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SPANNING GRAPHS AND THE AXIOM OF CHOICE

A b s t r a c t. We show in set-theory **ZF** that the *axiom of choice* is equivalent to the statement *every bipartite connected graph has a spanning sub-graph omitting some complete finite bipartite graph* $K_{n,n}$, and in particular it is equivalent to the fact that every connected graph has a spanning cycle-free graph (possibly non connected).

1. Introduction

We consider simple undirected loop-free graphs. A *forest* is a graph with no cycles, a *tree* is a connected forest. A graph $\mathfrak{G}'$ is a *sub-graph* of $\mathfrak{G}$ if all its edges (and vertices) belong to $\mathfrak{G}$; say that such a sub-graph is *spanning* if every vertex of $\mathfrak{G}$ belongs to an edge of $\mathfrak{G}'$.

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We work in Zermelo-Fraenkel set-theory **ZF** (without the axiom of choice). A spanning tree of a connected graph is precisely a maximal spanning sub-graph with no cycles, thus **AC** implies that *every connected graph has a spanning tree* (**ST**). The converse is easily seen to hold (see Proposition 1). Indeed, the axiom of choice follows from the fact ($\mathbf{SB}_{even}$) that *every connected graph has a connected spanning sub-graph without cycles of all even lengths* (see Remark 1 of Section 3), whereas no choice is needed to establish that *every connected graph has connected spanning sub-graphs with no odd cycles at all* (see Proposition 2). In fact the proofs of **AC** from **ST** or $\mathbf{SB}_{even}$, which statements involve *connected* spanning sub-graphs, hold in systems weaker than **ZF** (see Section 7). We show that (in **ZF**) this connectedness restriction can be relaxed, thus in particular that **AC** follows from the fact (**SF**) that *every connected graph has a spanning forest.* Indeed we prove (Section 4) that the axiom of choice is equivalent to the statement

> **SC**, **Spanning Coppice** : *Every bipartite connected graph has a spanning subgraph omitting some finite complete bipartite graph* $K_{n,n}$.

We also establish a correspondance between restricted choice principles and spanning tree in restricted classes of graphs (Section 6).

2. Definitions

2.1. Graphs

The graphs that we consider are simple undirected graphs ; loops or isolated vertices are irrelevant to the present purpose ; thus these graphs can just be specified by their edge-sets : let us call *graph* any set of two-element sets. Then the *edges* of a graph $\mathfrak{G}$ are just its elements, the *vertices* of $\mathfrak{G}$ are the elements of its union-set $\cup\mathfrak{G}$ and the *subgraphs* of $\mathfrak{G}$ are just its subsets. Two vertices are *adjacent* (or *linked*) if they make-up an edge ; the *neighbourhood* of a vertex x is the set of vertices adjacent to x. A subgraph $\mathfrak{H}$ of $\mathfrak{G}$ is *spanning* if its edges cover all vertices of $\mathfrak{G}$, *i.e.* if $\cup\mathfrak{H} = \cup\mathfrak{G}$.

Given a non-negative integer n, a *path* of *length* n in the graph $\mathfrak{G}$ is a one-to-one finite sequence $(x_i)_{0\leq i\leq n}$ of vertices such that for each $i < n$,

$\{x_i, x_{i+1}\} \in \mathfrak{G}$; such a path *joins* x_0 to x_n. The graph $\mathfrak{G}$ is *connected* if any two vertices are joined by a path. A *cycle* of $\mathfrak{G}$ (or an *n-cycle*) is a path $(x_i)_{0 \leq i < n}$ such that $\{x_{n-1}, x_0\} \in \mathfrak{G}$ and $n \geq 3$; it is an *induced cycle* when $\{x_i, x_j\} \in \mathfrak{G} \Leftrightarrow |j - i| = 1 \pmod{n}$. An *odd* (resp. *even*) cycle is a cycle of odd (resp. even) length. A *forest* is a graph with no cycles ; a *tree* is a connected forest.

A graph is *bipartite* if there exists an equivalence relation on its vertex-set, with at most two classes, and such that no adjacent vertices are equivalent. Such a graph has no odd cycles (*cf.* Remark 5 Section 5). Given two disjoint sets A and B, denote by $\mathfrak{K}_{A,B} := \{\{a,b\} : (a,b) \in A \times B\}$ the *complete bipartite graph with parts A and B*. Now say that a graph $\mathfrak{G}$ is a *coppice* if there is an integer n such that $\mathfrak{G}$ does not admit any complete bipartite subgraphs with both parts of size n (in which case say that $\mathfrak{G}$ is an *n-coppice*).

2.2. About AC

In this paper, we work in Zermelo-Fraenkel set-theory **ZF without** the *axiom of choice* :

> **AC (Axiom of Choice)** : *For every family $(X_i)_{i \in I}$ of non-empty sets, there is a function f of domain I such that for each $i \in I$, $f(i) \in X_i$.*

Notice that this statement is equivalent to its restriction to families of pair-wise disjoint sets : replace each X_i by $X_i \times \{i\}$.

As usual, when stating that a sentence is not provable from **ZF**, it is to be understood *unless* **ZF** *is inconsistent.*

The set $\{0, 1, 2, \dots\}$ of integers is denoted by $\mathbb{N}$. Such notation as $X = \dot\cup_{i \in I} X_i$ (resp. $X = X_1 \dot\cup X_2$) stresses the fact that the set X is the union of the *pair-wise disjoint* X_i's.

3. *SPANNING TREE* yields AC

Although **ST** $\Rightarrow$ **AC** of Proposition 1 below can be considered as a corollary of **SC** $\Rightarrow$ **AC** of the main Theorem 1 (next Section), it admits an easy specific proof which holds in weak systems (see Remark 8).

Proposition 1. AC $\Longleftrightarrow$ ST.

Proof. AC $\Rightarrow$ ST : Given a connected graph, observe that any maximal subgraph with no cycles must be a spanning tree ; the existence of such a maximal subgraph follows from *Zorn's Lemma*.

ST $\Rightarrow$ AC : Given a non-empty family $(X_i)_{i\in I}$ of pair-wise disjoint non-empty sets, consider a one-to-one family $(O_i)_{i\in I}$ such that no O_i belongs to any X_j, and some $r \notin \dot\bigcup_{i\in I}(X_i \dot\cup \{O_i\})$ (see Remark 9). Let $V := \{r\}\dot\cup\big(\dot\bigcup_{i\in I}(X_i\dot\cup\{O_i\})\big)$. Define a connected graph $\mathfrak{G}$ with vertex-set V as follows : for each $i \in I$ and $x \in X_i$, link x to O_i and to r. (See the left-hand side of Figure 1.) Then, using **ST**, consider a spanning tree $\mathfrak{T}$ in the graph $\mathfrak{G}$. For each $i \in I$, every path in $\mathfrak{G}$ joining O_i to r "does pass through a unique element of X_i" : it is of the form (O_i, x, r) for some $x \in X_i$. As a spanning connected subgraph of $\mathfrak{G}$, $\mathfrak{T}$ has at least one such path, and at most one, since it has no 4-cycles ; let x_i denote the element of X_i this path goes through. Now $(x_i)_{i\in I}$ belongs to $\prod_{i\in I} X_i$. □

Remark 1. For each integer $n \geq 3$, call *n-bush* any connected graph with no n-cycles ; call *even-bush* a graph which is an n-bush for some even $n \geq 4$. So a tree is a graph which is a n-bush for every n, and it is also an even-bush. In particular $\mathbf{SB}_{even}$ and each of the statements $\mathbf{SB}_n$ $(n \geq 3)$ below follows from **ST** :

> $\mathbf{SB}_n$ **(Spanning n-Bush)** : *Every connected graph admits a spanning n-bush.*

> $\mathbf{SB}_{even}$ **(Spanning Even Bush)** : *Every connected graph admits a spanning even-bush.*

Notice that the proof of **ST $\Rightarrow$ AC** above in fact shows that $\mathbf{SB}_4 \Rightarrow$ **AC**. More generally, for every even integer $n \geq 4$, **AC** $\Leftrightarrow \mathbf{SB}_n$. Indeed, given the non-empty family $(X_i)_{i\in I}$ of pair-wise non-empty disjoint sets and an integer $n = 2(k+1) \geq 4$, consider a graph $\mathfrak{G}_k$ obtained by slightly modifying the graph $\mathfrak{G}$ of the above proof : Instead of directly linking each $x \in X_i$ to O_i, join x to O_i *via* some path of length k ; in other words replace the vertex x by a path of length $k-1$. (See Figure 1.) We also claim that still **AC** $\Leftrightarrow \mathbf{SB}_{even}$. Indeed, given the family $(X_i)_{i\in I}$, "amalgamate" the $\mathfrak{G}_k$'s

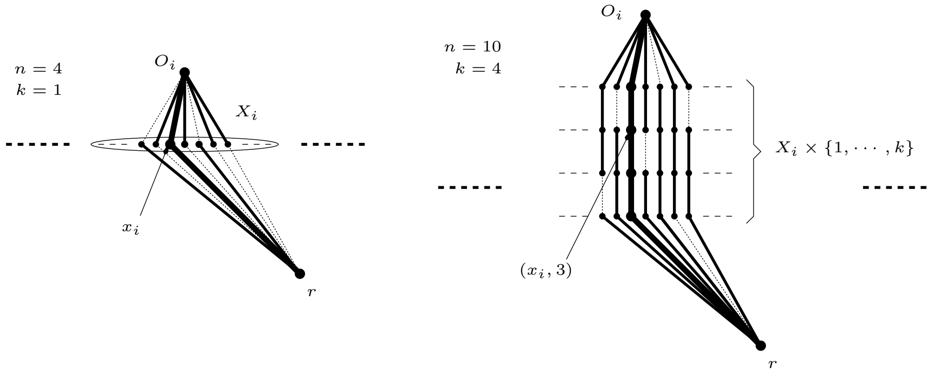

Figure 1:

above on their "roots" and each O_i, formally getting the graph

$$\mathfrak{G}_\omega := \cup_{i\in I} \cup_{k\in\mathbb{N}\setminus\{0\}} \cup_{x\in X_i} \big\{\{r, (x, 1, k)\}, \{(x, 1, k), (x, 2, k)\}, \ldots, \\ \{(x, k-1, k), (x, k, k)\}, \{(x, k, k), (O_i, k)\}\big\},$$

where the O_i's are pair-wise distinct and belong to no $X_j \times \{1, \cdots, k\}$, and the root r belongs to no $\big((X_i \times \{1, \cdots, k\})\dot\cup\{O_i\}\big) \times \{k\}$. A spanning $2(k+1)$-bush $\mathfrak{H}$ in the connected graph $\mathfrak{G}_\omega$ yields a choice function for the family $(X_i)_{i\in I}$: for each $i \in I$, there is a unique $x \in X_i$, let it be x_i, and a unique k such that $\big((O_i, k), (x, k, k), \cdots, (x, 1, k), r\big)$ is a path in $\mathfrak{H}$.

Remark 2. In the case of finite graphs, there are polynomial algorithms which compute spanning trees in connected graphs (for example, Prim and Kruskal's algorithms). These algorithms extend to the case of well-orderable graphs, and in particular they yield **WO** $\Rightarrow$ **ST**. As for Kruskal's algorithm for instance, given a connected graph $\mathfrak{G}$ endowed with a well ordering $<$, define a spanning tree $\mathfrak{T}$ as follows : an edge e of $\mathfrak{G}$ belongs to $\mathfrak{T}$ if and only if the graph it makes-up together with the previously selected edges, namely $\{d \in \mathfrak{T} : d < e\} \cup \{e\}$, has no cycles.

4. ***SPANNING COPPICE*** yields the ***AXIOM OF CHOICE***

In this section, we prove that the statement *Spanning Coppice* **SC** is equivalent to **AC** ; thus *Spanning Forest* **SF** also is equivalent to **AC**. Indeed, we show that **SC** is equivalent to some choice principle **sMC**, itself equivalent to **AC** in **ZF** (but not in **ZFA** for instance). For a "quicker and direct" proof of **SF** $\Rightarrow$ **AC**, see Remark 4 below.

For every set A and positive integer p, $[A]^p$ denotes the set of p-element subsets of A.

Lemma 1. *Given a set X and a set A which is the range of no mapping with domain X, consider a mapping $f : A \longrightarrow \mathcal{P}(X)\backslash\{\varnothing\}$ (with values non-empty subsets of X). Then*

1. *There are distinct a and b in A such that $f(a) \cap f(b) \neq \varnothing$.*
2. *If, in addition, the set A is infinite and well-orderable, then for every positive integer p, there is an $F \in [A]^p$ such that $\cap f[F] := \cap_{a \in F} f(a)$ is non-empty.*

Remark 3. Notice that, in **ZF**, for every set X there is indeed an ordinal onto which there is no mapping with domain X : A mapping φ of domain X and range an ordinal is characterized by the binary relation $\prec_\varphi$ on X given by "$y \prec_\varphi z \Leftrightarrow \varphi(y) < \varphi(z)$" ; indeed the relation $\prec_\varphi$ is well-founded and φ is its rank function ; so the class of those φ's is a set, then the class of their ranges is a set too (we use the power-set axiom and the replacement schema here, *cf.* Remark 8 Section 7) ; hence not every ordinal is the range of such a φ.

Proof. [Proof of the Lemma]

1. Otherwise, given some $c \in A$, X would be mapped onto A by

$$x \longmapsto \begin{cases} \text{the (only) } a \text{ for which } x \in f(a) & \text{if } x \in f[A] \\ c & \text{otherwise} \end{cases}$$

2. For every set B and positive integer p, consider the property $R(B,p)$: *For every mapping $g : B \to \mathcal{P}(X)\backslash\{\varnothing\}$, there exists $G \in [B]^p$ such that $\cap g[G] \neq \varnothing$.* The previous point asserts that $R(A,2)$ holds and we have to prove that when A is well-orderable, $R(A,p)$ holds for every p. Clearly $R(B,p)$ implies $R(C,r)$ for every set C equipotent to B and $r \leq p$. Observe that for every B, C, p and q such that $R(B,p)$ and $R(C,q)$ hold, if B is well-orderable, then $R(B \times C, pq)$

holds too : Given a mapping $g : B \times C \to \mathcal{P}(X)\backslash\{\varnothing\}$, for each $c \in C$ let $g_c : B \to \mathcal{P}(X)\backslash\{\varnothing\}$ map each $b \in B$ to $g(b,c)$. By assumption on B, for each $c \in C$ there is a p-subset F of B such that $\cap g_c[F] \neq \varnothing$; let F_c denote the first such F *w.r.t.* some fixed well-ordering of $[B]^p$ (notice that since B is well-orderable, $[B]^p$ is also well-orderable) and let

$$h(c) := \cap g_c[F_c] = \cap g[\{c\} \times F_c] \in \mathcal{P}(X)\backslash\{\varnothing\}$$

Then, by assumption on C, there is some q-subset $H \subseteq C$ such that $\cap h[H] \neq \varnothing$, *i.e.* such that $\cap g[G] \neq \varnothing$ where G denotes the pq-subset $\dot\cup_{c\in H}(\{c\} \times F_c)$ of $B \times C$.

Now assume that A is well-orderable and infinite. In that case, for every positive integer n, the set A^n is equipotent to A. Then it easily follows from the discussion above that $R(A,p)$ holds for every p.

□

The following *strong multiple choice principle* is known to be equivalent to **AC** in **ZF** ([4]), see also [7] p. 8 :

> **sMC (strong Multiple Choice)** : *For every family $(X_i)_{i\in I}$ of non-empty sets, there exist an integer $n \geq 1$ and a family $(F_i)_{i\in I}$ such that for each i, F_i is a non-empty subset of X_i with at most n elements.*

Theorem 1. SC $\Longrightarrow$ sMC.

Proof. Given a non-empty family $(X_i)_{i\in I}$ of pair-wise disjoint sets, let $X := \dot\cup_{i\in I} X_i$. We consider a family $(A_i, <_i)_{i\in I}$ of well-ordered sets such that, for each $i \in I$, A_i is disjoint from X and the other A_j's, and there is no mapping with domain X_i and range A_i : let for instance A be some well-orderable set which is the range of no mapping with domain X (see Remark 3), and let $A_i := (A \times \{i\}) \times \{\infty\}$ for some $\infty \notin \cup\cup X$ (see Remark 9) together with the obvious well-ordering $<_i$. Then given some $r \notin X \dot\cup (\dot\cup_{i\in I} A_i)$ let $V := \{r\} \dot\cup X \dot\cup (\dot\cup_{i\in I} A_i)$. We define a connected graph $\mathfrak{G}$ with vertex-set V : for each $i \in I$ and $x \in X_i$, link x to r and to every element of A_i.

With **SC**, there exists an integer $n \geq 2$ such that $\mathfrak{G}$ admits a spanning n-coppice $\mathfrak{F}$. We now build a family $(F_i)_{i\in I}$ with each F_i a non-empty finite subset of X_i with less than n elements : For each $i \in I$, let $f_i : A_i \to \mathcal{P}(X_i)\backslash\{\varnothing\}$ map each element of A_i to its neighbourhood in $\mathfrak{F}$. A family of well orderings on the $[A_i]^n$'s is definable from the family $(<_i)_{i\in I}$. So one can choose, for every $i \in I$, a finite n-subset $H_i \subseteq A_i$ such that $\cap f_i[H_i] \neq \varnothing$. Each $F_i := \cap f_i[H_i]$ has less than n elements, since $\mathfrak{K}_{H_i,F_i}$ is a subgraph of the n-coppice $\mathfrak{F}$. □

Corollary 1. **AC** $\Longleftrightarrow$ **SC**.

Remark 4. Notice that the argument in the proof of Theorem 1 above directly shows **SF** $\Rightarrow$ **AC**, not relying on **sMC** $\Rightarrow$ **AC** nor on the second Point of Lemma 1 : Indeed, given a non-empty family $(X_i)_{i\in I}$ of pair-wise disjoint sets, consider the very same graph $\mathfrak{G}$ as in the proof above, then, using **SF**, a spanning forest $\mathfrak{F}$. Now, given any $i \in I$, notice that, for each element ζ of A_i, its neighbourhood in the graph $\mathfrak{F}$ is a subset of X_i (since $\mathfrak{F}$ is a subgraph of $\mathfrak{G}$) and is non-empty (since $\mathfrak{F}$ is spanning and $\cup E = V$) ; then let $\varphi_i : A_i \to \mathcal{P}(X_i)\backslash\{\varnothing\}$ map each element of A_i to its neighbourhood in $\mathfrak{F}$; observe that, for distinct ζ and ξ in A_i, $\varphi_i(\zeta) \cap \varphi_i(\xi)$ has at most one element, because two of them, say y and z, would yield a cycle (ζ, y, ξ, z) in the forest $\mathfrak{F}$; now, with the first Point of Lemma 1 above, there are tuples (ζ, ξ) of distinct elements of A_i such that $\varphi_i(\zeta) \cap \varphi_i(\xi)$ is non-empty, then a singleton ; let x_i be the element of $\varphi_i(\zeta) \cap \varphi_i(\xi)$, for the first such tuple *w.r.t.* the lexicographical ordering on $A_i \times A_i$ (see Figure 2).

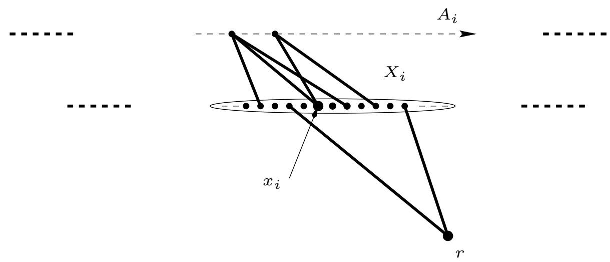

Figure 2:

Problem 1. *From the proof of Theorem 1 above, it also follows that* **AC** *is implied by :* every connected graph admits a spanning subgraph without *induced* 4-cycles. *On the other hand* **SC** *follows from :* every connected graph admits a spanning subgraph without cycles of all even lengths *(a* *weakening of* $\mathbf{SB}_{even}$*). Do the following weakenings respectively of* $\mathbf{SB}_{2k}$ *(*$k > 1$ *fixed) and of* $\mathbf{SB}_{even}$ *(see Remark 1) imply* **AC** *?* $\mathbf{SB}'_{2k}$: Every connected graph admits a spanning subgraph without any *induced* cycles of length $2k$ (resp. $\mathbf{SB}'_{even}$: without *induced* cycles of all even lengths).

Notice that the answer is positive for $k = 2$, since $\mathbf{SB}'_4 \Rightarrow \mathbf{SC}$: given a bipartite graph, a spanning graph without any induced square is a 2-coppice.

5. Bipartite graphs

Let us observe that in **ZF**, *every connected graph admits a spanning subgraph with no odd cycles.*

Proposition 2. *Every* connected *graph has a connected spanning bipartite subgraph.*

In particular, for every odd integer $n \geq 3$, $\mathbf{ZF} \models \mathbf{SB}_n$ (see Remark 1).

Proof. Given a connected graph $\mathfrak{G}$, fix some vertex $r \in V := \cup\mathfrak{G}$ (unless V is empty, in which case nothing has to be done) and let $\delta : V \to \mathbb{N}$ map each vertex to the least length of a path from r to that vertex. Consider the binary relation $\mathcal{B}$ on V given by

$$x\mathcal{B}y :\Longleftrightarrow \delta(x) = \delta(y) \bmod 2$$

This is an equivalence relation with at most two classes neither of which includes any edge of the following subgraph of $\mathfrak{G}$ (which graph is then bipartite) :

$$\mathfrak{H} := \big\{\{x, y\} \in \mathfrak{G} : x \not\mathcal{B} y\big\}.$$

To conclude that $\mathfrak{H}$ is a connected spanning subgraph of $\mathfrak{G}$, it remains to check that every vertex of $\mathfrak{G}$ is joined to r by a path in $\mathfrak{H}$: Given a vertex x and a path $(x_i)_{0 \leq i \leq \delta(x)}$ of minimal length joining r to x in $\mathfrak{G}$, observe that for each $j \leq \delta(x)$, $(x_i)_{0 \leq i \leq j}$ is still a path of minimal length joining r to x_j, and in particular $\delta(x_j) = j$; hence for any $j < \delta(x)$, the two vertices x_j and x_{j+1} are non $\mathcal{B}$-equivalent vertices making-up an edge of $\mathfrak{G}$, so they make-up an edge of $\mathfrak{H}$ also ; thus $(x_i)_{0 \leq i \leq \delta(x)}$ is indeed a path in $\mathfrak{H}$. □

Remark 5. *No bipartite graph has odd cycles.* Conversely, *every connected graph without any odd cycles is bipartite.* However, the fact that *every graph with no odd cycles is bipartite* is equivalent to the *axiom of choice for families of pairs* ([6]), and therefore is not provable from **ZF**.

6. "Sparse" graphs

6.1. Locally finite and thin graphs

A graph is *locally finite* if every vertex has only finitely many neighbours. Say that a graph is *strongly thin* if, between any two vertices, there are only finitely many paths ; say that it is *weakly thin* if it is empty or admits a vertex r with the property that for any vertex x, there are only finitely many minimal paths from r to x.

Notice that every forest is strongly thin, but a forest need not be locally finite. On the other hand, in a locally finite graph, from any given point, there start only finitely many paths of a given finite length ; in particular such a graph is weakly thin ; but it may fail to be strongly thin.

Now, consider the following restriction of **ST** to "sparse" graphs :

STℓF (Spanning Tree for Locally Finite connected graphs)
Every locally finite connected graph has a spanning tree.

STsT (Spanning Tree for Strongly Thin connected graphs)
Every connected strongly thin graph has a spanning tree.

STwT (Spanning Tree for Weakly Thin connected graphs)
Every connected weakly thin graph has a spanning tree.

Then consider the axiom of choice restricted to families of finite sets and further to countable such families :

AC^{fin} (Axiom of Choice for Finite sets) *For every non-empty family $(X_i)_{i\in I}$ of finite non-empty sets, the product set $\prod_{i\in I} X_i$ is non-empty.*

> $\mathbf{AC}_\omega^{fin}$ **(Axiom of Countable Choice for Finite sets)** *For every non-empty sequence $(X_n)_{n\in\mathbb{N}}$ of finite non-empty sets, the product set $\prod_{n\in\mathbb{N}} X_n$ is non-empty.*

Clearly $\mathbf{AC}^{fin}$ implies $\mathbf{AC}_\omega^{fin}$; but the converse does not hold (see [2]). We prove :

Theorem 2. $\mathbf{STwT} \iff \mathbf{STsT} \iff \mathbf{AC}^{fin} \Longrightarrow \mathbf{AC}_\omega^{fin} \iff \mathbf{ST\ell F}$.

In fact, the left-hand equivalences, which may be considered as properties of the class of finite sets, rely essentially on two *ad hoc* properties of that class, namely on its closure under image and product ; in particular the analogue will hold with the class of countable sets, and more generally with the class of sets *sub-potent* (resp. *strictly sub-potent*) to a given ordinal (see Remark 7-5 below). These equivalences will follow from Proposition 3 below, so stated as to include all these cases (Corollary 2). Let us prove right now the rightmost equivalence :

Proof. [Proof of the rightmost equivalence] $\mathbf{AC}_\omega^{fin} \Rightarrow \mathbf{ST\ell F}$ **:** First, let us mention the equivalence between $\mathbf{AC}_\omega^{fin}$ and the fact that *every union of a countably many finite sets is countable* (see [2]). Now, let $\mathfrak{G} = (V, E)$ be some non-empty locally finite connected graph. Then V is countable : consider some $r \in V$; for each integer $n \geq 1$, let $V_n := \{v \in V : d_{\mathfrak{G}}(r, v) = n\}$; each V_n is finite (by local finiteness) and $V = \cup_{n\in\mathbb{N}} V_n$ (by connectedness) ; then from $\mathbf{AC}_\omega^{fin}$ it follows that V is countable, so that $\mathfrak{G}$ is countable too, hence it has a spanning tree (see Remark 2).

$\mathbf{ST\ell F} \Rightarrow \mathbf{AC}_\omega^{fin}$ **:** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of non-empty finite sets. Without loss of generality, assume that the X_n's are pair-wise disjoint, and, letting $X := \dot\cup_{n\in\mathbb{N}} X_n$, consider two one-to-one sequences $(O_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ such that $O := \{O_n : n \in \mathbb{N}\}$ and $R := \{r_n : n \in \mathbb{N}\}$ are disjoint from one another and from X. Let $V := X \dot\cup O \dot\cup R$. For each $n \in \mathbb{N}$, link r_n to r_{n+1} and to every element of X_n ; also link O_n to every element of X_n. The graph thus obtained is connected and locally finite ; a spanning tree in this graph yields a choice function for $(X_n)_{n\in\mathbb{N}}$. *Cf.* Theorem 1. □

Remark 6. In the proof of $\mathbf{ST\ell F} \Rightarrow \mathbf{AC}_\omega^{fin}$, the point O_n can be replaced by $(A_n, <_n)$ a copy of $card(X_n) + 1$, as in the proof of $\mathbf{SF} \Rightarrow \mathbf{AC}$ (Remark 4), yielding a proof of $\mathbf{SF\ell F} \Rightarrow \mathbf{AC}_\omega^{fin}$ ($\mathbf{SF\ell F}$ is the statement *Spanning Forest for Locally Finite connected graphs*), whence the equivalence between these statements.

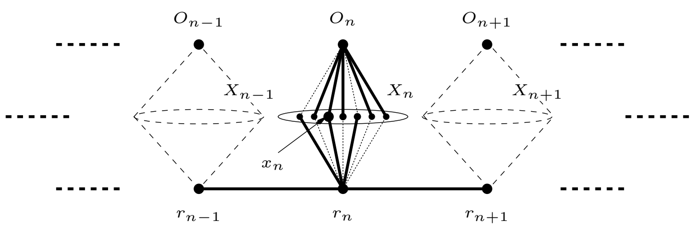

Figure 3:

6.2. K-thin graphs

Consider a class K of sets *not containing the empty set.*

1. Say that K is *closed under image* if $K \ni x \twoheadrightarrow y \Rightarrow y \in K$, where $x \twoheadrightarrow y$ means "*there is a mapping from x onto y*".
2. Say that K is *closed under product* if $(x \in K \wedge y \in K) \Rightarrow x \times y \in K$.

Remark 7.

1. Closure under image reformulates "*if the domain of a function belongs to K, then so does its range.*"
2. From closure under image, it follows closure under equi-potence, *i.e.* "*if a set admits a bijection with an element of K, then it too belongs to K*".
3. From closure under image, it follows closure under non-empty subsets *i.e.* $\varnothing \neq y \subseteq x \in K \Rightarrow y \in K$,
4. More generally, from closure under image, it follows $\varnothing \neq y \hookrightarrow x \in K \Rightarrow y \in K$, where $y \hookrightarrow x$ means "*there is an one-to-one mapping from y into x*". Indeed this follows from closure under equi-potence and non-empty subsets, or more directly from the observation that $\varnothing \neq y \hookrightarrow x \Rightarrow x \twoheadrightarrow y$.
5. As mentioned just above, $\varnothing \neq y \hookrightarrow x$ implies $x \twoheadrightarrow y$, but in **ZF**, the converse need not hold. However *when x is well-orderable*, $x \twoheadrightarrow y$ does imply $y \hookrightarrow x$. So, for a well-orderable non-empty set x, $\varnothing \neq y \hookrightarrow x$ if and only if $x \twoheadrightarrow y$, in which case, say that y is *sub-potent* to x, and say that it is *strictly sub-potent* when in addition x is not sub-potent

to it (notice that if y is sub-potent to the well-orderable set x, then it is well-orderable too).

Examples.

1. For every non-empty set x, $K_x := \{y : x \twoheadrightarrow y\}$ is closed under image ; so is $K_x^- := \{y : x \twoheadrightarrow y \not\twoheadrightarrow x\}$.
2. In general $\{y : y \hookrightarrow x\}$ need not be closed under image. However *when x is well-orderable*, $\{y : \varnothing \neq y \hookrightarrow x\} = K_x$ and $\{y : \varnothing \neq y \hookrightarrow x \not\hookrightarrow y\} = K_x^-$, so both classes are closed under image ; furthermore, when in addition of being well-orderable, x is infinite, the classes K_x and K_x^- are closed under product.
3. For a non-zero integer n, K_n is the class of non-empty sets with at most n elements ; it is closed under image, but not under product.
4. The classes below are closed under image and product.
    - The class of all non-empty well-orderable sets.
    - The class of all non-empty sets sub-potent (resp. strictly sub-potent) to a given infinite well-orderable set. In particular :
    - K_ω^- is the class of finite non-empty sets.
    - K_ω is the class of countable non-empty sets. (By *countable* we mean *finite or denumerable i.e.* or countably infinite.)

Given a graph $\mathfrak{G}$, for every vertices x and y, let $P_{\mathfrak{G}}(x,y)$ denote the set of paths from x to y, and let $M_{\mathfrak{G}}(x,y)$ denote the set of minimal paths from x to y. Also for every vertex x, let $N_{\mathfrak{G}}(x)$ denote its neighbourhood.

Now say that a graph $\mathfrak{G}$ is *strongly K-thin* if, for any vertices x and y, $P_{\mathfrak{G}}(x,y) \in K$; say that it is *weakly K-thin* if it is empty or admits a vertex r with the property that for any vertex x, $M_{\mathfrak{G}}(r,x) \in K$. Observe that a weakly K-thin graph is connected, since $\varnothing \notin K$. Now consider **ST** restricted to K-thin graphs and **AC** restricted to families of sets in K :

STsK (Spanning Tree for Strongly K-Thin graphs)
Every strongly K-thin graph has a spanning tree.

STwK (Spanning Tree for Weakly K-Thin graphs)
Every weakly K-thin graph has a spanning tree.

$\mathbf{AC}^K$ (Axiom of Choice in sets in K) *For every non-empty family $(X_i)_{i\in I}$ of elements of the class K, the product set $\prod_{i\in I} X_i$ is non-empty.*

Proposition 3. *When the class K is closed under image, $\mathbf{AC}^K \iff \mathbf{STs}K$. When, in addition, K is closed under product, $\mathbf{AC}^K \iff \mathbf{STw}K$.*

Thus from the remarks and examples above it will follow :

Corollary 2. *For every infinite ordinal α, $\mathbf{AC}^{K_\alpha} \iff \mathbf{STs}K_\alpha \iff \mathbf{STw}K_\alpha$ and $\mathbf{AC}^{K_\alpha^-} \iff \mathbf{STs}K_\alpha^- \iff \mathbf{STw}K_\alpha^-$ (recall that K_α and K_α^- denote the classes of sets respectively sub-potent and stricly sub-potent to α).*

Proof. [Proof of the proposition] The graph $\mathfrak{G}$ introduced in the proof of Proposition 1 is weakly K-thin when the X_i's belong to K. Furthermore, the graph $\mathfrak{G}$ is strongly K-thin when, in addition, K is closed under product : Given $i \in I$, $M_{\mathfrak{G}}(r, O_i)$ is equipotent to X_i, and for every $x \in X_i$, $M_{\mathfrak{G}}(r, x)$ is a singleton (hence an image of X_i). Furthermore, for two distinct vertices x and y in a same X_i, $P_{\mathfrak{G}}(x, y)$ has two elements, then is an image of the X_i ; for $x \in X_i$, $P_{\mathfrak{G}}(r, x)$ and $P_{\mathfrak{G}}(O_i, x)$ are both equipotent to that X_i (they contains one path of length one and the others have length three) ; as for an $x \in X_i \cup \{O_i\}$ and a $y \in X_j \cup \{O_j\}$ with $i \neq j$, $P_{\mathfrak{G}}(x, y)$ is equipotent to $X_i \times X_j$.

Conversely, it remains to prove that, given a weakly K-thin graph $\mathfrak{G} = (V, E)$, it follows from $\mathbf{AC}^K$ that $\mathfrak{G}$ admits a spanning tree : Assume that $\mathfrak{G}$ is non-empty and consider some vertex r witnessing its being weakly K-thin. For each $n \in \mathbb{N}$, let $V_n := \{v \in V : d_{\mathfrak{G}}(r, v) = n\}$ (*cf.* the proof of $\mathbf{AC}_\omega^{fin} \Rightarrow \mathbf{ST}\ell\mathbf{F}$) ; so $V_0 = \{r\}$ and $V = \dot\cup_{n\in\mathbb{N}} V_n$; for each $x \in V$, let $\rho(x)$ denote the unique n such that $x \in V_n$. Now for any $x \in V \backslash \{r\}$, observe that $(x_0, \cdots, x_{\rho(x)}) \longmapsto x_{\rho(x)-1}$ maps $M_{\mathfrak{G}}(r, x)$ onto $N_{\mathfrak{G}}(x) \cap V_{\rho(x)-1}$. So, by assumption, $N_{\mathfrak{G}}(x) \cap V_{\rho(x)-1}$ belongs to K (notice however that $N_{\mathfrak{G}}(x)$ need not belong to K). Then, with $\mathbf{AC}^K$, consider some

$$\pi \in \prod_{x\in V\backslash\{r\}} (N_{\mathfrak{G}}(x) \cap V_{\rho(x)-1})$$

The following graph

$$\mathfrak{G}' := \big(V, \big\{\{\pi(x), x\} : x \in V\backslash\{r\}\big\}\big)$$

is a spanning tree of $\mathfrak{G}$: On the one hand $\mathfrak{G}'$ is connected since for each $x \in V$, $\left(\pi^0(x), \cdots, \pi^{\rho(x)}(x)\right) \in P_{\mathfrak{G}'}(x, r)$. On the other hand, $\mathfrak{G}'$ has no cycle ; indeed observe that for every edge $\{x, y\}$ of $\mathfrak{G}'$, either $y = \pi(x)$, or $\rho(y) > \rho(x)$; it follows that, if $(x_i)_{0 \leq i < k}$ were a cycle in $\mathfrak{G}'$, then letting $\ell < k$ such that $\rho(x_\ell)$ is maximum, $x_{\ell-1}$ and $x_{\ell+1}$ (the sums being considered modulo k) would both have to equal $\pi(x_\ell)$, contradicting the assumption that cycles are one-to-one sequences (and $k \geq 3$). □

7. Remarks about spanning trees in weaker systems

Remark 8. The proofs of **ST** ⇒ **AC** and of $\mathbf{SB}_n$ ⇒ **AC** (for each even integer $n \geq 4$) above are valid in theories weaker than **ZF**, namely in Kripke-Platek system **KP**, and the proof of $\mathbf{SB}_{even}$ ⇒ **AC** holds in **KPI** (**KP** + *axiom of infinity*), see [1] or [5] ; indeed the proofs do not rely on the *power-set axiom* and only involve the Δ_0-fragments of *separation* and *collection schemata*.

Indeed observe that the following Remark holds in **KP** :

Remark 9.

1. Given a set X, the set $\infty_X := \{x \in X : x \notin x\}$ does not belong to X. Notice that, with the *axiom of foundation* **AF**, which belongs to **ZF**, one can just consider $\infty_X := X$.
2. For every sets X and I, there is a one-to-one family $(O_i)_{i \in I}$ with the property that no O_i belongs to X : let $a \notin \cup \cup X$, for example $a := \infty_{\cup \cup X}$; then $O_i := (a, i)$ suits. Likewise, with **AF**, one can just consider $O_i := (X, i)$.

Remark 10. (See Remarks 3 and 8.) Notice that in **KPI**, it is not even provable that *for every set X there exists a well-orderable set, or just simply a set, which is the range of no mapping with domain X* : indeed, in **ZFC**, the class **HC** of hereditarily countable sets is a model of **KPI**, in which every infinite set maps onto any set.

Problem 2. *Does* **KPI** *prove* **SC** ⇒ **AC** *? In particular does* **KPI** *prove* **SF** ⇒ **AC** *?*

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