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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: COORDINATIZATION IS ALGEBRAIZATION

A b s t r a c t. The methods of categorical abstract algebraic logic are employed to show that the classical process of the coordinatization of abstract (affine plane) geometry can be viewed under the light of the algebraization of logical systems. This link offers, on the one hand, a new perspective to the coordinatization of geometry and, on the other, enriches abstract algebraic logic by bringing under its wings a very well-known geometric process, not known hitherto to be related or amenable to its methods and techniques. The algebraization takes the form of a deductive equivalence between two institutions, one corresponding to affine plane geometry and the other to Hall ternary rings.

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1. Introduction

Around the end of the nineteenth and the beginning of the last century, several authors considered the process of coordinatization of various axiomatically defined abstract geometries. Among them were the affine plane geometry, affine plane geometry with the Desargues and Pappus properties, projective plane geometry etc. A classic introductory book to be consulted on this topic, which partly inspired our investigations and to which the reader will be referred for several constructions and results that will be used in the present paper, is Blumenthal's "A Modern View of Geometry" [3]. A more recent reference on this material, that is now available online in its second edition, is Peter Cameron's "Projective and Polar Spaces" [6]. Finally, a general reference pertaining to the topic that may prove of further help to the interested reader is the handbook [5].

The general process of coordinatization consists, generally speaking, of using the points and incidence structure of a given abstract geometry as a basis for constructing an abstract algebraic structure. Ordinarily, the points of the geometry serve as the building blocks for the universe of the algebraic structure. Then, the structural properties of the geometry are used to define one or more algebraic operations on this universe. These operations are shown to satisfy several algebraic properties that may lead to a characterization of the resulting algebraic structure. These algebraic structures are then used to assign coordinates to the points of the original abstract geometry. One of the benefits of this process of coordinatization is that, having the algebraic structure of the coordinates at hand, enables one to manipulate the points and lines of the geometry and study several of their properties by algebraic means, using the operations of the resulting algebras. For instance, one may define slopes, equations of lines, discover intersection points by solving systems of equations etc.

Taking the opposite point of view, another significant advantage of this process is that, given an abstract algebra of the kind used in the coordinatization, there may be a reverse construction for obtaining an abstract geometry by defining points and lines using the elements of the universe of the algebraic structure. Then, by studying the structure of the resulting geometry, and taking into account the interplay between algebra and geometry, one may develop useful geometric intuitions concerning aspects of the original algebraic structures. So, as it turns out, on several occasions, coordinatization has positive consequences not only in the study of the original abstract geometry, by making it amenable to algebraic techniques, but also in the study of a class of abstract algebras, by possibly endowing their operations with some geometric dimension that may make their presentation more intuitive and, thus, better understood and more easily and efficiently studied.

The initial motivation for the attempt to relate the coordinatization of geometry to the algebraization of logical systems arose from the fact that the two processes are very similar in intent and nature. As with the coordinatization of geometry, the goal of abstract algebraic logic, as crystalized in the seminal memoirs monograph "Algebraizable Logics" of Blok and Pigozzi [2], is to associate with a logical system a class of abstract algebras in such a way that the logical entailment of the system may be studied by taking advantage of the algebraic properties of the class of algebras. More precisely, the logical entailment is translated to the equational entailment of the corresponding class of algebras and, then, its study may be carried out by a detailed analysis of the structure of the corresponding algebraic congruences. This can be successfully carried out under the condition that a close enough relationship exists between the original logical and the induced algebraic entailment systems. Thus, leaving the apparent discrepancies in the details involved aside, there are essential similarities in the intention and the goals of coordinatization in geometry and algebraization in logic: both strive to associate with their objects of study (geometries and logics, respectively) an algebraic system or a class of algebraic systems, in such a way that the basic structures under consideration (incidence and consequence, respectively) are reflected in, or can be expressed in terms of, the algebraic properties of the associated algebraic systems. Then, by looking at the algebras (algebraic operations and equational consequence, respectively) one can draw conclusions about the original geometry or logic. The passage to abstract algebra in both cases is assumed to facilitate the study of the original objects, since the structure of algebraic objects is, in many instances, better understood and more readily analyzable than that of the original objects.

In this paper, we carry out this project of relating the coordinatization of abstract (affine plane) geometry, as presented in [3], with the process of algebraization of logics expressed as institutions, as it was developed in the original papers on this subject [15, 16]. The subfield of categorical abstract algebraic logic (CAAL), whose definitions and techniques we employ in this work, is the one responsible for making the original concepts, methods and techniques employed in abstract algebraic logic (see, e.g., [9, 7, 10]) to logics expressed as institutions [11, 12] or π-institutions [8]. In the sense of CAAL, the algebraization of an institution [16] takes the form of a deductive equivalence [15] of the institution with another institution whose classes of models consist of algebraic systems and whose sentences consist of equations. Thus, the way coordinatization is related to algebraization consists, roughly speaking, of expressing the abstract geometry as a logical system in the form of an institution, and, then, using the coordinatizing abstract algebraic structures as algebraic models of another algebraic institution. A critical selection of sentences of the geometric institution and of equations of the algebraic institution must be made, so that the coordinatization process may be accurately captured by the process of establishing a deductive equivalence between the geometric institution and the corresponding algebraic institution. This process involves the construction of translations between incidence relations and equations and vice-versa, which become interpretations between entailments on incidence relations and equational entailments, that are inverses of one another in the precise sense stipulated by CAAL (see [2] and [15]).

The link established in this way between the coordinatization of abstract geometry and the general process of algebraization of institutional logics is interesting for several reasons. First, it reveals a link between two seemingly unrelated processes: that of associating a class of algebras as an algebraic semantics to a logical system and that of associating a class of algebras as coordinate algebras to the models of an abstract geometry. Second, it sheds new light to the nature of coordinatization by linking it, for instance, to the association of Boolean algebras to classical propositional calculus through this very abstract channel. Finally, it enriches abstract algebraic logic by bringing under its umbrella a very well-known subfield of geometric investigations, not known previously to be amenable to its methods and techniques. Thus, the main formal result, Theorem 16, of our work, that spans both geometry and logic, justifies (at least to some extent) the motto:

> "The **coordinatization** of abstract geometry is a form of **algebraization**, in the formal sense attributed to this term by the modern theory of *abstract algebraic logic*."

The paper is organized as follows: In Section 2, abstract geometries, the geometries of affine planes, are introduced and morphisms between them, termed geometric morphisms, are defined. In Section 3, an account of the process of the coordinatization of the affine plane, as detailed in, e.g., [3], is presented.[1] Moreover, coordinate systems are added to abstract geometries and the resulting coordinated abstract geometries and coordinated geometric morphisms between them are formally defined. The resulting category is denoted by **AG**. In Section 4 our main work starts in earnest. An institution $\mathcal{AG}$ is built that formalizes the system of abstract (affine plane) geometry as a logical system. This involves the construction of its syntax, consisting of formulas made up using the incidence relation symbol of the geometry, and of its semantics, which consists of the models of the axioms of affine plane geometry. The two interact through the satisfaction relation, which, as is expected, involves an incidence formula being true if, intuitively speaking, the corresponding intended incidence relation holds in the chosen geometric model. In Section 5, the properties of the coordinatizing rings of the abstract geometries proven in [3] are revisited together with the process by which, given such a ring, called a Hall ternary ring, an abstract geometry may be constructed. If that geometry is endowed with a canonically associated coordinate system, then the resulting coordinate ring coincides with the originally given Hall ternary ring. In Section 6, the institution $\mathcal{GA}$ of geometric algebra is constructed. It is essentially the institution of the equational theory of Hall ternary rings. The fact that Hall ternary rings provide the coordinate rings for abstract geometry is the connecting link between this institution and that of abstract geometry, introduced in Section 4. Indeed, in Section 7, it is shown that $\mathcal{GA}$ is an algebraic semantics for $\mathcal{AG}$ by exhibiting an interpretation from the sentences of $\mathcal{AG}$ into the equations of $\mathcal{GA}$. Finally, in Section 8, it is shown that an inverse interpretation from the equations of $\mathcal{GA}$ into the sentences of $\mathcal{AG}$ exists. The existence of these two mutually inverse interpretations establish the fact that $\mathcal{AG}$ is algebraizable in the sense of CAAL with $\mathcal{GA}$ as its equivalent algebraic semantics. Since $\mathcal{AG}$ is the institution of affine plane geomeries whereas its equivalent algebraic semantics $\mathcal{GA}$ is the institution

[1] A referee has pointed out that the technical details of the treatment of the algebraization of abstract geometry might become less tedious if a Tarski-Szmielew style presentation of affine plane geometry [14], using points as the only primitive objects, is chosen in place of the presentation used in [3], employing both points and lines.

of the coordinate rings of affine plane geometries, the deductive equivalence established between these two institutions reveals the close relationship between coordinatization in geometry and algebraization in abstract algebraic logic that was displayed in the boxed motto above.

For all unexplained categorical terminology and notation the reader is encouraged to consult any of the standard references [1, 4, 13].

2. Abstract Geometry and Geometric Morphisms

The notion of *abstract geometry* (see, e.g., Chapter IV of [3], where this affine plane geometry is defined) abstracts the very basic features common in many applied plane geometries, in particular those of many finite geometries and of the Euclidean plane geometry. An **abstract geometry** $\mathbf{G} = \langle P, L, I\rangle$ consists of a set P of abstract **points**, a set L of abstract **lines**, and an **incidence relation** $I \subseteq P \times L$, which are subject to three postulates (the term **parallel** is used to characterize two lines $l, l' \in L$, when there does not exist any point $p \in P$ incident to both):

Postulate 1 If $p_0, p_1 \in P$ with $p_0 \neq p_1$, then there exists a unique line $l \in L$ incident to both p_0 and p_1;

Postulate 2 If $p \in P$ and $l \in L$, such that p is not incident to l, then there exists a unique line l' parallel to l, such that p is incident to l';

Postulate 3 There exists at least one quadruple of distinct points in P, no three of which are incident to the same line.

Given two abstract geometries $\mathbf{G} = \langle P, L, I\rangle$ and $\mathbf{G}' = \langle P', L', I'\rangle$, a **morphism of abstract geometries** (or a **geometric morphism**, for short) $f : \mathbf{G} \to \mathbf{G}'$ is a pair $f = \langle f_0, f_1\rangle$, such that $f_0 : P \to P'$ and $f_1 : L \to L'$, preserving the incidence relations, i.e., satisfying:

$$p \; I \; l \quad \text{implies} \quad f_0(p) \; I' \; f_1(l).$$

A geometric morphism $f : \mathbf{G} \to \mathbf{G}'$ is said to be **strict** if the implication above is an equivalence.

It is shown in Theorem IV.1.2 of [3] that, given an abstract geometry $\mathbf{G} = \langle P, L, I\rangle$, every line $l \in L$ of $\mathbf{G}$ is incident to the same number[2] n of

[2] possibly an infinite cardinal

points and, also, in Corollary IV.1.1 of [3], that every point $p \in P$ of $\mathbf{G}$ is incident to the same number $n+1$ of lines, i.e., one more than the number of points incident with any line of $\mathbf{G}$. Moreover, the parallel class of any one line $l \in L$ of $\mathbf{G}$ (i.e., the set containing l together with all lines in L parallel to l) has exactly n members, as many as the number of points on any one line of the abstract geometry.

3. Coordinatization of an Abstract Geometry

Given an abstract geometry $\mathbf{G} = \langle P, L, I \rangle$, there is a well-known process that can be used to coordinatize its points. We briefly recall this process here, but the reader is referred to Chapter IV of [3] for many more details.

By Postulate 3 of an abstract geometry, there exists at least one quadruple of distinct points, no three of which are incident with the same line. We select such a quadruple of points (O, I, X, Y). The point O is referred to as the **origin**, I as the **unit point**, the unique line incident to O and X is called the x-**line**, the unique line incident to O and Y is called the y-**line** and the unique line incident to O and I is called the **unit line**. Next, an arbitrary set of elements R with the same cardinality n as that of the set of points on the unit line is picked and a one-to-one correspondence ρ between the points on the unit line and the elements of R is established. Labels are assigned to the points of R in such a way that two labels 0 and 1 are reserved for the images of the origin O and the unit point I, respectively, i.e., $\rho(O) = 0$ and $\rho(I) = 1$. Once this selection has been fixed, a point $p \in P$ can be assigned coordinates $c(p) = (a, b) \in R^2$ in the following way:

- If p is incident to the unit line, then $c(p) = (\rho(p), \rho(p))$. Thus, for instance, $c(O) = (0, 0)$ and $c(I) = (1, 1)$.
- If p is not incident to the unit line, then, if the unique line incident to p and parallel to the y-line intersects the unit line at u, with $c(u) = (a, a)$, and the unique line incident to p and parallel to the x-line intersects the unit line at v, with $c(v) = (b, b)$, we set $c(p) = (a, b)$.

Furthermore, given a pair $(a, b) \in R^2$, it can be shown that there exists a unique point $p \in P$ of $\mathbf{G}$, such that $c(p) = (a, b)$. We denote that point by $p(a, b)$. Finally, given $m, b \in R$, it can be shown that there exists a unique line $l \in L$, with slope m and y-intercept b in a way very similar to the ordinary process used in the Euclidean plane. That line will be denoted by $l(m, b)$. And, conversely, given a line l, not belonging to the same parallel class as the y-line, there exist unique $m, b \in R$, such that l has slope m and y-intercept b. We write $d(l) = (m, b)$.

Given an abstract geometry $\mathbf{G} = \langle P, L, I \rangle$, we write $\mathcal{S} = \langle (O, I, X, Y), R, \rho \rangle$ to denote the **coordinate system** of $\mathbf{G}$, including the coordinate set R and the bijection ρ from the points incident to the unit line to R, described in this section. The pair $\langle \mathbf{G}, \mathcal{S} \rangle$ is called a **coordinated abstract geometry**.

Let $\langle \mathbf{G}, \mathcal{S} \rangle$ and $\langle \mathbf{G}', \mathcal{S}' \rangle$ be two coordinated abstract geometries. A **morphism of coordinated abstract geometries** (or, more simply a **coordinated geometric morphism**) $\bar{f} : \langle \mathbf{G}, \mathcal{S} \rangle \to \langle \mathbf{G}', \mathcal{S}' \rangle$ is a pair $\bar{f} = \langle f, f^* \rangle$, where $f : \mathbf{G} \to \mathbf{G}'$ is a geometric morphism and $f^* : R \to R'$ is a mapping between the corresponding coordinate sets, such that

1. f_0 maps O, I, X, Y to O', I', X', Y', respectively;
2. $f^*(\rho(p)) = \rho'(f_0(p))$, for all p incident to the unit line of $\langle \mathbf{G}, \mathcal{S} \rangle$.

The morphism $\bar{f}$ is called **strict** if $f : \mathbf{G} \to \mathbf{G}'$ is strict.

Proposition 1. *Let $\langle \mathbf{G}, \mathcal{S} \rangle$ and $\langle \mathbf{G}', \mathcal{S}' \rangle$ be two coordinated abstract geometries and $\bar{f} : \langle \mathbf{G}, \mathcal{S} \rangle \to \langle \mathbf{G}', \mathcal{S}' \rangle$ be a strict coordinated geometric morphism. Then*

1. *$c'(f_0(p)) = (f^*)^2(c(p))$, for all $p \in P$, where $(f^*)^2$ denotes coordinate-wise application of f^* on an order pair;*
2. *$d'(f_1(l)) = (f^*)^2(d(l))$, for all $l \in L$, not in the parallel class of the y-line.*

Proof. Suppose that $c(p) = (a, b)$. Thus, the unique line through p that is parallel to OY intersects OI at $p(a, a)$ and the unique line through p that is parallel to OX intersects OI at $p(b, b)$. By the definition and the strictness of $\bar{f}$, we get that $p'(f^*(a), f^*(a))$ is the point of intersection of the line through $f_0(p)$ that is parallel to the line $O'Y'$ and, similarly, that $p'(f^*(b), f^*(b))$ is the point of intersection of the line through $f_0(p)$ that is parallel to $O'X'$. Therefore, we obtain that $c'(f_0(p)) = (f^*(a), f^*(b))$, as was to be shown. The second part may be proven similarly. $\square$

Proposition 1 shows that the entire strict coordinated geometric morphism $\bar{f}$ can be reconstructed from knowledge of f^* and the pairs $\langle \mathbf{G}, \mathcal{S} \rangle$ and $\langle \mathbf{G}', \mathcal{S}' \rangle$ alone. Thus, in the sequel we will identify a strict coordinated geometric morphism $\bar{f} = \langle f, f^* \rangle$ with its second component f^* and express this by saying that $f : \langle \mathbf{G}, \mathcal{S} \rangle \to \langle \mathbf{G}', \mathcal{S}' \rangle$ is a strict coordinated geometric morphism, where $f : R \to R'$ (i.e., we identify $\bar{f}$ with $f^* : R \to R'$, but, then drop the * from the notation). Moreover, since in what follows only strict coordinated geometric morphisms will enter our discussion, we will drop the qualifiers "strict coordinated" and use *geometric morphism* to refer to strict coordinated geometric morphisms.

Let **AG** denote the category with objects all coordinated abstract geometries and morphisms all (strict coordinated) geometric morphisms between them.

For the formulation of the institution of abstract geometry, which will constitute the cornerstone in the algebraization process that will be presented in the last section of the paper, we will need the following proposition (for the proof see Sections IV.5 and IV.6 of [3]):

Proposition 2. *Let $\langle \mathbf{G}, \mathcal{S} \rangle$ be a coordinated abstract geometry. Then, for all $x, m, b \in R$, there exists a unique $y \in R$, such that $p(x, y) \ I \ l(m, b)$.*

Following Section IV.5 of [3], we introduce the notation $I(x, m, b)$ to denote the unique y, such that $p(x, y) \ I \ l(m, b)$, whose existence is postulated in the conclusion of Proposition 2. Note that for this notation to be meaningful, one must have available not only the abstract geometry $\mathbf{G} = \langle P, L, I \rangle$, but also a chosen fixed coordinate system $\mathcal{S} = \langle (O, I, X, Y), R, \rho \rangle$ on $\mathbf{G}$, even though this is not made explicit in the notation.

4. The Institution $\mathcal{AG}$ of Abstract Geometry

In this section, we construct an institution that formalizes the logical system of abstract (affine plane) geometry. In this institution, the sentences are constructed using the incidence of the abstract geometry and the models are coordinated affine planes, accompanied by intended interpretations of the signature variables in the coordinate spaces of the geometries. Satisfaction of an incidence formula by a model is defined based on whether the intended incidence relation denoted by the formula holds in the corresponding model under the stipulated interpretation of its variables in the coordinate space, which are, subsequently, translated to points and lines of the geometric model.

Let X be a set. The set of **terms** $\mathrm{Tm}(X)$ over X is recursively defined as the smallest set, such that

- $X \subseteq \mathrm{Tm}(X)$, and
- $T(t_0, s_0, s_1) \in \mathrm{Tm}(X)$, for all $t_0, s_0, s_1 \in \mathrm{Tm}(X)$.

The set of **formulas** $\mathrm{Fm}(X)$ over X, on the other hand, is the set

$$\mathrm{Fm}(X) = \{(t_0, t_1)\ I\ (s_0, s_1) : t_0, t_1, s_0, s_1 \in \mathrm{Tm}(X)\}.$$

A function $f : X \to \mathrm{Tm}(Y)$ can be uniquely extended to $f^* : \mathrm{Tm}(X) \to \mathrm{Tm}(Y)$ by setting

- $f^*(x) = f(x)$, for all $x \in X$, and
- $f^*(T(t_0, s_0, s_1)) = T(f^*(t_0), f^*(s_0), f^*(s_1))$.

Moreover, given such an $f : X \to \mathrm{Tm}(Y)$, we define $f^* : \mathrm{Fm}(X) \to \mathrm{Fm}(Y)$ by

$$f^*((t_0, t_1)\ I\ (s_0, s_1)) = (f^*(t_0), f^*(t_1))\ I\ (f^*(s_0), f^*(s_1)),$$

for all $t_0, t_1, s_0, s_1 \in \mathrm{Tm}(X)$.

Define $\mathbf{Sign}$, the signature category of the institution under construction, as the category with objects all small sets and mappings $f \in \mathbf{Sign}(X, Y)$ all set mappings $f : X \to \mathrm{Tm}(Y)$. Composition in this category is defined by setting, given $f \in \mathbf{Sign}(X, Y)$ and $g \in \mathbf{Sign}(Y, Z)$,

$$g \circ f = g^* f : X \to \mathrm{Tm}(Z).$$

Let $\mathrm{SEN} : \mathbf{Set} \to \mathbf{Set}$ be the functor that maps a set X to the set of formulas $\mathrm{Fm}(X)$ and maps a function $f : X \to \mathrm{Tm}(Y)$ to its extension $\mathrm{SEN}(f) = f^* : \mathrm{Fm}(X) \to \mathrm{Fm}(Y)$.

Construct, next, the functor $\mathrm{MOD} : \mathbf{Sign} \to \mathbf{Cat}^{\mathrm{op}}$ as follows: Given $X \in |\mathbf{Sign}|$, the category $\mathrm{MOD}(X)$ has objects all pairs of the form $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle$, where $\langle \mathbf{G}, \mathcal{S}\rangle$ is a coordinated abstract geometry, i.e., $\mathbf{G} = \langle P, L, I\rangle$ is an abstract geometry and $\mathcal{S} = \langle (O, I, X, Y), R, \rho\rangle$ is a coordinate system on $\mathbf{G}$, and $\sigma : X \to R$ is a mapping from X to the underlying coordinate set R of $\mathcal{S}$. Notice that such a mapping $\sigma : X \to R$ extends in a unique way to a mapping $\sigma^* : \mathrm{Tm}(X) \to R$ as follows:

- $\sigma^*(x) = \sigma(x)$, for all $x \in X$, and
- $\sigma^*(T(t_0, s_0, s_1)) = I(\sigma^*(t_0), \sigma^*(s_0), \sigma^*(s_1))$, for all $t_0, s_0, s_1 \in \mathrm{Tm}(X)$.

The morphisms $h : \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \to \langle\langle \mathbf{G}', \mathcal{S}'\rangle, \sigma'\rangle$ of $\mathrm{MOD}(X)$ are (strict coordinated) geometric morphisms $h : \langle \mathbf{G}, \mathcal{S}\rangle \to \langle \mathbf{G}', \mathcal{S}'\rangle$, that make the following diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ R \arrow[rr, "h"'] & & R' \end{tikzcd}$$

Given a mapping $f : X \to Y$ in **Sign** (i.e., $f : X \to \mathrm{Tm}(Y)$ in **Set**), the functor $\mathrm{MOD}(f) : \mathrm{MOD}(Y) \to \mathrm{MOD}(X)$ is defined, for all $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(Y)|$, by

$$\mathrm{MOD}(f)(\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle) = \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma^* f\rangle,$$

and $\mathrm{MOD}(f)(h) = h : \langle(\mathbf{G}, \mathcal{S}), \sigma^* f\rangle \to \langle(\mathbf{G}', \mathcal{S}'), \sigma'^* f\rangle$, for all $h : \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \to \langle\langle \mathbf{G}', \mathcal{S}'\rangle, \sigma'\rangle$ in $\mathrm{MOD}(Y)$. It is clear that this definition is sound, since the commutativity of the triangle displayed above implies the commutativity of

$$\begin{tikzcd} & X \arrow[dl, "\sigma^* f"'] \arrow[dr, "\sigma'^* f"] & \\ R \arrow[rr, "h"'] & & R' \end{tikzcd}$$

Finally, define, for all $X \in |\mathbf{Set}|$, the relation $\models_X \subseteq |\mathrm{MOD}(X)| \times \mathrm{SEN}(X)$, by setting, for all $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, with $\mathbf{G} = \langle P, L, I\rangle$, and all $(t_0, t_1)\ I\ (s_0, s_1) \in \mathrm{SEN}(X)$,

$$\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) \quad \text{iff} \quad \sigma^*(t_1) = \sigma^*(T(t_0, s_0, s_1)).$$

Let $\mathcal{AG} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$. $\mathcal{AG}$ is called the **institution of abstract geometry**. This terminology is justified by the following

Theorem 3. *The quadruple* $\mathcal{AG} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ *is an institution.*

Proof. Suppose $f : X \to Y$ be in **Sign**, $(t_0, t_1)\ I\ (s_0, s_1) \in \mathrm{SEN}(X)$ and $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(Y)|$, with $\mathbf{G} = \langle P, L, I\rangle$. Then

$$\begin{array}{ll} \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_Y \mathrm{SEN}(f)((t_0, t_1)\ I\ (s_0, s_1)) & \\ \quad \text{iff} & \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_Y (f^*(t_0), f^*(t_1))\ I\ (f^*(s_0), f^*(s_1)) \\ \quad \text{iff} & \sigma^*(f^*(t_1)) = \sigma^*(T(f^*(t_0), f^*(s_0), f^*(s_1))) \\ \quad \text{iff} & \sigma^*(f^*(t_1)) = I(\sigma^*(f^*(t_0)), \sigma^*(f^*(s_0)), \sigma^*(f^*(s_1))) \\ \quad \text{iff} & (\sigma^* f)^*(t_1) = I((\sigma^* f)^*(t_0), (\sigma^* f)^*(s_0), (\sigma^* f)^*(s_1)) \\ \quad \text{iff} & \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma^* f\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) \\ \quad \text{iff} & \mathrm{MOD}(f)(\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle) \models_X (t_0, t_1)\ I\ (s_0, s_1). \end{array}$$

Thus, the satisfaction condition holds and $\mathcal{AG}$ is an institution. $\square$

5. The Ternary Ring $\mathbf{R} = \langle R, T, 0, 1\rangle$

In this section, we review the abstract properties of the ternary ring **R** that is formed by the coordinatization of an abstract geometry **G**, as presented in Section IV.6 of [3]. Moreover, we describe the reverse process of coordinatization, presented in Section IV.7 of [3], by which an abstract geometry **G** is associated with a ternary ring. The abstract properties of the coordinatizing ternary rings as well as this construction of an abstract geometry using the elements and the algebraic properties of a ternary ring will prove useful when we study the process of algebraization of abstract geometry in the final sections of this paper.

Let $\mathbf{G} = \langle P, L, I\rangle$ be an abstract geometry and let $\mathcal{S} = \langle (O, I, X, Y), R, \rho\rangle$ be a coordinate system for **G**. Let $T : R^3 \to R$ be the ternary operation on R defined by setting $T(x, m, b) = I(x, m, b)$, for all $x, m, b \in R$. Then, as is shown in Section IV.6 of [3], the operation T on R satisfies the following properties:

1. $T(0, m, b) = T(x, 0, b) = b$, for all $x, b, m \in R$.
2. $T(x, 1, 0) = T(1, x, 0) = x$, for all $x \in R$.
3. The equation $T(x, m, b) = T(x, m', b')$ has a unique solution in R, for all $m, m', b, b' \in R$, with $m \neq m'$.

4. The system of equations

$$\left\{ \begin{array}{lcl} T(a,x,y) & = & b \\ T(a',x,y) & = & b' \end{array} \right\}$$

has a unique solution in R, for all $a.a', b, b' \in R$, with $a \neq a'$.

5. The equation $T(a,m,x) = c$ has a unique solution in R, for all $a, m, c \in R$.

The ternary ring associated with $\mathbf{G}$ and the coordinatization $\mathcal{S}$ of $\mathbf{G}$ will be denoted by $\mathsf{R}(\mathbf{G}, \mathcal{S})$ or $\mathsf{R}_{\mathcal{S}}(\mathbf{G})$. More generally, an algebraic structure $\mathbf{R} = \langle R, T, 0, 1\rangle$, where T is a ternary operation satisfying properties 1-5 above, will be called a **Hall ternary ring**.

Suppose, next, that a Hall ternary ring $\mathbf{R}$ is given. Then an abstract geometry $\mathbf{G} = \langle P, L, I\rangle$ may be constructed as follows:

- $P = \{(x,y) : x, y \in R\}$;
- $L = \{\{(a,y) : y \in R\} : a \in R\} \cup \{\{(x, T(x,m,b)) : x \in R\} : m, b \in R\}$;
- The incidence relation is simply the membership relation, i.e., for all $(x,y) \in P$ and all $l \in L$, we have $(x,y) \; I \; l$ iff $(x,y) \in l$.

The following may now be established:

Theorem 4. *Given a Hall ternary ring, the structure* $\mathbf{G}$ *is an abstract geometry. Moreover, the Hall ternary ring associated with* $\mathbf{G}$ *under the coordinatization* $\mathcal{S} = \langle((0,0),(1,1),(1,0),(0,1)), R, i_R\rangle$ *coincides with the Hall ternary ring* $\mathbf{R}$*.*

The coordinated abstract geometry that is obtained in this fashion out of the given Hall ternary ring $\mathbf{R}$ will be denoted by $\mathsf{G}(\mathbf{R})$. According to Theorem 4, we have that $\mathsf{R}(\mathsf{G}(\mathbf{R})) = \mathbf{R}$.

6. The Institution $\mathcal{GA}$ of Geometric Algebra

In this section, the algebraic institution $\mathcal{GA}$, that will serve as the algebraic semantics of the institution $\mathcal{AG}$ of abstract geometry, will be constructed. Its models will be essentially Hall ternary rings and its sentences will be equations of terms over the language of Hall ternary rings. The fact that Hall ternary rings serve as coordinate rings for affine plane geometries is used in subsequent sections to construct mutually inverse interpretations between these two institutions.

Let **Sign** be the category of signatures of the institution $\mathcal{AG}$, defined in Section 4.

Define the functor EQ : **Sign** $\to$ **Set** as follows: Given a set X,

$$\mathrm{EQ}(X) = \{t_0 \approx t_1 : t_0, t_1 \in \mathrm{Tm}(X)\}$$

and, given $f : X \to \mathrm{Tm}(Y)$ in **Sign**, define

$$\mathrm{EQ}(f)(t_0 \approx t_1) = f^*(t_0) \approx f^*(t_1), \text{ for all } t_0, t_1 \in \mathrm{Tm}(X),$$

where $f^* : \mathrm{Tm}(X) \to \mathrm{Tm}(Y)$ is the unique extension of f on terms that was also defined in Section 4.

Furthermore, define the functor ALG : **Sign** $\to$ $\mathbf{Cat}^{\mathrm{op}}$ as follows: Given a set X, the category $\mathrm{ALG}(X)$ has as its objects all pairs $\langle \mathbf{R}, \sigma\rangle$, where $\mathbf{R}$ is a Hall ternary ring and $\sigma : X \to R$ an assignment of elements from the universe R of $\mathbf{R}$ to the variables in X. A morphism $h : \langle \mathbf{R}, \sigma\rangle \to \langle \mathbf{R}', \sigma'\rangle$ in $\mathrm{ALG}(X)$ is a Hall ternary ring homomorphism $h : \mathbf{R} \to \mathbf{R}'$, that makes the following diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ R \arrow[rr, "h"'] & & R' \end{tikzcd}$$

Moreover, given an $f : X \to \mathrm{Tm}(Y)$ in **Sign**, the corresponding functor $\mathrm{ALG}(f) : \mathrm{ALG}(Y) \to \mathrm{ALG}(X)$ sends an object $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(Y)|$ to the object $\mathrm{ALG}(f)(\langle \mathbf{R}, \sigma\rangle) = \langle \mathbf{R}, \sigma^* f\rangle$ and a morphism $h : \langle \mathbf{R}, \sigma\rangle \to \langle \mathbf{R}', \sigma'\rangle$ in $\mathrm{ALG}(Y)$ to $\mathrm{ALG}(f)(h) = h : \langle \mathbf{R}, \sigma^* f\rangle \to \langle \mathbf{R}', \sigma'^* f\rangle$ in $\mathrm{ALG}(X)$.

Finally, for every set X, define the **satisfaction relation**

$$\models_X \subseteq |\mathrm{ALG}(X)| \times \mathrm{EQ}(X)$$

by setting, for all $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$ and all $t_0, t_1 \in \mathrm{Tm}(X)$,

$$\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1 \quad \text{iff} \quad \sigma^*(t_0) = \sigma^*(t_1).$$

The quadruple $\mathcal{GA} = \langle \mathbf{Sign}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$ is called the **institution of geometric algebra**, which is justified by the following:

Theorem 5. *The quadruple* $\mathcal{GA} = \langle \mathbf{Sign}, \mathrm{EQ}, \mathrm{ALG}, \models \rangle$ *is an institution.*

Proof. We check the satisfaction condition. Suppose that $f : X \to \mathrm{Tm}(Y)$ is a morphism in **Sign**, $\langle \mathbf{R}, \sigma \rangle \in |\mathrm{ALG}(Y)|$ and $t_0 \approx t_1 \in \mathrm{EQ}(X)$. Then, we have

$$\begin{array}{rll} \langle \mathbf{R}, \sigma \rangle \models_Y \mathrm{EQ}(f)(t_0 \approx t_1) & \text{iff} & \langle \mathbf{R}, \sigma \rangle \models_Y f^*(t_0) \approx f^*(t_1) \\ & \text{iff} & \sigma^*(f^*(t_0)) = \sigma^*(f^*(t_1)) \\ & \text{iff} & (\sigma^* f)^*(t_0) = (\sigma^* f)^*(t_1) \\ & \text{iff} & \langle \mathbf{R}, \sigma^* f \rangle \models_X t_0 \approx t_1 \\ & \text{iff} & \mathrm{ALG}(f)(\langle \mathbf{R}, \sigma \rangle) \models_X t_0 \approx t_1. \end{array}$$

□

7. $\mathcal{GA}$ is an Algebraic Semantics of $\mathcal{AG}$

In this section, it is shown that the institution $\mathcal{GA}$ of the equational logic of the Hall ternary rings, introduced in Section 6, constitutes an algebraic semantics of the institution $\mathcal{AG}$ of abstract geometry, introduced in Section 4. According to the theory of categorical abstract algebraic logic, an algebraic institution $\mathcal{A} = \langle \mathbf{Sign}', \mathrm{SEN}', \mathrm{MOD}', \models^{\mathcal{A}} \rangle$ is an algebraic institution semantics of an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models^{\mathcal{I}} \rangle$ if the corresponding π-institution $\pi(\mathcal{I}) = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathcal{I}} \rangle$ is interpretable in $\pi(\mathcal{A}) = \langle \mathbf{Sign}', \mathrm{SEN}', C^{\mathcal{A}} \rangle$. Note that $C^{\mathcal{I}}$ is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$ by $\phi \in C^{\mathcal{I}}_{\Sigma}(\Phi)$ iff, for every model $M \in |\mathrm{MOD}(\Sigma)|$,

$$M \models^{\mathcal{I}}_{\Sigma} \Phi \quad \text{implies} \quad M \models^{\mathcal{I}}_{\Sigma} \phi.$$

A similar definition applies for $C^{\mathcal{A}}$, i.e., both $\pi(\mathcal{I})$ and $\pi(\mathcal{A})$ are the π-institutions whose closure systems are the closure systems induced by the semantical entailment systems of the corresponding institutions. The π-institution $\pi(\mathcal{I})$ is interpretable in $\pi(\mathcal{A})$ if there exists an interpretation $\langle F, \alpha \rangle : \mathcal{I} \to \mathcal{A}$, i.e., a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{SEN}' \circ F$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C^{\mathcal{I}}_{\Sigma}(\phi) \quad \text{iff} \quad \alpha_{\Sigma}(\Phi) \subseteq C^{\mathcal{A}}_{F(\Sigma)}(\alpha_{\Sigma}(\phi)).$$

To show that $\mathcal{GA}$ is an algebraic semantics of $\mathcal{AG}$, we define the pair $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle : \mathcal{AG} \to \mathcal{GA}$ as follows: $\mathrm{I}_{\mathbf{Sign}} : \mathbf{Sign} \to \mathbf{Sign}$ is the identity functor on the common signature category of the two institutions. The natural transformation $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{EQ}$ is defined by setting, for every set X and all $t_0, t_1, s_0, s_1 \in \mathrm{Tm}(X)$,

$$\alpha_\Sigma((t_0, t_1)\ I\ (s_0, s_1)) = \{t_1 \approx T(t_0, s_0, s_1)\}.$$

Lemmas 6 and 7, that follow, are supporting lemmas for showing the main equivalence establishing the interpretation property of $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle$. This equivalence is shown in Proposition 8. The main theorem, Theorem 9, simply restates the equivalence of Proposition 8 in the language of abstract algebraic logic.

Lemma 6. *Let X be a set, $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$ and $t_0, t_1, s_0, s_1 \in \mathrm{Tm}(X)$. Then*

$$\langle \mathbf{R}, \sigma\rangle \models_X t_1 \approx T(t_0, s_0, s_1) \quad \textit{iff} \quad \langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1).$$

Proof.

$$\begin{aligned}
&\langle \mathbf{R}, \sigma\rangle \models_X t_1 \approx T(t_0, s_0, s_1)\\
&\qquad \text{iff} \quad \sigma^*(t_1) = \sigma^*(T(t_0, s_0, s_1))\\
&\qquad \text{iff} \quad \sigma^*(t_1) = I(\sigma^*(t_0), \sigma^*(s_0), \sigma^*(s_1))\\
&\qquad \text{iff} \quad (\sigma^*(t_0), \sigma^*(t_0))\ I\ (\sigma^*(s_0), \sigma^*(s_1))\\
&\qquad \text{iff} \quad \langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1).
\end{aligned}$$

□

Lemma 7. *Let X be a set, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$ and $t_0, t_1, s_0, s_1 \in \mathrm{Tm}(X)$. Then*

$$\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) \quad \textit{iff} \quad \langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_1 \approx T(t_0, s_0, s_1).$$

Proof.

$$\begin{aligned}
&\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1)\\
&\qquad \text{iff} \quad \sigma^*(t_1) = \sigma^*(T(t_0, s_0, s_1))\\
&\qquad \text{iff} \quad \langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_1 \approx T(t_0, s_0, s_1).
\end{aligned}$$

□

Proposition 8. *Let X be a set. For all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(X)$,*

$$\phi \in C_X^{\mathcal{AG}}(\Phi) \quad \textit{iff} \quad \alpha_X(\phi) \subseteq C_X^{\mathcal{GA}}(\alpha_X(\Phi)).$$

Proof. Assume, first, that $\phi \in C_X^{\mathcal{AG}}(\Phi)$. This means that, for all $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \Phi$ implies $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \phi$. Suppose, now, that $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$, and $\langle \mathbf{R}, \sigma\rangle \models_X \alpha_X(\Phi)$. Thus, $\langle \mathbf{R}, \sigma\rangle \models_X t_1 \approx T(t_0, s_0, s_1)$, for all $(t_0, t_1)\ I\ (s_0, s_1) \in \Phi$. Therefore, by Lemma 6, we have $\langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1)$, for all $(t_0, t_1)\ I\ (s_0, s_1) \in \Phi$. Hence, by hypothesis, we get that $\langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X \phi$. Again, using Lemma 6, we obtain that $\langle \mathbf{R}, \sigma\rangle \models_X \alpha_X(\phi)$. This proves that $\alpha_X(\phi) \subseteq C_X^{\mathcal{GA}}(\alpha_\Sigma(\Phi))$.

Assume, conversely, that $\alpha_X(\phi) \subseteq C_X^{\mathcal{GA}}(\alpha_X(\Phi))$. This means that, for all $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$, $\langle \mathbf{R}, \sigma\rangle \models_X \alpha_X(\Phi)$ implies $\langle \mathbf{R}, \sigma\rangle \models_X \alpha_X(\phi)$. Suppose, now, that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, such that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \Phi$. Thus, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1)$, for all $(t_0, t_1)\ I\ (s_0, s_1) \in \Phi$. Therefore, by Lemma 7, $\langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_1 \approx T(t_0, s_0, s_1)$, for all $(t_0, t_1)\ I\ (s_0, s_1) \in \Phi$. Hence, by hypothesis, we get that $\langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X \alpha_X(\phi)$. Again, using Lemma 7, we obtain that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \phi$. This proves that $\phi \subseteq C_X^{\mathcal{AG}}(\Phi)$.

□

Theorem 9. *The institution of Hall ternary rings $\mathcal{GA}$ is an algebraic semantics of the institution $\mathcal{AG}$ of abstract geometry.*

Proof. This is just a restatement of Proposition 8. □

8. Algebraization of $\mathcal{AG}$

In this section, it is shown that not only is $\mathcal{GA}$ an algebraic semantics of $\mathcal{AG}$, but, moreover, $\mathcal{AG}$ is an algebraizable institution with $\mathcal{GA}$ its algebraic counterpart. We do this by showing that there exists an interpretation $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle$ from $\mathcal{GA}$ into $\mathcal{AG}$, which is inverse to the interpretation $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle$. Thus, the two institutions $\mathcal{AG}$ and $\mathcal{GA}$ are deductively equivalent institutions, as is required for algebraizability.

To this end, define the natural transformation $\beta : \mathrm{EQ} \to \mathcal{P}\mathrm{SEN}$, by setting, for every set X and all terms $t_0, t_1 \in \mathrm{Tm}(X)$,

$$\beta_X(t_0 \approx t_1) = \{(t_0, t_1)\ I\ (1, 0)\}.$$

Then we have the following analogs of Lemmas 6 and 7 and of Proposition 8 establishing that $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle$ is an interpretation from $\mathcal{GA}$ to $\mathcal{AG}$:

Lemma 10. *Let X be a set, $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$ and $t_0, t_1 \mathrm{Tm}(X)$. Then*

$$\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1 \quad \textit{iff} \quad \langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (t_0, t_1)\ I\ (1,0).$$

Proof.

$$\begin{array}{rll} \langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1 & \text{iff} & \sigma^*(t_0) = \sigma^*(t_1) \\ & \text{iff} & \sigma^*(t_0) = I(\sigma^*(t_1), 1, 0) \\ & \text{iff} & \sigma^*(t_0) = \sigma^*(T(t_1, 1, 0)) \\ & \text{iff} & \langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (t_0, t_1)\ I\ (1,0). \end{array}$$

□

Lemma 11. *Let X be a set, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$ and $t_0, t_1 \in \mathrm{Tm}(X)$. Then*

$$\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (1,0) \quad \textit{iff} \quad \langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_0 \approx t_1.$$

Proof.

$$\begin{array}{rll} \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (1,0) & \text{iff} & \sigma^*(t_0) = \sigma^*(T(t_1, 1, 0)) \\ & \text{iff} & \sigma^*(t_0) = I(\sigma^*(t_1), 1, 0) \\ & \text{iff} & \sigma^*(t_0) = \sigma^*(t_1) \\ & \text{iff} & \langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_0 \approx t_1. \end{array}$$

□

Proposition 12. *Let X be a set. For all $E \cup \{t_0 \approx t_1\} \subseteq \mathrm{EQ}(X)$,*

$$t_0 \approx t_1 \in C_X^{\mathcal{GA}}(E) \quad \textit{iff} \quad \beta_X(t_0 \approx t_1) \subseteq C_X^{\mathcal{AG}}(\beta_X(E)).$$

Proof. Assume, first, that $t_0 \approx t_1 \in C_X^{\mathcal{GA}}(E)$. This means that, for all $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$, $\langle \mathbf{R}, \sigma\rangle \models_X E$ implies $\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1$. Suppose, now, that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, and $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \beta_X(E)$. Thus, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (\epsilon_0, \epsilon_1)\ I\ (1,0)$, for all $\epsilon_0 \approx \epsilon_1 \in E$. Therefore, by Lemma 11, $\langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X \epsilon_0 \approx \epsilon_1$, for all $\epsilon_0 \approx \epsilon_1 \in E$. Hence, by hypothesis, we have $\langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_0 \approx t_1$. Again, using Lemma 11, we obtain that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \beta_X(t_0 \approx t_1)$. This proves that $\beta_X(t_0 \approx t_1) \subseteq C_X^{\mathcal{AG}}(\beta_\Sigma(E))$.

Assume, conversely, that $\beta_X(t_0 \approx T_1) \subseteq C_X^{\mathcal{AG}}(\beta_X(E))$. This means that, for all $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \beta_X(E)$ implies $\langle\langle \mathbf{G}, \mathcal{S}\rangle,$

$\sigma\rangle \models_X \beta_X(t_0 \approx t_1)$. Suppose, now, that $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$, such that $\langle \mathbf{R}, \sigma\rangle \models_X E$. Thus, $\langle \mathbf{R}, \sigma\rangle \models_X \epsilon_0 \approx \epsilon_1$, for all $\epsilon_0, \approx \epsilon_1 \in E$. Therefore, by Lemma 10, $\langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (\epsilon_0, \epsilon_1)\ I\ (1,0)$, for all $\epsilon_0 \approx \epsilon_1 \in E$. Hence, by hypothesis, we get that $\langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X \beta_X(t_0 \approx t_1)$. Again, using Lemma 10, we obtain that $\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1$. This proves that $t_0 \approx t_1 \in C_X^{\mathcal{GA}}(E)$. $\square$

Theorem 13. *The pair $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle$ forms an interpretation from the institution $\mathcal{GA}$ of Hall ternary rings to the institution $\mathcal{AG}$ of abstract geometry.*

Proof. This is simply a restatement of Proposition 12. $\square$

To complete our demonstration that $\mathcal{AG}$ is algebraizable and the institution of Hall ternary rings $\mathcal{GA}$ is its equivalent algebraic semantics, it suffices now to show that the two interpretations $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle : \mathcal{AG} \to \mathcal{GA}$ and $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle : \mathcal{GA} \to \mathcal{AG}$ are inverse of one another in the precise technical sense of [15] (see also [2]). In other words, it must be shown that composing α and β results in interderivable sets of geometric formulas in $\mathcal{AG}$ and composing β and α results in interderivable sets of equations in $\mathcal{GA}$. The following two lemmas pave the way for the final results:

Lemma 14. *Let X be a set, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$ and*

$$(t_0, t_1)\ I\ (s_0, s_1) \in \mathrm{SEN}(X).$$

Then

$$\begin{aligned}
&\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) \\
&\qquad \textit{iff} \quad \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_1, T(t_0, s_0, s_1))\ I\ (1, 0).
\end{aligned}$$

Proof.

$$\begin{aligned}
&\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) \\
&\qquad \text{iff} \quad \sigma^*(t_1) = \sigma^*(T(t_0, s_0, s_1)) \\
&\qquad \text{iff} \quad \sigma^*(T(t_0, s_0, s_1)) = I(\sigma^*(t_1), 1, 0) \\
&\qquad \text{iff} \quad \sigma^*(T(t_0, s_0, s_1)) = \sigma^*(T(t_1, 1, 0)) \\
&\qquad \text{iff} \quad \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_1, T(t_0, s_0, s_1))\ I\ (1, 0).
\end{aligned}$$

$\square$

Lemma 15. *Let X be a set, $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$ and $t_0, t_1 \in \mathrm{Tm}(X)$. Then*

$$\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1 \quad \textit{iff} \quad \langle \mathbf{R}, \sigma\rangle \models_X t_1 \approx T(t_0, 1, 0).$$

Proof.

$$\begin{array}{rll}\langle \mathbf{R}, \sigma \rangle \models_X t_0 \approx t_1 & \text{iff} & \sigma^*(t_0) = \sigma^*(t_1) \\ & \text{iff} & \sigma^*(t_1) = I(\sigma^*(t_0), 1, 0) \\ & \text{iff} & \sigma^*(t_1) = \sigma^*(T(t_0, 1, 0)) \\ & \text{iff} & \langle \mathbf{R}, \sigma \rangle \models_X t_1 \approx T(t_0, 1, 0).\end{array}$$

□

Theorem 16. *The institution $\mathcal{AG}$ of abstract geometry and the institution $\mathcal{GA}$ of Hall ternary rings are deductively equivalent institutions. Thus, $\mathcal{AG}$ is algebraizable and the institution $\mathcal{GA}$ is its equivalent algebraic semantics.*

Proof. We have already proven in Theorems 9 and 13 that $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle : \mathcal{AG} \to \mathcal{GA}$ and $\langle \mathrm{I}_{\mathbf{Sign}}, \beta \rangle : \mathcal{GA} \to \mathcal{AG}$ are interpretations. Thus, it suffices to show that they are inverse to one another. Suppose, that X is a set and $(t_0, t_1)\ I\ (s_0, s_1) \in \mathrm{SEN}(X)$. Then, for all $\langle \langle \mathbf{G}, \mathcal{S} \rangle, \sigma \rangle \in |\mathrm{MOD}(X)|$, we have, by Lemma 14, that $\langle \langle \mathbf{G}, \mathcal{S} \rangle, \sigma \rangle \models_X (t_0, t_1)\ I\ (s_0, s_1)$ iff $\langle \langle \mathbf{G}, \mathcal{S} \rangle, \sigma \rangle \models_X \beta_X(\alpha_X((t_0, t_1)\ I\ (s_0, s_1)))$. Therefore, we get that $C_X^{\mathcal{AG}}((t_0, t_1)\ I\ (s_0, s_1)) = C_X^{\mathcal{AG}}(\beta_X(\alpha_X((t_0, t_1)\ I\ (s_0, s_1))))$. Similarly, if X is a set and $t_0 \approx t_1 \in \mathrm{EQ}(X)$, then, for all $\langle \mathbf{R}, \sigma \rangle \in |\mathrm{ALG}(X)|$, we have, by Lemma 15, that $\langle \mathbf{R}, \sigma \rangle \models_X t_0 \approx t_1$ iff $\langle \mathbf{R}, \sigma \rangle \models_X \alpha_X(\beta_X(t_0 \approx t_1))$. Therefore, we get that $C_X^{\mathcal{GA}}(t_0 \approx t_1) = C_X^{\mathcal{GA}}(\alpha_X(\beta_X(t_0 \approx t_1)))$. This concludes the proof that $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle$ and $\langle \mathrm{I}_{\mathbf{Sign}}, \beta \rangle$ are indeed inverse to one another in the precise technical sense of abstract algebraic logic □

Theorem 16, which is the main theorem of the paper, shows that the institution of abstract (affine plane) geometry is deductively equivalent to the institution of geometric algebra. Thus, the class of all Hall ternary rings forms an equivalent algebraic semantics of affine plane geometry in the precise technical sense of abstract algebraic logic. It is in this sense that one may say that the coordinate rings of modern abstract geometry form an equivalent algebraic semantics of the logic of abstract geometry. Therefore, the process of coordinatization in geometry may be viewed as a special case of the formal process of algebraization of logical systems.

References

[1] M. Barr and C. Wells, Category Theory for Computing Science, Third Edition, Les Publications CRM, Montréal, 1999.

[2] W.J. Blok and D. Pigozzi, Algebraizable Logics, Memoirs of the American Mathematical Society, Vol. 77, No. 396, 1989.

[3] L.M. Blumenthal, A Modern View of Geometry, Dover Publications, Inc., New York, 1980.

[4] F. Borceux, Handbook of Categorical Algebra, Vol. I, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1994.

[5] F. Buekenhout (ed.), Handbook of Incidence Geometry, Elsevier, Amsterdam, 1995.

[6] P.J. Cameron, Projective and Polar Spaces, Second Online Edition, `http://www.maths.qmul.ac.uk/ pjc/pps/`, September 2000.

[7] J. Czelakowski, Protoalgebraic Logics, Trends in Logic-Studia Logica Library 10, Kluwer, Dordrecht, 2001.

[8] J. Fiadeiro and A. Sernadas, *Structuring Theories on Consequence*, in Recent Trends in Data Type Specification, Donald Sannella and Andrzej Tarlecki, Eds., Lecture Notes in Computer Science, **332** (1988), pp. 44–72.

[9] J.M. Font and R. Jansana, A General Algebraic Semantics for Sentential Logics, Lecture Notes in Logic, **332**:7 (1996), Springer-Verlag, Berlin Heidelberg, 1996.

[10] J.M. Font, R. Jansana, and D. Pigozzi, *A Survey of Abstract Algebraic Logic*, Studia Logica **74**:1/2 (2003), pp. 13–97.

[11] J.A. Goguen and R.M. Burstall, *Introducing Institutions*, in Proceedings of the Logic of Programming Workshop, E. Clarke and D. Kozen, Eds., Lecture Notes in Computer Science **164** (1984), pp. 221–256.

[12] J.A. Goguen and R.M. Burstall, *Institutions: Abstract Model Theory for Specification and Programming*, Journal of the Association for Computing Machinery **39** (1992), pp. 95–146.

[13] S. Mac Lane, Categories for the Working Mathematician, Springer-Verlag, New York, 1971.

[14] W. Szmielew, From Affine to Euclidean Geometry An Axiomatic Approach, Kluwer, Dordtrecht, 1983.

[15] G. Voutsadakis, *Categorical Abstract Algebraic Logic: Equivalent Institutions*, Studia Logica **74** (2003), pp. 275–311.

[16] G. Voutsadakis, *Categorical Abstract Algebraic Logic: Algebraizable Institutions*, Applied Categorical Structures **10** (2002), pp. 531–568.

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