$. Since μ satisfies $(\alpha \wedge \neg\beta)$, μ must be a model of α and $\neg\beta$. But μ cannot be a model of $\neg\beta$ because μ is a model of β.

Next, consider M' to be the set of all models that do not satisfy α. We know that any such $\mu' \in M'$ cannot satisfy $(\alpha \wedge \neg\beta)$ because a model must satisfy both sides of a conjunction in order to satisfy a conjunction. Therefore, $\alpha \models \beta$ implies that $(\alpha \wedge \neg\beta)$ is unsatisfiable.

Second, we show that if $(\alpha \wedge \neg\beta)$ is unsatisfiable, then $\alpha \models \beta$. Since $(\alpha \wedge \neg\beta)$ is unsatisfiable, then any model must:

1. Satisfy α but not satisfy $\neg\beta$, or
2. Satisfy $\neg\beta$ but not satisfy α, or
3. Satisfy neither $\neg\beta$ nor α

This is trivially proven via contradiction: assume there was a model m that satisfied both α and $\neg\beta$. Then m satisfies $(\alpha \wedge \neg\beta)$, which contradicts that $(\alpha \wedge \neg\beta)$ is unsatisfiable.

From here, it is easy to show that all models that satisfying α must entail β. Since all models that satisfy α must not satisfy $\neg\beta$, we know that any model that satisfies α must satisfy β by the excluded middle. Therefore, if $(\alpha \wedge \neg\beta)$ is unsatisfiable, then $\alpha \models \beta$. $\square$