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DEFINABILITY IN INFINITARY LANGUAGES AND INVARIANCE BY AUTOMORPHISMS

A b s t r a c t. Given a $\mathcal{L}^{E}_{\alpha\beta}$-structure E, where $\mathcal{L}^{E}_{\alpha\beta}$ is an infinitary language, we show that α and β can be chosen in such way that every orbit of the group G of automorphisms of E is $\mathcal{L}^{E}_{\alpha\beta}$-definable. It follows that two sequences of elements of the domain D of E satisfy the same set of $\mathcal{L}_{\alpha\beta}$-formulas if and only if they are in the same orbit of G.

1. Introduction

This paper is about two notions of model theory, the concepts of definability in a given language and invariance by automorphisms of a structure. If a

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first-order structure E is an interpretation of a language $\mathcal{L}$, every relation definable in $\mathcal{L}$ is invariant by the group of automorphisms of E. In general, the converse is not true but it may be asked whether a strongly enough infinitary language $\mathcal{L}$ may be chosen for the converse to be true.

Let E be a first-order structure defined on a set D and let $\mathcal{R}$ be the set of primitive relations of E. Given an ordinal number γ and a relation $R \subseteq D^\gamma$, invariant by the group G of automorphisms of E, we show that R is definable in a suitable infinitary language $\mathcal{L}^E_{\alpha\beta}$. The parameters α and β are determined by γ and by the cardinals of D and $\mathcal{R}$ (corollaries 3.8, 3.9, 3.10 of theorem 3.7). An easy consequence of theorem 3.7 is that given two points $p_1, p_2 \in D^\gamma$ satisfying the same set of formulas of the language mentioned above, there exists an automorphism g of E which sends p_1 into p_2, that is, $g \cdot p_1 = p_2$.

Although it is possible to show that invariant relations are definable in a suitably strong infinitary language, by means of explicit formulas [5], our proof of theorem 3.7 follows a much more conceptual path. Let δ be the cardinal of D. A point $q \in D^\delta$ is normal if $q : \delta \to D$ is a bijection of δ onto D. We start proving that the orbit $G \cdot q$ of any normal point q is definable in a suitable language $\mathcal{L}^E_{\alpha\beta}$ (proposition 3.6). Next, we show that any orbit of G operating on D^γ can be obtained from $G \cdot q$ applying operators ξ^*, ξ^{*-1} and remaining definable in $\mathcal{L}^E_{\alpha\beta}$ (theorem 3.7).

The operators ξ^*, ξ^{*-1} are very natural from a set-theoretical point of view (section 2). They behave nicely with respect to compositions of maps and commute with the extensions of bijections of D to relations. However, they are not immediate translations to relations of the syntactic rules of definition of formulas. Part of our work is to translate the action of operators ξ^*, ξ^{*-1} on relations into the action of the syntactic rules (theorems 3.1 and 3.2).

We think appropriate to mention here the work of two authors M. Krasner and J. Sebastião e Silva who may be considered forerunners of the introduction of infinitary languages in model theory. Krasner has shown how to construct invariant relations of a first order structures from the primitive relations by means of set-theoretical operations [3]. In a previous paper [4], the authors proved that the operators ξ^*, ξ^{*-1} together with intersections and the complement operation are enough to generate all invariant relations from the primitive ones. Sebastião e Silva [6] has treated the same problem for higher order structures. For an exposition of ideas and results of Sebastião e Silva see [1].

For the sake of simplicity, the theorems in this paper are proved for relational languages only. In the last section, we show how to extend the results to structures associated to complete first order languages.

2. $\mathcal{L}^E_{\alpha\beta}$-structures

Let α, β, π be infinite cardinal numbers, α being a regular cardinal and $\beta, \pi \leq \alpha$. By $\mathcal{L}_{\alpha\beta\pi}$ we denote an *infinitary first order relational language*, whose formulas are sequences of less than α symbols and admitting sequences of less than α conjunctions and blocks of sequences of less than β instantiations. The arity of every predicate symbol of $\mathcal{L}_{\alpha\beta\pi}$ is less than π. $\mathcal{L}_{\alpha\beta\pi}$ is essentially the infinitary relational language with parameters α, β, π as defined by C. R. Karp [2].

In order to fix our notation, we recall the rules of formation of formulas of $\mathcal{L}_{\alpha\beta\pi}$. We use the notion of concatenation of sequences (Karp [2]) and denote by $|A|$ the cardinal of a set A and by $\wp(A)$ its power set. The letters γ and δ always denote ordinal numbers.

The symbols of $\mathcal{L}_{\alpha\beta\pi}$ are the logical symbols $\neg, \bigwedge, \exists$, predicate symbols including $=$ and symbols of variables. We use the standard abbreviations $\forall, \bigvee$. We denote by $\bar{\mathcal{R}}$ the set of predicates symbols and by $\mathcal{V}$ the infinite set of variables. To each predicate symbol $\bar{R} \in \bar{\mathcal{R}}$ is associated an ordinal number $\gamma < \pi$; γ is the arity of $\bar{R}$. We assume that we have fixed an enumeration $\chi : i \in |\mathcal{V}| \mapsto x_i \in \mathcal{V}$ of the set of variables and define in $\mathcal{V}$ the order: $x_i \leq x_j \Leftrightarrow i \leq j$.

Definition 2.1. The set of formulas of $\mathcal{L}_{\alpha\beta\pi}$ is the smallest set of sequences of symbols of length less than α satisfying conditions 1) to 4) below:

1) If $\bar{R}$ is a predicate symbol of arity $\gamma < \pi$ and $\tau : \gamma \to \mathcal{V}$ is a sequence of variables, then $\bar{R}\tau$ is a formula.

2) If φ is a formula, $\neg\varphi$ is a formula.

3) If φ is a formula, $\gamma < \beta$ and $\eta : \gamma \to \mathcal{V}$ is a sequence of variables, then $\exists\eta\varphi$ is a formula.

4) If $\gamma < \alpha$ and $(\varphi_i)_{i<\gamma}$ is a sequence of formulas, then $\bigwedge(\varphi_i)_{i<\gamma}$ is a formula.

We use the notation $\mathcal{L}_{\alpha\beta\pi}$ also to denote the set of formulas of the language $\mathcal{L}_{\alpha\beta\pi}$ and denote by $\mathcal{V}(\varphi)$ the set of *free* variables of a formula φ, endowed with the order induced by the order of $\mathcal{V}$. The *arity* of φ is the ordinal number of $\mathcal{V}(\varphi)$. If φ is a formula of arity γ, $\sigma_\varphi : \gamma \to \mathcal{V}(\varphi)$ denotes the unique order preserving isomorphism.

Consider a set D and an ordinal γ. A *γ-tuple defined on D* is a sequence $p : \gamma \to D$ of elements of D defined on γ. We refer to γ-tuples also as *γ-points* or *points of arity γ defined on D*. Let D^γ be the set of all γ-points defined on D. A *relation of arity γ* or a *γ-relation defined on D* is a subset of D^γ.

Given a map $g : D \to D$, we denote by g_γ its extension to γ-points. If $p \in D^\gamma$, $g_\gamma(p) = g \circ p$. We use the same notation g_γ to denote the extension of g_γ to γ-relations. If $R \subseteq D^\gamma$, $g_\gamma(R)$ is the set of points $g \circ p$ for all $p \in R$. Whenever there is no danger of confusion, we write g instead of g_γ.

A *first order structure* E is a couple $\langle D, \mathcal{R} \rangle$ where D is a non empty set and $\mathcal{R}$ is a set of relations defined on D containing the diagonal of D^2. $\mathcal{R}$ is the set of *primitive relations* and D is the *domain* of E.

An *automorphism* of E is a bijection $g : D \to D$ which preserves every primitive relation, that is, $g(R) = R$ for all $R \in \mathcal{R}$. An *invariant relation* of E is a relation which is invariant by all elements of the group G of automorphisms of E.

An *interpretation of a language* $\mathcal{L}_{\alpha\beta\pi}$ on a non empty set D is a map which assigns to each predicate symbol $\bar{R} \in \bar{\mathcal{R}}$ of arity γ a relation R of arity γ defined on D, the relation assigned to $=$ being the diagonal of D^2.

Consider a structure $E = \langle D, \mathcal{R} \rangle$ and a language $\mathcal{L}_{\alpha\beta\pi}$. We say that E is a *$\mathcal{L}_{\alpha\beta\pi}$-structure* when we have fixed an interpretation of $\mathcal{L}_{\alpha\beta\pi}$ on D mapping $\bar{\mathcal{R}}$ onto $\mathcal{R}$. In this case, considering that π may be taken as the least upper bound of the arities of the primitive relations of E, we denote both the language $\mathcal{L}_{\alpha\beta\pi}$ and its set of formulas by $\mathcal{L}^E_{\alpha\beta}$ and we call E a *$\mathcal{L}^E_{\alpha\beta}$-structure*. When considering the language $\mathcal{L}^E_{\alpha\beta}$, we assume that the cardinal of the set $\mathcal{V}$ of variables is greater than $max\{|D|, \alpha\}$.

An *interpretation of the variables* of the language $\mathcal{L}^E_{\alpha\beta}$ is a map $\mathcal{I} : \mathcal{V} \to D$. The notion of an interpretation $\mathcal{I}$ satisfying a formula φ is defined by induction on rules 1) to 4) of construction of formulas. When $\mathcal{I}$ satisfies a formula $\varphi \in \mathcal{L}^E_{\alpha\beta}$, we write $\mathcal{I} \vDash \varphi$; otherwise, we write $\mathcal{I} \nvDash \varphi$. We recall the definition of satisfaction with the same notation as in the definition of formulas.

Definition 2.2. 1) If φ is $\bar{R}\tau$, then $\mathcal{I} \vDash \varphi \Leftrightarrow \mathcal{I} \circ \tau \in R$;

2) If φ is $\neg\psi$, then $\mathcal{I} \vDash \varphi \Leftrightarrow \mathcal{I} \nvDash \psi$;

3) If φ is $\exists\eta\psi$, then $\mathcal{I} \vDash \varphi$ if and only if there exists $\mathcal{I}' : \mathcal{V} \to D$ such that $\mathcal{I}' \vDash \psi$ and $\mathcal{I} \circ \sigma_\varphi = \mathcal{I}' \circ \sigma_\varphi$;

4) If φ is $\bigwedge(\varphi_i)_{i<\gamma}$, then $\mathcal{I} \vDash \varphi \Leftrightarrow \mathcal{I} \vDash \varphi_i$ for all $i < \gamma$.

The definition implies that if $\mathcal{I}, \mathcal{I}'$ are interpretations of variables and $\mathcal{I}|\mathcal{V}(\varphi) = \mathcal{I}'|\mathcal{V}(\varphi)$, then $\mathcal{I} \vDash \varphi$ if and only if $\mathcal{I}' \vDash \varphi$.

Given a formula $\varphi \in \mathcal{L}^E_{\alpha\beta}$ of arity γ, $||\varphi||$ denotes the set of all γ-points $\mathcal{I} \circ \sigma_\varphi$ for which $\mathcal{I} \vDash \varphi$. Sometimes, we say that $||\varphi||$ is the *relation defined* by φ. We say also that $||\varphi||$ is the *set of solutions* of the formula φ. A relation R on D is *definable* in the $\mathcal{L}^E_{\alpha\beta}$-structure, or is $\mathcal{L}^E_{\alpha\beta}$-*definable*, when there is a formula $\varphi \in \mathcal{L}^E_{\alpha\beta}$ which defines R.

We shall give an alternative definition of the relation $||\varphi||$, independently of the notion of interpretation of variables. For this purpose, we need to introduce a very natural operator on the set of relations defined on D.

Let γ, δ be ordinal numbers and let $\xi : \gamma \to \delta$ be a map. ξ induces a map $\xi^* : D^\delta \to D^\gamma$ defined by $\xi^*(p) = p \circ \xi, p \in D^\delta$. For $R \subseteq D^\delta$, $\xi^*(R) = \{p \circ \xi : p \in R\}$. As usual, ξ^{*-1} denotes the inverse map of ξ^* from $\wp(D^\gamma)$ into $\wp(D^\delta)$. For $S \subseteq D^\gamma$, $\xi^{*-1}(S) = \{p \in D^\delta : p \circ \xi \in S\}$. Consider a third ordinal ε and a map $\eta : \delta \to \varepsilon$, then $(\eta \circ \xi)^* = \xi^* \circ \eta^*$. When ξ is bijective, ξ^* is also bijective and $(\xi^*)^{-1} = (\xi^{-1})^*$. For a map $g : D \to D$, $\xi^* \circ g_\delta = g_\gamma \circ \xi^*$. We denote by $\mathcal{C}_\gamma : \wp(D^\gamma) \to \wp(D^\gamma)$ the map associating to a relation $R \subseteq D^\gamma$ its complement $D^\gamma - R$.

Consider again a language $\mathcal{L}^E_{\alpha\beta}$. We define independently of interpretation of variables, a relation $[\![\varphi]\!]$ associated to every formula $\varphi \in \mathcal{L}^E_{\alpha\beta}$. Later, we shall prove that $[\![\varphi]\!] = ||\varphi||$.

We remark the following easy facts using the same notation of definition 2.1:

1) If φ is $\bar{R}\tau$, then $\mathcal{V}(\varphi) = range(\tau) \subseteq \mathcal{V}$;
2) If φ is $\neg\psi$, then $\mathcal{V}(\varphi) = \mathcal{V}(\psi)$;
3) If φ is $\exists\eta\psi$, then $\mathcal{V}(\varphi) = \mathcal{V}(\psi) - range(\eta)$;
4) If φ is $\bigwedge(\varphi_i)_{i<\gamma}$, then $\mathcal{V}(\varphi) = \bigcup_{i<\gamma} \mathcal{V}(\varphi_i)$.

Definition 2.3. The notation being the same as in the definition of formulas,

1) If φ is $\bar{R}\tau$, then $[\![\varphi]\!] = (\sigma_\varphi^{-1} \circ \tau)^{*-1}(R)$;
2) If φ is $\neg\psi$ and the arity of ψ is γ, then $[\![\varphi]\!] = \mathcal{C}_\gamma([\![\psi]\!])$;
3) If φ is $\exists\eta\psi$, then $[\![\varphi]\!] = (\sigma_\psi^{-1} \circ \sigma_\varphi)^*([\![\psi]\!])$;
4) If φ is $\bigwedge(\varphi_i)_{i<\gamma}$, then $[\![\varphi]\!] = \bigcap_{i<\gamma}(\sigma_\varphi^{-1} \circ \sigma_{\varphi_i})^{*-1}([\![\varphi_i]\!])$.

Proposition 2.4. *For every* $\varphi \in \mathcal{L}^E_{\alpha\beta}$, $||\varphi|| = [\![\varphi]\!]$.

Proof. Let $\varphi \in \mathcal{L}^E_{\alpha\beta}$ be a formula of arity γ. For every $p \in D^\gamma$ there exists $\mathcal{I} : \mathcal{V} \to D$ satisfying $p = \mathcal{I} \circ \sigma_\varphi$. Hence, it is enough to show that: $\mathcal{I} \vDash \varphi \Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in [\![\varphi]\!]$. The proof is by induction on rules 1) to 4) of definition of formulas.

1) If φ is the formula $\bar{R}\tau$, $\tau : \gamma \to \mathcal{V}$, then,

$$\begin{aligned}
\mathcal{I} \models \varphi &\Leftrightarrow \mathcal{I} \circ \tau \in R \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \circ (\sigma_\varphi^{-1} \circ \tau) \in R \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in (\sigma_\varphi^{-1} \circ \tau)^{*-1}(R) \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in [\![\varphi]\!].
\end{aligned}$$

2) Assume that the proposition holds for ψ and φ is $\neg\psi$. Then, $\sigma_\varphi = \sigma_\psi$ and

$$\begin{aligned}
\mathcal{I} \models \varphi &\Leftrightarrow \mathcal{I} \not\models \psi \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\psi \notin ||\psi|| \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \notin [\![\psi]\!] \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in \mathcal{C}_\gamma([\![\psi]\!]) \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in [\![\neg\psi]\!] \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in [\![\varphi]\!].
\end{aligned}$$

3) Assume that φ is the formula $\exists\eta\psi$ and that the proposition holds for ψ. By definition, $\mathcal{I} \models \varphi$ if and only if there exists $\mathcal{I}' : \mathcal{V} \to D$, $\mathcal{I}' \models \psi$ and $\mathcal{I} \circ \sigma_\varphi = \mathcal{I}' \circ \sigma_\varphi$. Hence,

$$\begin{aligned}
\mathcal{I} \models \varphi &\Leftrightarrow \mathcal{I}' \models \psi \\
&\Leftrightarrow \mathcal{I}' \circ \sigma_\psi \in ||\psi|| \\
&\Leftrightarrow \mathcal{I}' \circ \sigma_\psi \in [\![\psi]\!] \\
&\Leftrightarrow \mathcal{I}' \circ \sigma_\varphi \circ (\sigma_\varphi^{-1} \circ \sigma_\psi) \in [\![\psi]\!] \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in (\sigma_\varphi^{-1} \circ \sigma_\psi)^{*-1}([\![\psi]\!]) \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in (\sigma_\psi^{-1} \circ \sigma_\varphi)^{*}([\![\psi]\!]) \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in [\![\varphi]\!].
\end{aligned}$$

4) Assume that φ is the formula $\bigwedge(\varphi_i)_{i<\gamma}$ and that the proposition holds for each φ_i. Then,

$$\begin{aligned}
\mathcal{I} \models \varphi &\Leftrightarrow \mathcal{I} \models \varphi_i, \text{ for all } i < \gamma \\
&\Leftrightarrow \mathcal{I} \circ \sigma_{\varphi_i} \in ||\varphi_i||, \text{ for all } i < \gamma \\
&\Leftrightarrow \mathcal{I} \circ \sigma_{\varphi_i} \in [\![\varphi_i]\!], \text{ for all } i < \gamma \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \circ (\sigma_\varphi^{-1} \circ \sigma_{\varphi_i}) \in [\![\varphi_i]\!], \text{ for all } i < \gamma \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in (\sigma_\varphi^{-1} \circ \sigma_{\varphi_i})^{*-1}([\![\varphi_i]\!]), \text{ for all } i < \gamma \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in \textstyle\bigcap_{i<\gamma} (\sigma_\varphi^{-1} \circ \sigma_{\varphi_i})^{*-1}([\![\varphi_i]\!]) \\
&\Leftrightarrow \mathcal{I} \circ \sigma_\varphi \in [\![\varphi]\!].
\end{aligned}$$

End of the proof. □

3. Definability of invariant relations

The following notation will be used in the statements of theorems 3.1 and 3.2 bellow. Given a map $\xi : \gamma \to \delta$, consider the equivalence relation defined on γ: $i \sim j$ if and only if $\xi(i) = \xi(j)$, $i, j \in \gamma$. Denote by I the set of first elements of the equivalence classes. Let γ' and δ' be respectively the ordinals of the sets $\gamma - I$ and $\delta - \xi(\gamma)$, the orders being the orders induced by γ and δ.

Consider a $\mathcal{L}^E_{\alpha\beta}$-structure E, a $\mathcal{L}^E_{\alpha\beta}$-definable relation R and a map $\xi : \gamma \to \delta$ whit $\gamma < \alpha$. We have the following results:

Theorem 3.1. *If $R \subseteq D^\delta$ and assuming that $\delta' < \beta$, then $\xi^*(R)$ is $\mathcal{L}^E_{\alpha\beta}$-definable.*

Theorem 3.2. *If $R \subseteq D^\gamma$ and assuming that $\gamma' < \beta$ and $\delta' < \alpha$, then $\xi^{*-1}(R)$ is $\mathcal{L}^E_{\alpha\beta}$-definable.*

Theorems 3.1 and 3.2 play an important role in the proof of our main result, theorem 3.7. Their proves being not related to the ideas involved in theorem 3.7 are left to section 4.

In the remaining of this section, we denote by δ the cardinal number of D. If δ is infinite, δ^+ denotes the first cardinal greater than δ. If δ is finite, δ^+ is ω. Observe that δ^+ is regular. We also remark that if γ is an ordinal, we use the notation δ^γ instead of $\delta^{|\gamma|}$ to express *cardinal* exponentiation.

A point $p \in D^\delta$ is a *normal point* if the map $p : \delta \to D$ is a bijection onto D. The main idea in the proof of theorem 3.7 consists in proving first that orbits of normal points are $\mathcal{L}^E_{\delta^+\delta^+}$-definable and then, showing that any orbit of arity $\gamma < \delta^+$ can be obtained from an orbit of a normal point by means of an operator ξ^* suitably chosen.

Proposition 3.3. *The set N of normal points is $\mathcal{L}^E_{\delta^+\delta^+}$-definable.*

Proof. Let y denote a variable distinct of all x_i, $i < \delta$. Then,

$$\varphi = \forall y[\bigvee_{i<\delta}(y = x_i)] \wedge [\bigwedge_{i,j<\delta, i\neq j}(x_i \neq x_j)]$$

is a formula of $\mathcal{L}^E_{\delta^+\omega} \subseteq \mathcal{L}^E_{\delta^+\delta^+}$ and clearly $||\varphi|| = N$. □

Given a relation R of arity γ and a normal point q, let Θ_R be the set of maps $\xi : \gamma \to \delta$ satisfying $\xi^*(q) \in R$. Define

$$\bar{M}_R = \bigcap_{\xi\in\Theta_R} \xi^{*-1}(R).$$

Proposition 3.4. *Assume that γ satisfies $\delta^\gamma < \delta^+$. Then, for any relation $\mathcal{L}^E_{\delta^+\delta^+}$-definable R of arity γ and any normal point q, $\bar{M}_R$ is $\mathcal{L}^E_{\delta^+\delta^+}$-definable. Moreover, for any map $g : D \to D$, $g(R) \subseteq R$ if and only if $g(q) \in \bar{M}_R$.*

Proof. If $\delta = 1$ the proposition is trivial. If $\delta > 1$, the condition on γ yields $\gamma < \delta^+$. By theorem 3.2, $\xi^{*-1}(R)$ is $\mathcal{L}^E_{\delta^+\delta^+}$-definable for all maps $\xi : \gamma \to \delta$. By definition of Θ_R, $|\Theta_R| \leq \delta^\gamma < \delta^+$. Therefore, $\bar{M}_R$ is $\mathcal{L}^E_{\delta^+\delta^+}$-definable. Since q is bijective, for every $p \in R$ there is $\xi \in \Theta_R$ satisfying $p = q \circ \xi$. Hence,

$$\begin{aligned} g(R) \subseteq R &\Leftrightarrow \forall p \in R(g_\gamma \circ p \in R) \\ &\Leftrightarrow \forall \xi \in \Theta_R(g_\gamma(\xi^*(q)) \in R) \\ &\Leftrightarrow \forall \xi \in \Theta_R(\xi^*(g_\delta(q)) \in R) \\ &\Leftrightarrow \forall \xi \in \Theta_R(g_\delta(q) \in \xi^{*-1}(R)) \\ &\Leftrightarrow g_\delta(q) \in \bar{M}_R. \end{aligned}$$

End of the proof. □

Proposition 3.5. *Assume that γ satisfies $\delta^\gamma < \delta^+$ and let q be a normal point. For any relation $R \subseteq D^\gamma$ there exists a relation $M_R \subseteq D^\delta$, $\mathcal{L}^E_{\delta^+\delta^+}$-definable satisfying the condition: for any bijection $g : D \to D$, $g_\gamma(R) = R$ if and only if $g_\delta(q) \in M_R$.*

Proof. Let R' be the complement of R in D^γ and let

$$M_R = \bar{M}_R \cap \bar{M}_{R'} \cap N.$$

By propositions 3.3 and 3.4, M_R is $\mathcal{L}^E_{\delta^+\delta^+}$-definable. Since $g_\delta(N) = N$, $g_\delta(q) \in M_R$ if and only if $g_\gamma(R) \subseteq R$ and $g_\gamma(R') \subseteq R'$. Since g_γ is a bijection, the last two conditions are equivalent to $g_\gamma(R) = R$. □

Proposition 3.6. *Let $E = \langle D, \mathcal{R} \rangle$ be a first order structure and let $\alpha = sup\{|\mathcal{R}|, \delta\}$. If the arity γ of every primitive relation $R \in \mathcal{R}$ satisfies $\delta^\gamma < \delta^+$, the orbit of every normal point q by the group G of automorphisms of E is $\mathcal{L}^E_{\alpha^+\delta^+}$-definable.*

Proof. Let $M = \bigcap_{R\in\mathcal{R}} M_R$. By proposition 3.5 and the definition of α, M is $\mathcal{L}^E_{\alpha^+\delta^+}$-definable. Let $g : D \to D$ be an automorphism of E. For every $R \in \mathcal{R}$ of arity γ, $g_\gamma(R) = R$. Hence, $g_\delta(q) \in M$. Conversely, if $p \in M$, $g = q^{-1} \circ p$ is a bijection of D onto D and $g(q) = p$. Since $g(q) \in M_R$, it follows from proposition 3.5 that $g(R) = R$, for all $R \in \mathcal{R}$. Hence, g is an automorphism of E and M is the orbit of q by G. □

Theorem 3.7. *Let $E = \langle D, \mathcal{R} \rangle$ be a first order structure and let $\alpha = sup\{|\mathcal{R}|, \delta\}$. Assume that the arity γ of every primitive relation $R \in \mathcal{R}$ satisfies $\delta^\gamma < \delta^+$. Then, every orbit of the group G of automorphisms of E of arity less than δ^+ is $\mathcal{L}^E_{\alpha^+\delta^+}$-definable.*

Proof. Let S be an orbit of arity $\gamma < \delta^+$. Choose a normal point q and consider a map $\xi : \gamma \to \delta$ such that $\xi^*(q) \in S$. By proposition 3.6, the orbit M of q is $\mathcal{L}^E_{\alpha^+\delta^+}$-definable. As $\gamma < \delta^+$ (since $\delta^\gamma < \delta^+$) and $\delta' \leq \delta < \delta^+$, by theorem 3.1, $\xi^*(M)$ is $\mathcal{L}^E_{\alpha^+\delta^+}$-definable. The hypothesis on S and the commutativity $g_\delta \circ \xi = \xi \circ g_\gamma$ yield $S = \{g_\gamma(\xi^*(q)) : g \in G\}$ and $M = \{g_\delta(q) : g \in G\}$, hence $\xi^*(M) = \{\xi^*(g_\delta(q)) : g \in G\} = S$. Therefore, S is $\mathcal{L}^E_{\alpha^+\delta^+}$-definable. $\square$

Corollary 3.8. *Every invariant relation of E of arity γ satisfying $\delta^\gamma < \delta^+$ is $\mathcal{L}^E_{\alpha^+\delta^+}$-definable.*

Proof. An invariant relation R of arity γ is the union of orbits of G. Since $|D^\gamma| = \delta^\gamma < \delta^+$, the cardinality of the union is at most δ^γ. Therefore, R is $\mathcal{L}^E_{\alpha^+\delta^+}$-definable. $\square$

Corollary 3.9. *If D is finite, every invariant relation of finite arity is $\mathcal{L}^E_{\omega\omega}$-definable.*

Corollary 3.10. *If $|D| = \omega_1$ and $|\mathcal{R}| \leq \omega_1$, every invariant relation of arity less or equal to ω is $\mathcal{L}^E_{\omega_2\omega_2}$-definable.*

Let E be a $\mathcal{L}^E_{\alpha\beta}$-structure. Two points $p_1, p_2 \in D^\gamma$ which are in the same orbit of the action of G on D^γ satisfy the same set of formulas of $\mathcal{L}^E_{\alpha\beta}$. The converse of this statement is not true in general. However, it is natural to ask whether α and β can be so chosen for the converse to be true. Keeping the notation as in theorem 3.7, corollary 3.11 below answers this question.

Corollary 3.11. *Two points $p_1, p_2 \in D^\gamma$ which satisfy the same formulas of $\mathcal{L}^E_{\alpha^+\delta^+}$ are in the same orbit of G.*

Proof. Let $G \cdot p_1$ be the orbit of p_1 under the action of G on D^γ. By theorem 3.7, there exists $\varphi \in \mathcal{L}^E_{\alpha^+\delta^+}$ which defines $G \cdot p_1$, that is, $||\varphi|| = G \cdot p_1$. If $p_2 \notin G \cdot p_1$, then p_2 does not satisfies φ, against the hypothesis that p_1 and p_2 satisfy the same formulas of $\mathcal{L}^E_{\alpha^+\delta^+}$. $\square$

4. Proofs of theorems 3.1 and 3.2

Given a formula $\varphi \in \mathcal{L}^E_{\alpha\beta}$, let $Var(\varphi)$ be the set of all variables of φ and let λ be a bijection of $Var(\varphi)$ onto a subset $\mathcal{U}$ of the set of variables $\mathcal{V}$; $\lambda^*(\varphi)$ denotes the formula obtained replacing each variable of φ by its image by λ. The set of free variables of $\lambda^*(\varphi)$ is $\lambda(\mathcal{V}(\varphi))$. Let γ and δ be the arities of φ and $\lambda^*(\varphi)$ respectively and let $\sigma' : \delta \to \lambda(\mathcal{V}(\varphi))$ be the order preserving bijection. Consider the map $\xi = \sigma'^{-1} \circ \lambda \circ \sigma_\varphi : \gamma \to \delta$.

Proposition 4.1. $||\lambda^*(\varphi)|| = \xi^{*-1}(||\varphi||)$.

Proof. By proposition 2.4 it is sufficient to prove that $[\![\lambda^*(\varphi)]\!] = \xi^{*-1}([\![\varphi]\!])$. The proof is by induction on rules 1) to 4) of definition of formulas. By definition,

$$[\![\varphi]\!] = (\sigma_\varphi^{-1} \circ \tau)^{*-1}(R)$$

and

$$[\![\lambda^*(\varphi)]\!] = (\sigma'^{-1} \circ \lambda \circ \tau)^{*-1}(R)$$

Then,

$$\begin{aligned} p \in \xi^{*-1}([\![\varphi]\!]) &\Leftrightarrow p \circ (\sigma'^{-1} \circ \lambda \circ \sigma_\varphi) \in [\![\varphi]\!] \\ &\Leftrightarrow p \circ (\sigma'^{-1} \circ \lambda \circ \sigma_\varphi) \in (\sigma_\varphi^{-1} \circ \tau)^{*-1}(R) \\ &\Leftrightarrow p \circ (\sigma'^{-1} \circ \lambda \circ \sigma_\varphi) \circ (\sigma_\varphi^{-1} \circ \tau) \in R \\ &\Leftrightarrow p \circ (\sigma'^{-1} \circ \lambda \circ \tau) \in R \\ &\Leftrightarrow p \in [\![\lambda^*(\varphi)]\!]. \end{aligned}$$

The other steps of the induction are straightforward. □

If $\lambda|\mathcal{V}(\varphi)$ preserves the order, then $\gamma = \delta$ and $\xi : \gamma \to \delta$ is the identity map. Hence, by proposition 4.1, $||\lambda^*(\varphi)|| = ||\varphi||$.

Let $\xi : \gamma \to \delta$ be a map. The notation being the same as in section 3, consider the ordinals γ', δ' and let $\bar{\xi}'_1 : \gamma' \to \gamma - I$ and $\bar{\xi}'_2 : \delta' \to \delta - \xi(\gamma)$ be the order preserving bijections. Finally, consider the maps $\xi'_1 = \bar{\xi}'_1 \circ id_1$ and $\xi'_2 = \bar{\xi}'_2 \circ id_2$ where id_1 and id_2 are respectively the identical maps of $\gamma - I$ into γ and of $\delta - \xi(\gamma)$ into δ.

Proposition 4.2. *Let $R \subseteq D^\delta$ be a relation $\mathcal{L}^E_{\alpha\beta}$-definable and let $\xi : \gamma \to \delta$ be an order preserving map. If $\delta' < \beta$, then $\xi^*(R)$ is $\mathcal{L}^E_{\alpha\beta}$-definable.*

Proof. Consider a formula φ of $\mathcal{L}^E_{\alpha\beta}$ defining R, i.e. $||\varphi|| = R$, and let $\eta = \sigma_\varphi \circ \xi'_2 : \delta' \to \mathcal{V}(\varphi)$.

$$\begin{tikzcd}
 & \mathcal{V}(\varphi) \\
\delta' \arrow[ur, "\eta"] \arrow[r, "\xi_2'"] \arrow[dr, "\bar{\xi}_2'"'] & \delta \arrow[u, "\sigma_\varphi"'] \\
 & \delta - \xi(\gamma) \arrow[u, "id_2"']
\end{tikzcd}$$

Since $\delta' < \beta$, $\psi = \exists\eta\varphi$ is a formula of $\mathcal{L}^E_{\alpha\beta}$. By proposition 2.4, $||\psi|| = (\sigma_\varphi^{-1} \circ \sigma_\psi)^*(R)$. Considering that $\mathcal{V}(\psi) = \mathcal{V}(\varphi) - range(\eta)$, then $\sigma_\varphi^{-1}(\mathcal{V}(\varphi)) = \delta - \xi(\gamma)$; since ξ is order preserving, then $\xi|\gamma - I : (\gamma - I) \to (\delta - \xi(\gamma))$ is an order preserving map and then $\sigma_\psi = \sigma_\varphi \circ \xi$. Hence,

$$||\psi|| = (\sigma_\varphi^{-1} \circ (\sigma_\varphi \circ \xi))^*(R) = \xi^*(R).$$

End of the proof. □

Proposition 4.3. *Let $R \subseteq D^\gamma$ be a relation $\mathcal{L}^E_{\alpha\beta}$-definable and let $\xi : \gamma \to \delta$ be an order preserving map. If $\delta' < \alpha$, then $\xi^{*-1}(R)$ is also $\mathcal{L}^E_{\alpha\beta}$-definable.*

Proof. Let φ be a formula of $\mathcal{L}^E_{\alpha\beta}$ satisfying $||\varphi|| = R$ and let $Var(\varphi)$ be the set of all variables of φ. Consider the map $\lambda' = (\chi|\delta) \circ \xi \circ \sigma_\varphi^{-1} : \mathcal{V}(\varphi) \to \mathcal{V}$ and extend λ' to a bijection λ of $Var(\varphi)$ onto a subset $\mathcal{U}$ of $\mathcal{V}$. Since ξ is order preserving, so is λ'. Hence, by proposition 4.1, $||\lambda^*(\varphi)|| = ||\varphi||$. Let ψ_1 be the formula

$$\bigwedge (x_{\xi_2'(i)} = x_{\xi_2'(i)})_{i \in \delta'}.$$

Since $\delta' < \alpha$, ψ_1 is a formula of $\mathcal{L}^E_{\alpha\beta}$ and $||\psi_1|| = D^{\delta'}$. Then, the formula $\lambda^*(\varphi) \wedge \psi_1$, denoted by ψ, is also a formula of $\mathcal{L}^E_{\alpha\beta}$. By construction, $\mathcal{V}(\psi) = \mathcal{V}(\lambda^*(\varphi)) \cup \mathcal{V}(\psi_1) = range(\chi|\xi(\gamma)) \cup range(\chi|\delta - \xi(\gamma)) = range(\chi|\delta)$, and then we have

$$\sigma_\psi = \chi|\delta, \qquad \sigma_{\lambda^*(\varphi)} = (\chi|\delta) \circ \xi, \qquad \sigma_{\psi_1} = (\chi|(\delta - \xi(\gamma))) \circ \xi_2'.$$

Hence,

$$\sigma_\psi^{-1} \circ \sigma_{\lambda^*(\varphi)} = \xi, \qquad \sigma_\psi^{-1} \circ \sigma_{\psi_1} = \xi_2'.$$

By proposition 2.4,

$$\begin{aligned}
||\psi|| &= \xi^{*-1}(||\lambda^*(\varphi)|| \bigcap (\xi_2')^{*-1}(||\psi_1||)) \\
&= \xi^{*-1}(||\varphi|| \bigcap D^\delta) \\
&= \xi^{*-1}(R).
\end{aligned}$$

End of the proof. $\square$

We now prove theorems 3.1 and 3.2.

Proof of theorem 3.1. Let $\bar{\gamma}$ and $\bar{\delta}$ be the ordinals of I and $\xi(\gamma)$ respectively, the notation being as in section 3. Let also $\bar{\xi}_1 : \bar{\gamma} \to I$, $\bar{\xi}_2 : \bar{\delta} \to \xi(\gamma)$ be the order preserving bijections. Then,

$$\bar{\xi} = \bar{\xi}_2^{-1} \circ (\xi|I) \circ \bar{\xi}_1 : \bar{\gamma} \to \bar{\delta}$$

is a bijection.

Consider the maps $\xi_1 = id_1 \circ \bar{\xi}_1$ and $\xi_2 = id_2 \circ \bar{\xi}_2$ where $id_1 : I \to \gamma$ and $id_2 : \xi(\gamma) \to \delta$ are the identical maps.

$$\begin{tikzcd}
\bar{\delta} \arrow[r, "\bar{\xi}_2"] & \xi(\gamma) \arrow[r, "id_2"] & \delta \arrow[r, "\sigma_\varphi"] & \mathcal{V}(\varphi) \\
\bar{\gamma} \arrow[u, "\bar{\xi}"] \arrow[r, "\bar{\xi}_1"'] & I \arrow[u, "\xi|I"] \arrow[r, "id_1"'] & \gamma \arrow[u, "\xi"]
\end{tikzcd}$$

Let S be the set of points $p \in D^\gamma$ satisfying the condition: for all $i, j \in \gamma$, if $\xi(i) = \xi(j)$, then $p(i) = p(j)$. For $i \in I$, let P_i be the set of elements $j \in \gamma$ satisfying $\xi(j) = \xi(i)$ and let γ_i be the ordinal of P_i. Denote by $\sigma_i : \gamma_i \to P_i$ the order preserving bijection. Since $|\gamma_i| \leq |\gamma| \leq \gamma < \alpha$, for each $i \in I$, the formula

$$\bigwedge (x_{\sigma_i(0)} = x_{\sigma_i(j)})_{j \in \gamma_i}$$

is a $\mathcal{L}^E_{\alpha\beta}$-formula. Denoting the last formula by ψ_i, the formula

$$\bigwedge (\psi_{\bar{\xi}_1(k)})_{k \in \bar{\gamma}},$$

denoted by ψ, is also a $\mathcal{L}^E_{\alpha\beta}$-formula. Clearly, $||\psi|| = S$, proving that S is $\mathcal{L}^E_{\alpha\beta}$-definable.

Since ξ_2 is order preserving and $\delta' < \beta$, proposition 4.2 applies yielding that $\xi_2^*(R)$ is $\mathcal{L}^E_{\alpha\beta}$-definable. Remarking that $\bar{\xi}$ is a bijection from $\bar{\gamma}$ onto $\bar{\delta}$, by proposition 4.1, $\bar{\xi}^*(\xi_2^*(R))$ is also $\mathcal{L}^E_{\alpha\beta}$-definable. Since ξ_1 is order preserving and $|\gamma'| \leq |\gamma| \leq \gamma < \alpha$, by proposition 4.3, $\xi_1^{*-1}(\bar{\xi}^*(\xi_2^*(R)))$ is $\mathcal{L}^E_{\alpha\beta}$-definable.

To complete the proof it is enough to show that

$$\xi^*(R) = \xi_1^{*-1}(\bar{\xi}^*(\xi_2^*(R))) \bigcap S.$$

Denote by A the second member of the equality above and let $p \in D^\gamma$. Then, we have

$$\begin{aligned} p \in A \quad &\Leftrightarrow \quad p \circ \xi_1 \in (\xi_2 \circ \bar{\xi})^*(R) \text{ and } p \in S \\ &\Leftrightarrow \quad p \circ \xi_1 = q \circ (\xi_2 \circ \bar{\xi}) \text{ and } q \in R,\ p \in S \\ &\Leftrightarrow \quad p \circ \xi_1 = (q \circ \xi) \circ \xi_1 \text{ and } q \in R,\ p \in S. \end{aligned}$$

Since p and $q \circ \xi$ belong to S, the last equality yields

$$p \in A \Leftrightarrow p = q \circ \xi \text{ and } q \in R,\ p \in S.$$

Hence, $p \in A \Leftrightarrow p \in \xi^*(R)$. □

Let $R \subseteq D^\delta$ be a relation $\mathcal{L}^E_{\alpha\beta}$-definable and let $\xi : \gamma \to \delta$ be a map. If $\gamma < \alpha$ and $\delta < \beta$, it follows from theorem 3.1 that $\xi^*(R)$ is also $\mathcal{L}^E_{\alpha\beta}$-definable.

Proof of theorem 3.2. Since $\gamma' < \beta$, by proposition 4.2, we have that $\xi_1^*(R)$ is $\mathcal{L}^E_{\alpha\beta}$-definable. By proposition 4.1, $(\bar{\xi}^{-1})^*(\xi_1^*(R))$ is also $\mathcal{L}^E_{\alpha\beta}$-definable. By proposition 4.3 and the hypothesis $\delta' < \alpha$, $\xi_2^{*-1}((\bar{\xi}^{-1})^*(\xi_1^*(R)))$ is $\mathcal{L}^E_{\alpha\beta}$-definable. The notation being as in the proof of theorem 3.1, the hypothesis $\gamma < \alpha$ yields that S is $\mathcal{L}^E_{\alpha\beta}$-definable. To complete the proof it is enough to show that

$$\xi^{*-1}(R) = \xi_2^{*-1}((\bar{\xi}^{-1})^*(\xi_1^*(R \cap S))).$$

Let B be the second member of the equality above and let $p \in D^\gamma$. Then,

$$\begin{aligned} p \in B \quad &\Leftrightarrow \quad p \circ \xi_2 \in (\xi_1 \circ \bar{\xi}^{-1})^*(R \cap S) \\ &\Leftrightarrow \quad p \circ \xi_2 = q \circ (\xi_1 \circ \bar{\xi}^{-1}) \text{ and } q \in R \cap S \\ &\Leftrightarrow \quad p \circ \xi_2 \circ \bar{\xi} = q \circ \xi_1 \text{ and } q \in R \cap S \\ &\Leftrightarrow \quad p \circ \xi \circ \xi_1 = q \circ \xi_1 \text{ and } q \in R \cap S. \end{aligned}$$

Since $p \circ \xi$ and q belong to S, $p \circ \xi \circ \xi_1 = q \circ \xi_1 \Leftrightarrow p \circ \xi = q$. Hence,

$$p \in B \Leftrightarrow p \circ \xi = q \text{ and } q \in R \cap S.$$

But, $p \circ \xi = q$ and $q \in R \cap S \Leftrightarrow p \in \xi^{*-1}(R)$. Therefore, $B = \xi^{*-1}(R)$, completing the proof. □

A consequence of theorem 3.2 is that if R is $\mathcal{L}^E_{\alpha\beta}$-definable and $\gamma < \alpha,\ \gamma < \beta,\ \delta < \alpha$, then $\xi^{*-1}(R)$ is $\mathcal{L}^E_{\alpha\beta}$-definable.

5. Complete first order languages

In this section, we extend the results we have obtained to complete first order languages. Let $\mathcal{L}^E_{\alpha\beta}$ be a language associated to a structure E over a domain D. We add to the symbols of $\mathcal{L}^E_{\alpha\beta}$ a set $\bar{\mathcal{C}}$ of constant symbols and a set $\bar{\mathcal{F}}$ of functions symbols; each function symbol $\bar{f} \in \bar{\mathcal{F}}$ has an ordinal arity $\gamma < \alpha$. To each constant symbol $\bar{c} \in \bar{\mathcal{C}}$ it is associated an element $c \in D$ which is added to the structure E and to each function symbol $\bar{f} \in \bar{\mathcal{F}}$ of arity γ it is associated a function f of arity γ defined on D with values in D which is added to E also. We denote by $\bar{\mathcal{L}}^E_{\alpha\beta}$ the complete first order language obtained in this way and we say that E is an $\bar{\mathcal{L}}^E_{\alpha\beta}$-structure.

The set $\mathcal{T}$ of terms of $\bar{\mathcal{L}}^E_{\alpha\beta}$ is defined by induction as usual. $\mathcal{T}$ is the least set of sequences satisfying the following conditions:

1) If x_i is a variable, the sequence $\langle x_i \rangle \in \mathcal{T}$;

2) If $\bar{c}$ is a symbol of constant, $\langle \bar{c} \rangle \in \mathcal{T}$;

3) If $\bar{f} \in \bar{\mathcal{F}}$ has arity γ, and $\tau : \gamma \to \mathcal{T}$ is a sequence of terms, the concatenation $\bar{f}\tau \in \mathcal{T}$.

We add to the set of formulas of $\mathcal{L}^E_{\alpha\beta}$ the formulas obtained by the following rule: If $\bar{R}$ is a predicate symbol of $\mathcal{L}^E_{\alpha\beta}$ of arity $\gamma < \pi$ and γ, and $\tau : \gamma \to \mathcal{T}$ is a sequence of terms, then $\bar{R}\tau$ is a formula of $\bar{\mathcal{L}}^E_{\alpha\beta}$. Since α is a regular cardinal, the arity of $\bar{R}\tau$ is less then α.

Given an interpretation of the variables $\mathcal{I} : \mathcal{V} \to D$, we extend $\mathcal{I}$ to an interpretation of the terms $\mathcal{I}^* : \mathcal{T} \to D$ in the following way:

1) If τ is $\langle x_i \rangle$, then $\mathcal{I}^*(\tau) = \mathcal{I}(\langle x_i \rangle)$;

2) If τ is $\langle \bar{c} \rangle$, then $\mathcal{I}^*(\tau) = c$;

3) If τ is $f\sigma$ where f is a function symbol of arity γ, then $\mathcal{I}^*(\tau) = f((\mathcal{I}^*(\sigma(i)))_{i<\gamma})$.

It remains to define when a formula $\bar{R}\tau$, where τ is a sequence of terms, is satisfied by an interpretation of variables. We say that:

$$\mathcal{I} \vDash \bar{R}\tau \Leftrightarrow \mathcal{I}^* \circ \tau \in R.$$

The notion of relation $||\varphi||$ defined by a formula $\varphi \in \bar{\mathcal{L}}^E_{\alpha\beta}$ remains unchanged.

Let $\tilde{E}$ be the relational structure over D obtained adding to the primitive relations of E the following relations:

1) For each constant symbol $\bar{c} \in \bar{\mathcal{C}}$, the unary relation $\{c\}$;

2) For each function symbol $\bar{f} \in \bar{\mathcal{F}}$ of arity γ, the $\gamma + 1$-relation $\tilde{f}$ which defines the function f.

Let $\mathcal{L}^{\tilde{E}}_{\alpha\beta}$ the relational language obtained adding to the predicate symbols of $\mathcal{L}^{E}_{\alpha\beta}$ a predicate symbol for every relation which was added to the relations of the structure E. The structure $\tilde{E}$ is a $\mathcal{L}^{\tilde{E}}_{\alpha\beta}$-relational structure.

A straightforward induction on the rules of construction of formulas shows that the set of definable relations of the complete $\bar{\mathcal{L}}^{E}_{\alpha\beta}$-structure is the same as the set of definable relations of the relational $\mathcal{L}^{\tilde{E}}_{\alpha\beta}$-structure $\tilde{E}$. Thus, theorem 3.7 and its corollaries apply without change to the $\bar{\mathcal{L}}^{E}_{\alpha\beta}$-structure.

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